Invent. math. 87, 477–483 (1987)

Inventiones mathematicae

On the Hausdorff dimension of harmonic measure in higher dimension

J. Bourgain

Institut des Hautes Etudes Scientifiques, 35, Route de Chartres
F-91440 Bures-sur-Yvette, France

Summary. For a given dimension $d \geqq 2$, there exists $\rho < d$ such that if ω is the harmonic measure of a domain in $\mathbb{R}^d$, then there is a set S satisfying $\omega(S)=1$ and $h_\rho(S)=0$. This improves the result of B. Øksendal, according to which ω is always singular with respect to d-dimensional Lebesgue measure (see [O]).

1. Introduction

Let ω be a compactly supported positive measure. We say that the support $S(\omega)$ of ω has dimension at most α if for every $\beta > \alpha$, and every $\varepsilon > 0$, we can find balls D_ν of radii r_ν so that

$$\sum r_\nu^\beta < \varepsilon \quad \text{and} \quad \omega(\mathbb{R}^d \setminus \bigcup D_\nu) < \varepsilon.$$

Assume $\Lambda = \mathbb{R}^d \setminus E$ a domain in $\mathbb{R}^d$, where E is a compact set. Denote $\omega(\Lambda, A, x)$ the harmonic measure for Λ of A, evaluated at $x \in \mathbb{R}^d$. According to Øksendal's theorem [O], $\omega_E = \omega(\Lambda, ., x)$ is singular with respect to d-dimensional Lebesgue measure. For $d > 2$ and general domains, this result seemed to be so far the only known localization property. Recently for $d=2$, it has been shown by P. Jones and T. Wolff [J-W] that $S(\omega_E)$ has dimension at most 1. This result completes previous work due to N.G. Makarov and L. Carleson (see [M] and [C2]). Let g be the Green's function of $\Lambda \subset \mathbb{C}^*$ with pole at some point of $\mathbb{C}^* = \mathbb{C} \cup \{\omega\}$. In both the Carleson and Jones-Wolff arguments, the integral

$$\int_{\text{boundary}} \frac{\partial g}{\partial n} \log \frac{\partial g}{\partial n}\, ds$$

plays an essential role. The evaluations involved for the latter integral rely on specific 2-dimensional phenomena which do not seem to be conclusive in higher dimension. Our purpose is to prove the following fact:

Theorem. *If $A=\mathbb{R}^d\setminus E$ is a domain in $\mathbb{R}^d$, then $S(\omega_E)$ has dimension at most $d-\tau(d)$, where $\tau(d)>0$ is some positive number only dependent on d.*

There seems to be some evidence that for $d>2$, one cannot take $\tau(d)=1$ as in the 2-dimensional case [W]. The proof of the theorem is elementary and easy to visualize. In particular, the Green's function of the domain will not be used. In the argument, the dimension d plays essentially no role, besides when writing explicitly potentials down. To fix ideas, let $d=3$, $E\subset[0,1]^3$ and $\omega(.)$ the harmonic measure for A evaluated at 0. Adaptation of the argument given below to other situations is straightforward.

Let us next recall some simple facts which will be exploited in the proof given below. We think about harmonic measure in terms of hitting probability of Brownian motion.

Let $A\subset S$, $A\subset\tilde{A}$. By the strong Markov property of Brownian motion (cf. [K-W])

$$\omega(\mathbb{R}^3\setminus S, A, x)=\int_{\tilde{A}}\omega(\mathbb{R}^3\setminus S, A, y)\,\omega(\mathbb{R}^3\setminus(S\cup\tilde{A}), dy, x).$$

Hence

$$\omega(\mathbb{R}^3\setminus S, A, x)\leqq\omega(\mathbb{R}^3\setminus(S\cup\tilde{A}),\tilde{A},x)\sup_{y\in\partial\tilde{A}}\omega(\mathbb{R}^3\setminus S, A, y)\leqq\omega(\mathbb{R}^3\setminus(S\cup\tilde{A}),\tilde{A},x) \tag{$*$}$$

$$\omega(\mathbb{R}^3\setminus S, A, x)\geqq\omega(\mathbb{R}^3\setminus(S\cup\tilde{A}),\tilde{A},x)\inf_{y\in\partial\tilde{A}}\omega(\mathbb{R}^3\setminus S, A, y) \tag{$**$}$$

We will make repeated use of this principle.

If I is an interval in $\mathbb{R}^3$, denote $|I|$ its Lebesgue measure. Let $h_\rho(A)=\inf\{\sum_\alpha|I_\alpha|^{\rho/3};\ I_\alpha \text{ cube}, A\subset\bigcup_\alpha I_\alpha\}$. We also need the next lemma.

