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Unipotent Characters of the Symplectic and Odd Orthogonal Groups Over a Finite Field

George Lusztig*

Department of Mathematics, M.I.T., Cambridge, MA 02139, USA

The purpose of this paper is to give explicit formulas for the character of the unipotent representations of the symplectic or odd special orthogonal groups over a finite field F_q on any regular semisimple element, provided that q is sufficiently large. These formulas (which were conjectured in [11, 4.3]) involve a Fourier transform on a certain symplectic vector space over a field with two elements.

Some of the main ingredients in the proof are:

a) Kawanaka's theorem [7] on the existence of lifting for certain field extensions of odd degree in case of finite classical groups, and its application, due to Asai [2], to the zeta functions of the varieties X_w of [5].

b) The use of the Deligne-Goresky-Macpherson cohomology [4] for the closure $\bar{X}_w$ of X_w. (This depends on results of [9] concerning singularities of Schubert varieties.)

c) The results of [10] on classification and degrees of unipotent representations of classical groups.

The case of even orthogonal groups will be considered in a sequel to this paper.

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1. Characters of Hecke Algebras

1.1. Let (W, S) be a Weyl group and let H be the corresponding Hecke algebra (see for example [8, §1]) with coefficients in $\mathbb{Q}[u^{1/2}, u^{-1/2}]$; here $u^{1/2}$ is an indeterminate.

Let E be an irreducible $\mathbb{Q}[W]$-module. We associate to E an H-module $\tilde{E}$ by the method of [13]. Let $C \subset W$ be the two-sided cell of W (see [8, §1]) corresponding to E. The free $\mathbb{Q}[u^{1/2}, u^{-1/2}]$-module $\mathcal{M}_C$ with basis $e_z (z \in C)$ is a

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left H-module and a right W-module:

$$T_x \cdot e_z = \sum_{z' \underset{L}{\sim} z} N_{x,z,z'} e_{z'}$$

$$e_z \cdot w = \sum_{z'' \underset{R}{\sim} z} n_{w,z,z''} e_{z''}.$$

(Here $T_x (x \in W)$ is the standard basis of H, $\underset{L}{\sim}$, $\underset{R}{\sim}$ are defined as in [8, §1] and $N_{x,z,z'} \in \mathbb{Z}[u^{1/2}]$, $n_{w,z,z''} \in \mathbb{Z}$ are defined by the formulas

$$T_x C_z = \sum_{z'} N_{x,z,z'} C_{z'}$$

$$C_z|_{u=1} \cdot w = \sum_{z''} n_{w,z,z''} C_{z''}|_{u=1}$$

and

$$C_z = \sum_{y \leqq z} (-1)^{l(z)-l(y)} u^{-l(y)+l(z)/2} P_{y,z}(u^{-1}) T_y$$

is defined in [8, 1.1].)

1.2. It has been proved in [13] that the left H-module structure and right W-module structure on $\mathcal{M}_C$ commute with each other. It follows that $\tilde{E} = (\mathcal{M}_C \otimes_{\mathbb{Q}} E)^W$ is in a natural way an H-module, free as a $\mathbb{Q}[u^{1/2}, u^{-1/2}]$-module. It also follows from the results in [13] that, for any $x \in W$, we have

(1.2.1) $$\operatorname{Tr}(T_x, \tilde{E}) \in \mathbb{Z}[u^{1/2}].$$

We have also:

$$\operatorname{Tr}(T_x, \tilde{E}) = |W|^{-1} \operatorname{Tr}(m \to T_x m w \text{ on } \mathcal{M}_C) \operatorname{tr}(w, E)$$
$$= |W|^{-1} \sum_{z \in C} \sum_{\substack{z' \in C \\ z' \underset{L}{\sim} z \\ z' \underset{R}{\sim} z}} \sum_{w \in W} N_{x,z,z'} n_{w,z',z} \operatorname{tr}(w, E).$$

(Here Tr means trace over $\mathbb{Q}[u^{1/2}, u^{-1/2}]$ and tr means trace over $\mathbb{Q}$.) The coefficients $N_{x,z,z'}$, $n_{w,z',z}$ can be computed as follows.

Let

$$D_z = \sum_{\substack{y \in W \\ z \leqq y}} u^{-l(z)/2} Q_{z,y}(u^{-1}) T_y \in H$$

where $Q_{z,y}(z \leqq y)$ are polynomials in u defined by

$$\sum_{z \leqq y \leqq w} (-1)^{l(w)-l(z)} Q_{z,y} P_{y,w} = \delta_{z,w} \qquad (\forall z \leqq w).$$

Let $\tau: H \to \mathbb{Q}[u^{1/2}, u^{-1/2}]$ be the $\mathbb{Q}[u^{1/2}, u^{-1/2}]$-linear map such that $\tau(T_w) = 0$ if $w \neq e$, $\tau(T_e) = 1$. It is well known that

(1.2.2) $$\tau(T_x T_{y^{-1}}) = \begin{cases} u^{l(x)} & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

It follows immediately that

$$\tau(C_x D_{y^{-1}}) = \delta_{x,y}$$

and, hence, that for any $x, z, z', z'' \in W$, we have:

$$N_{x,z,z'} = \tau(T_x C_z D_{z'^{-1}}), \qquad n_{x,z,z''} = \tau(D_{z''^{-1}} C_z T_x)|_{u=1}.$$

Thus, we have the following.

1.3. **Proposition.** *For all $x \in W$, we have:*

$$\text{(1.3.1)} \qquad \operatorname{Tr}(T_x, \tilde{E}) = |W|^{-1} \sum_{w \in W} \sum_{\substack{z \approx z' \\ \text{in } C}} \tau(T_x C_z D_{z'^{-1}}) \cdot \tau(D_{z^{-1}} C_{z'} T_w)|_{u=1} \cdot \operatorname{tr}(w, E)$$

where the second sum is over all ordered pairs z, z' of elements in C such that $z \underset{L}{\sim} z'$, $z \underset{R}{\sim} z'$. (We write $z \approx z'$ instead of $z \underset{L}{\sim} z'$, $z \underset{R}{\sim} z'$.)

1.4. It follows from the results of [13] that $\tilde{E} \otimes \mathbb{Q}(u^{1/2})$ is an absolutely irreducible H-module and that this gives a $1-1$ correspondence between irreducible W-modules and irreducible H-modules. Note that under the specialization $u^{1/2} \to 1$, $\tilde{E}$ becomes the W-module E.

1.5. Let $a \to \bar{a}$ be the involution of the ring $\mathbb{Q}[u^{1/2}, u^{-1/2}]$ such that $\overline{u^{i/2}} = u^{-i/2}$. It extends to an involution $h \to \bar{h}$ of the ring H, such that $\bar{T}_x = T_{x^{-1}}^{-1}$ for all $x \in W$ (see [8, §1]).

1.6. **Corollary.** $\operatorname{Tr}(T_{x^{-1}}^{-1}, \tilde{E}) = \overline{\operatorname{Tr}(T_x, \tilde{E})}$, *for all $x \in W$.*

Proof. It is enough to prove that, for all $x, z, z' \in W$, we have

$$\tau(T_{x^{-1}}^{-1} C_z D_{z'^{-1}}) = \overline{\tau(T_x C_z D_{z'^{-1}})}.$$

Let

$$C'_y = u^{-l(y)/2} \sum_{y' \leqq y} P_{y',y}(u) T_{y'}.$$

By [8, (1.1.c)] we have $\overline{C'_y} = C'_y$. It follows that

$$D_{z'^{-1}} = u^{-\nu/2} C'_{z'^{-1} w_0} T_{w_0}$$

where w_0 is the longest element in W and ν is its length. Since $\overline{C_z} = C_z$, we see that it is enough to prove the identity

$$\text{(1.6.1)} \qquad \tau(\bar{h} \cdot u^{-\nu/2} T_{w_0}) = \overline{\tau(h \cdot u^{-\nu/2} T_{w_0})}$$

for all $h \in H$. We may assume that $h = T_x$. If $x \neq w_0$, then both sides of (1.6.1) are zero. If $x = w_0$, then (1.6.1) is equivalent to the identity $\tau(T_{w_0}^2) = u^\nu$ which is a special case of (1.2.2).

1.7. **Lemma.** *For any $x \in W$, we have*

$$\operatorname{Tr}(T_x, \tilde{E}) = \operatorname{Tr}(T_{x^{-1}}, \tilde{E}).$$

Proof. It is enough to show that there exists a non-singular symmetric bilinear form $(\,,\,): (\tilde{E}\otimes\mathbb{Q}(u^{1/2}))\times(\tilde{E}\otimes\mathbb{Q}(u^{1/2}))\to\mathbb{Q}(u^{1/2})$ such that $(T_x e, e')=(e, T_{x^{-1}} e')$ for all $x\in W$, $e, e'\in\tilde{E}\otimes Q(u^{1/2})$. Let $e_1, \dots, e_n$ be a basis of the vector space $\tilde{E}\otimes\mathbb{Q}(u^{1/2})$. Consider the bilinear form $(\,,\,)_0$ on $\tilde{E}\otimes\mathbb{Q}(u^{1/2})$ given by $(e_i, e_j)_0=\delta_{ij}$ and define $(e, e')=\sum_{w\in W} u^{-l(w)}(T_w e, T_w e')_0$. One checks immediately that $(T_s e, e')=(e, T_s e')$ for any simple reflection s, and hence $(T_x e, e')=(e, T_{x^{-1}} e')$ for any $x\in W$. It remains to show that $(\,,\,)$ is non-singular. But if e is a non-zero vector, then (e, e) is a sum of squares of elements in $\mathbb{Q}(u^{1/2})$, at least one of which is non-zero. It follows that $(e, e)\neq 0$ and the lemma is proved.

1.8. Let $\operatorname{Dim}(\tilde{E})$ be the "formal dimension" of $\tilde{E}$. It is an element of $\mathbb{Q}[u]$, satisfying the identity (see [3]):

$$\sum_{x\in W} u^{-l(x)}\operatorname{Tr}(T_x, \tilde{E})\operatorname{Tr}(T_{x^{-1}}, \tilde{E})=\frac{\sum_{w\in W} u^{l(w)}}{\operatorname{Dim}(\tilde{E})}\cdot\dim(E). \tag{1.8.1}$$

Let $A(E)$ be the degree of the polynomial (in u) $\operatorname{Dim}(\tilde{E})$ and let $u^{a(E)}$ be the largest power of u dividing this polynomial. Since $\sum_{w\in W} u^{l(w)}$ is a product of cyclotomic polynomials in u, and the left hand side of (1.8.1) is in $\mathbb{Z}[u^{1/2}, u^{-1/2}]$, it follows that $\operatorname{Dim}(\tilde{E})$ is of the form $\gamma_E u^{a(E)}$ times a product of cyclotomic polynomials $\neq u-1$ (where γ_E is a strictly positive rational number). It follows that

$$\overline{\operatorname{Dim}(\tilde{E})}=u^{-a(E)-A(E)}\operatorname{Dim}(\tilde{E}). \tag{1.8.2}$$

1.9. **Proposition.** *For any $x\in W$, we have*

$$\operatorname{Tr}(T_x, \tilde{E})=\begin{cases} c_x u^{\frac{l(x)-a(E)}{2}} + \text{higher powers of } u^{1/2} \\ c'_x u^{\frac{l(x)-A(E)+\nu}{2}} + \text{lower powers of } u^{1/2}\end{cases}$$

where c_x, c'_x are integers.

Moreover, for given E, there is at least one $x\in W$ with $c_x\neq 0$ and there is at least one $x\in W$ with $c'_x\neq 0$.

Proof. Using Lemma 1.7 and (1.8.1) we see that

$$\sum_{x\in W} u^{-l(x)}\operatorname{Tr}(T_x, \tilde{E})^2=\begin{cases}\dim(E)\,\gamma_E^{-1} u^{-a(E)} + \text{higher powers of } u \\ \dim(E)\,\gamma_E^{-1} u^{\nu-A(E)} + \text{lower powers of } u.\end{cases}$$

Since $\operatorname{Tr}(T_x, \tilde{E})\in\mathbb{Z}[u^{1/2}]$, (see (1.2.1)), the proposition follows.

1.10. **Corollary.** *$a(E)$ is the smallest integer α such that $u^{(-l(x)+\alpha)/2}\operatorname{Tr}(T_x, \tilde{E})\in\mathbb{Z}[u^{1/2}]$ for all $x\in W$; $A(E)$ is the largest integer β such that $u^{(-l(x)+\beta-\nu)/2}\operatorname{Tr}(T_x, \tilde{E})\in\mathbb{Z}[u^{-1/2}]$ for all $x\in W$.*

1.11. **Lemma.** *Assume that w_0 is in the centre of W and let $\varepsilon_E=\pm 1$ be the scalar by which w_0 acts on E. Then T_{w_0} acts on $\tilde{E}$ as $\varepsilon_E\cdot u^{\nu-(a(E)+A(E))/2}$ times identity* *and, for all $x\in W$, we have*

$$\operatorname{Tr}(T_{w_0 x}, \tilde{E})=\varepsilon_E u^{\nu-\frac{a(E)+A(E)}{2}}\,\overline{\operatorname{Tr}(T_x, \tilde{E})}.$$

Proof. Our assumption implies that T_{w_0} is in the centre of H, so it acts on E as λ times identity where $\lambda\in\mathbb{Q}[u^{1/2}, u^{-1/2}]$. We have

$$\operatorname{Tr}(T_{w_0 x}, \tilde{E})=\operatorname{Tr}(T_{w_0} T_{x^{-1}}^{-1}, \tilde{E})=\lambda \operatorname{Tr}(T_{x^{-1}}^{-1}, \tilde{E})=\lambda\, \overline{\operatorname{Tr}(T_x, \tilde{E})}.$$

(The last equality follows from Corollary 1.6.) It follows that

$$\begin{aligned}\sum_x u^{-l(x)} \operatorname{Tr}(T_x, \tilde{E})^2&=\sum_x u^{-l(w_0 x)} \operatorname{Tr}(T_{w_0 x}, E)^2=\sum_x u^{-\nu+l(x)} \lambda^2\, \overline{\operatorname{Tr}(T_x, E)^2}\\ &=u^{-\nu}\lambda^2\, \overline{\sum_x u^{-l(x)} \operatorname{Tr}(T_x, \tilde{E})^2}\end{aligned}$$

hence, using (1.8.1):

$$\sum_w u^{l(w)}\cdot \operatorname{Dim}(\tilde{E})^{-1}=u^{-\nu}\lambda^2\, \overline{\sum_w u^{l(w)}}\; \overline{\operatorname{Dim}(\tilde{E})^{-1}}.$$

Using now (1.8.2) and the identity $\sum_w u^{l(w)}=u^{\nu}\,\overline{\sum_w u^{l(w)}}$, we see that

$$1=u^{-2\nu}\lambda^2\cdot u^{a(E)+A(E)}$$

hence

$$\lambda=\pm u^{\nu-\frac{a(E)+A(E)}{2}}$$

If we specialize $u^{1/2}$ to 1, λ must specialize to ε_E, hence

$$\lambda=\varepsilon_E u^{\nu-\frac{a(E)+A(E)}{2}}.$$

The lemma is proved.

1.12. *Remark.* Without the assumption that w_0 is in the centre of W, it is still true that $T_{w_0}^2=u^{2\nu-a(E)-A(E)}$ on $\tilde{E}$. The proof is similar to that of Lemma 1.11. Springer (see [3]) has shown that, on $\tilde{E}$, $T_{w_0}^2=u^{\nu+\dim(E)^{-1}\Sigma\operatorname{tr}(r,E)}$ (sum over all reflections r in W).

