Invent. math. 63, 159–179 (1981)

Inventiones mathematicae

Generalized Exponents of a Free Arrangement of Hyperplanes and Shepherd-Todd-Brieskorn Formula

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1. Introduction

We define an n-arrangement as a finite family of hyperplanes through the origin in $\mathbf{C}^{n+1}$. In [11] and [12] we studied the free arrangement and defined its structure sequence (their definitions will be given again in Sect. 2). In this article we say the generalized exponents instead of the structure sequence. Let $(d_0, d_1, \ldots, d_n)$ be the generalized exponents of a free n-arrangement X. Let $|X| = \bigcup_{H \in X} H$. Our main result is

Main Theorem. $\prod_{i=0}^{n} (1+d_i t)$ *equals the Poincaré polynomial of* $\mathbf{C}^{n+1} \setminus |X|$.

Let $G \subset \mathrm{GL}(n+1; \mathbf{C})$ be a finite unitary reflection group acting on $\mathbf{C}^{n+1}$. Then the set of the reflecting hyperplanes of the unitary reflections in G makes an n-arrangement X. Such an arrangement is called a Shepherd-Todd arrangement. We can show that a Shepherd-Todd arrangement is free. Moreover its generalized exponents coincide with the generalized exponents of G which were recently defined by Orlik-Solomon [6]. In this special case our Main Theorem is nothing other than the main result in [6]. For details see [13].

Especially when $G \subset \mathrm{GL}(n+1; \mathbf{R})$, the arrangement X is called a Coxeter arrangement which is of course free [7]. In this case our Main Theorem is known as the Shepherd-Todd-Brieskorn formula ([10], [1] Theorem 6(ii)).

Remark. The class of free arrangements is far wider than that of Shepherd-Todd arrangement. In fact many examples show that the freeness of arrangement is a combinatorial property [11].

In Sect. 3, we briefly review the combinatorial formula for the Poincaré polynomial proved by Orlik-Solomon [5]. Next we compute some Hilbert polynomial by two different methods. One method is by the generalized ex-

* The author gratefully acknowledges support by the Grant in Aid for Scientific Research of the Ministry of Education of Japan No. 574047

0020-9910/81/0063/0159/$04.20

ponents (Sect. 4) and the other is combinatorial (Sect. 5). And in Sect. 6 we prove Main Theorem by comparing the two computations and by applying the Orlik-Solomon formula.

Assume that $Q \in \mathbf{R}[z_0, \ldots, z_n]$, a product of real linear forms, is a defining equation of a free n-arrangement X. By combining the Main Theorem with the Orlik-Solomon formula and Zaslavsky's result ([14] p. 18 Theorem A), we have

$$\begin{aligned}&\#\{\text{connected component of } \mathbf{R}^{n+1} \setminus \{Q=0\}\}\\&= \sum_{i=0}^{n+1} b_i(\mathbf{C}^{n+1} \setminus |X|) = \prod_{i=0}^{n} (1+d_i),\end{aligned}$$

where $b_i(\mathbf{C}^{n+1} \setminus |X|)$ stands for the i-th Betti number of $\mathbf{C}^{n+1} \setminus |X|$.

This equality is called the Coxeter equality and was proved when $n=2$ in [12]. In [2] Coxeter proved this equality when X is a Coxeter arrangement. K. Saito proved

$$\#\{\text{connected component of } \mathbf{R}^{n+1} \setminus \{Q=0\}\} \leqq \prod_{i=0}^{n} (1+d_i)$$

in [8].

The author would like to express his thanks to Professor K. Saito who has encouraged him constantly and gave him many helpful suggestions.

2. The Generalized Exponents of a Free n-Arrangement

(2.1) *Definition.* Let X be an n-arrangement in $\mathbf{C}^{n+1}$. We call X to be *free* if

$$\begin{aligned}\operatorname{Der}(\log |X|)_0 := \{&\text{a germ } \theta \text{ of holomorphic vector field}\\&\text{at } 0 \text{ such that } \theta \cdot Q \in Q \cdot \mathcal{O}_{\mathbf{C}^{n+1}, 0}\},\end{aligned}$$

is a free $\mathcal{O}_{\mathbf{C}^{n+1}, 0}$-module, where 0 is the origin of $\mathbf{C}^{n+1}$ and $Q \in \mathbf{C}[z_0, \ldots, z_n]$ is a defining equation of $|X|$.

A germ θ of holomorphic vector field at 0 is said to be homogeneous of degree d, denoted by $\deg \theta = d$, if θ has a local expression

$$\theta = \sum_{i=0}^{n} f_i \frac{\partial}{\partial z_i}$$

at the origin such that f_i's are homogeneous polynomials and all non-zero f_i's have the same degree d. A little observation leads us to the existence of a system of homogeneous free basis $\{\theta_0, \ldots, \theta_n\}$ for $\operatorname{Der}(\log |X|)_0$ if X is a free n-arrangement. It is easy to see that a set $(\deg \theta_0, \ldots, \deg \theta_n)$ of integers depend only on X.

(2.2) *Definition.* We call $(\deg \theta_0, \ldots, \deg \theta_n)$ *the generalized exponents* of a free n-arrangement X.

3. The Möbius Function and the Poincaré Polynomial

Let X be an n-arrangement throughout this article.

In this section we briefly review the combinatorial formula for the Poincaré polynomial of $\mathbf{C}^{n+1} \setminus |X|$.

(3.1) *Definition.* Let $L(X)=\{\bigcap_{H\in A} H;\ A\subset X\}$, where we interpret that $\mathbf{C}^{n+1} = \bigcap_{H\in\phi} H$. Define the join and meet operations in $L(X)$ by

$$s\vee t=s\cap t,$$

and

$$s\wedge t=\cap H \quad (H \text{ runs over a set } \{L\in X;\ L\supset s\cup t\}) \quad \text{for } s,t\in L(X).$$

Then $L(X)$ becomes a lattice which is called *the lattice associated with* X.

Write $s\prec t$ if $s\vee t=t(s,t\in L(X))$.

(3.2) *Definition.* Define the *Möbius function* μ on $L(X)$ inductively by

$$\mu(\mathbf{C}^{n+1})=1$$
$$\mu(s)=-\sum_{\substack{t\prec s\\ t\neq s}}\mu(t).$$

(3.3) *Definition.* The *rank* of $s\in L(X)$, denoted by $r(s)$, is the length of the longest chain in $L(X)$ below s. Then $r(\mathbf{C}^{n+1})=0$.

(3.4) *Definition.* For any integer $i\geqq 0$, put

$$\mu_i(L(X)) := \sum_{\substack{s\in L(X)\\ r(s)=i}} |\mu(s)|.$$

(3.5) **Theorem** (Orlik-Solomon [5]).

$$\mu_i(L(X))=b_i(\mathbf{C}^{n+1}\setminus|X|)$$

for any integer $i\geqq 0$, *where the right handside stands for the i-th Betti number of* $\mathbf{C}^{n+1}\setminus|X|$.

(3.6) *Remark.* In the first draft of this article the proof of (3.5) was contained, but we recently found that (3.5) had already been proved by Orlik-Solomon, so we omit the proof here.

4. The Computation of the Hilbert Polynomial by the Generalized Exponents

From now on we denote $\mathcal{O}_{\mathbf{C}^{n+1},0}$ and $\mathrm{Der}(\log|X|)_0$ simply by $\mathcal{O}$ and $\mathrm{D}(X)$ respectively. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and $Q\in\mathbf{C}[z_0,\dots,z_n]$ be a defining equation of $|X|$. By ∂Q denote we the Jacobian ideal of Q in $\mathcal{O}$ (i.e., $\partial Q = (\partial Q/\partial z_0,\dots,\partial Q/\partial z_n)\cdot\mathcal{O}$). Then ∂Q depends only on X. Thus we sometimes write $J(X)$ instead of ∂Q.