Lemma 1. *Let Q be a cube in $\mathbb{R}^3$ and Q_* the cube with same center as Q and of $\frac{1}{100}$-size. Then one of the following alternatives holds*

$$\omega(Q\setminus E, Q\cap E, a)\geqq\delta \quad \textit{if } a\in Q_* \tag{1}$$

$$h_\rho(E\cap Q_*)<C(\rho)\delta h_\rho(Q) \quad \textit{for } \rho>1. \tag{2}$$

Proof. Fix $\rho>1$. By the result in Carleson's book [C1] (p. 7, Th. 1), there is a positive measure μ supported by $E\cap Q_*$ such that

$$\mu(I)\leqq h_\rho(I) \quad \textit{for any cube } I \tag{3}$$

and

$$\mu(E\cap Q_*)\geqq ch_\rho(E\cap Q_*) \tag{4}$$

Define now the harmonic function

$$u(x)=\int|x-y|^{-1}\mu(dy).$$

By (3), we have

$$u\leqq C(\rho-1)^{-1}|Q_*|^{\frac{\rho-1}{3}} \tag{5}$$

$$u(a)\geqq|Q_*|^{-1/3}\mu(E\cap Q_*) \quad \textit{for } a\in Q_* \tag{6}$$

$$u(a)\leqq\tfrac{1}{100}|Q_*|^{-1/3}\mu(E\cap Q_*) \quad \textit{if } a\in\partial Q \tag{7}$$

Define

$$\bar{u} = \frac{1}{\operatorname{Sup} u}\,(u - \sup_{a \in \partial Q} u(a)) \leqq 1$$

Since $\bar{u} \leqq 0$ on ∂Q and $\bar{u} \leqq 1$, it follows from the maximum principle for $x \in Q$

$$\bar{u}(x) \leqq \omega(Q \setminus (E \cap Q_*), E \cap Q_*, x) \leqq \omega(Q \setminus E, E \cap Q, x)$$

In particular, for $a \in Q_*$, by (5), (6), (7) above and by (4)

$$\begin{aligned}\omega(Q \setminus E, Q \cap E, a) &\geqq \bar{u}(a) \geqq c(\rho - 1)|Q_*|^{\frac{1-\rho}{3}} |Q_*|^{-1/3} \mu(E \cap Q_*) \\ &\geqq c(\rho - 1)|Q_*|^{-\rho/3} h_\rho(E \cap Q_*)\end{aligned}$$

from where the alternative follows.

2. Proof of the Theorem

Let thus $E \subset [0,1]^3$ be a compact set and $\omega(.)$ the harmonic measure for $\Lambda = \mathbb{R}^3 \setminus E$ corresponding to the point 0. Use the letter c for various constants. Let l be a fixed integer to be defined later. Partition $[0,1]^3$ incubes by successive l-adic refinements. Let $\mathscr{E}_j$ be the j^{th} generation of cubes, thus of size l^{-j}. Let $\mathscr{E} = \bigcup_{j=1}^{\infty} \mathscr{E}_j$.

It will be useful to define the following additional Hausdorff measures

$$h_\rho(A, \varepsilon) = \inf\{\sum_\alpha |I_\alpha|^{\rho/3}, I_\alpha \text{ is a cube of size } |I_\alpha|^{1/3} < \varepsilon \text{ and } A \subset \bigcup_\alpha I_\alpha\}$$

and

$$m_\rho(A, \varepsilon) = \inf\{\sum_\alpha |I_\alpha|^{\rho/3}; I_\alpha \text{ is an } \mathscr{E}\text{-cube of size } |I_\alpha|^{1/3} < \varepsilon \text{ and } A \subset \bigcup_\alpha I_\alpha\}.$$

The proof is based on the following

Lemma 2. *There is $\rho < 3$ such that for each $I \in \mathscr{E}_j$ one of the following properties hold*

(D) $m_\rho(E \cap I, l^{-j-1}) < |I|^{\rho/3}$

(L) $\sum_{J \in \mathscr{E}_{j+1}, J \subset I} \omega(J)^{1/2} |J|^{1/2} \leqq \frac{1}{10} \omega(I)^{1/2} |I|^{1/2}$

(D) = local estimate of the Hausdorff measure of E

(L) = localization of harmonic measure.

Proof. Let $Q \in \mathscr{E}_{j+1}$ be a subcube of I. Denote Q_* the cube with same center as Q and $\frac{1}{100}$-size. According to lemma 1, we thus have the following alternative

(1) $\omega(Q \setminus E, E \cap Q, a) \geqq \delta$ if $a \in Q_*$

(2) $h_\rho(E \cap Q_*) < \delta' h_\rho(Q)$.