1.13. **Lemma.** $\operatorname{Tr}(T_x, \widetilde{E\otimes\operatorname{sign}})=(-u)^{l(x)}\,\overline{\operatorname{Tr}(T_x, \tilde{E})}$.

Proof. See [6].

This Lemma, together with Corollary 1.10, imply

1.14. **Lemma.** $a(E\otimes\operatorname{sign})=\nu-A(E)$.

1.15. For any integer i we define $\operatorname{Tr}(T_x, \tilde{E}; i/2)\in\mathbb{Z}$ to be the coefficient of $u^{i/2}$ in $\operatorname{Tr}(T_x, \tilde{E})$ (see (1.2.1)). For any $x\in W$, we define two virtual representations α_x, $\mathscr{A}_x$ of W by

(1.15.1) $$\alpha_x=(-1)^{l(x)}\sum_E \operatorname{Tr}\left(T_x, \tilde{E}; \frac{l(x)-a(E)}{2}\right) E,$$

(1.15.2) $$\mathscr{A}_x = \sum_E \operatorname{Tr}\left(T_x, \tilde{E}; \frac{l(x) - A(E) + v}{2}\right) E.$$

(Both sums are over all irreducible $\mathbb{Q}[W]$-modules E, up to isomorphism.) We have

1.16. **Lemma.** $\mathscr{A}_x = \alpha_x \otimes \text{sign}$.

Proof. From Lemmas 1.13, 1.14, it follows that for each irreducible E,

$$\operatorname{Tr}\left(T_x, \tilde{E}; \frac{l(x) - A(E) + v}{2}\right) = (-1)^{l(x)} \operatorname{Tr}\left(T_x, \widetilde{E \otimes \text{sign}}; \frac{l(x) - a(E \otimes \text{sign})}{2}\right),$$

whence the required identity.

1.17. In the case where w_0 is in the centre of W, we define a $\mathbb{Z}$-linear map ζ of the group of virtual representations of W into itself, by the requirement that $\zeta(E) = \varepsilon_E E$ for E irreducible ($\varepsilon_E = \pm 1$ is as in 1.11). We have

1.18. **Lemma.** *If w_0 is in the centre of W, we have* $\mathscr{A}_{w_0 x} = (-1)^{l(x)} \zeta(\alpha_x)$, $(x \in W)$.

Proof. This follows immediately from Lemma 1.11.

1.19. Now let $W_I (I \subset S)$ be a standard parabolic subgroup of W and let H_I be the corresponding subalgebra of H. For each irreducible $\mathbb{Q}[W_I]$-module E' we define a $\mathbb{Q}[W]$-module $J_{W_I}^W(E')$ by the formula

(1.19.1) $$J_{W_I}^W(E') = \Sigma [E' : E]_{W_I} E$$

sum over all irreducible $\mathbb{Q}[W]$-modules E such that $a(E) = a(E')$. Here $[E' : E]_{W_I}$ denotes the multiplicity of E' in the restriction of E to W_I; it is equal to $[\tilde{E}' : \tilde{E}]_{H_I}$, the multiplicity of E' in the restriction of E to H_I (over the field $\mathbb{Q}(u^{1/2})$). Note that, for any irreducible $\mathbb{Q}[W]$-module E, we have (cf. [12, Lemma 4]):

(1.19.2) $$[E' : E]_{W_I} \neq 0 \Rightarrow a(E') \leqq a(E),$$

(1.19.3) $$\gamma_{E'} = \Sigma [E' : E]_{W_I} \cdot \gamma_E$$

(sum ranges over the same set as in (1.19.1)). With these notations we have

1.20. **Lemma.** *Let $z \in W_I$. Consider the virtual W_I-module $\alpha_z^{(W_I)}$ ($= \alpha_z$ with respect to W_I) and the virtual W-module $\alpha_z^{(W)}$ ($= \alpha_z$ with respect to W). We have*

$$\alpha_z^{(W)} = J_{W_I}^W(\alpha_z^{(W_I)})$$

(where $J_{W_I}^W$ is extended to virtual representations by $\mathbb{Z}$-linearity).

Proof.

$$\alpha_z^{(W)} = (-1)^{l(z)} \sum_{\substack{E:\\ \text{irred.}\\ W\text{-mod.}}} \operatorname{Tr}\left(T_z, \tilde{E}; \frac{l(x) - a(E)}{2}\right) E$$

$$= \sum_{\substack{E\\ \text{irred.}\\ W\text{-mod.}}} \sum_{\substack{E'\\ \text{irred.}\\ W_I\text{-mod.}}} [\tilde{E}' : \tilde{E}]_{H_I} \cdot \operatorname{Tr}\left(T_z, \tilde{E}'; \frac{l(z) - a(E)}{2}\right) E.$$

By 1.9, we have $\mathrm{Tr}\left(T_z, \tilde{E}', \frac{l(z)-a(E)}{2}\right)=0$ unless $\frac{l(z)-a(E)}{2} \geqq \frac{l(z)-a(E')}{2}$ i.e. $a(E) \leqq a(E')$. On the other hand by (1.19.2), we have $[\tilde{E}': \tilde{E}]_{H_I}=[E': E]_{W_I}=0$ unless $a(E') \leqq a(E)$. Thus,

$$\alpha_z^{(W)}=\sum_{E} \sum_{\substack{E' \\ a(E')=a(E)}} [E': E]_{W_I} \mathrm{Tr}\left(T_z, \tilde{E}', \frac{l(z)-a(E')}{2}\right) E=J_{W_I}^W(\alpha_z^{(W_I)}).$$

1.21. *Remark.* We shall apply the previous lemma in the case where W_I is a product of two commuting standard parabolic subgroups $W_{I'} \times W_{I''}$, and $z=z' \cdot z''$ $(z' \in W_{I'}, z'' \in W_{I''})$. In that case, we have clearly

$$\alpha_z^{(W_I)}=\alpha_{z'}^{(W_{I'})} \otimes \alpha_{z''}^{(W_{I''})}.$$

1.22. **Lemma.** *If w_0 is the longest element of W, then α_{w_0} is the sign representation of W.*

Proof. If E appears with non-zero coefficient in α_{w_0}, then, by 1.12, we have

$$\frac{\nu-a(E)}{2}=\nu-\frac{a(E)+A(E)}{2}$$

hence $A(E)=\nu$. Applying 1.9 with $x=s \in S$, we see that $\mathrm{Tr}(T_s, \tilde{E})=c_s' u^{1/2}+$ constant. But the eigenvalues fo T_s must be -1 or u and the previous equality shows that u is not an eigenvalue of T_s on $\tilde{E}$. Hence $T_s=-1$ on $\tilde{E}$. This shows that E is the sign representation. Conversely, it is clear that the sign representation appears with coefficient 1 in α_{w_0}, since T_{w_0} acts as $(-1)^\nu$ on $\widetilde{\text{sign}}$.

2. Irreducible Representations of a Weyl Group of Type B_n

2.1 Let W_n be the group of all permutations of the set

$$\mathscr{S}_n=\{1, 2, \ldots, n, n', \ldots 2', 1'\}$$

which commute with the involution $i \to i'$, $i' \to i$ of $\mathscr{S}_n$. A permutation in W_n defines a permutation of the n element set consisting of the pairs $(1, 1')$, $(2, 2')$, …, (n, n'). Thus we have a natural homomorphism of W_n onto $\mathfrak{S}_n$, the symmetric group in n letters.

Let $\chi\colon W_n \to \{\pm 1\}$ be the homomorphism defined by

$$\chi(\sigma)=\begin{cases} 1 & \text{if } \{\sigma(1), \sigma(2), \ldots, \sigma(n)\} \cap \{1', 2', \ldots, n'\} \\ & \qquad \text{has even cardinality} \\ -1 & \text{, otherwise.} \end{cases}$$

The irreducible $\mathbb{Q}[W_n]$-modules are in $1-1$ correspondence with ordered pairs σ_1, σ_2 of irreducible representations of $\mathfrak{S}_k$, $\mathfrak{S}_l$ $(k+l=n)$. The correspondence is defined as follows. We identify $W_k \times W_l$ with the subgroup of W_n consisting of all permutations in W_n which map $\{1, 2, \dots, k, k', \dots, 2', 1'\}$ into itself and hence also may $\{k+1, \dots, n, n', \dots, (k+1)'\}$ into itself. As before, we have natural homomorphisms $W_k \to \mathfrak{S}_k$, $W_l \to \mathfrak{S}_l$. We can regard σ_1, σ_2 as representations σ_1, σ_2 of W_k, W_l, via these homomorphisms. Consider the representation $\bar{\sigma}_1 \otimes (\bar{\sigma}_2 \otimes \chi|_{W_l})$ of $W_k \times W_l$. We induce it to W_n; the resulting representation of W_n is irreducible; it is the representation corresponding to the ordered pair (σ_1, σ_2). Now σ_1 corresponds to a partition $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \dots \leqq \alpha_{m'}$ of k ($\sum \alpha_i = k$), in the following way: it is the unique irreducible representation of $\mathfrak{S}_k$ whose restriction to $\mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \dots \times \mathfrak{S}_{\alpha_{m'}} \subset \mathfrak{S}_k$ contains the unit representation and its restriction to $\mathfrak{S}_{\alpha_1^*} \times \mathfrak{S}_{\alpha_2^*} \times \dots \subset \mathfrak{S}_k$ (where $\alpha_1^* \leqq \alpha_2^* \leqq \dots$ is the dual partition) contains the sign representation. Similarly, σ_2 corresponds to a partition $0 \leqq \beta_1 \leqq \dots \leqq \beta_{m''}$ of l. Since m', m'' can be increased at our will (by adding zeroes) we may assume that $m' = m+1$, $m'' = m$. We now set $\lambda_i = \alpha_i + i - 1$, $(1 \leqq i \leqq m+1)$, $\mu_i = \beta_i + i - 1$ $(1 \leqq i \leqq m)$. Let Λ denote the tableau

$$\begin{pmatrix} \lambda_1, \lambda_2, \dots, \lambda_{m+1} \\ \mu_1, \mu_2, \dots, \mu_m \end{pmatrix}.$$

Then Λ is a symbol of *rank* n and *defect* 1 in the sense of [10, 3.1]. In other words, $\{\lambda_1, \lambda_2, \dots, \lambda_{m+1}\}$ is a set of $m+1$ distinct, $\geqq 0$ integers, $\{\mu_1, \mu_2, \dots, \mu_m\}$ is a set of m distinct, $\geqq 0$ integers and $\sum \lambda_i + \sum \mu_i = n + m^2$. Since m can be increased at our will, we must regard Λ as being equivalent to the symbol

$$\begin{pmatrix} 0, \lambda_1+1, \lambda_2+1, \dots, \lambda_{m+1}+1 \\ 0, \mu_1+1, \dots, \mu_m+1 \end{pmatrix}$$

obtained from Λ by a shift and also to the symbols obtained from Λ by iterating such shifts. We shall often identify a symbol with its equivalence class under shift. We shall denote by $[\Lambda]$ the irreducible representation of W_n corresponding to (σ_1, σ_2), constructed above.

Thus, we have a $1-1$ correspondence $[\Lambda] \leftrightarrow \Lambda$ between irreducible $\mathbb{Q}(W_n]$-modules and symbols Λ of rank n and defect 1, modulo shift.

We shall regard W_n as a Weyl group of type B_n, with simple reflections as described in [10, 2.1]. The longest element of W_n is the permutation $1 \leftrightarrow 1'$, $2 \leftrightarrow 2', \dots, n \leftrightarrow n'$.

2.2. **Lemma.** *The longest element of W_n acts on $[\Lambda]$ as multiplication by* $\varepsilon_{[\Lambda]} = (-1)^{\sum_1^m (\mu_i - i + 1)}$ *(with the previous notations).*

Proof. Assume that $[\Lambda]$ corresponds to (σ_1, σ_2) as above where σ_1 is a representation of $\mathfrak{S}_k$ and σ_2 is a representation of $\mathfrak{S}_l$, $k+l=n$. Using the definitions, we are immediately reduced to the case where either $k=n$ or $l=n$. If $k=n$, we have $\varepsilon_{[\Lambda]}=1$, since $[\Lambda]$ factors through $\mathfrak{S}_n$ and the longest element of W_n is in the kernel of $W_n \to \mathfrak{S}_n$. Similarly, if $l=n$, we have $\varepsilon_{[\Lambda]} = \varepsilon_\chi = (-1)^n$ and the lemma is proved.

2.3. The sign character of W_n is the tensor product of the sign character of $\mathfrak{S}_n$ lifted to W_n, with the character χ, defined in 2.1.

Let $\Lambda=\begin{pmatrix}\lambda_1, \ldots, \lambda_{m+1}\\ \mu_1, \ldots, \mu_m\end{pmatrix}$ be a symbol of rank n and defect one.
Let t be an integer, $t \geqq \max(\lambda_i, \mu_i)$. We consider

$$\bar{\Lambda}=\begin{matrix}\{t-i \mid 0 \leqq i \leqq t,\ i \neq \mu_1, \ldots, \mu_m\}\\ \{t-i \mid 0 \leqq i \leqq t,\ i \neq \lambda_1, \ldots, \lambda_{m+1}\}.\end{matrix}$$

Then $\bar{\Lambda}$ is again a symbol of rank n and defect one. Its equivalence class is independent of the choice of t.

2.4. **Lemma.** $[\Lambda] \otimes \mathrm{sign}=[\bar{\Lambda}]$.

Proof. This can be immediately reduced to a statement on representations of symmetric groups, for which we can appeal to [14, (1.7)].

2.5. Let $\Lambda=\begin{pmatrix}\lambda_1, \ldots, \lambda_{m+1}\\ \mu_1, \ldots, \mu_m\end{pmatrix}$ be a symbol of rank n and defect one. The following formulas follow from the results in [10, (2.8.1), 8.2].

(2.5.1) $\quad a[\Lambda] \equiv A[\Lambda] \pmod 2$.

(2.5.2) $$a[\Lambda]=\sum_{1 \leqq i<j \leqq m+1} \inf(\lambda_i, \lambda_j)+\sum_{1 \leqq i<j \leqq m} \inf(\mu_i, \mu_j) + \sum_{\substack{1 \leqq i \leqq m+1\\ 1 \leqq j \leqq m}} \inf(\lambda_i, \mu_j)-\tfrac{1}{6} m(m-1)(4m+1).$$

(Note that this expression is invariant under shift.)

(2.5.3) $\gamma_{[\Lambda]}=2^{-d}$, where $2d+1$ is the number of "singles" in Λ (entries which appear in exactly one row of Λ).

(Recall that γ was defined in 1.8.)