(4.1) *Definition.* Define an $\mathcal{O}$-submodule

$$\mathrm{Ann}(X)=\{\theta\in \mathrm{D}(X);\, \theta\cdot Q=0\}$$

of $\mathrm{D}(X)$.

(4.2) *Definition.* Introduce a decreasing filtration

$$(\mathcal{O}^k)_m=\underbrace{\mathfrak{m}^m\oplus\ldots\oplus\mathfrak{m}^m}_{k} \qquad (m\geqq 0)$$

on an $\mathcal{O}$-module $\mathcal{O}^k$ $(k>0)$. Then this filtration $((\mathcal{O}^k)_m)_{m\geqq 0}$ makes $\mathcal{O}^k$ to be an $\mathfrak{m}$-bonne filtered $\mathcal{O}$-module (see [9]).

By the natural projection $\mathcal{O}\to\mathcal{O}/J(X)$, we can induce an $\mathfrak{m}$-bonne filtration on $\mathcal{O}/J(X)$.

On the other hand, $\mathrm{D}(X)$ can be embedded in $\mathcal{O}^{n+1}$ by the correspondence

$$\sum_{i=0}^{n} f_i(\partial/\partial z_i)\mapsto(f_0,\ldots,f_n) \qquad (f_i\in\mathcal{O}\,(i=0,\ldots,n)).$$

So one can induce an $\mathfrak{m}$-bonne filtration on $\mathrm{D}(X)$. Since $\mathrm{Ann}(X)$ is an $\mathcal{O}$-submodule, $\mathrm{Ann}(X)$ can also be an $\mathfrak{m}$-bonne filtered $\mathcal{O}$-module.

From now on we regard $\mathcal{O}^{n+1}$, $\mathcal{O}$, $\mathcal{O}/J(X)$, $\mathrm{D}(X)$ and $\mathrm{Ann}(X)$ as $\mathfrak{m}$-bonne filtered $\mathcal{O}$-modules in the above manners.

(4.3) *Definition.* Let $M=(M_n)_{n\geqq 0}$ be a filtered $\mathcal{O}$-module. Then $M(k)=(M(k)_n)_{n\geqq 0}$ is another filtered $\mathcal{O}$-module defined by $M(k)_n=M_{k+n}$ for $k\in\mathbf{Z}$, $k\geqq 0$.

In [11] we dealt with an exact sequence of filtered $\mathcal{O}$-modules

(4.4) $$0\to\mathrm{Ann}(X)\to\mathcal{O}^{n+1}\to\mathcal{O}(m-1)\to(\mathcal{O}/J(X))(m-1)\to 0,$$

where each morphism is strictly compatible (e.g., see [3] p. 7) with each filtration and $m=\#X=\deg Q$.

(4.5) *Definition.* Let $M=(M_n)_{n\geqq 0}$ be an $\mathfrak{m}$-bonne (decreasingly) filtered $\mathcal{O}$-module. A polynomial $H(M;v)$ is characterized by the property that:

$H(M;v)\in\mathbf{Q}[v]$ equals the dimension of $\mathcal{O}/\mathfrak{m}\simeq\mathbf{C}$-vector space M_v/M_{v+1} for sufficiently large v.

For our convenience, we put

$$f^{(m)}=\frac{(f+1)\ldots(f+m)}{m!} \quad\text{and}\quad f^{(0)}=1$$

for any polynomial f and $m>0$.

Let X be free with its generalized exponents $(d_0,\ldots,d_n)$ throughout this section. Then we have

(4.6) **Proposition.**

(1) $H(\mathrm{D}(X);v)=\sum_{i=0}^{n}(v-d_i)^{(n)}$,

and

(2) $H(\mathrm{Ann}(X); v) = \sum_{i=1}^{n} (v-d_i)^{(n)}$ *if* $d_0 = 1$.

Proof. By the very definition of $(d_0, \ldots, d_n)$ we have an isomorphism

$$\mathrm{D}(X) \xrightarrow{\sim} \bigoplus_{i=0}^{n} \mathcal{O}(-d_i),$$

from which we deduce (1).

If $d_0 = 1$, then X is not void, so we have

$$\mathrm{D}(X) = \mathrm{Ann}(X) \oplus \mathcal{O} \cdot \theta_0,$$

where $\theta_0 = \sum_{i=0}^{n} z_i(\partial/\partial z_i)$ is the Euler vector field [11]. Thus we obtain (2).

(4.7) *Definition.* $H(X; v) \in \mathbf{Q}[v]$ is defined by:

$$H(X; v) = \begin{cases} H(\mathcal{O}/J(X); v) & \text{if } X \neq \phi, \\ 0 & \text{if } X = \phi. \end{cases}$$

We call $H(X; v)$ the *Hilbert polynomial of* X.

Define $P_i(X)(i = 1, \ldots, n) \in \mathbf{Z}$ by

$$H(X; v) = \sum_{i=1}^{n} P_i(X)\, v^{(n-i)}.$$

By (4.4) and (4.6), we can calculate the Hilbert polynomial of X through a little bit complicated but easy computation;

(4.8) **Theorem.** *If* $d_0 = 1$, *then*

$$H(X; v+m-1) = \sum_{i=1}^{n} \left\{ \binom{m+i-2}{i} + (-1)^i \sum_{j=1}^{n} \binom{d_j}{i} \right\} v^{(n-i)},$$

where we interpret $\binom{a}{b} = 0$ *if* $a < b$.

(4.9) *Definition.* Let $i \geq 0$. The polynomials

$$\pi_i(t_0, \ldots, t_n) \in \mathbf{C}[t_0, \ldots, t_n]$$

are defined by

$$\prod_{i=0}^{n} (1 + t_i t) = \sum_{j \geq 0} \pi_j(t_0, \ldots, t_n)\, t^j.$$

Then $\pi_0 = 1$ and $\pi_1(t_0, \ldots, t_n) = t_0 + \ldots + t_n$.

Define

$$\pi_i(X) = \pi_i(d_0, \ldots, d_n),$$

and

$$\pi_I(X) = \prod_{j=1}^{k} \pi_{I(j)}(X)$$

for any multi-index $I = (I(1), \ldots I(k))$ composing of k non-negative integers.

(4.10) **Proposition.**

1) $\pi_0(X) = \mu_0(L(X)) = 1$,
2) $\pi_1(X) = \mu_1(L(X)) = m$.

Proof. 1) is obvious.

If $X = \phi$, then $\pi_1(X) = 0 = \mu_1(L(X))$. So we can assume that $X \neq \phi$ and $d_0 = 1$ [11]. On the other hand we know that

$$\deg H(X; \nu) = \deg H(\mathcal{O}/\partial Q; \nu) = \dim \operatorname{Spec}(\mathcal{O}/\partial Q) - 1 \leqq n - 2$$

(see [9]).