Actually ρ will be taken < 3 but close to 3, according to certain needs that will appear in what follows

First alternative. There is a subcube $Q\in\mathscr{E}_{j+1}$ of I satisfying (2). Notice that replacing a general cube by a union of $\mathscr{E}$-cubes (bounded in number) and letting $\rho<3$ be close enough to 3 (depending on l), (2) yields also

$$m_\rho(E\cap Q_*, l^{-j-1})\leqq 2\delta'|Q|^{\rho/3}.$$

Hence, for ρ close enough to 3

$$\begin{aligned} m_\rho(E\cap I, l^{-j-1}) &\leqq m_\rho(I\backslash Q, l^{-j-1})+m_\rho(Q\backslash Q_*, l^{-j-1})+m_\rho(E\cap Q_*, l^{-j-1})\\ &\leqq (l^3-1)l^{-(j+1)\rho}+(1-c)l^{-(j+1)\rho}+2\delta' l^{-(j+1)\rho}\\ &\leqq l^{-j\rho}+(l^3-l^\rho)l^{-j\rho}-\frac{c}{2}l^{-(j+1)\rho} \text{ for } \delta' \text{ small enough}\\ &< l^{-j\rho}=|I|^{\rho/3}\end{aligned}$$

Second alternative. Any $(j+1)$-cube $Q\subset I$ satisfies (1).

The point is that for l large enough (depending on δ), the Q's which lie deeper inside I almost don't catch any harmonic measure. This can be formalized by defining suitable stopping times on the Brownian paths penetrating I and using the strong Markov property (see Introduction). Define

$$I_1 = I\backslash \text{outer } Q\text{'s in } I$$

$$I_2 = I_1\backslash \text{outer } Q\text{'s in } I_1$$

$$I_{\bar{l}}$$

for $\bar{l}=[10^{-6}l]$ say.

Thus by (*)

$$\omega(I_{\bar{l}})\leqq\omega(\mathbb{R}^3\backslash(E\cup I_{\bar{l}}), I_{\bar{l}}, 0)\leqq\omega(\mathbb{R}^3\backslash(E\cup I_1), I_1, 0)\cdot\sup_{a\in\partial I_1}\omega(\mathbb{R}^3\backslash(E\cup I_{\bar{l}}), I_{\bar{l}}, a) \quad (3)$$

By (**) and (1), clearly

$$\omega(I)=\omega(\mathbb{R}^3\backslash E, E\cap I, 0)\geqq\omega(\mathbb{R}^3\backslash(E\cup I_1), I_1, 0)\cdot\inf_{y\in\partial I_1}\omega(\mathbb{R}^3\backslash E, E\cap I, y)$$

$$\omega(I)\geqq c\delta\omega(\mathbb{R}^3\backslash(E\cup I_1), I_1, 0)$$

considering the outer Q's in I only.

Estimate second factor in (3), again by the strong Markov property, as

$$\operatorname*{Sup}_{a\in\partial I_1}\omega(\mathbb{R}^3\backslash(E\cup I_2), I_2, a)\sup_{a\in\partial I_2}\omega(\mathbb{R}^3\backslash(E\cup I_3), I_3, a)\ldots\sup_{a\in\partial I_{\bar{l}-1}}\omega(\mathbb{R}^3\backslash(E\cup I_{\bar{l}}), I_{\bar{l}}, a) \quad (4)$$

Now $\omega(\mathbb{R}^3\backslash(E\cup I_2), I_2, a)\leqq 1-\omega(\mathbb{R}^3\backslash(E\cup I_2), E, a)\leqq 1-c\delta$ as a consequence of (1) and the same holds for the next factors in (4). Hence

$$(4)\leqq(1-c\delta)^{\bar{l}}=\exp(-c\delta\bar{l}).$$

If l is sufficiently large, we get

$$\omega(I_{\bar{l}})\leqq C' 1/\delta\exp(-c\delta\bar{l})\,\omega(I)<10^{-6}\,\omega(I).$$

Writing

$$\sum_{J\subset\mathscr{E}_{j+1}} \omega(J)^{1/2}|J|^{1/2} = \sum_{J\subset I_{\bar{l}}} + \sum_{J\not\subset I_{\bar{l}}} \leqq \omega(I_{\bar{l}})^{1/2}|I_{\bar{l}}|^{1/2} + C10^{-3}|I|^{1/2}\omega(I)^{1/2}$$

(L) is obtained. This proves Lemma 2.

Remark that $\delta' \Rightarrow \delta \Rightarrow l \Rightarrow \rho < 3$ in above considerations.