2.6. We identify $\mathfrak{S}_r \times W_s$ $(r+s=n)$ with the subgroup of W_n consisting of all permutations in W_n which map $\{1, 2, \ldots, r\}$ into itself, (hence also map $\{1', 2', \ldots, r'\}$ and $\{r+1, \ldots, n, n', \ldots, (r+1)'\}$ into themselves. This is a standard parabolic subgroup of W_n. We consider an irreducible representation of $\mathfrak{S}_r \times W_s$ of the form $\varepsilon(r) \otimes [\Lambda']$ where $\varepsilon(r)$ is the sign representation of $\mathfrak{S}_r$ and

$$\Lambda'=\begin{pmatrix}\lambda'_1, \ldots, \lambda'_{m+1}\\ \mu'_1, \ldots, \mu'_m\end{pmatrix}$$

is a symbol of rank s and defect one. Since m can be increased at our will, we may take it so that $2m+1 \geqq r$. We want to associate to Λ' a symbol Λ of rank n and defect one. We try to define Λ as the symbol obtained by increasing each of the r largest entries in Λ' by one and leaving the others unchanged. However, it may happen that the set of r largest entries in Λ' is not uniquely defined but there are two choices for it. Then the same procedure gives rise to two distinct symbols Λ_{I}, Λ_{II} of rank n and defect one. For example, if $\Lambda'=\begin{pmatrix}1 & 2 & 5\\ & 2 & 3 & \end{pmatrix}$ and $r=2$, the r largest entries in Λ' are $\begin{pmatrix}\cdot & \cdot & 5\\ & \cdot & 3 & \end{pmatrix}$, hence $\Lambda=\begin{pmatrix}1 & 2 & 6\\ & 2 & 4 & \end{pmatrix}$ is defined. If however $r=3$, the r largest entries in Λ' could be taken as $\begin{pmatrix} \cdot & 2 & 5 \\ & \cdot & 3 \end{pmatrix}$ or as $\begin{pmatrix} \cdot & \cdot & 5 \\ & 2 & 3 \end{pmatrix}$. Accordingly, we have $\Lambda_{\mathrm{I}}=\begin{pmatrix} 1 & 3 & 6 \\ & 2 & 4 \end{pmatrix}$, $\Lambda_{\mathrm{II}}=\begin{pmatrix} 1 & 2 & 6 \\ & 3 & 4 \end{pmatrix}$.

In general, we have $a(\varepsilon(r)\otimes[\Lambda'])=\binom{r}{2}+a[\Lambda']=\begin{cases} a[\Lambda] \\ \text{or } a[\Lambda_{\mathrm{I}}]=a[\Lambda_{\mathrm{II}}]\end{cases}$, as one sees immediately from (2.5.2). Moreover, by (2.5.3), we have

$$\gamma_{\varepsilon(r)\otimes[\Lambda']}=\gamma_{[\Lambda']}=\begin{cases}\gamma_{[\Lambda]} \\ \text{or } \gamma_{[\Lambda_{\mathrm{I}}]}+\gamma_{[\Lambda_{\mathrm{II}}]}.\end{cases}$$

2.7. **Proposition.** *With the previous notations, we have*

$$J_{\mathfrak{S}_r\times W_s}^{W_n}(\varepsilon(r)\otimes[\Lambda'])=\begin{cases}[\Lambda] \\ \text{or } [\Lambda_{\mathrm{I}}]+[\Lambda_{\mathrm{II}}].\end{cases}$$

The remarks just proceding the proposition, in conjunction with (1.19.2), (1.19.3), show that the proposition is a consequence of the following

2.8. **Lemma.** *If Λ is defined, then $[[\Lambda]:\varepsilon(r)\otimes[\Lambda']]_{\mathfrak{S}_r\times W_s}\geqq 1$. If Λ is not defined, then $[[\Lambda_{\mathrm{I}}]:\varepsilon(r)\otimes[\Lambda']]_{\mathfrak{S}_r\times W_s}\geqq 1$ and $[[\Lambda_{\mathrm{II}}]:\varepsilon(r)\otimes[\Lambda']]_{\mathfrak{S}_r\times W_s}\geqq 1$.*

Proof. Assume that $[\Lambda']$ corresponds to the pair σ_1,σ_2 of irreducible representations of $\mathfrak{S}_a$, $\mathfrak{S}_b$ (respectively), $a+b=s$ (see 2.1). Similarly, we assume that $[\Lambda]$ (or $[\Lambda_{\mathrm{I}}]$, or $[\Lambda_{\mathrm{II}}]$, if defined) corresponds to the pair τ_1,τ_2 of irreducible representations of $\mathfrak{S}_c$, $\mathfrak{S}_d$ (respectively), $c+d=n$. We must prove that

$$\text{(2.8.1)}\qquad \begin{aligned}&\langle \operatorname{Ind}_{\mathfrak{S}_r\times W_a\times W_b}^{W_n}(\varepsilon(r)\otimes\bar\sigma_1\otimes\bar\sigma_2\cdot\chi),\\ &\operatorname{Ind}_{W_c\times W_d}^{W_n}(\bar\tau_1\otimes\bar\tau_2\cdot\chi)\rangle_{W_n}\geqq 1\end{aligned}$$

where $\bar\sigma_i,\bar\tau_i$ are defined as in 2.1, $\bar\sigma_2\cdot\chi$ means $\bar\sigma_2\otimes\chi\mid W_{b'}$, $\bar\tau_i\chi$ means $\bar\tau_2\otimes\chi\mid W_d$; $W_c\times W_d$ is identified with the subgroup of W_n consisting of all permutations in W_n which map $\{1,2,\ldots,c,c',\ldots,2',1'\}$ into itself and hence also map $\{c+1,\ldots,n,n',\ldots,(c+1)'\}$ into itself; $\mathfrak{S}_r\times W_a\times W_b$ is identified with the subgroup of W_n consisting of all permutations in W_n which map $\{1,2,\ldots,a,a',\ldots,1'\}$ into itself, $\{n-b+1,\ldots,n,n',\ldots,(n-b+1)'\}$ into itself, $\{a,a+1,\ldots,n-b\}$ into itself, hence also map $\{(n-b)',\ldots,(a+1)',a'\}$ into itself. The intersection of the two subgroups of W_n appearing in (2.8.1) is the subgroup $\mathfrak{S}_{c-a}\times W_a\times\mathfrak{S}_{d-b}\times W_b$ of W_n consisting of all permutations in W_n which map $\{1,2,\ldots,a,a',\ldots,1'\}$ into itself, $\{n-b+1,\ldots,n,n',\ldots,(n-b+1)'\}$ into itself, $\{a+1,\ldots,c\}$ into itself, $\{c+1,\ldots,n-b\}$ into itself, hence also $\{c',\ldots,(a+1)'\}$ into itself and $\{(n-b)',\ldots,(c+1)'\}$ into itself. (Note that $a\leqq c$, $b\leqq d$.) The inner product (2.8.1) is a sum of contributions ($\geqq 0$) from the various double cosets of W_n with respect to the two subgroups in (2.8.1). It is enough to show that the contribution of the double coset of the identity element is $\geqq 1$. That contribution is an inner product of two representations of the intersections of these two subgroups. Thus, it is enough to show that

$$\begin{aligned}&\langle\varepsilon(c-a)\otimes\bar\sigma_1\otimes\varepsilon(d-b)\otimes\bar\sigma_2\chi,\\ &(\bar\tau_1\mid\mathfrak{S}_{c-a}\times W_a)\otimes(\bar\tau_2\mid\mathfrak{S}_{d-b}\times W_b)\rangle_{\mathfrak{S}_{c-a}\times W_a\times\mathfrak{S}_{d-b}\times W_b}\geqq 1\end{aligned}$$

or, equivalently, that

$$\langle \varepsilon(c-a)\otimes\bar{\sigma}_1, \bar{\tau}_1\rangle_{\mathfrak{S}_{c-a}\times W_a}\cdot\langle \varepsilon(d-b)\otimes\bar{\sigma}_2\chi, \bar{\tau}_2\chi\rangle_{\mathfrak{S}_{d-b}\times W_b}\geqq 1$$

or, equivalently, that

$$\langle \varepsilon(c-a)\otimes\sigma_1, \tau_1\rangle_{\mathfrak{S}_{c-a}\times\mathfrak{S}_a}\cdot\langle \varepsilon(d-b)\otimes\sigma_2, \tau_2\rangle_{\mathfrak{S}_{d-b}\times\mathfrak{S}_b}\geqq 1.$$

We have

$$\langle \varepsilon(c-a)\otimes\sigma_1, \tau_1\rangle_{\mathfrak{S}_{c-a}\times\mathfrak{S}_a}=1, \qquad \langle \varepsilon(d-b)\otimes\sigma_2, \tau_2\rangle_{\mathfrak{S}_{d-b}\times\mathfrak{S}_b}=1$$

as it is well known in the representation theory of the symmetric group. This completes the proof of the Lemma.

2.9. Let $Z=\begin{pmatrix} z_0, z_2, \ldots, z_{2m} \\ z_1, z_3, \ldots, z_{2m-1}\end{pmatrix}$ be a symbol of rank n and defect one. We arrange the z's in such a way that $z_0<z_2<\ldots<z_{2m}$, $z_1<z_3<\ldots<z_{2m-1}$. We say that Z is a *special symbol*, if the inequalities $z_0\leqq z_1\leqq z_2\leqq z_3\leqq\ldots\leqq z_{2m-1}\leqq z_{2m}$ are satisfied. This concept is clearly invariant under shift. The following result is immediate from (2.5.2).

2.10. **Lemma.** *Let Z be as above. Assume that Z is special. Then*

$$a[Z]\equiv\sum_{i=1}^{m}(z_{2i-1}-i+1)\,(\mathrm{mod}\,2).$$

In other words, we have $\varepsilon_{[Z]}=(-1)^{a[Z]}$ (see Lemma 2.2).

2.11. Assume now that Z is special. Let Φ be an arrangement of the $2d+1$ "singles" in Z into d disjoint pairs and one isolated element, such that each pair in Φ contains one single in the first row of Z and one in the second row of Z. We want to define what it means for Φ to be an *admissible arrangement* for Z. We use induction on d. If $d=1$ there is a unique arrangement, the empty one; it is, by definition, admissible. Assume now that $d>1$. An arrangement Φ for Z is admissible if Φ contains a pair of singles $(z_i<z_j)$ in different rows of Z such that there are no singles z' in Z with $z_i<z'<z_j$ and if the corresponding arrangement for the special symbol obtained from Z by removing z_i, z_j is admissible. For example, the special symbol $\begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix}$ has two admissible arrangements: one of them consists of the pair (0, 1), the other one consists of the pair (1, 2). As another example, the symbol $\begin{pmatrix} 0 & & 2 & & 4 \\ & 1 & & 3 & \end{pmatrix}$ has 5 admissible arrangements:

the first one consists of (0, 1), (2, 3)
the second one consists of (1, 2), (3, 4)
the third one consists of (0, 1), (3, 4)
the fourth one consists of (1, 2), (0, 3)
the fifth one consists of (2, 3), (1, 4).

2.12. If Ψ is a subset of Φ, we denote by Ψ^* the set of singles in the first row of Z which appear in a pair of Ψ; we denote by Ψ_* the set of singles in the second row of Z which appear in a pair of Ψ. Let $(Z_0)^*$ (resp. $(Z_0)_*$) be the set of elements in the first row of Z (resp. the second row of Z) which don't appear in any pair of Φ. In particular any entry of Z which is not a single is in $(Z_0)^*$ or $(Z_0)_*$.

Let Φ be an admissible arrangement for the special symbol Z as above. For any subset $\hat{\Phi}\subset\Phi$ we define a virtual representation of W_n (n=rank of Z) by the formula

$$\underline{c}(Z,\Phi,\hat{\Phi})=\sum_{\Psi\subset\Phi}(-1)^{e(\Psi)}\begin{pmatrix}(Z_0)^*\amalg\Psi_*\amalg(\Phi-\Psi)^*\\(Z_0)_*\amalg\Psi^*\amalg(\Phi-\Psi)_*\end{pmatrix}\tag{2.12.1}$$

where $e(\Psi)=|\hat{\Phi}\cap\Psi^*|$. Note that $\underline{c}(Z,\Phi,\hat{\Phi})$ is a sum of 2^d terms (d=number of pairs in Φ) each of which is $\pm$ an irreducible representation of W_n corresponding to a symbol of rank n and defect one. The term corresponding to $\Psi=\emptyset$ is $[Z]$ itself; all other terms are of form $[\Lambda]$ with Λ non-special. The $\underline{c}(Z,\Phi,\hat{\Phi})$ are called the *virtual cells* of W_n.

2.13. Consider a virtual cell $\underline{c}(Z,\Phi,\hat{\Phi})$ as in 2.12. We define a new virtual cell $\underline{c}(\bar{Z},\bar{\Phi},\bar{\hat{\Phi}})$ as follows. Choose an integer $t\geqq z_{2m}$. Let

$$\bar{Z}=\begin{pmatrix}\{t-i\,|\,0\leqq i\leqq t,\ i\neq z_1,z_3,\dots,z_{2m-1}\}\\\{t-i\,|\,0\leqq i\leqq t,\ i\neq z_0,z_2,\dots,z_{2m}\}\end{pmatrix}.$$

This is again a special symbol of rank n and defect one. There is a $1-1$ correspondence $z\leftrightarrow t-z$ between the singles in Z and the singles in $\bar{Z}$. Using this $1-1$ correspondence we transport Φ, $\hat{\Phi}$ to $\bar{Z}$ and we get an arrangement $\bar{\Phi}$ of $\bar{Z}$ and a subset $\bar{\hat{\Phi}}$ of $\bar{\Phi}$. It is easy to see that $\bar{\Phi}$ is admissible. Using now Lemma 2.4, we see that the following result holds

2.14. **Lemma.** $\underline{c}(Z,\Phi,\hat{\Phi})\otimes\text{sign}=\underline{c}(\bar{Z},\bar{\Phi},\bar{\hat{\Phi}})$.

2.15. We now consider the standard parabolic subgroup $\mathfrak{S}_r\times W_s$ of W_n ($r+s=n$) as in 2.6. Let

$$Z'=\begin{pmatrix}z'_0 & & z'_2 & & \cdots & & z'_{2m}\\ \leqq & \leqq & \leqq & \leqq & \leqq & & \leqq \\ & z'_1 & & z'_3 & & z'_{2m-1} & \end{pmatrix}$$

be a special symbol of rank s and defect 1; we shall assume, as we may, that $2m+1\geqq r$. We associate to Z' the special symbol

$$Z=\begin{pmatrix}z_0 & & z_2 & & & z_{2m}\\ \leqq & \leqq & \leqq & & \leqq & \\ & z_1 & & z_3\dots z_{2m-1} & & \end{pmatrix}$$

defined by $z_i=z'_i\,(0\leqq i\leqq 2m-r)$, $z_i=z'_i+1\,(2m+1-r\leqq i\leqq 2m)$. Then Z has rank n. Suppose we are given an admissible arrangement Φ' for Z' and a subset $\hat{\Phi}'$ of Φ'. We transport these to Z using the natural bijection $z'_i\leftrightarrow z_i$ between Z' and Z. In the case where $r=2m+1$ or $r\leqq 2m$ and $z'_{2m-r}<z'_{2m+1-r}$ (so that Z, Z' have the same number of singles) we thus get an admissible arrangement

Φ for Z and a subset $\hat{\Phi}\subset\Phi$. In the case where $z'_{2m-r}=z'_{2m+1-r}$ (so that Z has 2 new singles in addition to those coming from Z'), the set of pairs in Z coming from those in Φ' together with the new pair (z_{2m-r}, z_{2m+1-r}) from an admissible arrangement for Z. It has a subset $\hat{\Phi}$ corresponding to the pairs in $\hat{\Phi}'$ (the new pair is not in $\hat{\Phi}$). Using now Proposition 2.7 we see that the following result holds.

2.16. **Lemma.** $J^{W_n}_{\mathfrak{S}_r\times W_s}(\varepsilon(r)\otimes\underline{c}(Z', \Phi', \hat{\Phi}'))=\underline{c}(Z, \Phi, \hat{\Phi})$.