Thus we have

$$0 = P_1(X) = \binom{m-1}{1} - \sum_{j=1}^{n} \binom{d_j}{1} = m - 1 - \sum_{j=1}^{n} d_j,$$

which implies that

$$m = 1 + \sum_{j=1}^{n} d_j = \sum_{j=0}^{n} d_j = \pi_1(X).$$

By a direct computation we obtain

$$\mu_1(L(X)) = m.$$ **Q.E.D.**

In Sect. 6, we shall prove that

(4.11) $$\pi_i(X) = \mu_i(L(X)) \qquad (i \geqq 0).$$

(4.12) **Proposition.** *The alternating sum of $\pi_0(X), \ldots, \pi_{n+1}(X)$ is zero if $X \neq \phi$.*

Proof. We can assume that $d_0 = 1$, thus

$$\sum (-1)^i \pi_i(X) = \prod_{i=0}^{n} (1 - d_i) = 0.$$

(4.13) **Proposition.** *Let $X \neq \phi$. For each integer i $(2 \leqq i \leqq n)$, there exist real numbers $c_i \neq 0$ and $c(I; i)$ $(I \in I[i])$, which are independent of X, such that*

$$P_i(X) - c_i \pi_i(X) = \sum_{I \in I[i]} c(I; i) \pi_I(X).$$

Here

$$I[i] := \left\{ I = (I(1), \ldots, I(k)); 0 \leqq I(j) < i \, (j = 1, \ldots, k), \sum_{j=1}^{k} I(j) \leqq i \right\}.$$

Proof. By combining (4.8) with the fact that $P_1(X) = 0$ we have

$$H(X;v)=\sum_{i=2}^{n}\left\{\binom{m+i-2}{i}+(-1)^i\sum_{j=0}^{n}\binom{d_j}{i}\right\}(v-m+1)^{(n-i)}.$$

We simply write $\pi_j(t)$ instead of $\pi_j(t_0,\dots,t_n)$ $(j=0,\dots,n+1)$. Let $2\leqq i\leqq n$. Since

$$F_i(t):=\binom{t_0+\dots+t_n+i-2}{i}+(-1)^i\sum_{j=0}^{n}\binom{t_j}{i}\in\mathbf{C}[t_0,\dots,t_n]$$

is a symmetric polynomial of degree i of $t_0,\dots,t_n$, it can be written as a polynomial of $\pi_0(t),\dots,\pi_i(t)$. Notice that $\left\{\sum_{j=0}^{n}\binom{t_j}{k}\right\}_{0\leqq k\leqq n}$ are algebraically independent over $\mathbf{C}$ and generate the $\mathbf{C}$-algebra of symmetric polynomials in $\mathbf{C}[t_0,\dots,t_n]$. Therefore one knows that there exists a real number $c_i\neq 0$ such that

$$(-1)^i\sum_{j=0}^{n}\binom{t_j}{i}-c_i\pi_i(t)\quad\text{and thus}\quad F_i(t)-c_i\pi_i(t)$$

are polynomials of $\pi_0(t),\dots,\pi_{i-1}(t)$ because $\binom{t_j}{i}$ is of degree i. $\left(\text{In fact } c_i=-\frac{1}{(i-1)!}\right)$.

On the other hand, we have

$$(v-m+1)^{(n-i)}=v^{(n-i)}-\binom{m-1}{1}v^{(n-i-1)}+\binom{m-1}{2}v^{(n-i-2)}-\dots.$$

Thus if we define

$$P_i(t):=\sum_{j=0}^{i-2}(-1)^jF_{i-j}(t)\binom{\pi_1(t)-1}{j}\in\mathbf{C}[t_0,\dots,t_n],$$

then $P_i(d_0,\dots d_n)=P_i(X)$. It is easy to see that $P_i(t)-c_i\pi_i(t)$ is of degree i and is a polynomial of $\pi_0(t),\dots,\pi_{i-1}(t)$. And replace t_j by $d_j (j=0,\dots,n)$. **Q.E.D.**

(4.14) The observation so far shows that the following two data concerning a free arrangement X are equivalent:

(1) The set of the generalized exponents $(d_0,\dots,d_n)$ of X, which is equivalent to

$$\sum_{i=0}^{n}\pi_i(X)t^i=\prod_{i=0}^{n}(1+d_it),$$

(2) The Hilbert polynomial $H(X;v)$ of X together with $\#X$, which is of course equivalent to the data

$$(\#X,P_2(X),\dots,P_n(X)).$$

If (4.11) holds true, then the additional two data are also equivalent:

(3) The polynomial $\sum_{i=0}^{n}\mu_i(L(X))t^i$,

(4) The Poincaré polynomial of $\mathbf{C}^{n+1} \setminus |X|$.

The author would like to thank the referee for his comment on (4.14).

5. The Combinatorial Computation of the Hilbert Polynomial

(5.1) *Definition.* The *essential dimension* of X, denoted by ess.dim X, is defined by

$$\text{ess.dim}\, X = \operatorname*{Max}_{x \in L(X)} r(x).$$

Then $0 \leqq \text{ess.dim}\, X \leqq n+1$.

(5.2) *Definition.* Define an n-arrangement

$$X_s = \{H \in X;\, H \supset s\}$$

for any $s \in L(X)$. Then it is easy to see that $\text{ess.dim}(X_s) = r(s)$.

(5.3) *Definition.* Let L be a lattice and $x \in L$. Define a sublattice

$$L \setminus x = \{y \in L;\, y \prec x\}$$

of L.

Under the same notation as in (5.2) we have the following two Propositions:

(5.4) **Proposition.** *There is a lattice isomorphism*

$$L(X_s) \xrightarrow{\sim} L(X) \setminus s.$$

Proof. Easy.

(5.5) **Proposition.** *An n-arrangement X_s is free if X is free.*

Proof. In [11] we introduced a coherent sheaf $\text{Der}(\log |X|)$ on $\mathbf{C}^{n+1}$. Thus $\text{Der}(\log |X|)$ is locally free at the origin. If we take a generic point P on s such that P is sufficiently near to the origin, then $\text{Der}(\log |X|)_P$ is a free $\mathcal{O}_{\mathbf{C}^{n+1}, P}$-module. And it is clear that

$$\text{Der}(\log |X|)_P \xrightarrow{\sim} \text{Der}(\log |X_s|)_0. \qquad \textbf{Q.E.D.}$$

(5.6) *Definition.* Define

$$\mathscr{A}(X) = \{X_s;\, s \in L(X)\},$$

and

$$\mathscr{L}(X) = \{L(Y);\, Y \in \mathscr{A}(X)\}.$$

Then there exists the natural surjective map

$$L\colon \mathscr{A}(X) \to \mathscr{L}(X).$$

We denote $\{s \in L(X);\, r(s) = i\}$ by $L(X)(i)$ $(i \geqq 0)$.

The concepts defined in the following (5.7) are essential:

(5.7) *Definition.* Let G be an abelian group. For a map

$$q;\ \mathscr{A}(X)\to G(\mathscr{L}(X)\to G),$$

define a new map

$$r_i q\colon \mathscr{A}(X)\to G(\text{resp. } \mathscr{L}(X)\to G)$$

by

$$(r_i q)(Y)=q(Y)-\sum_{s\in L(Y)(i)} q(Y_s)$$

$$(\text{resp. } (r_i q)(L(Y))=q(L(Y))-\sum_{s\in L(Y)(i)} q(L(Y)\setminus s))$$

for any $Y\in\mathscr{A}(X)$ and any integer $i\geqq 0$. Denote $r_i r_{i-1}\dots r_0 q$ by $R_i q$.
We say that q is *i-cumulative* ($i\geqq 0$) on X (resp. on $L(X)$) if

$$(R_i q)(X)=0 \qquad (\text{resp. } (R_i q)(L(X))=0).$$

(5.8) *Remark.* For an i-cumulative map $q;\ \mathscr{L}(X)\to G$, a map

$$q\circ L\colon \mathscr{A}(X)\to G$$

is i-cumulative on X because of (5.4). Thus we may say that the i-cumulativeness on $L(X)$ implies the i-cumulativeness on X.

The following Proposition is easy:

(5.9) **Proposition.** *If two maps*

$$q_1, q_2\colon \mathscr{A}(X)\to G(\mathscr{L}(X)\to G)$$

are i-cumulative on X (resp. on L(X)), then

(1) q_1+q_2, q_1-q_2 *are both i-cumulative on X (resp. on L(X)),*
(2) q_1 *is j-cumulative on X (resp. on L(X)) for any integer* $j\geqq i$.