Construction of tree

If $I\in\mathscr{E}_j$ is an (L)-cube, we associate to I its l^3 subcubes in $\mathscr{E}_{j+1}$.

If I is a (D)-cube, we associate a family $\{I_\alpha\}\subset\mathscr{E}$ of cubes satisfying

(.) $I_\alpha \subsetneqq I$

(..) $E\cap I \subset \bigcup I_\alpha$

(...) $\sum |I_\alpha|^{\rho/3} \leqq |I|^{\rho/3}$

Starting with $I_0=[0,1]^3$, the previous procedure yields a tree $\mathscr{T}$ which elements we label by complexes $c=(k_1,k_2,\dots,k_s)$. If c is of type (L), then c has exactly l^3 successors. The cubes on level s of this tree belong to $\bigcup_{j\geqq s}\mathscr{E}_j$ and hence are of size $\leqq l^{-s}$.

Fix a number δ. We stop $\mathscr{T}$ when the cube is of size $<\delta$. Thus each branch in $\mathscr{T}$ is at most $\log 1/\delta$ long. Let $\mathscr{T}^*$ stand for the maximal elements of $\mathscr{T}$.

Given a (maximal) branch $c\in\mathscr{T}^*$, we enumerate the consecutive L-cubes on c by stopping times

$$\tau_1<\tau_2<\dots$$

(which we let take the value ∞ on the $c\in\mathscr{T}^*$ where corresponding τ is not defined naturally). Thus given $c\in\mathscr{T}^*$ and denoting $c|k$ the restriction of c to the k first digits

$$I_{c|\tau_1(c)} \supset I_{c|\tau_2(c)} \supset \dots$$

is the sequence of L-type cubes appearing on c. By construction

$$E\subset \bigcup_{c\in\mathscr{T}^*} I_c \text{ (disjoint union)}$$

and

$$\text{Size } I_c<\delta \quad \text{for } c\in\mathscr{T}^*.$$

Choose an integer $\bar{s}\sim c\log 1/\delta$ to be specified later. Let

$$\mathscr{C}_1=\{c\in\mathscr{T}^* | \tau_{\bar{s}}(c)=\infty\};\qquad \mathscr{C}_2=\mathscr{T}^*\backslash\mathscr{C}_1$$

and further

$$\mathscr{L}_1=\{I_{c|\tau_1(c)}; c\in\mathscr{C}_2\},\dots,\mathscr{L}_{\bar{s}}=\{I_{c|\tau_{\bar{s}}(c)}; c\in\mathscr{C}_2\}.$$

Thus $\bigcup_{c\in\mathscr{L}_s} I_c$ $(s=1,2,\dots)$ decreases and

$$E\subset \bigcup_{c\in\mathscr{C}_1} I_c \cup \bigcup_{c\in\mathscr{L}_{\bar{s}}} I_c. \tag{5}$$

These cubes are of size δ^c. We estimate

$$\sum_{c\in\mathscr{C}_1} |I_c|^{\rho'/3} \quad \text{for some } \rho'<3 \text{ (fixed)}$$

and will further reduce $\mathscr{L}_{\bar{s}}$ to a subclass $\mathscr{L}$.

Estimates. If $c\in\mathscr{T}$ is of type (D), then

$$\sum_{k=1,2,\dots,(c,k)\in\mathscr{T}} |I_{c,k}|^{\rho/3} \leqq |I_c|^{\rho/3}.$$

If $c\in\mathscr{T}$ of type (L), then

$$I_c = \bigcup_{k=1}^{l^3} I_{c,k}$$

and for ρ close enough to 3

$$\sum_{(c,k)\in\mathscr{T}} |I_{c,k}|^{\rho/3} \leqq 2|I_c|^{\rho/3}.$$

Thus

$$\begin{aligned} |I_0|^{\rho/3} &\leqq \sum \{|I_{c|\tau_1(c)}|^{\rho/3}; c\in\mathscr{C}_1\} \\ &\geqq 1/2 \sum \{|I_{c|\tau_1(c)+1}|^{\rho/3}; c\in\mathscr{C}_1\} \\ &\geqq 1/2 \sum \{|I_{c|\tau_2(c)}|^{\rho/3}; c\in\mathscr{C}_1\} \\ &\geqq 1/4 \sum \{|I_{c|\tau_2(c)+1}|^{\rho/3}; c\in\mathscr{C}_1\} \\ &\geqq 2^{-\bar{s}} \sum_{c\in\mathscr{C}_1} |I_c|^{\rho/3} \end{aligned}$$

from where

$$\sum_{c\in\mathscr{C}_1} |I_c|^{\frac{3+\rho}{6}} \leqq 2^{\bar{s}} \delta^{\frac{3-\rho}{6}} \leqq 1 \qquad \text{for suitable } \bar{s}\sim\log 1/\delta. \tag{6}$$