2.17. Let $Z, \Phi, \hat{\Phi}$ be as in 2.12. Let Φ_1 be the set of pairs (z_i, z_j) in Φ such that z_i+z_j is odd. Let $\Phi_2\subset\Phi$ be defined by $\Phi_2=(\hat{\Phi}\cup\Phi_1)-(\hat{\Phi}\cap\Phi_1)$. We have

2.18. **Lemma.** $(-1)^{a[Z]}\zeta(\underline{c}(Z, \Phi, \hat{\Phi}))=\underline{c}(Z, \Phi, \Phi_2)$ *where* ζ *is defined as in* 1.17. *(Note that the longest element in* W_n *is central.)*

Proof. By Lemmas 2.2 and 2.10, the left hand side of the identity to be proved equals

$$\sum_{\Psi\subset\Phi}(-1)^{e'(\Psi)}\begin{pmatrix}(Z_0)^*\amalg\Psi_*\amalg(\Phi-\Psi)^*\\(Z_0)_*\amalg\Psi^*\amalg(\Phi-\Psi)_*\end{pmatrix}$$

where

$$\begin{aligned}e'(\Psi)&=e(\Psi)+\sum_{i\text{ odd}}z_i+\sum_{z_i\in(Z_0)_*}z_i+\sum_{z_i\in\Psi^*}z_i+\sum_{z_i\in(\Phi-\Psi)_*}z_i\\&\equiv e(\Psi)+\sum_{z_i\in\Psi^*}z_i+\sum_{z_i\in\Psi_*}z_i\quad(\text{mod }2)\\&\equiv e(\Psi)+|\Phi_1\cap\Psi|\quad(\text{mod }2)\\&=|\hat{\Phi}^*\cap\Psi^*|+|\Phi_1^*\cap\Psi^*|\\&\equiv|\Phi_2^*\cap\Psi^*|\quad(\text{mod }2)\end{aligned}$$

and the lemma is proved.

Remark. If, for example $z_i\equiv i$ (mod 2) for all i, we have $\Phi_1=\Phi$ and $\Phi_2=\Phi-\hat{\Phi}$.

2.19. We now define by induction on n a certain set of involutions $\Omega_n\subset W_n$. For $n=0$, we take $\Omega_0=W_0=\{e\}$. Assume now that $n\geqq 1$ and that $\Omega_s\subset W_s$ is already defined for $s<n$. We say that $w\in W_n$ is in Ω_n if and only if there exists a partition $n=r+s$ $(0\leqq s<n)$ and an element $z\in\Omega_s\subset W_s$ such that w is either equal to $w_0^{(r)}\cdot z\in\mathfrak{S}_r\times W_s\subset W_n$ ($w_0^{(r)}$ is the logest element of $\mathfrak{S}_r$ and $\mathfrak{S}_r\times W_s$ is the standard parabolic subgroup as in 2.6) or it is equal to $w_0\cdot(w_0^{(r)}\cdot z)$ where $w_0^{(r)}z$ is as before and w_0 is the longest element of W_n.

2.20. **Proposition.** *The following* 3 *sets of virtual representations of* W_n *coincide:*

(a) $\{\alpha_w\mid w\in\Omega_n\}$,
(b) $\{\mathscr{A}_w\mid w\in\Omega_n\}$,
(c) *the set of virtual cells of* W_n.

Moreover, if $\underline{c}(Z, \Phi, \hat{\Phi})=\mathscr{A}_w=\alpha_{w'}$ $(w, w'\in\Omega_n, Z, \Phi, \hat{\Phi}$ *as in* 2.12), *then*

(2.20.1) $$a[Z]\equiv l(w')\ (\text{mod }2).$$

(2.20.2) $$\nu-A[Z]\equiv l(w)\ (\text{mod }2).$$

Proof. This is obvious when $n=0$. Assume now that $n \geqq 1$ and that the proposition is already known for $n' < n$.

First, we show that if $w \in \Omega_n$ then α_w is a virtual cell. If $w = w_0^{(r)} \cdot z \in \mathfrak{S}_r \times W_s \subset W_n$ ($z \in \Omega_s$, $0 \leqq s \leqq n$, $r+s=n$) then by 1.20, 1.21 and 1.22 we have

$$\alpha_w = J_{\mathfrak{S}_r \times W_s}^{W_n}(\varepsilon(r) \otimes \alpha_z^{(W_s)}).$$

By the induction hypothesis, $\alpha_s^{(W_s)}$ is a virtual cell, hence by Lemma 2.15, $\alpha_{w'}$ is also a virtual cell. At the same time we deduce from the induction hypothesis that (2.20.1) holds for our w. Now let $w' = w_0 w$ where w is the element we have just considered. We have

$$\begin{aligned}
\alpha_{w'} &= (-1)^{l(w')} \zeta(\mathscr{A}_{w_0 w'}), && \text{by Lemma 1.18}\\
&= (-1)^{l(w')} \zeta(\alpha_w \otimes \text{sign}), && \text{by Lemma 1.16}\\
&= (-1)^{l(w')} \zeta(\underline{c}(Z, \Phi, \hat{\Phi}) \otimes \text{sign}), && \text{by first part of proof}\\
&= (-1)^{l(w')} \zeta(\underline{c}(\bar{Z}, \bar{\Phi}, \bar{\hat{\Phi}})), && \text{by Lemma 2.14}\\
&= (-1)^{l(w') + a[\bar{Z}]} \cdot \text{virtual cell}, && \text{by Lemma 2.17.}
\end{aligned}$$

But as (2.20.1) holds for w, we have $a(Z) \equiv l(w) \pmod 2$. It follows that

$$\begin{aligned}
l(w') + a[\bar{Z}] &\equiv l(w_0 w) + a[Z] + n = l(w_0) - l(w) + n + a[Z]\\
&\equiv l(w_0) + n = n^2 + n \equiv 0 \pmod 2.
\end{aligned}$$

Thus $\alpha_{w'}$ is a virtual cell. We have at the same time verified that (2.20.1) holds for w'. We have verified that for all $w \in \Omega_n$, α_w is a virtual cell and (2.20.1) is satisfied.

We shall now prove that any virtual cell $\underline{c}(Z, \Phi, \hat{\Phi})$ of W_n (notations as in 2.12) is of the form α_w for some $w \in \Omega_n$. We may assume that 0 doesn't occur twice in Z. Let t_0 be the largest entry in Z. If some number i, $0 \leqq i \leqq t_0$, doesn't appear in Z, then there is an $r \geqq 1$ such that z_{2m-r+1} is $\geqq 1$ and appears in Z, but $z_{2m-r+1}+1$ doesn't appear in Z (which implies $r \leqq n$); let $s = n - r$. Then Z, Φ is obtained from a Z', Φ' for W_s as in 2.15. Moreover, since the number of singles in Z is exactly the same as the number of singles in Z', $\hat{\Phi}$ is also obtained from a subset $\hat{\Phi}' \subset \Phi'$ as in 2.15. By Lemma 2.16, we have

$$\begin{aligned}
\underline{c}(Z, \Phi, \hat{\Phi}) &= J_{\mathfrak{S}_r \times W_s}^{W_n}(\varepsilon(r) \otimes \underline{c}(Z', \Phi', \hat{\Phi}'))\\
&= J_{\mathfrak{S}_r \times W_s}^{W_n}(\varepsilon(r) \otimes \alpha_z^{(W_s)}), (z \in \Omega_n), && \text{by induction hypothesis}\\
&= \alpha_{w_0^{(r)} z}.
\end{aligned}$$

Consider now $\bar{Z}$, $\bar{\Phi}$ defined with respect to $t = t_0$ as in 2.13. Then 0 doesn't appear twice in $\bar{Z}$ and t_0 is the largest number in $\bar{Z}$. By 2.14 and 2.18, there is a unique subset $\bar{\hat{\Phi}} \subset \bar{\Phi}$ such that

$$\underline{c}(Z, \Phi, \hat{\Phi}) = (-1)^{a[Z]} \zeta(\underline{c}(\bar{Z}, \bar{\Phi}, \bar{\hat{\Phi}}) \otimes \text{sign}).$$

If some number i, $0 \leqq i < t_0$ doesn't appear in $\bar{Z}$, then by the previous argument we have $\underline{c}(\bar{Z}, \bar{\Phi}, \bar{\hat{\Phi}}) = \alpha_{w'}$ for some $w' \in \Omega_n$. Then $\underline{c}(Z, \Phi, \hat{\Phi}) = (-1)^{a[Z]} \zeta(\alpha_{w'} \otimes \text{sign})$

$=(-1)^{a[Z]+l(w_0 w')}\alpha_{w' w_0}$. If $a[Z]+l(w_0 w')$ was odd, $\alpha_{w' w_0}$ would be equal to minus a virtual cell. By the first part of the proof it is also equal to a virtual cell. But minus a virtual cell cannot be equal to a virtual cell, since a virtual cell has a unique component corresponding to a special symbol and that component appears with coefficient $+1$. It follows that $a[Z]+l(w_0 w')$ is even and $\underline{c}(Z, \Phi, \hat{\Phi})=\alpha_{w' w_0}$.

Thus we may assume that both Z and $\bar{Z}$ contain all numbers between 0 and t_0. It follows that each of these numbers is a single in Z, hence $t_0=2d$ and

$$Z=\begin{pmatrix} 0, & 2, & 4, \dots, 2d & \\ & 1, & 3, \dots, 2d-1 & \end{pmatrix}.$$

(This is a symbol of rank $n=d^2+d$.) By definition of an admissible arrangement there exists at least one pair $(i, i+1)\in\Phi$. Assume first that $(i, i+1)\notin\hat{\Phi}$. Let Z' be the special symbol obtained by replacing $i+1, i+2, \dots, 2d$ in Z by i, $i+1, \dots, 2d-1$ and keeping the other entries unchanged. Then Φ, $\hat{\Phi}$ come from corresponding objects Φ', $\hat{\Phi}'$ for Z' as in 2.13, and hence

$$\begin{aligned}\underline{c}(Z, \Phi, \hat{\Phi}) &= J^{W_n}_{\mathfrak{S}_{2d-i}\times W_{n-2d+i}}(\varepsilon(2d-i)\otimes\underline{c}(Z', \Phi', \hat{\Phi}')) \\ &= J^{W_n}_{\mathfrak{S}_{2d-i}\times W_{n-2d+i}}(\varepsilon(2d-i)\otimes\alpha_z) \\ &\quad (z\in\Omega_{n-2d+i}, \text{ by induction hypothesis}) \\ &= \alpha_{w_0^{(2d-i)}z}.\end{aligned}$$

Assume next that $(i, i+1)\in\hat{\Phi}$. We have

$$\underline{c}(Z, \Phi, \hat{\Phi})=(-1)^{a[Z]}\zeta(\underline{c}(\bar{Z}, \bar{\Phi}, \bar{\Phi}-\bar{\hat{\Phi}}).$$

(For our Z, by the remark following Lemma 2.18, we have $(-1)^{a[Z]}\zeta(\underline{c}(Z, \Phi, \hat{\Phi}))$ $=\underline{c}(Z, \Phi, \Phi-\hat{\Phi})$.) Now $(t_0-i-1,\ t_0-i)\in\bar{\hat{\Phi}}$ hence $(t_0-i-1,\ t_0-i)\notin\bar{\Phi}-\bar{\hat{\Phi}}$. By the previous argument it follows that $\underline{c}(\bar{Z}, \bar{\Phi}, \bar{\Phi}-\bar{\hat{\Phi}})=\alpha_w$ for some $w\in\Omega_n$, hence $\underline{c}(Z, \Phi, \hat{\Phi})=(-1)^{a[Z]}\zeta(\alpha_w\otimes\text{sign})=\pm\alpha_{w_0 w}$. As earlier in the proof, we see that the sign is $+1$. Thus, we have proved that each virtual cell of W_n is of the form α_w for some $w\in\Omega_n$. Hence the sets (a), (c) coincide. Under tensor product with sign, the set (c) remains stable (Lemma 2.14) while the sets (a), (b) are switched among them (1.16). It follows that the set (b) coincides with the sets (a) and (c). Finally (2.20.2) follows from (2.20.1) together with 1.14, and 2.4. This completes the proof of the proposition.

2.21. **Corollary.** *Let $\underline{c}$ be a virtual cell of W_n. There exist two integers $a(\underline{c})\leq A(\underline{c})$ such that $a(E)=a(\underline{c})$, $A(E)=A(\underline{c})$ for each irreducible representation E of W_n which appears with non-zero coefficient in $\underline{c}$. If $w\in\Omega_n$ is such that $\underline{c}=\mathscr{A}_w$, then*

$$\underline{c}=\sum \operatorname{Tr}\left(T_w, E; \frac{l(w)-A(E)+v}{2}\right) E$$

(sum over all irreducible $Q[W]$-modules E such that $A(E)=A(\underline{c})$).

Proof. The definition of a virtual cell and the formula (2.5.2) show that $a(E)$ is the same for all irreducible E appearing in $\underline{c}$ with non-zero coefficient. Applying this statement to the virtual cell $\underline{c}\otimes$sign and using Lemma 1.14, we see that $A(E)$ is the same for all irreducible E appearing in $\underline{c}$. The Corollary follows.

2.22. **Lemma.** *Let Λ be a symbol of rank n and defect one. Then there exists a special symbol Z of rank n and an admissible arrangement Φ for Z such that $[\Lambda]$ is the component of $\underline{c}(Z, \Phi, \Phi)$ corresponding to a subset $\Psi \subset \Phi$ in the sum* (2.12.1) *defining $\underline{c}(Z, \Phi, \Phi)$. We then have*

$$[\Lambda] = 2^{-d} \sum_{\hat{\Phi} \subset \Phi} (-1)^{e'(\hat{\Phi})} c(Z, \Phi, \hat{\Phi}) \tag{2.22.1}$$

where $2d+1$ is the number of singles in Z, and $e'(\hat{\Phi}) = |\hat{\Phi}^* \cap \Psi^*|$.

Proof. We may assume that $d \geqq 1$. We take the entries in both rows of Λ and arrange them in increasing order. We get a monotonic sequence of integers in which there may be equalities but no two consecutive equalities. The first, third, fifth, etc. term of this sequence will be the first row of Z while the second, fourth, etc. term of this sequence will be the second row. It is clear that Z is a special symbol of rank n. We can form a sequence of singles in $\Lambda: x_1, x_2, \ldots, x_g, x_{g+1}$ such that x_1 is the smallest single, x_2 is the next smallest single, etc, and such that $x_1, x_2, \ldots, x_g$ are in the same row of x_1 but x_{g+1} is in another row. (Not all singles can be in the same row of Λ.) Thus we have found the pair of singles (x_g, x_{g+1}) in different rows of Λ such that there are no singles of Λ in between x_g, x_{g+1}. We set $x^1 = x_g$, $x^2 = x_{g+1}$. We remove (x^1, x^2) from Λ. We get a symbol Λ' with only $2d-1$ singles. If $2d-1 \geqq 3$, we do the same procedure for Λ' as we did for Λ and we thus find a new pair (x^3, x^4). We iterate this procedure as long as it is possible. We find d pairs, which can be regarded as an admissible arrangement Φ for Z and it is then easy to see that $[\Lambda]$ is one of the components of $c(Z, \Phi, \Phi)$. (See also the proof of Lemma 3.4.)

The formula (2.22.1) follows immediately from (2.12.1).