Assume that X is free in the rest of this section.
The Main result in this section is

(5.10) **Theorem.** *A mapping* $P_i\colon \mathscr{A}(X)\to \mathbf{Z}$ (see (4.7)) *is i-cumulative on X* ($i=1,\dots,n$).

(5.10) easily reduces to

(5.11) **Proposition.** *A mapping* $H\colon \mathscr{A}(X)\to \mathbf{Q}[v]$ *defined by* $H(Y)=H(Y;v)$ (see (4.7)) *for* $Y\in\mathscr{A}(X)$ *satisfies*

$$\deg(R_i H)(X)\leqq n-i-1 \qquad (1\leqq i\leqq n).$$

The rest of this section is devoted to the proof of (5.11). We need some preparations for the algebraic geometry on $\operatorname{Spec}\mathcal{O}$ which is the main tool for the proof.

Let $t=\mathbf{V}(\mathfrak{P})$ ($\mathfrak{P}\in\operatorname{Spec}\mathbf{C}[z_0,\dots,z_n]$) be a plane through the origin in $\mathbf{C}^{n+1}$. Then $\tilde{t}:=\mathbf{V}(\mathfrak{P}\cdot\mathcal{O})$ is a closed subvariety of $\operatorname{Spec}\mathcal{O}$. For any n-arrangement Y, define

$$\tilde{L}(Y)=\{\tilde{t};\, t\in L(Y)\},$$
$$\tilde{L}(Y)(i)=\{\tilde{t}\in\tilde{L}(Y);\, t\in L(Y)(i)\}.$$

Induce the lattice structure on $\tilde{L}(Y)$ from the lattice structure on $L(Y)$ through the one-to-one correspondence $t\mapsto\tilde{t}$.

Let s be a closed subvariety of $\operatorname{Spec}\mathcal{O}$. Then there exists an element $\widetilde{\langle s\rangle}\in\tilde{L}(Y)$ which is the largest (with respect to the order in $\tilde{L}(Y)$) element among a set $\{\tilde{t}\in\tilde{L}(Y);\, \tilde{t}\supset s\}$. Define $Y_s=Y_{\langle s\rangle}$. Then notice that

$$\text{ess.dim}\, Y_s=r(\langle s\rangle) \quad (5.2)$$

and

$$Y_{\tilde{t}}=Y_t \quad \text{if } t\in L(Y).$$

Define a morphism in $\mathscr{A}(X)$ as an inclusion map, then $\mathscr{A}(X)$ becomes a category.

Define another category $\mathscr{C}$ by

$$\mathcal{O}b\!j(\mathscr{C})=\{\text{m-bonne decreasingly filtered } \mathcal{O}\text{-module}\},$$
$$\mathcal{M}or(\mathscr{C})=\{\mathcal{O}\text{-homomorphism which is compatible with the filtrations}\}.$$

Then $\mathscr{C}$ is an abelian category.

Let $1\leqq i\leqq n$. Then we shall construct a contravariant functor $M(i)$ from $\mathscr{A}(X)$ to $\mathscr{C}$ satisfying the following three conditions:

(A_i): The morphism $M(i)(Y)\to M(i)(Z)$ induced from the inclusion $Z\hookrightarrow Y$ is surjective and strictly compatible with the filtrations for any $Y, Z\in\mathscr{A}(X)$, $Z\subset Y$,

(B_i): Put $s=\mathbf{V}(\mathfrak{P})$ for any $\mathfrak{P}\in\operatorname{Spec}\mathcal{O}$, then

$$M(i)(Y)_{\mathfrak{P}}\xrightarrow{\sim} M(i)(Y_s)_{\mathfrak{P}}$$

for any $Y\in\mathscr{A}(X)$, where the $\mathcal{O}$-isomorphism above is the localization (by $\mathfrak{P}$) of the morphism

$$M(i)(Y)\to M(i)(Y_s)$$

induced from the inclusion map $Y_s\hookrightarrow Y$,

(C_i): $M(i)(Y)$ is either 0 or a Cohen-Macaulay $\mathcal{O}$-module of dimension $(n-i)$ and

$$\operatorname{Ass}(M(i)(Y))\subset\mathfrak{P}(Y)(i+1),$$

where by $\mathfrak{P}(Y)(i+1)$ denote we a set

$$\{\mathfrak{P}\in\operatorname{Spec}\mathcal{O};\, \mathbf{V}(\mathfrak{P})\in\tilde{L}(Y)(i+1)\}.$$

In the rest of this section we simply write $|\tilde{L}(Y)(i)|$ instead of $\bigcup_{s\in\tilde{L}(Y)(i)} s$. Then notice that

$$\operatorname{Supp}(M(i)(Y))\subset|\tilde{L}(Y)(i+1)| \quad (5.12)$$

for any $Y\in\mathscr{A}(X)$ if (C_i) holds.

Our aim is to define $M(i)(1 \leqq i \leqq n)$ inductively.

Let $2 \leqq i \leqq n$ and $M(i-1)$ be as above. We shall construct $M(i)$ satisfying (A_i), (B_i) and (C_i). Let $Y \in \mathscr{A}(X)$, $t \in \tilde{L}(Y)(i)$ and $\mathfrak{P} \in \operatorname{Spec} \mathcal{O}$. Then we have

(5.13) **Lemma.**

$$\bigoplus_{t \in \tilde{L}(Y)(i)} M(i-1)(Y_t)_{\mathfrak{P}} = \bigoplus_{\substack{t \in \tilde{L}(Y)(i) \\ t \supset \mathbf{V}(\mathfrak{P})}} M(i-1)(Y_t)_{\mathfrak{P}},$$

in other words, $M(i-1)(Y_t)_{\mathfrak{P}} = 0$ *if* $t \not\supset \mathbf{V}(\mathfrak{P})$.

Proof. Notice that

$$\tilde{L}(Y_t)(i) = \{t\}$$

and thus

$$\operatorname{Supp} M(i-1)(Y_t) \subset |\tilde{L}(Y_t)(i)| = t.$$

by (C_{i-1}). Therefore (5.13) is a result in the theory of commutative algebra (see [4]). **Q.E.D.**

There is a morphism

$$\varphi_t(Y)\colon M(i-1)(Y) \to M(i-1)(Y_t)$$

induced from the inclusion. Define K, $M(i)(Y) \in \mathcal{O}bj(\mathscr{C})$ such that every morphism (with a possible exception of $\varphi(Y)$) of an exact sequence

$$(5.14) \qquad 0 \to K \to M(i-1)(Y) \xrightarrow{\varphi(Y)} \bigoplus_{t \in \tilde{L}(Y)(i)} M(i-1)(Y_t) \to M(i)(Y) \to 0,$$

is strictly compatible, where $\varphi(Y) = \bigoplus_{t \in \tilde{L}(Y)(i)} \varphi_t(Y)$.

(5.15) **Lemma.** $K = 0$.

Proof. Let $\mathfrak{P} \in \mathfrak{P}(Y)(i)$, then the localization of $\varphi(Y)$ by $\mathfrak{P}$ is an isomorphism because of (5.13) and (B_{i-1}). Thus we have

$$K_{\mathfrak{P}} = M(i)(Y)_{\mathfrak{P}} = 0,$$

which implies that

$$\operatorname{Ass}(K) \cap \mathfrak{P}(Y)(i) = \operatorname{Ass}(M(i)(Y)) \cap \mathfrak{P}(Y)(i) = \phi.$$

On the other hand

$$\operatorname{Ass}(K) \subset \operatorname{Ass}(M(i-1)(Y)) \subset \mathfrak{P}(Y)(i) \qquad (C_{i-1}),$$

therefore we obtain

$$\operatorname{Ass}(K) = \phi,$$

which implies that $K = 0$.