It remains to consider $\bigcup_{c\in\mathscr{L}_{\bar{s}}} (I_c\cap E)$. We reduce $\mathscr{L}_{\bar{s}}$ to a family $\mathscr{L}$ satisfying

$$\sum_{c\in\mathscr{L}} |I_c|^{\rho'/3} \leqq 1 \qquad (\rho'<3) \tag{7}$$

and

$$\omega\Big(\bigcup_{c\in\mathscr{L}_{\bar{s}}\setminus\mathscr{L}} I_c\Big) < \kappa \tag{7}$$

where $\kappa>0$ will be a small number.

Intersect E with

$$\bigcup_{c\in\mathscr{C}_1\cup\mathscr{L}} I_c \tag{8}$$

to obtain E_1 such that $\omega(E\setminus E_1)<\kappa$ and $h_{\rho'}(E_1,\delta^c)<1$.

To prove the existence of $\mathscr{L}$, we make the following computation

$$\sum_{c\in\mathscr{L}_{\bar{s}}} |I_c|^{1/2}\omega(I_c)^{1/2} = \sum_{c'\in\mathscr{L}_{\bar{s}-1}} \sum_{k=1}^{l^3} \sum_{\substack{c\in\mathscr{L}_{\bar{s}} \\ c>(c',k)}} |I_c|^{1/2}\omega(I_c)^{1/2}$$

$$\leqq \sum_{c'\in\mathscr{L}_{\bar{s}-1}} \sum_{k=1}^{l^3} |I_{c',k}|^{1/2}\omega(I_{c',k})^{1/2} \quad \text{(Hölder)}$$

$$\leqq \tfrac{1}{10} \sum_{c'\in\mathscr{L}_{\bar{s}-1}} |I_{c'})^{1/2}\omega(I_{c'})^{1/2} \quad (c' \text{ is (L)-cube})$$

etc. ...

$$\leqq 10^{-\bar{s}} \leqq \delta^c \tag{9}$$

Define

$$\mathscr{L} = \{c\in\mathscr{L}_{\bar{s}} | \omega(I_c) > \delta^{-c}|I_c|\}$$

Since

$$|I_c| \geqq \delta$$

We have

$$\omega(I_c) \leqq |I_c|^{1-c} \qquad \text{if } c\in\mathscr{L}$$

$$\sum_{c\in\mathscr{L}} |I_c|^{1-c} \leqq \sum_{c\in\mathscr{L}} \omega(I_c) \leqq 1.$$

Also by (9)

$$\sum_{c\in\mathscr{L}_{\bar{s}}\setminus\mathscr{L}} \omega(I_c) \leqq \sum_{c\in\mathscr{L}_{\bar{s}}} \delta^{-c/2}|I_c|^{1/2}\omega(I_c)^{1/2} \leqq \delta^{c/2} < \kappa$$

for δ small enough. Hence (7) holds.

We satisfy (8). Indeed $\omega(E\setminus E_1) < \kappa$ holds as consequence of (5) and (7). Also for appropriate numerical $\rho' < 3$ and $c > 0$, we have $h_{\rho'}(E_1, \delta^c) < 1$ as a consequence of (6), (7) and the fact that the cubes involved are of size $< \delta^c$. Consider $\rho' < \rho'' < 3$. Taking $\delta > 0$ small enough, it is clear from the preceding that for any $\varepsilon > 0$ there is a subset E_ε of E fulfilling the conditions

$$h_{\rho''}(E_\varepsilon) < \varepsilon \quad \text{and} \quad \omega(E\setminus E_\varepsilon) < \varepsilon.$$

Since ρ'' is a fixed constant < 3, the theorem is proved.

References

[C1] Carleson, L.: Selected problems on exceptional sets. Princeton, N-J: Van Nostrand 1967

[C2] Carleson, L.: Preprint 1985

[J-W] Jones, P., Wolff, T.: Preprint 1985

[K-W] Kaufman, R., Wu, J-M: On the snow flake domain. Stud. Math. **75**, 177–182 (1983)

[M] Makarov, N.: On the distorsion of boundary sets under conformal mappings. Proc. London Math. Soc. **51**, 369–384 (1985)

[O] Øksendal, B.: Brownian motion and sets of harmonic measure zero. Pac. J. Math. **95**, 179–192 (1981)

[W] Wolff, T.: Private communications

Oblatum 11-III-1986