3. Lagrangian Subspaces over F_2

3.1. Let V be a vector space over the field F_2, endowed with a basis $e_1, e_2, \ldots, e_{2d}$ and with a symplectic form $(,): V \times V \to F_2$ such that $(e_i, e_j) = \begin{cases} 1 & \text{if } |i-j|=1 \\ 0 & \text{otherwise} \end{cases}$. It is clear that $(,)$ is non-singular. If $d \geqq 1$, we consider for each i, $1 \leqq i \leqq 2d$, the vector space $V_i = \langle e_i \rangle^{\perp} / \langle e_i \rangle$ of dimension $2d-2$. It inherits a symplectic form $(,)$ from V, and also a basis $e'_1, \ldots, e'_{2d-2}$ given by $e'_h = e_h (h \leqq i-2)$, $e'_{i-1} = e_{i-1} + e_{i+1}$, $e'_h = e_{h+2}$ $(i \leqq h \leqq 2d-2)$, if $1 < i < 2d$ $e'_h = e_{h+2} (1 \leqq h \leqq 2d-2)$, if $i=1$, $e'_h = e_h$ $(1 \leqq h \leqq 2d-2)$, if $i = 2d$. We then have again $(e'_h, e'_k) = \begin{cases} 1 & \text{if } |h-k|=1 \\ 0 & \text{otherwise} \end{cases}$.

3.2. We define, by induction on d, a family $\mathscr{I}(V)$ of lagrangian (= maximal isotropic) subspaces of V, depending on the given basis (e_i) of V. If $d=0$, we set $\mathscr{I}(V) = \{0\}$. Assume now that $d \geqq 1$ and that $\mathscr{I}(V_i)$ has been already defined for

$1 \leqq i \leqq 2d$ (with respect to the basis of V_i described above). By definition, a lagrangian subspace C of V is in $\mathscr{I}(V)$ if and only if there exists i, $(1 \leqq i \leqq 2d)$ such that $e_i \in C$ and such that the image of C under the natural map $\langle e_i \rangle^{\perp} \to \langle e_i \rangle^{\perp} / \langle e_i \rangle$ is in $\mathscr{I}(V_i)$. For example, if $d=1$, $\mathscr{I}(V)$ consists of two subspaces; the first is spanned by e_1, e_3, the second is spanned by e_2, e_4, the third is spanned by e_1, e_4, the fourth is spanned by e_2, e_1+e_3 and the fifth is spanned by e_3, e_2+e_4.

3.3. We now define a function $f_V \colon V \to \mathbb{Z}$, as follows. Any $v \in V$ can be written uniquely in the form

$$v = \sum_{1 \leqq \alpha \leqq r} (e_{i_\alpha} + e_{i_\alpha+1} + \dots + e_{j_\alpha - 1})$$

where $1 \leqq i_1 < j_1 < i_2 < j_2 < \dots < j_r \leqq 2d$. We then set

$$f_V(v) = \#\{\alpha \mid i_\alpha \equiv j_\alpha \equiv 0 \,(\mathrm{mod}\, 2)\} - \#\{\alpha \mid i_\alpha \equiv j_\alpha \equiv 1 \,(\mathrm{mod}\, 2)\}.$$

3.4. **Lemma.** *Let $v \in V$. Then $f_V(v)=0$ if and only if there exists $C \in \mathscr{I}(V)$ such that $v \in C$.*

Proof. If $(v, e_h) \neq 0$ for all h, then v is uniquely determined: it is $(e_2+e_3)+(e_6+e_7)+(e_{10}+e_{11})+\dots$ if d is even and it is $(e_1+e_2)+(e_5+e_6)+(e_9+e_{10})+\dots$ if d is odd. In both cases, $f_V(v) \neq 0$ (if $s \geqq 1$). It is also clear that such v cannot be contained in any $C \subset \mathscr{I}(V)$. Hence we may assume that $d \geqq 1$ and that $(v, e_h) = 0$, for some h, and that the lemma is already proved for V_h. Then either $i_\alpha < h < j_\alpha - 1$ for some α, $(1 \leqq \alpha \leqq r)$, or $j_\alpha < h < i_{\alpha+1} - 1$ for some α, $(0 \leqq \alpha \leqq r)$, or $j_\alpha = h = i_{\alpha+1} - 1$ for some α, $(1 \leqq \alpha \leqq r-1)$. (We agree to set $j_0 = -1$, $i_{r+1} = 2d+1$.) A simple computation shows that in each of these three cases, the image $\bar{v}$ of v in $V_h = \langle e_h \rangle^{\perp} / \langle e_h \rangle$ satisfies $f_{V_h}(\bar{v}) = f_V(v)$. If $f_V(v)=0$, then $f_{V_h}(\bar{v})=0$, hence by the induction hypothesis, we have $\bar{v} \in C'$, $(C' \in \mathscr{I}(V_h))$. Then v is contained in the inverse image C of C' under $\langle e_h \rangle^{\perp} \to \langle e_h \rangle^{\perp} / \langle e_h \rangle = V_h$, and C is in $\mathscr{I}(V)$, by definition of $\mathscr{I}(V)$. Conversely, if $v \in C (C \in \mathscr{I}(V))$ then there exists an h such that $e_h \in C$ and C is the inverse image of $C' \in \mathscr{I}(V_h)$ under $\langle e_h \rangle^{\perp} \to \langle e_h \rangle^{\perp} / \langle e_h \rangle$. The image $\bar{v}$ of v in V_h is contained in C' hence, by the induction hypothesis it satisfies $f_{V_h}(\bar{v})=0$. But then $f_V(v) = f_{V_h}(\bar{v}) = 0$ and the Lemma is proved.

3.5. Let $\tilde{V} = \{v \in V \mid f_V(v) = 0\}$. The proof of Lemma 3.4 gives at the same time:

3.6. **Lemma.** *Let $v \in \tilde{V}$ be such that $\langle v, e_h \rangle = 0$ for some h, $1 \leqq h \leqq 2d$. Then there exists $C \in \mathscr{I}(V)$ containing v, e_h and $v + e_h$. In particular, we have also $v + e_h \in \tilde{V}$.*

We shall now prove

3.7. **Lemma.** *Assume that $d \geqq 1$. Given i, $(1 \leqq i \leqq 2d)$ and two elements $v, v' \in \tilde{V}$ such that $(v, e_i) = (v', e_i) = 1$, there exists a sequence of elements $v = v_1, v_2, \dots, v_m = v'$ in $\tilde{V}$ and a sequence of subspaces $C_1, C_2, \dots, C_{m-1}$ in $\mathscr{I}(V)$ such that v_h, $v_{h+1} \in C_h$ $(1 \leqq h \leqq m-1)$ and $(v_h, e_i) = 1$, $(1 \leqq h \leqq m)$.*

Proof. When $d=1$, we must have $v = v' =$ basis vector $e_{i'}$ other then e_i, hence the lemma is obvious in this case. We now assume $d \geqq 2$, and that the lemma is already proved for all $V_h (1 \leqq h \leqq 2d)$. We shall make the additional assumption that v' is one of the basis elements e_{i+1} or e_{i-1} and that basis element appears with coefficient 1 in v. This will certainly imply the general case, since e_{i-1}, e_{i+1} (if both are defined) are contained in the same $C \in \mathscr{I}(V)$, by Lemma 3.6. To be definite, we assume that $v'=e_{i-1}$ hence $i \geqq 2$ and that e_{i-1} appears with coefficient 1 in v. (The other case is entirely similar.) Since $v \in \tilde{V}$, it satisfies $(v, e_h)=0$ for some h. Since, by assumption, $h \neq i$, we are in one the four cases below.

Case 1. There exists $h \neq i-2, i-1, i, i+1$ such that $(v, e_h)=0$. Then v, e_{i-1}, e_i are in $\langle e_h \rangle^{\perp}$ and their images $\bar{v}, \bar{e}_{i-1}$ in V_h are not orthogonal to the image $\bar{e}_i$ in V_h. Moreover $\bar{e}_i$ is one of the elements in the standard basis of V_h. Applying the induction hypothesis to V_h, we find a sequence of elements $v=v_1, v_2, \ldots, v_m=e_{i-1}$ in $\langle e_h \rangle^{\perp}$ and a sequence of subspaces $C_1, \ldots, C_{m-1} \subset \mathscr{I}(V)$ such that each C_j contains e_h, v_j, v_{j+1} and $(v_j, e_i)=1$, for all j, as required.

Case 2. $(v, e_{i-1})=0$. In this case, by Lemma 3.6, there exists $C \subset \mathscr{I}(V)$ such that v, e_{i-1} are both in C.

Case 3. $(v, e_{i-2})=0$. Since e_{i-1} appears with coefficient 1 in v, and $(v, e_{i-2})=0$, it follows that e_{i-3} also appears with coefficient 1 in v (and, in particular, $i \geqq 4$). We may assume that $(v, e_{i-3})=1$, otherwise we are in Case 1 and we are done. But then we must have $(v+e_{i-2}, e_{i-3})=1+1=0$ hence $v+e_{i-2}$ satisfies the assumption of Case 1 with $h=i-3$. (Note that $v+e_{i-2} \in \tilde{V}$ (by Lemma 3.6), $(v+e_{i-2}, e_i)=1$ and e_{i-1} appears with coefficient 1 in $v+e_{i-2}$.) Applying Case 1 to $v+e_{i-2}$ and using the fact that there exists $C \in \mathscr{I}(V)$ containing v and $v+e_{i-2}$ (see Lemma 3.6), we are again done.

Case 4. $(v, e_{i+1})=0$. By Lemma 3.6, there exists a $C \in \mathscr{I}(V)$ containing v and e_{i+1} and also a $C' \in \mathscr{I}(V)$ containing e_{i+1} and e_{i-1}. Since $(e_{i+1}, e_i)=1$, we are done. The Lemma is proved.

3.8. **Proposition.** *Let $x, y \rightarrow [x, y]$ be a map $V \times \tilde{V} \rightarrow F_2$ with the following properties.*

a) *For any $x \in V$, and any $C \in \mathscr{I}(V)$, the function $y \rightarrow [x, y]$ $(C \rightarrow F_2)$ is F_2-linear.*

b) *For any $x \in V$, $y \in \tilde{V}$ and e_j such that $(x, e_j)=0$, we have*

$$(-1)^{[x, y]}+(-1)^{[x+e_j, y]}=(-1)^{(x, y)}+(-1)^{(x+e_j, y)}.$$

c) *For any $y \in \tilde{V}$, we have*

$$\sum_{x \in V}(-1)^{[x, y]}=\sum_{x \in V}(-1)^{(x, y)}.$$

Then $[x, y]=(x, y)$ for all $x \in V$, $y \in \tilde{V}$.

Proof. Let us fix $x, x' \in V$ and e_j such that $x'=x+e_j$, $(x, e_j)=0$. Let $C \in \mathscr{I}(V)$ be such that $e_j \in C$. Then $(e_j, y)=0$ for all $y \in C$, hence, by b), $(-1)^{[x, y]}+(-1)^{[x', y]} =2(-1)^{(x, y)}$ for all $y \in C$. It follows that $(-1)^{[x, y]}=(-1)^{[x', y]}=(-1)^{(x, y)} =(-1)^{(x', y)}$, for all $y \in C$ hence $[x, y]=[x', y]=(x, y)=(x', y)$ for all $y \in C$. Now let $C \in \mathscr{I}(V)$ be such that $e_j \notin C$. From b) we have

$$(-1)^{[x, y]+(x, y)}+(-1)^{[x', y]+(x, y)}=1+(-1)^{(e_j, y)}$$

hence

$$2^{-s}\sum_{y\in C}(-1)^{[x,y]+(x,y)}+2^{-s}\sum_{y\in C}(-1)^{[x',y]+(x,y)}=1$$

(since $y\to(e_j, y)$ is a linear function on C, not identically zero.) It follows then from a) that exactly one of the linear functions $y\to[x,y]+(x,y)$, $y\to[x',y]+(x,y)$ on C is zero. Similarly, exactly one of the linear functions $y\to[x,y]+(x',y)$, $y\to[x',y]+(x',y)$ on C is zero. Thus there are 2 possibilities:

1) $[x,y]=(x,y)$ and $[x',y]=(x',y)$ for all $y\in C$ (we then say that C is of the 1st kind).

2) $[x,y]=(x',y)$ and $[x',y]=(x,y)$ for all $y\in C$ (we then say that C is of the 2nd kind).

We shall now show that all $C\in\mathscr{I}(V)$ such that $e_j\notin C$ are of the same kind. If this is not the case, we could find $C, C'\in\mathscr{I}(V)$, such that $e_j\notin C$, $e_j\notin C'$, with C of the 1st kind and C' of the 2nd kind. We can find vectors $v\in C$, $v'\in C'$ such that $(v,e_j)=1$, $(v',e_j)=1$ (since $e_j\notin C$, $e_j\notin C'$). By Lemma 3.7, there exists a sequence of elements $v=v_1, v_2, \ldots, v_m=v'$ in $\tilde{V}$ and a sequence of subspaces $C_1, C_2, \ldots, C_{m-1}$ in $\mathscr{I}(V)$ such that $(v_i,e_j)=1$ $(1\leqq i\leqq m)$, and $v_i, v_{i+1}\in C_i$ $(1\leqq i\leqq m-1)$. We set $C=C_0$, $C'=C_m$. Then $v_i\in C_{i-1}\cap C_i$ $(1\leqq i\leqq m)$. Since $(v_i,e_j)=1$ $(1\leqq i\leqq m)$, we have $e_j\in C_i$ $(0\leqq i\leqq m)$. Now C_0 is of the 1st kind and C_m is of the 2nd kind. Hence there exists $i(1\leqq i\leqq m)$ such that C_{i-1} is of the 1st kind and C_i is of the 2nd kind. The vector $v_i\in C_{i-1}\cap C_i$ will then satisfy simultaneously the equations:

$$[x,v_i]=(x,v_i)\quad(\text{since } v_i\in C_{i-1})$$
$$[x,v_i]=(x',v_i)\quad(\text{since } v_i\in C_i).$$

It follows that $(x,v_i)=(x',v_i)$ hence $(v_i,e_j)=(v_i,x-x')=0$. This is a contradiction.

We have proved that, given any $x\in V$ and e_j such that $(x,e_j)=0$, we have either

(3.8.1) $$[x,y]=(x,y)\quad\text{for all } y\in\tilde{V}$$

or

(3.8.2) $$[x,y]=(x+e_j,y)\quad\text{for all } y\in\tilde{V}.$$

We shall consider three cases for a vector $x\in V$.

Case 1. $x\in V$ is such that there exist $e_j\neq e_k$ with $(x,e_j)=(x,e_k)=0$. Then the function $y\to[x,y]$ (on $\tilde{V}$) must be equal to one of the functions $y\to(x,y)$, $y\to(x+e_j,y)$ and it must be also equal to one of the functions $y\to(x,y)$, $y\to(x+e_k,y)$. The functions $y\to(x,y)$, $y\to(x+e_j,y)$, $y\to(x+e_k,y)$ (on $\tilde{V}$) are distinct, since $\tilde{V}$ spans V. It follows that $y\to[x,y]$ is equal to $y\to(x,y)$ (on $\tilde{V}$).

Case 2. $x\in V$ is such that there exists e_j with $(x,e_j)=0$, *but* $(x,e_k)=1$ *for all* $k\neq j$. The vector $x'=x+e_j$ satisfies $(x',e_j)=0$ and also $(x',e_{j\pm1})=1+1=0$. (At least one of e_{j+1}, e_{j-1} is defined.) Hence, by Case 1, we have $[x',y]=(x',y)$ for all $y\in\tilde{V}$. Using now the identity b), it follows that $[x,y]=(x,y)$ for all $y\in\tilde{V}$.

Case 3. $x\in V$ is such that $(x, e_j)=1$ for all j. (There is exactly one such vector x in V.) Since for all vectors $x'\neq x$, the identity $[x', y]=(x, y)$ $(y\in\tilde{V})$ is already known, the identity c) shows that $[x, y]=(x, y)$ for all $y\in\tilde{V}$.

The Proposition is proved.