Next let $Z \in \mathscr{A}(X)$, $Z \subset Y$, then we have a commutative diagram

$$(5.16)\qquad \begin{array}{ccccccc} 0\to M(i-1)(Y) & \to & \bigoplus_{t\in\tilde{L}(Y)(i)} M(i-1)(Y_t) & \to & M(i)(Y) & \to & 0\\ \downarrow & & \downarrow & & & & \\ 0\to M(i-1)(Z) & \to & \bigoplus_{u\in\tilde{L}(Z)(i)} M(i-1)(Z_u) & \to & M(i)(Z) & \to & 0, \end{array}$$

where the rows are exact and the vertical morphisms are induced from $Z\hookrightarrow Y$ and $Z_u\hookrightarrow Y_u$ respectively ($u\in\tilde{L}(Z)(i)$). From (5.16), we can define a surjective and strictly compatible morphism

$$M(i)(Y)\to M(i)(Z)$$

commuting (5.16). This implies (A_i).

Replace Z by Y_s ($s=\mathbf{V}(\mathfrak{P})$, $\mathfrak{P}\in\operatorname{Spec}\mathcal{O}$) in (5.16) and localize by $\mathfrak{P}$. Then we obtain another commutative diagram

$$(5.17)\qquad \begin{array}{ccccccc} 0\to M(i-1)(Y)_{\mathfrak{P}} & \to & \bigoplus_{t\in\tilde{L}(Y)(i)} M(i-1)(Y_t)_{\mathfrak{P}} & \to & M(i)(Y)_{\mathfrak{P}} & \to & 0\\ \downarrow\wr & & \downarrow\wr & & & & \\ 0\to M(i-1)(Y_s)_{\mathfrak{P}} & \to & \bigoplus_{\substack{t\in\tilde{L}(Y)(i)\\ t\supset s}} M(i-1)(Y_t)_{\mathfrak{P}} & \to & M(i)(Y_s)_{\mathfrak{P}} & \to & 0 \end{array}$$

because of (B_{i-1}) and (5.13). Thus we have (B_i).

Let $\mathfrak{P}\in\operatorname{Spec}\mathcal{O}$ and $s=\mathbf{V}(\mathfrak{P})$. Assume that $s\not\subset|\tilde{L}(X)(i+1)|$, then $r(\langle s\rangle)\leqq i$. If $r(\langle s\rangle)\leqq i-1$, then by (5.13) we obtain

$$\bigoplus_{t\in\tilde{L}(Y)(i)} M(i-1)(Y_t)_{\mathfrak{P}}=0$$

and thus

$$M(i)(Y)_{\mathfrak{P}}=0.$$

If $r(\langle s\rangle)=i$, then

$$\{t\in\tilde{L}(Y)(i);\, t\supset s\}=\{\widetilde{\langle s\rangle}\}.$$

Therefore we have an exact sequence

$$0\to M(i-1)(Y)_{\mathfrak{P}}\to M(i-1)(Y_{\widetilde{\langle s\rangle}})_{\mathfrak{P}}\to M(i)(Y)_{\mathfrak{P}}\to 0,$$

and thus $M(i)(Y)_{\mathfrak{P}}=0$ by (B_{i-1}). (Notice that $Y_{\widetilde{\langle s\rangle}}=Y_s$.) These facts imply that

$$\operatorname{Supp}(M(i)(Y))\subset|\tilde{L}(X)(i+1)|.$$

Take the cohomology long exact sequence of the first row in (5.16) by the functor $\bigotimes_{\mathcal{O}}\mathcal{O}/\mathfrak{m}$, then we have

$$\operatorname{Tor}^{\mathcal{O}}_{i+2}(M(i)(Y),\mathcal{O}/\mathfrak{m})=0,$$

because

$$\begin{aligned}&\text{homolog. dim}(M(i-1)(Y))\\ &\quad=\text{homolog. dim}\Big(\bigoplus_{t\in\tilde{L}(Y)(i)} M(i-1)(Y_t)\Big)=i\end{aligned}$$

(see [4] p. 129). This implies that

$$\begin{aligned} i+1 &\geqq \text{homolog.}\dim(M(i)(Y)) \\ &= n+1-\operatorname{depth}(M(i)(Y)) \\ &\geqq n+1-\dim\operatorname{Supp}(M(i)(Y)) \\ &\geqq n+1-(n-i)=i+1, \end{aligned}$$

and thus $M(i)(Y)$ is Cohen-Macaulay of dimension $n-i$. So $M(i)(Y)$ has no embedded primes and its associated primes are all of height $(i+1)$. Thus we have (C_i).

Next we have to construct $M(1)$. Define

$$M(1)(Y)=\begin{cases}\mathcal{O}/J(Y) & \text{if } Y\neq\phi,\\ 0 & \text{if } Y=\phi.\end{cases}$$

Then the $\mathcal{O}$-homomorphism

$$M(1)(Y)\to M(1)(Z)$$

is naturally defined when $Z\subset Y$, because $J(Z)\supset J(Y)$ unless $Z=\phi$. This map is surjective and strictly compatible with each filtration and $M(1)$ is a functor from $\mathscr{A}(X)$ to $\mathscr{C}$. This is (A_1).

If $Y=\phi$, then (A_1), (B_1) and (C_1) trivially hold true.

So we assume $Y\neq\phi$, then in [11] (Prop. 2.6) we proved (C_1).

Let

$$J(Y)=\mathfrak{Q}(1)\cap\ldots\cap\mathfrak{Q}(k) \qquad \text{(primary decomposition)},$$

then $\{\mathbf{V}(\mathfrak{Q}(1)),\ldots,\mathbf{V}(\mathfrak{Q}(k))\}=\tilde{L}(Y)(2)$ and $\mathbf{V}(\mathfrak{Q}(i))\neq\mathbf{V}(\mathfrak{Q}(j))$ $(i\neq j)$.

What remains to be proved is (B_1). Let $\mathfrak{P}\in\operatorname{Spec}\mathcal{O}$ and $s=\mathbf{V}(\mathfrak{P})$, then we must show that the $\mathcal{O}$-homomorphism

$$(\mathcal{O}/J(Y))_{\mathfrak{P}}\to(\mathcal{O}/J(Y_s))_{\mathfrak{P}}$$

is an isomorphism. Thus the whole constructions of $M(i)$ $(i\geqq 1)$ reduce to the following

(5.18) **Lemma.**

(1) $J(Y_s)=\bigcap_{\mathfrak{Q}(i)\subset\mathfrak{P}}\mathfrak{Q}(i)$,

(2) *there is a natural $\mathcal{O}$-isomorphism*

$$(\mathcal{O}/\mathfrak{Q}(1)\cap\ldots\cap\mathfrak{Q}(k))_{\mathfrak{P}}\xrightarrow{\sim}(\mathcal{O}/\bigcap_{\mathfrak{Q}(i)\subset\mathfrak{P}}\mathfrak{Q}(i))_{\mathfrak{P}}.$$

Proof. (1); Assume that

$$\mathbf{V}(\mathfrak{P})=s=\mathbf{V}(\mathfrak{Q}(1))\in\tilde{L}(Y)(2),$$

then in this case we must prove that

$$J(Y_s)=\mathfrak{Q}(1).$$

Since $J(Y_s)$ has no embedding primes (5.5) and $\tilde{L}(Y_s)(2)=\{s\}$, we deduce that $J(Y_s)$ itself is $\mathfrak{P}$-primary. Notice that $J(Y_s)\cap\mathfrak{Q}(1)$ is also $\mathfrak{P}$-primary. Because

$J(Y_s) \supset J(Y)$, we have

$$\begin{aligned}\mathfrak{Q}(1)\cap\ldots\cap\mathfrak{Q}(k) &= J(Y)\\ &= J(Y_s)\cap J(Y)\\ &= (J(Y_s)\cap\mathfrak{Q}(1))\cap\mathfrak{Q}(2)\cap\ldots\cap\mathfrak{Q}(k).\end{aligned}$$

By the uniqueness of the primary decomposition of $J(Y)$, we have

$$\mathfrak{Q}(1) = J(Y_s)\cap\mathfrak{Q}(1),$$

and thus $J(Y_s)\supset\mathfrak{Q}(1)$.

By an appropriate coordinate change, we can assume that

$$\sqrt{\mathfrak{Q}(1)} = \sqrt{J(Y_s)} = (z_0, z_1)\cdot\mathcal{O}.$$

The product $Q_1 Q_2$ of a defining equation Q_1 of Y_s and a defining equation Q_2 of $Y\setminus Y_s$ is a defining equation of Y, so

$$\mathfrak{Q}(1)\ni\partial(Q_1 Q_2)/\partial z_i = Q_1(\partial Q_2/\partial z_i) + Q_2(\partial Q_1/\partial z_i) \quad (i=0, 1).$$

Notice that

$$Q_2\notin(z_0, z_1)\cdot\mathcal{O} = \sqrt{\mathfrak{Q}(1)} \quad\text{and}\quad Q_1\cdot Q_2\in J(Y)\subset\mathfrak{Q}(1).$$

Then we have $Q_1\in\mathfrak{Q}(1)$, $Q_2(\partial Q_1/\partial z_i)\in\mathfrak{Q}(1)$ and thus

$$\partial Q_1/\partial z_i\in\mathfrak{Q}(1) \quad (i=0, 1).$$

This proves that $J(Y_s)=\mathfrak{Q}(1)$. In other words

$$J(Y) = \bigcap_{i=1}^{k} J(Y_{s(i)}),$$

where $s(i)=\mathbf{V}(\mathfrak{Q}(i))$ $(i=1, \ldots, k)$.

For a general $s=\mathbf{V}(\mathfrak{P})$,

$$\begin{aligned}J(Y_s) &= \bigcap_{t\in\tilde{L}(Y_s)(2)} J((Y_s)_t)\\ &= \bigcap_{\substack{t\in\tilde{L}(Y)(2)\\ t\supset s}} J(Y_t) = \bigcap_{\mathfrak{Q}(i)\subset\mathfrak{P}} \mathfrak{Q}(i).\end{aligned}$$

This proves (1).

(2): Let

$$\begin{aligned}&\mathfrak{Q}(1)\cap\ldots\cap\mathfrak{Q}(l)\supset\mathfrak{P},\\ &\mathfrak{Q}(l+1), \ldots, \mathfrak{Q}(k)\not\supset\mathfrak{P} \quad (0\leqq l\leqq k).\end{aligned}$$

What we have to show is that

$$(5.19)\qquad (\mathfrak{Q}(1)\cap\ldots\cap\mathfrak{Q}(l)/\mathfrak{Q}(1)\cap\ldots\cap\mathfrak{Q}(k))_{\mathfrak{P}} = 0.$$

Take $a(j)\in\mathfrak{Q}(j)\setminus\mathfrak{P}$ $(j=l+1, \ldots, k)$ and put $a=a(l+1)\ldots a(k)$. Then $a\notin\mathfrak{P}$ and

$$a(\mathfrak{Q}(1)\cap\ldots\cap\mathfrak{Q}(l))\subset\mathfrak{Q}(1)\cap\ldots\cap\mathfrak{Q}(k),$$

which shows (5.19).

Thus we have finished to construct $M(i)$ ($i\geqq 1$) as desired.
We shall continue the proof of (5.11).

(5.20) *Definition.* Let $M\in\mathcal{O}bj(\mathscr{C})$ and $\nu\geqq 0$. By $(M_\nu)_{\nu\in\mathbf{Z}}$ denote we the decreasing filtration of M. Define

$$\mathrm{Gr}^\nu M=M_\nu/M_{\nu+1},$$

then $\mathrm{Gr}^\nu M$ is an $\mathcal{O}/\mathfrak{m}\simeq\mathbf{C}$-vector space. A $\mathbf{C}$-homomorphism

$$\sigma\colon \mathrm{Gr}^\nu M\to M_\nu$$

is called a *ν-section of M* if $p\circ\sigma=\mathrm{id}$, where $p\colon M_\nu\to\mathrm{Gr}^\nu M$ is the natural projection.

(5.21) **Lemma.** *Let* $\nu\geqq 0$ *and* $i\geqq 1$. *Then*

$(1)_i$: *There exists a ν-section* $\sigma(i)$ *of each* $M(i)(Y)$ $(Y\in\mathscr{A}(X))$ *commuting a diagram*

$$\begin{array}{ccc} M(i)(Y)_\nu & \longrightarrow & M(i)(Z)_\nu \\ {\scriptstyle\sigma(i)}\uparrow & \circlearrowleft & \uparrow{\scriptstyle\sigma(i)} \\ \mathrm{Gr}^\nu(M(i)(Y)) & \longrightarrow & \mathrm{Gr}^\nu(M(i)(Z)), \end{array}$$

where the two row morphisms are induced from the inclusion $Z\hookrightarrow Y$ *for any* $Y, Z\in\mathscr{A}(X)$, $Z\subset Y$,

$(2)_i$: *In the exact sequence*

$$0\to M(i)(Y)\xrightarrow{\varphi(Y)}\bigoplus_{t\in L(Y)(i+1)} M(i)(Y_t)\to M(i+1)(Y)\to 0$$

(*see* (5.16)), $\varphi(Y)$ (*thus every morphism*) *is strictly compatible for any* $Y\in\mathscr{A}(X)$.

Proof. As for $(1)_1$, we have

$$M(1)(Y)_\nu=\mathfrak{m}^\nu/\mathfrak{m}^\nu\cap J(Y)$$

and

$$\mathrm{Gr}^\nu(M(1)(Y))=\mathfrak{m}^\nu/(\mathfrak{m}^{\nu+1}+\mathfrak{m}^\nu\cap J(Y)).$$

(If $Y=\phi$, then $(1)_1$ trivially holds true.) Define $\sigma(1)$ by

$$\sigma(1)([f])=\text{the class of } f_\nu \text{ in } \mathfrak{m}^\nu/\mathfrak{m}^\nu\cap J(Y),$$

where $f\in\mathfrak{m}^\nu$, $[f]\in\mathfrak{m}^\nu/(\mathfrak{m}^{\nu+1}+\mathfrak{m}^\nu\cap J(Y))$ and f_ν is a homogeneous polynomial of degree ν satisfying

$$f\equiv f_\nu(\mathrm{mod}\ \mathfrak{m}^{\nu+1}).$$

Then $\sigma(1)$ is well-defined because $J(Y)$ is generated by homogeneous polynomials. It is easy to see that $(1)_1$ holds true.