4. Some Results on Reductive Groups

4.1. Let X be a (possibly singular) algebraic variety over $\bar{F}_p$ whose connected components are irreducible, of the same dimension. Deligne, Goresky and Macpherson define a canonical complex ${}^{\pi}Q_l$ of l-adic sheaves on X (l = prime other than p), defined in the derived category; its cohomology sheaves are denoted $\mathscr{H}^i(X)$. (For a definition, see [4], [9, §3]. Let $\mathbb{H}_c^i(X)$ denote the hypercohomology with compact support of X with coefficients in ${}^{\pi}Q_l$.

4.2. We shall apply this construction to the varieties X_w (defined in [5, 1.4]). Let G be a connected reductive algebraic group defined over a finite field $F_q\subset\bar{F}_p$ and let $F: G\to G$ be the corresponding Frobenius map. For each element w in the Weyl group of G, let X_w be the variety of all Borel subgroups of G such that B and FB are in relative position w. The finite group G^F acts naturally on X_w (by conjugation) and we thus have a virtual representation of G^F defined by $R_w=\sum_i (-1)^i H_c^i(X_w, Q_l)$ (see [5, 1.5].) Let $\bar{X}_w$ be the closure of X_w in the variety of all Borel subgroups of G. The following result is closely related to [9, 4.2, 4.3].

4.3. **Lemma.** *$\bar{X}_w$ is the union of all $X_y (y\leqq w)$ where $\leqq$ is the standard partial order on W. The sheaf $\mathscr{H}^i(\bar{X}_w)$ is constant over each X_y $(y\leqq w)$ and is zero if i is odd. If $B\in X_y$ $(y\leqq w)$ then the stalks $\mathscr{H}_B^{2i}(\bar{X}_w)$ satisfy*

$$\sum_i \dim \mathscr{H}_B^{2i}(\bar{X}_w)\, u^i = P_{y,w}(u)$$

where $P_{y,w}$ are the polynomials introduced in [8, 1.1]. *If $F^n B=B$ and if F^n acts trivially on the Weyl group of G, then all eigenvalues of F^n on $\mathscr{H}_B^{2i}(\bar{X}_w)$ are equal to q^{ni}.*

Proof. Let $B_0\supset T_0$ be a Borel subgroup and a maximal torus of G, both defined over F_q. We identify the Weyl group of W with $N(T_0)/T_0$, in the usual way. For each $w\in W$, let $\dot{w}$ be a representative for w in $N(T_0)$, let $G_w=B_0\dot{w}B_0\subset G$ and let $G'_w=\mathscr{L}^{-1}(G_w)\subset G$, where $\mathscr{L}: G\to G$ is the Lang map $\mathscr{L}(g)=g^{-1}F(g)$. Let $\pi: G\to G/B_0$ be the natural projection. Note that both G_w and G'_w are stable under right multiplication by elements of B_0. We identify X_w with $\pi(G'_w)\subset G/B_0$ under $gB_0\to gB_0g^{-1}$. Then $\bar{X}_w$ becomes $\overline{\pi(G'_w)}=\pi(\overline{G'_w})\subset G/B_0$.

We consider the diagram

$$\begin{array}{ccc} \overline{G'_w} & \xrightarrow{\mathscr{L}} & \overline{G_w} \\ \downarrow & & \downarrow \\ \overline{\pi(G'_w)} & & \overline{\pi(G_w)} \end{array}$$

where the vertical maps are locally trivial fibrations with smooth fibre ($\approx B_0$) and the horizontal map is étale. The assertions of the lemma about $\overline{\pi(G'_w)}$ are then a consequence of the analogous assertions about the Schubert variety $\overline{\pi(G_w)}$ which were proved in [9, §4].

4.4. The finite group G^F acts naturally on $\bar{X}_w$ and on the corresponding complex of sheaves ${}^{\pi}Q_l$, hence it also acts on the hypercohomology spaces $\mathbb{H}_c^i(\bar{X}_w)$. Using the filtration of $\bar{X}_w$ by $X_y (y \leqq w)$ we see that we have an equality of virtual (G^F, F^n)-modules

$$(4.4.1) \qquad \sum_i (-1)^i \mathbb{H}_c^i(\bar{X}_w) = \sum_{y \leqq w} \sum_{i,j} (-1)^i \mathbb{H}_c^i(X_y, \mathscr{H}^{2j}(\bar{X}_w))$$

where n is such that F^n acts trivially on the Weyl group. Let $\mathbb{H}_c^i(\bar{X}_w)^{(h)}$, $\mathbb{H}_c^i(X_y)^{(h)}$ be the part of weight h of $\mathbb{H}_c^i(\bar{X}_w)$, $\mathbb{H}_c^i(X_y)$, i.e. the part on which the eigenvalues of F^n have all their complex absolute values equal to $q^{nh/2} (h \in \mathbb{Z})$. Taking the part of weight h in (4.4.1), we get an equality of virtual G^F-modules

$$\sum_i (-1)^i \mathbb{H}_c^i(\bar{X}_w)^{(h)} = \sum_{y \leqq w} \sum_{i,j} (-1)^i \cdot P_{y,w,j} \mathbb{H}_c^i(X_y)^{(h/2-j)}$$

where $P_{y,w,j}$ is the coefficient of u^j in $P_{y,w}$. But according to a version of the Weil conjectures, due to Deligne (see [4], [9, 4.4]) we have $\mathbb{H}_c^i(\bar{X}_w)^{(i)} = \mathbb{H}_c^i(\bar{X}_w)$. (The assumption in [loc. cit.] is verified by $\bar{X}_w$, see Lemma 4.3.) Moreover, $\mathbb{H}_c^i(X_y) = H_c^i(X_y, Q_l)$ since X_y is non-singular. Hence, we have:

4.5. **Lemma.** *Given $w \in W$ and $h \in \mathbb{Z}$, the virtual G^F-module*

$$(-1)^h \sum_{y \leqq w} \sum_{i,j} (-1)^i P_{y,w,j} H_c^i(X_y, Q_l)^{(h/2-j)}$$

is an actual G^F-module: it is equal to $\mathbb{H}_c^h(\bar{X}_w)$.

4.6. We now assume that F acts trivially on the Weyl group W of G. For each virtual $\mathbb{Q}[W]$-module M, we define (cf. [11, (3.17.1)].

$$(4.6.1) \qquad R(M) = |W|^{-1} \sum_{w \in W} \operatorname{Tr}(w, M) R_w$$

(an element of the Grothendieck group $\mathscr{R}(G^F) \otimes \mathbb{Q}$ of virtual $\bar{\mathbb{Q}}_l$-representations of G^F tensored with $\mathbb{Q}$. It is not in general, in $\mathscr{R}(G^F)$.) We now state some simple properties of $R(M)$.

$$(4.6.2) \qquad \langle R(M), R(M') \rangle_{G^F} = \langle M, M' \rangle_W.$$

(This follows from the orthogonality formula for R_w [5, 6.8].)

Let P be an F-stable parabolic subgroup of G with unipotent radical U_P, and let W_P be the corresponding standard parabolic subgroup of W. Let M' be a virtual $\mathbb{Q}[W_P]$-module. Then $R(M')$ is a well defined element of $\mathscr{R}(L^F) \otimes \mathbb{Q}$ where $L = P/U_P$; we regard $R(M')$ as an element of $\mathscr{R}(P^F) \otimes \mathbb{Q}$ (via the natural map $\mathscr{R}(P^F) \to \mathscr{R}(L^F)$). We have

$$(4.6.3) \qquad \operatorname{Ind}_{P^F}^{G^F}(R(M')) = R(\operatorname{Ind}_{W_P}^{W}(M')) \quad (\text{in } \mathscr{R}(G^F) \otimes \mathbb{Q}).$$

If ρ is an irreducible representation of G^F, the space of its U_P^F invariant vectors $\rho^{U_P^F}$ is in a natural way an L^F-module. This extends by $\mathbb{Q}$-linearity to a homomorphism $\rho \to \rho^{U_P^F}: \mathscr{R}(G^F)\otimes\mathbb{Q} \to \mathscr{R}(L^F)\otimes\mathbb{Q}$. If M is a virtual $\mathbb{Q}[W]$-module, and $M \mid W_P$ is its restriction to W_P, we have

(4.6.4) $$(R(M))^{U_P^F} = R(M \mid W_P) \quad (\text{in } \mathscr{R}(L^F)\otimes\mathbb{Q}).$$

Let $D: \mathscr{R}(G^F) \to \mathscr{R}(G^F)$ be defined by

(4.6.5) $$D(\rho) = \sum_{\substack{P \\ P \supset B_0}} (-1)^{r(P)} \operatorname{Ind}_{P^F}^{G^F}(\rho^{U_P^F})$$

where $r(P)$ is the semisimple F_q-rank of P/U_P; D extends to a $\mathbb{Q}$-linear map $D: \mathscr{R}(G^F)\otimes\mathbb{Q} \to \mathscr{R}(G^F)\otimes\mathbb{Q}$. From (4.6.3), (4.6.4) we have for any virtual $\mathbb{Q}[W]$-module M:

$$D(R(M)) = \sum_{\substack{P \\ P \supset B_0}} (-1)^{r(P)} R(\operatorname{Ind}_{W_P}^{W}(M \mid W_P))$$
$$= R(M \otimes (\sum_{\substack{P \\ P \supset B_0}} (-1)^{r(P)} \operatorname{Ind}_{W_P}^{W}(1)))$$

hence

(4.6.6) $$D(R(M)) = R(M \otimes \text{sign}).$$

The following result is due to Asai [2]; its proof depends on the result of [11, 3.9] concerning eigenvalues of Frobenius on $H_c^i(X_w, \mathbb{Q}_l)$ and on the recent results of Kawanaka [7] concerning lifting for field extensions of odd degree in the case of classical groups.

4.7. **Theorem.** [2, 2.4.7]. *Assume that* $G = Sp_{2n}$, SO_{2n+1} *or* SO_{2n}^+. *(+ stands for split). Then for any* $h \in \mathbb{Z}$ *we have*

$$\sum (-1)^i H_c^i(X_w, \mathbb{Q}_l)^{(h)} = \sum_E \operatorname{Tr}(T_w, \tilde{E}; h/2)\, R(E)$$

(sum over all irreducible $\mathbb{Q}[W]$*-modules* E.*)*

Combining Lemma 4.5 with the previous Theorem we get

4.8. **Proposition.** *Let* G *be as in Theorem* 4.7, *let* $w \in W$ *and let* $h \in \mathbb{Z}$. *Then the element of* $\mathscr{R}(G^F)\otimes\mathbb{Q}$ *given by*

(4.8.1) $$(-1)^h \sum_E \sum_{y \leqq w} \sum_j P_{y,w,j} \operatorname{Tr}(T_y, \tilde{E}; h/2 - j)\, R(E)$$

is a linear combination with integral positive coefficients of irreducible representations of G^F.

4.9. **Corollary.** *Assume that* $G = Sp_{2n}$ *or* SO_{2n+1}. *Let* $\underline{c}$ *be a virtual cell of* $W = W_n$ *and let* $w, w' \in \Omega_n$, *be such that* $\underline{c} = \mathscr{A}_w$, $\underline{c} \otimes \text{sign} = \mathscr{A}_{w'}$ *(see* 2.20). *Let* $A = A(\underline{c})$, $a = a(\underline{c})$, *(see* 2.21), $h = l(w) - A + v$, $h' = l(w') - a$. *Then*

(4.9.1) $$R(\underline{c}) + \sum_{\substack{E \\ A(E) < A}} \sum_{y \leqq w} \sum_j P_{y,w,j} \operatorname{Tr}(T_y, \tilde{E}; h/2 - j)\, R(E)$$

and

$$(4.9.2)\qquad R(\underline{c})+\sum_{\substack{E\\ a(E)>a}}\ \sum_{y\leqq w'}\sum_{j} P_{y,w',j}\,\mathrm{Tr}\,(T_y,\widetilde{E\otimes\mathrm{sign}},h'/2-j)\,R(E)$$

are linear combinations with integral positive coefficients of irreducible representations of G^F.

Proof. First note that by (2.20.1), (2.20.2), h and h' are even. It is known [8, 1.1] that for $y\leqq w$, we have $P_{y,w,j}=0$ unless $j\leqq\frac{1}{2}(l(w)-l(y))$; moreover, for $y<w$, we have $P_{y,w,j}=0$ unless $j\leqq\frac{1}{2}(l(w)-l(y)-1)$. By 1.9, we have $\mathrm{Tr}(T_y,\tilde{E};h/2-j)=0$ unless $h/2-j\leqq\frac{l(y)-A(E)+v}{2}$. Thus, $P_{y,w,j}\,\mathrm{Tr}(T_y,\tilde{E};h/2-j)\neq0$ implies $h/2=j+(h/2-j)\leqq\frac{1}{2}(l(w)-l(y))+\frac{l(y)-A(E)+v}{2}=\frac{l(w)-A(E)+v}{2}$ if $y\leqq w$ and similarly, $h/2\leqq\frac{l(w)-A(E)-1+v}{2}$, if $y<w$; or, in other words, that $A(E)\leqq A$ if $y\leqq w$ and $A(E)<A$ if $y<w$. Thus, for our particular h, there are no non-zero terms in the sum (4.8.1) corresponding to E with $A(E)>A$; the non-zero terms corresponding to E with $A(E)=A$ must have $y=w$ and $j=0$. Their contribution to the sum is

$$\sum_{\substack{E\\ A(E)=A}}\mathrm{Tr}\left(T_w,\tilde{E},\frac{l(w)-A(E)+v}{2}\right)R(E)=R(\mathscr{A}_w)=R(\underline{c}).$$

and hence (4.8.1) coincides with (4.9.1).

The expression (4.9.2) is obtained by applying the operator D to the expression (4.9.1) with $\underline{c}$ replaced by $\underline{c}\otimes\mathrm{sign}$. But when D is applied to an integral positive combination of unipotent representations (=irreducible representations of G^F which appear as components of some R_w), then the result is again an integral positive combination of irreducible representations. Indeed, by a result of Alvis and Kawanaka (see [1]), D applied to an irreducible representation ρ of G^F is $(-1)^{r(P)}$ times an irreducible representation of G^F, where P is an F-stable parabolic subgroup of G such that $\rho^{U_P^F}$ contains a cuspidal representation of P^F/U_P^F. In our case, $r(P)$ is necessarily even, since unipotent cuspidal representations of $Sp_{2n'}$, $SO_{2n'+1}$ can only occur for even values of n'. (cf. [10].) This completes the proof of the Corollary.

5. The Main Results

5.1. Let $G=G_n$ be either Sp_{2n} or SO_{2n+1} (defined over $\mathbb{F}_q$). For each partition $n=r+s$ $(0\leqq s<n)$ we denote by $P_{r,s}$ a maximal parabolic subgroup of G which is defined over $\mathbb{F}_q$ such that the corresponding standard parabolic subgroup of the Weyl group is $\mathfrak{S}_r\times W_s\subset W_n=W$ (see 2.6); then $P_{r,s}$ has a Levi subgroup $L_{r,s}$ defined over $\mathbb{F}_q$ and isomorphic to $GL_r\times G_s$.

The unipotent representations of G_n^F (i.e. irreducible representations of G_n^F appearing in some R_w, $w\in W$) have been classified in [10] in terms of symbols of rank n and odd defect.