Next we shall show that $(1)_i$ implies $(2)_i$. Assume $(1)_i$. Let $Y \in \mathscr{A}(X)$ and $t \in \tilde{L}(Y)(i+1)$. Define $K_t \in \mathcal{O}bj(\mathscr{C})$ such that every morphism of an exact sequence

$$0 \to K_t \to M(i)(Y) \xrightarrow{\varphi_t(Y)} M(i)(Y_t) \to 0 \tag{5.22}$$

is strictly compatible. Thus there exist three ν-sections (all denoted by $\sigma(i)$) of $M(i)(Y)$, $M(i)(Y_t)$ and K_t respectively because of $(1)_i$ and (5.22). Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & (K_t)_\nu & \longrightarrow & M(i)(Y)_\nu & \xrightarrow{\varphi_t(Y)} & M(i)(Y_t)_\nu & \to 0 \\
& \uparrow{\scriptstyle \sigma(i)} & \circlearrowleft & \uparrow{\scriptstyle \sigma(i)} & \circlearrowleft & \uparrow{\scriptstyle \sigma(i)} & \\
0 \to & \mathrm{Gr}^\nu(K_t) & \longrightarrow & \mathrm{Gr}^\nu(M(i)(Y)) & \longrightarrow & \mathrm{Gr}^\nu(M(i)(Y_t)) & \to 0.
\end{array}$$

If $x \in M(i)(Y)_\nu$ and $[\varphi_t(Y)](x) \in M(i)(Y_t)_{\nu+1}$ for all $t \in \tilde{L}(Y)(i+1)$, then we have

$$[\varphi_t(Y) \circ \sigma(i) \circ p](x) = 0$$

by chasing the diagram above, where

$$p \colon M(i)(Y)_\nu \to \mathrm{Gr}^\nu(M(i)(Y))$$

is the natural projection. This shows that

$$[\sigma(i) \circ p](x) \in \bigcap_{t \in \tilde{L}(Y)(i+1)} K_t = 0 \quad \text{(see (5.15))}$$

and thus $x \in M(i)(Y)_{\nu+1}$. This implies $(2)_i$.

Next assume $(1)_i$ and $(2)_i$. We shall prove $(1)_{i+1}$. Let $Y, Z \in \mathscr{A}(X)$ and $Z \subset Y$. Considering a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & M(i)(Y)_\nu & \to & \bigoplus_{t \in L(Y)(i+1)} M(i)(Y_t)_\nu & \to & M(i+1)(Y)_\nu & \to 0 \\
& \nearrow{\scriptstyle \sigma(i)} & & \nearrow{\scriptstyle \oplus\sigma(i)} & & & \\
0 \to & \mathrm{Gr}^\nu(M(i)(Y)) & \to & \mathrm{Gr}^\nu\big(\bigoplus_{t \in L(Y)(i+1)} M(i)(Y_t)\big) & \to & \mathrm{Gr}^\nu(M(i+1)(Y)) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & M(i)(Z)_\nu & \to & \bigoplus_{u \in L(Z)(i+1)} M(i)(Z_u)_\nu & \to & M(i+1)(Z)_\nu & \to 0 \\
& \nearrow{\scriptstyle \sigma(i)} & & \nearrow{\scriptstyle \oplus\sigma(i)} & & & \\
0 \to & \mathrm{Gr}^\nu(M(i)(Z)) & \to & \mathrm{Gr}^\nu\big(\bigoplus_{u \in L(Z)(i+1)} M(i)(Z_u)\big) & \to & \mathrm{Gr}^\nu(M(i+1)(Z)) & \to 0,
\end{array}$$

where all four rows are exact because of $(2)_i$. By using this diagram, we can define v-sections of $M(i+1)(Y)$ and $M(i+1)(Z)$ respectively and they satisfy the condition in $(1)_{i+1}$. **Q.E.D.**

(5.23) **Lemma.**

$$H(M(i)(X); v)=(-1)^{i-1}(R_i H)(X) \quad (i\geqq 1).$$

Proof. If $X=\phi$, then this is a trivial equality. So assume that $X\neq\phi$. We shall prove this by an introduction on i.

When $i=1$,

$$\begin{aligned}(R_1 H)(X)&=(R_0 H)(X)-\sum_{t\in L(X)(1)}(R_0 H)(X_t)\\ &=H(X)-\sum_{t\in L(X)(1)}H(X_t)\\ &=H(\mathcal{O}/J(X); v)-\sum_{t\in L(X)(1)}H(\mathcal{O}/J(X_t); v)\\ &=H(M(1)(X); v)\end{aligned}$$

because $\mathcal{O}/J(X_t)=0$ if $t\in L(X)(1)$.

Assume that

$$H(M(i-1)(X); v)=(-1)^{i-2}(R_{i-1}H)(X) \quad (i\geqq 2),$$

then

$$\begin{aligned}(-1)^i(R_i H)(X)&=(-1)^i(R_{i-1}H)(X)-\sum_{t\in L(X)(i)}(-1)^i(R_{i-1}H)(X_t)\\ &=H(M(i-1)(X); v)-\sum_{t\in L(X)(i)}H(M(i-1)(X_t); v)\\ &=H(M(i-1)(X); v)-\sum_{t\in \tilde{L}(X)(i)}H(M(i-1)(X_t); v)\\ &=-H(M(i)(X); v)\end{aligned}$$

because of $(5.21.\ 2)_{i-1}$. **Q.E.D.**

Since

$$\dim \operatorname{Supp}(M(i)(X))\leqq n-i$$

by (5.12), we have (see [9] III-7 Théorème 1)

$$\begin{aligned}\deg(R_i H)(X)&=\deg H(M(i)(X); v)\\ &=\dim \operatorname{Supp}(M(i)(X))-1\\ &\leqq n-i-1.\end{aligned}$$

This proves (5.11).

6. Proof of Main Theorem

Before the proof of Main Theorem we need some combinatorial preparations.

Let L be a lattice associated with some n-arrangement. Denote $\{s\in L; r(s)=i\}$ by $L(i)$ for $i\geqq 0$. Then $\mu_i(L)$ is defined by $\mu_i(L)=\sum_{x\in L(i)}|\mu(x)|$ for $i\geqq 0$ as in Sect. 3.

(6.1) *Definition.* Let $I=(I(1), \ldots, I(k))$ be an arbitrary multi-index composing of k non-negative integers. Define

$$\mu_I(L)=\prod_{j=1}^{k} \mu_{I(j)}(L).$$

(6.2) **Proposition.** *μ_I is $|I|$-cumulative on L.*

The proof of (6.2) is rather technical. We need the following two definitions and a lemma.

Fix a multi-index $I=(I(1), \ldots, I(k))$ with $I(j)\in \mathbf{Z}$, $I(j)\geqq 0$ $(j=1, \ldots, k)$.

(6.3) *Definition.* Define a set

$$\mathscr{T}(L)=\{(x_1, \ldots, x_k);\ x_j\in L(I(j))\ (j=1, \ldots, k)\}$$

and a function $f_I;\ \mathscr{T}(L)\to \mathbf{Z}$ by

$$f_I(\alpha)=\prod_{j=1}^{k} |\mu(x_j)|,$$

where $\alpha=(x_1, \ldots, x_k)\in \mathscr{T}(L)$.

For any integer $j\geqq 0$, define a set

$$\mathscr{T}(L;j)=\{\alpha=(x_1, \ldots, x_k)\in \mathscr{T}(L);\ r(|\alpha|)=j\},$$

where $|\alpha|$ stands for $x_1\cap \ldots \cap x_k=x_1 \vee \ldots \vee x_k\in L$.

(6.4) **Lemma.**

1) $\mu_I(L)=\sum_{\alpha\in \mathscr{T}(L)} f_I(\alpha)$,

2) $\mathscr{T}(L)=\bigcup_{j=0}^{|I|} \mathscr{T}(L;j)$ *(disjoint)*,

3) $\mathscr{T}(L;j)=\bigcup_{s\in L(j)} \mathscr{T}(L\setminus s;j)$ *(disjoint)* $(j\geqq 0)$.

Proof. Notice that $\mathscr{T}(L;j)=\phi$ for $j>|I|=I(1)+\ldots+I(k)$, then 2) is obvious.

1) is easy to see from the definitions of μ_I, f_I and $\mathscr{T}(L)$.