Recall that a symbol of rank n and odd defect is a pair $\Lambda=\binom{T'}{T''}$ consisting of two finite subsets T', T'' of $\{0,1,2,3,\dots\}$, such that $|T'|+|T''|=2m+1$, $|T'|\equiv m+1 \pmod 2$, $|T''|\equiv m \pmod 2$, $\sum_{\lambda\in T'}\lambda+\sum_{\mu\in T''}\mu=n+m^2$. There is an equivalence relation on such pairs generated by the shift $\binom{T'}{T''}\sim\binom{0\sqcup(T'+1)}{0\sqcup(T''+1)}$ and we shall often identify a symbol with its equivalence class (compare with 2.1 where a special case of this notion was considered). Any symbol Λ of rank n and odd defect gives rise to a special symbol Z of rank n, by exactly the same construction as in the proof of 2.22: Z is the unique special symbol whose set of entries (some of which may be repeated twice) coincides with the set of entries of Λ (union of T' and T'', with common elements repeated twice.) We shall then set $a_\Lambda=a[Z]$. (Note that, if Λ has defect 1, we have $a_\Lambda=a[\Lambda]$, see (1.8).)

5.2. **Lemma.** *There exists a $1-1$ correspondence $\Lambda\leftrightarrow\rho(\Lambda)$ between the set of symbols of rank n and odd defect (up to shift) and the set of unipotent representations (up to isomorphism) of G_n^F with the following properties.*

(i) *If Z is the special symbol corresponding to Λ, $a=a[Z]=a_\Lambda$ and $d=d[Z]$ is such that $2d+1$ is the number of singles of Z, then $2^d\dim(\rho)\equiv q^a \pmod{q^{a+1}}$.*

(ii) *Let $\Lambda=\binom{T'}{T''}$ be a symbol of rank n and odd defect. Let t be an integer, $t\geq$ all entries in Λ. Let $\bar T'=\{t-i\mid 0\leq i\leq t,\ i\notin T''\}$ $\bar T''=\{t-i\mid 0\leq i\leq t,\ i\notin T'\}$, and let $\bar\Lambda=\binom{\bar T'}{\bar T''}$. This is again a symbol of rank n odd defect and $D(\rho(\Lambda))=\rho(\bar\Lambda)$.*

(iii) *Let Λ' be a symbol of rank s and odd defect. We associate to Λ' a symbol Λ (or two symbols $\Lambda_{\rm I}$, $\Lambda_{\rm II}$) of rank n by increasing by 1 each of the $r=n-s$ largest entries in Λ' (we may assume that Λ' has $\geq r$ entries), as in 2.6, where the case of symbols of defect 1 was considered. (The discussion in 2.6 is applicable in the present, more general case.) Then*

$$\operatorname{Ind}_{P_{r,s}}^{G_n}(St_r\otimes\rho(\Lambda'))=\begin{cases}\rho(\Lambda)+\tau\\ \text{or } \rho(\Lambda_{\rm I})+\rho(\Lambda_{\rm II})+\tau\end{cases}$$

where τ is a $\mathbb{Z}$-linear combination of representation $\rho(\Lambda_i)$ such that $a_{\Lambda_i}>a_\Lambda$ (or $a_{\Lambda_i}>a_{\Lambda_{\rm I}}=a_{\Lambda_{\rm II}}$), and St_r is the Steinberg representation of $GL_r(F_q)$.

Proof. The description of unipotent representations given in [10] is in terms of irreducible representations of certain Hecke algebras of type $B_l(l\leq n)$ arising as endomorphism algebras of a representation induced by a unipotent cuspidal representation of a parabolic subgroup. That description allows one to reduce (ii), (iii) to statements about representation of Weyl groups of type $B_l(l\leq n)$ which follow from 2.4, 2.7 respectively. (i) follows from the explicit dimension formulas of [10].

We have:

5.3. **Lemma.** *For any unipotent representation ρ of G_n^F there exist integers $d=d(\rho)\geqq 0$, $a=a(\rho)\geqq 0$, such that*

$$d+d^2\leqq n \tag{5.3.1}$$

$$2^d \dim(\rho)\equiv q^a (\operatorname{mod} q^{a+1}). \tag{5.3.2}$$

Let $\sigma(n)$ be the largest integer such that $\sigma(n)+\sigma(n)^2\leqq n$. If $q>2^{\sigma(n)}$ then the conditions (5.3.1), (5.3.2) *determine $d(\rho)$, $a(\rho)$ uniquely.*

Proof. The existence of $s(\rho)$, $a(\rho)$ follows from Lemma 5.2. We now prove the uniqueness statement. We assume that $d, d'\leqq\sigma(n)$ and $2^d D\equiv q^a \pmod{q^{a+1}}$, $2^{d'} D=q^{a'} \pmod{q^{a'+1}}$, where D is an integer. If $q=p^e$ (p odd), the p-adic valuation of D is $ae=a'e$, hence $a=a'$; but then $2^d-2^{d'}\equiv 0 \pmod q$. Since $0<2^d$, $2^{d'}<q$, it follows that $2^d=2^{d'}$ hence $d=d'$. If $q=2^e$, the 2-adic valuation of D is $ae-d=a'e-d'$, hence $d-d'$ is divisible by e; but $0\leqq d$, $d'<e$ by assumption, hence $d=d'$ and $a=a'$.

5.4. **Lemma.** (a) *If Λ is a symbol of rank n and defect one, then*

$$\dim R[\Lambda]\equiv\begin{cases} q^{a[\Lambda]} \pmod{q^{a[\Lambda]+1}}, & \text{if } \Lambda \text{ is special}\\ 0 \pmod{q^{a[\Lambda]+1}}, & \text{if } \Lambda \text{ is non-special.}\end{cases}$$

(b) *If $\underline{c}$ is a virtual cell of W_n then*

$$\dim R[\underline{c}]\equiv q^{a[c]} \pmod{q^{a[c]+1}}.$$

Proof. (a) follows from [10, 2.7(i)] and (b) follows from (a).

5.5. **Lemma.** *Let Z be a special symbol of rank n with $2d+1$ singles, let Φ be an admissible arrangement for Z and let $\hat{\Phi}$, $\hat{\Phi}'$ be two subsets of Φ. Let $\underline{c}=\underline{c}(Z,\Phi,\hat{\Phi})$, $\underline{c}'=\underline{c}(Z,\Phi,\hat{\Phi}')$. Then*

$$\langle R(\underline{c}), R(\underline{c}')\rangle_{G_n^F}=\begin{cases} 2^d, & \text{if } \hat{\Phi}=\hat{\Phi}'\\ 0, & \text{if } \hat{\Phi}\neq\hat{\Phi}'.\end{cases}$$

Proof. Using (4.6.2), we have

$$\langle R(\underline{c}), R(\underline{c})\rangle_{G_n^F}=\langle \underline{c}, \underline{c}'\rangle_{W_n}=\sum_{\Psi\subset\Phi}(-1)^{f(\Psi)}$$

where $f(\Psi)=|\hat{\Phi}^*\cap\Psi^*|+|(\hat{\Phi}')^*\cap\Psi^*|=|J\cap\Psi^*| \pmod 2$ and $J=(\hat{\Phi}^*\cup(\hat{\Phi}')^*)-(\hat{\Phi}^*\cap(\hat{\Phi}')^*)\subset\Phi^*$. If $\hat{\Phi}=\hat{\Phi}$, then $J\neq\emptyset$ and so $f(\Psi)\equiv 0 \pmod 2$ for all $\Psi\subset\Phi$. If $\hat{\Phi}\neq\hat{\Phi}'$, then $J\neq\emptyset$ and $d\geqq 1$, and so $|J\cap(\Phi-\Psi)^*|$ is even for 2^{d-1} values of Ψ and is odd for the other 2^{d-1} values of Ψ. The Lemma is proved.

5.6. **Theorem.** *Let $\underline{c}=\underline{c}(Z,\Phi,\hat{\Phi})$ be a virtual cell of W_n. Let $a=a(\underline{c})$, be defined as in* 2.21 *and let $d=d(\underline{c})$ be such that $2d+1$ is the number of singles in Z. Assume that $q>2^{2\sigma(n)}$ ($\sigma(n)$ is as in Lemma* 5.3.) *Then*

$$R(\underline{c})=\sum_{i=1}^{2^d}\rho_i$$

where ρ_i $(1 \leqq i \leqq 2^d)$ *are distinct unipotent representations of* G_n^F *satisfying* $a(\rho_i) = a$, $d(\rho_i) = d$.

Proof. We may assume that the theorem is already proved for all virtual cells $\underline{c}'$ such that $a(\underline{c}') > a$ or such that $a(\underline{c}') = a$ and $d(c') < d$ (if such $\underline{c}'$ exist.) Let $\rho_1, \rho_2, \ldots, \rho_t$ be the set of all unipotent representations of G_n^F (up to isomorphism) such that $\langle \rho_i, R(\underline{c}) \rangle \neq 0$ and $\langle \rho_i, R[c'] \rangle = 0$ for any virtual cell such that $a(c') > a$, $(1 \leqq i \leqq t)$. If Λ is any symbol of rank n and defect 1 such that $a[\Lambda] > a$, then $[\Lambda]$ is a $\mathbb{Z}$-linear combination of such virtual cells $\underline{c}'$ (see Lemma 2.22) hence $\langle \rho_i, R[\Lambda] \rangle = 0$ $(1 \leqq i \leqq t)$. Taking inner product with the actual representation $\mathscr{R}$ of G_n^F given by (4.9.2), we see that $n_i = \langle \rho_i, R(\underline{c}) \rangle$ is an integer $\geqq 0$, and being $\neq 0$, it is an integer > 0, $(1 \leqq i \leqq t)$. We now show that $a(\rho_i) \leqq a$ $(1 \leqq i \leqq t)$. Indeed, on the one hand, from Lemma 5.2 we compute explicitely the number N of unipotent representations ρ satisfying $a(\rho) > a$: it is equal to $\sum_Z 2^{2d(Z)}$, sum over all special symbols Z of rank n with $a(Z) > a$. On the other hand, let us fix for each such Z an admissible arrangement Φ_Z. Then, when $\hat{\Phi}$ runs through the subsets of Φ_Z, we get $2^{d(Z)}$ representations $R(\underline{c}(Z, \Phi_Z, \hat{\Phi}))$ to which our induction hypothesis applies; these representations are disjoint and each contain $2^{d(Z)}$ distinct irreducible components ρ each satisfying $a(\rho) = a[Z]$, (see Lemma 5.5.) Thus, the number of unipotent representations ρ which satisfy $a(\rho) > a$ and appear in some $R(\underline{c}')$ with $a(c') > a$ is at least equal to N. We conclude that all unipotent representations ρ satisfying $a(\rho) > a$ must appear in some $R(\underline{c}')$ with $a(\underline{c}') > a$ and therefore cannot be in the set $\{\rho_1, \rho_2, \ldots, \rho_t\}$.

Next, we assume that $\min_{1 \leqq i \leqq t} a(\rho_i) = a' < a$; we shall reach a contradiction as follows.

We may assume that this minimum is reached precisely for $\rho_1, \rho_2, \ldots, \rho_{t'}$ $(t' \leqq t)$. We shall consider the dimension of the G_n^F-module $\mathscr{R}$ given by (4.9.2), in two different ways. On the one hand, using Lemma 5.4, we see that

$$\dim \mathscr{R} \equiv q^a (\mathrm{mod}\, q^{a+1});$$

on the other hand, by the induction hypothesis, all irreducible representations ρ appearing in $\mathscr{R}$, which are different from $\rho_1, \ldots, \rho_t$ satisfy $2^{\sigma(n)} \dim \rho \equiv 0 \,(\mathrm{mod}\, q^{a+1})$ and in particular, $2^{\sigma(n)} \dim \rho \equiv 0 \,(\mathrm{mod}\, q^{a'+1})$. Also $2^{\sigma(n)} \dim \rho_i \equiv 0$ $(\mathrm{mod}\, q^{a'+1})$ for $t' < i \leqq t$ and $2^{d(\rho_i)} \dim \rho_i \equiv q^{a'}$ $(\mathrm{mod}\, q^{a'+1})$ for $1 \leqq i \leqq t'$. Hence

$$\begin{aligned} 2^{\sigma(n)} \dim \mathscr{R} &\equiv \sum_{i=1}^{t'} 2^{d(n) - d(\rho_i)} n_i q^{a'} \,(\mathrm{mod}\, q^{a'+1}) \\ &\equiv 0 \,(\mathrm{mod}\, q^{a'+1}). \end{aligned}$$

It follows that

$$\sum_{i=1}^{t'} 2^{\sigma(n) - d(\rho_i)} n_i \equiv 0 \,(\mathrm{mod}\, q).$$

Therefore, if $t' \geqq 1$, we must have

$$\sum_{i=1}^{t'} 2^{\sigma(n) - d(\rho_i)} n_i \geqq q.$$

On the other hand by Lemma 5.5, we have:

$$\sum_{i=1}^{t} n_i^2 \leqq \langle R(\underline{c}), R(\underline{c})\rangle = 2^d$$

hence

$$q \leqq 2^{\sigma(n)} \sum_{i=1}^{t'} n_i \leqq 2^{\sigma(n)} \sum_{i=1}^{t} n_i^2 \leqq 2^{\sigma(n)+d} \leqq 2^{2\sigma(n)}$$

a contradiction.

We have thus proved that $a(\rho_i)=a$ for $i=1, \ldots, t$.

Next we show that $d(\rho_i)\geqq d$, $(1\leqq i\leqq t)$. Assume that $d(\rho_i)<d$, for some i, $1\leqq i\leqq t$; we have also $a(\rho_i)=a$. As before, we can count explicitely the number of unipotent representations ρ of G_n^F satisfying $a(\rho)=a$, $d(\rho)<d$: it is given by $v=\sum 2^{2d(Z)}$, sum over all special symbols Z of rank n satisfying $a(Z)=a$, $d(Z)<d$. On the other hand, as before, for each such Z we can construct $2^{d(Z)}$ representations $R(\underline{c}(Z, \Phi_Z, \hat{\Phi})$, ($\Phi_Z$ fixed) to which our induction hypothesis applies, so we see that the number of unipotent representations ρ which satisfy $a(\rho)=a$, $d(\rho)<d$ and which appear in some $R(\underline{c}')$ with $a(\underline{c}')=a$, $d(c')<d$, is at least equal to v. We conclude that all unipotent representations ρ satisfying $a(\rho)=a$, $s(\rho)<s$ must appear in some $R(\underline{c}')$ with $a(c')=a$, $d(c')<d$. In particular, our ρ_i must appear in some $R(\underline{c}')$ with $a(c')=a$, $d(c')<d$. The inner product of the actual representations $R(\underline{c}')$, $\mathscr{R}$ is strictly positive since ρ_i is a component of both. On the other hand, $R(\underline{c}')$ is clearly orthogonal to all terms of the sum (4.9.2) defining $\mathscr{R}$. This contradiction shows that $s(\rho_i)\geqq s$ for $i=1, \ldots, t$. We now consider, as before, the dimension of $\mathscr{R}$ in two different ways. On the one hand, $\dim \mathscr{R} \equiv q^a \pmod{q^{a+1}}$. On the other hand, as we have seen, we have $2^{\sigma(n)} \dim(\rho)\equiv 0 \pmod{q^{a+1}}$ for all irreducible components ρ of $\mathscr{R}$ other than $\rho_1, \rho_2, \ldots, \rho_t$, and $2^{d(\rho_i)} \dim(\rho_i)\equiv q^a \pmod{q^{a+1}}$ for $1\leqq i\leqq t$. Hence

$$\begin{aligned} 2^{\sigma(n)} \dim \mathscr{R} &\equiv \sum_{i=1}^{t} 2^{\sigma(n)-d(\rho_i)} n_i q^a \pmod{q^{a+1}} \\ &\equiv 2^{\sigma(n)} q^a \pmod{q^{a+1}}. \end{aligned}$$

It follows that

$$2^{\sigma(n)} - \sum_{i=1}^{t} 2^{\sigma(n)-d(\rho_i)} n_i \equiv 0 \pmod{q} \tag{5.6.1}$$

The left hand side of (5.6.1) cannot be >0 for then it is $\geqq q$, hence $2^{\sigma(n)}\geqq q$, a contradiction. We have

$$\begin{aligned} \sum_{i=1}^{t} 2^{\sigma(n)-d(\rho_i)} n_i &\leqq \sum_{i=1}^{t} 2^{\sigma(n)-d} n_i \quad (\text{since } d(\rho_i)\geqq d) \\ &\leqq \sum_{i=1}^{t} 2^{\sigma(n)-d} n_i^2 \\ &\leqq \sum_{i=1}^{t} 2^{\sigma(n)-d} \cdot 2^d \\ &= 2^{\sigma(n)} \end{aligned}$$

hence the left hand side of (5.6.1) is $\geqq 0$. Therefore it must be equal to 0; it follows that the last 3 inequalities are equalities so that $d(\rho_i)=d$ for all i, $1\leqq i\leqq t$, and $\sum_{i=1}^{t} n_i=\sum_{i=1}^{t} n_i^2=2^d$ i.e. $n_i=1$ for all i and $t=2^d$. Since $\langle R(\underline{c}), R(\underline{c})\rangle_{G_n^F}=2^d$, we must then have $R(\underline{c})=\sum_{i=1}^{2^d}\rho_i$. The theorem is proved.