As for 3), let $\alpha=(x_1, \ldots, x_k)\in \mathscr{T}(L;j)$, then $r(|\alpha|)=j$, thus $\alpha\in \mathscr{T}(L\setminus |\alpha|;j)$. Conversely it is easy to show $\mathscr{T}(L\setminus s;j)\subset \mathscr{T}(L;j)$ for any $s\in L(j)$, $j\geqq 0$. If $\alpha\in \mathscr{T}(L\setminus s;j)\cap \mathscr{T}(L\setminus t;j)$ for some $s, t\in L(j)$, then $|\alpha|\prec s$, $|\alpha|\prec t$ and $|\alpha|\in L(j)$, which implies that $s=|\alpha|=t$. **Q.E.D.**

(6.5) *Definition.* For any integer $j\geqq 0$, define

$$\mu_{I,j}(L)=\sum_{\alpha\in \mathscr{T}(L;j)} f_I(\alpha).$$

Then it is easy to see

(6.6)
$$\begin{aligned}\mu_I(L)&=\sum_{\alpha\in \mathscr{T}(L)} f_I(\alpha) \quad ((6.4)\,1))\\ &=\sum_{j=0}^{|I|}\sum_{\alpha\in \mathscr{T}(L;j)} f_I(\alpha) \quad ((6.4)\,2))\\ &=\sum_{j=0}^{|I|} \mu_{I,j}(L).\end{aligned}$$

Proof of (6.2). It is sufficient to prove that each $\mu_{I,j}$ $(0 \leqq j \leqq |I|)$ is j-cumulative on L because of (5.9) and (6.6). Assume that $r(s) < j$ $(s \in L)$, then $\mathscr{T}(L \setminus s; j) = \phi$, thus $\mu_{I,j}(L \setminus s) = 0$. This implies that

$$\begin{aligned}(R_j \mu_{I,j})(L) &= (r_j \mu_{I,j})(L) \\ &= \mu_{I,j}(L) - \sum_{s \in L(j)} \mu_{I,j}(L \setminus s) \\ &= \sum_{\alpha \in \mathscr{T}(L;\, j)} f_I(\alpha) - \sum_{s \in L(j)} \sum_{\alpha \in \mathscr{T}(L \setminus s;\, j)} f_I(\alpha) \\ &= 0\end{aligned}$$

because of (6.4) 3). **Q.E.D.**

Let X be an n-arrangement.

Let $i \geqq 0$. The following proposition gives a characterization of the map

$$\mu_i \circ L\colon \mathscr{A}(X) \to \mathbf{Z}.$$

(6.7) **Proposition.** *Assume that the maps*

$$q_j\colon \mathscr{A}(X) \to \mathbf{Z} \qquad (j \geqq 0)$$

satisfy the following conditions:

I. $q_0(\phi) = 1$.

II. $q_j(X_s) = 0$ *if* $s \in L(X)$ *and* $r(s) < j$ $(j \geqq 0)$.

III. *The alternating sum of* $q_j(Y)$ $(j \geqq 0)$ *is zero if* $Y \in \mathscr{A}(X) \setminus \{\phi\}$.

IV. q_j *is* j*-cumulative on any* $Y \in \mathscr{A}(X)$ $(j = 0, 1, \ldots, i)$. *Then* $q_j = \mu_j \circ L$ $(j = 0, 1, \ldots, i)$ *on* $\mathscr{A}(X)$.

Proof. It is easy to see that

$$q_i(\phi) = \begin{cases} 1 & i = 0 \\ 0 & i > 0, \end{cases}$$

thus $q_i(\phi) = \pi_i(\phi)$ $i \geqq 0)$. So we can assume that $\phi \neq Y \in \mathscr{A}(X)$.

We prove (6.7) by an induction on i.

When $i = 0$, we have

$$\begin{aligned}0 = r_0 q_0(Y) &= q_0(Y) - \sum_{s \in L(Y)(0)} q(Y_s) \\ &= q_0(Y) - q_0(\phi) = q_0(Y) - 1 \quad \text{(I)},\end{aligned}$$

thus

$$q_0(Y) = 1 = \mu_0 \circ L(Y).$$

Next assume that

$$q_j = \mu_j \circ L \qquad (j = 0, 1, \ldots, i-1). \tag{6.8}$$

Since $q_i(Y_t) = 0$ if $t \in L(Y)$ and $r(t) < i$ (II), we deduce

$$R_{i-1} q_i(Y) = q_i(Y).$$

Therefore we have

$$(6.9)\qquad \begin{aligned} 0&=(R_i q_i)(Y) \quad \text{(IV)}\\ &=q_i(Y)-\sum_{s\in L(Y)(i)} q_i(Y_s). \end{aligned}$$

Let $s\in L(Y)(i)$, then $q_j(Y_s)=0$ for $j>i$ (II). Thus we have

$$\begin{aligned} q_i(Y_s)&=q_{i-1}(Y_s)-q_{i-2}(Y_s)+\ldots+(-1)^{i-1}q_0(Y_s) \quad \text{(III)}\\ &=\mu_{i-1}\circ L(Y_s)-\mu_{i-2}\circ L(Y_s)+\ldots+(-1)^{i-1}\mu_0\circ L(Y_s) \quad (6.8)\\ &=\mu_i\circ L(Y_s) \end{aligned}$$

because the alternating sum of μ_i also vanishes ([14]). Therefore we obtain

$$\begin{aligned} q_i(Y)&=\sum_{s\in L(Y)(i)} q_i(Y_s) \quad (6.9)\\ &=\sum_{s\in L(Y)(i)} \mu_i\circ L(Y_s)\\ &=\sum_{s\in L(Y)(i)} \mu_i(L(Y)\setminus s) \quad (5.4)\\ &=\sum_{s\in L(Y)(i)} |\mu(s)|=\mu_i\circ L(Y). \end{aligned}$$

This completes the proof.

(6.10) *Remark.* We have already shown that π_j ($j\geqq 0$) satisfy the conditions I and III in (6.7) for any free n-arrangement X (4.10) (4.12).

Notice that $\pi_i(Y)=0$ for any free arrangement Y with ess. dim $Y<i$. In fact the number of non-zero generalized exponents of Y is less than i [11]. Thus π_j ($j\geqq 0$) satisfy the condition II in (6.7) for any free n-arrangement X because of (5.2).

The whole proof of Main Theorem reduces to the following (6.11) $1)_i$ ($i\geqq 0$) because of (3.5):

(6.11) **Proposition.** *Let $i\geqq 0$. Then we have*

$1)_i$ *$\pi_i(X)=\mu_i\circ L(X)$ for any free n-arrangement X,*

$2)_i$ *π_i: $\mathscr{A}(X)\to\mathbf{Z}$ is i-cumulative on X for any free n-arrangement X.*

Proof. When $i=0$ or $i=1$, it is easy to see because of (4.10).

Let $i\geqq 2$. Assume that $1)_j$ ($j=0,1,\ldots,i-1$) holds true. Let X be a free n-arrangement. Recall the fact that P_i is i-cumulative on X (5.10) and that $P_i-c_i\pi_i$ ($c_i\neq 0$) is a linear combination of $\{\pi_I\}$, where I runs over the set $I[i]$ (4.13). By the assumption, we know that $\pi_I(X)=\mu_I(L(X))$ ($I\in I[i]$). Since $\mu_I\circ L$ ($I\in I[i]$) is $|I|$-cumulative on X (6.2) (5.8), it is i-cumulative on X (5.9) (2). Thus π_I and P_i are both i-cumulative on X. Therefore $2)_i$ holds true in the light of (5.9) (1).

Next assume $2)_j$ ($j=0,1,\ldots,i$). Let X be a free n-arrangement. Then the assumption implies that the maps π_j: $\mathscr{A}(X)\to\mathbf{Z}$ ($j=0,1,\ldots,i$) satisfy the condition IV in (6.7) because $Y\in\mathscr{A}(X)$ is also free. Thus we can apply (6.7) (see (6.10)) and prove $1)_i$.

Therefore an induction proceeds and completes the proof.

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Received June 6 / September 3, 1980