5.7. Let Z be a special symbol of rank n, and let Z_1 be the set of singles of Z; let d be defined by $2d+1=|Z_1|$. We can write $Z_1=Z_1^* \amalg (Z_1)_*$, where Z_1^* is the set of entries of Z_1 appearing in the first row of Z and $(Z_1)_*$ is the set of entries of Z_1 appearing in the second row of Z. We have $|Z_1^*|=d+1$, $|(Z_1)_*|=d$. Let Z_2 be the set of elements which appear in both rows of Z. Thus, $Z=\begin{pmatrix} Z_2\amalg (Z_1)^* \\ Z_2\amalg (Z_1)_* \end{pmatrix}$.

Let $\mathscr{S}_Z$ be the set of all symbols of rank n and odd defect which contain the same entries as Z. There are exactly 2^{2d} such symbols, one for each subset $M\subset Z_1$ such that $|M|\equiv d\ (\mathrm{mod}\ 2)$: the symbol corresponding to M is $\Lambda_M=\begin{pmatrix} Z_2\amalg (Z_1-M) \\ Z_2\amalg M \end{pmatrix}$. If we associate to M the set $M^\#\subset Z_1$ defined by $M^\#=M\cup (Z_1)_*-(M\cap (Z_1)_*)$ we get a $1-1$ correspondence $\Lambda_M\leftrightarrow M^\#$ between $\mathscr{S}_Z$ and the set V_{Z_1} of subsets of Z_1 of even cardinality. The set V_{Z_1} has a natural structure of F_2-vector space of dimension $2d$: the sum of $M_1^\#$ and $M_2^\#$ is defined to be $(M_1^\#\cup M_2^\#)-(M_1^\#\cap M_2^\#)$. This allows us to regard $\mathscr{S}_Z$ as an F_2-vector space of dimension $2d$. The 0 element is Z itself. (Indeed, $(Z_1)_*^\#=\emptyset$.)

The vector space V_{Z_1} has also a natural non-singular symplectic form $(,)$: $V_{Z_1}\times V_{Z_1}\to F_2$: it is given by

$$(M_1^\#, M_2^\#)=|M_1^\#\cap M_2^\#| \bmod 2.$$

We shall regard this also as a symplectic form on $\mathscr{S}_Z$, via the bijection $\mathscr{S}_Z\leftrightarrow V_{Z_1}$.

The vector space V_{Z_1} has a natural basis $e_1, \dots, e_{2d}$ defined as follows: we arrange the elements in Z_1 in an increasing sequence; then e_i is the subset of Z_1 consisting of the i-th and $(i+1)$-th elements in this sequence. It is clear that

$$(e_i, e_j)=\begin{cases} 1 & \text{if } |i-j|=1 \\ 0 & \text{otherwise} \end{cases}.$$

Thus V_{Z_1} (hence $\mathscr{S}_Z$) is a symplectic vector space of the kind considered in 3.1.

The corresponding subset $\tilde{V}_{Z_1}\subset V_{Z_1}$ (see 3.5) consists of the subsets $M^\#\subset Z_1$ such that $|M^\#\cap Z_1^*|=|M^\#\cap (Z_1)_*|$, or equivalently, of the subsets $M^\#\subset Z_1$ such that $|M|=d$.

In other words, $\tilde{V}_{Z_1}$ corresponds to the subset of $\mathscr{S}_Z$ consisting of all symbols of defect one.

The lagrangian subspace in $\mathscr{I}(V_{Z_1})$ (see 3.2) are in $1-1$ correspondence with the admissible arrangements Φ for Z: the lagrangian corresponding to Ψ is: $\{\Psi_*\amalg \Psi^* \mid \Psi\subset\Phi\}\subset V_{Z_1}$. Under the bijection $V_{Z_1}\leftrightarrow\mathscr{S}_Z$ this lagrangian becomes the set of all

$$\begin{pmatrix}(Z_0)^* \amalg \Psi_* \amalg (\Phi - \Psi)^* \\ (Z_0)_* \amalg \Psi^* \amalg (\Phi - \Psi)_*\end{pmatrix} \in \mathscr{S}_Z, \quad (\Psi \subset \Phi).$$

Any subset $\hat{\Phi} \subset \Phi$, gives rise to an F_2-linear form on this lagrangian, sending the element corresponding to Ψ to $|\hat{\Phi}^* \cap \Psi^*| \bmod 2 \in F_2$. This gives a $1-1$ correspondence between subsets of Φ and linear forms on the corresponding lagrangian.

We can now state

5.8. **Theorem.** *Let $G = G_n$ be either Sp_{2n} or SO_{2n+1} (defined over $\mathbb{F}_q$). Let Z be a special symbol of rank n, and let $d = d[Z]$ be as in 5.2. Assume that $q \geqq 2^{2\sigma(n)}$ where $\sigma(n)$ is defined in Lemma 5.3. Then*

1) *For any $\Lambda \in \mathscr{S}_Z$ of defect one, we have*

(5.8.1) $$R[\Lambda] = 2^{-d} \sum_{\Lambda' \in \mathscr{S}_Z} (-1)^{(\Lambda, \Lambda')} \rho(\Lambda')$$

where $(,)$ is the symplectic form on $\mathscr{S}_Z$ described in 5.7.

2) *Let $\mathscr{L}$ be the subspace of $\mathscr{S}_Z$ corresponding to a lagrangian subspace in $\mathscr{I}(V_{Z_1})$, and let ξ: $\mathscr{L} \to F_2$ be a linear form. Then*

(5.8.2) $$\sum_{\Lambda \in \mathscr{L}} (-1)^{\xi(\Lambda)} R[\Lambda] = \sum_{\substack{\Lambda' \in \mathscr{S}_Z \\ \xi = (\,,\Lambda') \text{ on } \mathscr{L}}} \rho(\Lambda').$$

Remark. By the discussion in 5.7, the left hand side of (5.8.2) is of the form $R(\underline{c})$ where $\underline{c}$ is the most general virtual cell of W_n.

It is clear that (5.8.1) implies (5.8.2). Conversely, (5.8.2) implies (5.8.1) by Lemma 2.22.

5.9. **Corollary.** *We preserve the assumptions of* 5.8. *Let $\Lambda' \in \mathscr{S}_Z$ and let $w \in W_n$. Then*

$$\langle \rho(\Lambda'), R_w \rangle_{G_n^F} = 2^{-d} \sum_{\substack{\Lambda \in \mathscr{S}_Z \\ \text{of defect one}}} (-1)^{(\Lambda, \Lambda')} \operatorname{tr}(w, [\Lambda]).$$

5.10. We shall now prove Theorem 5.8. We may assume that $n \geqq 1$ and that the theorem is proved for $G_{n'}$, $n' < n$. We may also assume that 0 doesn't occur twice in Z. Let t_0 be the largest entry in Z.

A) If some number i, $0 \leqq i < t_0$ doesn't appear in Z, then Z is obtained from a special symbol Z' of rank $s < n$ by increasing each of the r largest entries of Z' by 1 $(r = n - s)$, and this set of r largest entries is unambiguously defined. There is a unique order preserving bijection between the set Z_1 of singles of Z and the set Z'_1 of singles in Z'. This gives rise to a bijection, h: $\mathscr{S}_{Z'_1} \approx \mathscr{S}_{Z_1}$ which preserves the symplectic forms and the subsets of symbols of defect 1. Let $\Lambda' \in \mathscr{S}_{Z'_1}$ and let Λ'' be a symbol of defect 1 in $\mathscr{S}_{Z'_1}$. By 5.2(iii), we have $\operatorname{Ind}_{P_{r,s}^F}^{G_n^F} \rho(\Lambda') = \rho(h(\Lambda')) +$ combination of ρ with $a(\rho) > a = a[Z]$.

By Theorem 5.6, all components of $R[h(\Lambda'')]$ are of form ρ with $a(\rho) = a$. It follows that

$$\begin{aligned}&\langle R[h(\Lambda'')], \rho(h(\Lambda'))\rangle_{G_n^F}\\&=\langle R[h(\Lambda'')], \mathrm{Ind}_{P_{r,s}^F}^{G_n^F}(St_r\otimes\rho(\Lambda'))\rangle_{G_n^F}\\&=\langle R[h(\Lambda'')]^{U_{P_{r,s}}^F}, St_r\otimes\rho(\Lambda')\rangle_{L_{r,s}^F}\\&=\langle R[h(\Lambda'')\,|\,\mathfrak{S}_r\times W_s], St_r\otimes\rho(\Lambda')\rangle_{L_{r,s}^F}\quad\text{(by (4.6.4)).}\end{aligned}$$

It follows from 2.7 that $[h(\Lambda'')]\,|\,\mathfrak{S}_r\times W_s=\varepsilon(r)\otimes[\Lambda'']+$combination of irreducible representations E with $a(E)<a$. Using again Theorem 5.6, we see that the last inner product is equal to

$$\begin{aligned}&\langle R(\varepsilon(r)\otimes[\Lambda'']), St_r\otimes\rho(\Lambda')\rangle_{L_{r,s}^F}\\&=\langle St_r\otimes R[\Lambda''], St_r\otimes\rho(\Lambda')\rangle_{L_{r,s}^F}\\&=\langle R[\Lambda''], \rho(\Lambda')\rangle_{G_s^F}\\&=(-1)^{(\Lambda'',\Lambda')}2^{-d}\quad\text{(by the induction hypothesis).}\\&=(-1)^{(h(\Lambda''),h(\Lambda'))}2^{-d}.\end{aligned}$$

Thus, we have 2^{2d} distinct irreducible representations of G_n^F which appear with coefficients $\pm 2^{-d}$ in $R[h(\Lambda'')]$. Since $\langle R[h(\Lambda'')], R[h(\Lambda'')]\rangle=1$ there cannot be other irreducible representations appearing in $R[h(\Lambda'')]$, and the Theorem follows for our Z.

B) Assume now that $\bar{Z}$ (defined with respect to $t=t_0$) has the property that some i, $0\leqq i<t_0$ doesn't appear in $\bar{Z}$. Then the Theorem is true for $\bar{Z}$. We shall deduce from this that it is also true for Z. We have a natural (order reversing) involution $z\leftrightarrow t-z$ between the sets of singles in Z and in $\bar{Z}$. This gives rise to the bijection $\Lambda\leftrightarrow\bar{\Lambda}$ between $\mathscr{S}_Z$ and $\mathscr{S}_{\bar{Z}}$ which preserves the symplectic forms and the subsets fo symbols of defect one.

Let Λ be a symbol of defect 1 in $\mathscr{S}_Z$. We have

$$R[\bar{\Lambda}]=2^{-d}\sum_{\Lambda'\in\mathscr{S}_Z}(-1)^{(\bar{\Lambda},\bar{\Lambda}')}\rho(\bar{\Lambda}').$$

We apply the operator D (see 4.6.5) to both sides of this equality. Using (4.6.6), (5.2(ii)) and the identity $(\bar{\Lambda},\bar{\Lambda}')=(\Lambda,\Lambda')$, the required identity (5.8.1) for Λ follows.

C) If Z is in neither case A) or B), then $t_0=2d$

$$Z=\begin{pmatrix}0,2,4,\dots,2d\\1,3,\dots,2d-1\end{pmatrix}.$$

We can still apply the method of A) to get information on the multiplicities $\langle\rho(\Lambda'), R[\Lambda]\rangle$ for Z, starting from information for smaller groups. We obtain the following weaker result: Let $\Lambda',\Lambda''\in\mathscr{S}_Z$ be such that for some j, $(1\leqq j\leqq 2d)$, we have that $j-1, j$ are in different rows of Λ' and Λ'' is obtained from Λ' by switching j with $j-1$ and leaving the other entries unchanged. Let $\Lambda\in\mathscr{S}_Z$ be of defect 1. Then

$$\langle\rho(\Lambda')+\rho(\Lambda''), R[\Lambda]\rangle_{G_n^F}=2^{-d}((-1)^{(\Lambda',\Lambda)}+(-1)^{(\Lambda'',\Lambda)}).\tag{5.10.1}$$

Now, let Φ be an admissible arrangement for Z. The 2^d representations $R(\underline{c}(Z, \Phi, \hat{\Phi}))$ of $G_n^F(\hat{\Phi} \subset \Phi)$ are disjoint (Lemma 5.5) and each contains precisely 2^d unipotent representations (with multiplicity one). These must be of the form $\rho(\Lambda')$, $\Lambda' \in \mathcal{S}_Z$, since all other unipotent representations of G_n^F are already accounted for by A) and B). It follows that, for Φ and $\Lambda' \in \mathcal{S}_Z$ fixed, $\rho(\Lambda')$ has multiplicity one in $R(\underline{c}(Z, \Phi, \hat{\Phi}_0))$ for a unique $\hat{\Phi}_0 \subset \Phi$ and has multiplicity zero in $R(\underline{c}(Z, \Phi, \hat{\Phi}))$ for all $\hat{\Phi} \subset \Phi$, $\hat{\Phi} \neq \hat{\Phi}_0$. Hence, if Λ_Ψ is a symbol of defect 1, such that $[\Lambda_\Psi]$ is the component of $\underline{c}(Z, \Phi, \hat{\Phi})$, corresponding to $\Psi \subset \Phi$ in the sum (2.12.1) defining $c(Z, \Phi, \hat{\Phi})$, we have:

(5.10.2) $$\langle \rho(\Lambda'), R[\Lambda_\Psi] \rangle_{G_n^F} = 2^{-d}(-1)^{|\hat{\Phi}_0^* \cap \Psi^*|}.$$

In particular, we have

$$\langle \rho(\Lambda'), R[\Lambda] \rangle_{G_n^F} = (-1)^{[\Lambda', \Lambda]} \cdot 2^{-d},$$

where $[\Lambda', \Lambda] \in F_2$ is an unknown function. If we identify $\mathcal{S}_Z$ with the symplectic vector space V_Z (see 5.7) then the function $[\Lambda', \Lambda]$ becomes a map $\tilde{V}_Z \times \tilde{V}_Z \to F_2$. This map satisfies the conditions of Proposition 3.8. Indeed condition (b) is just (5.10.1), condition (a) follows from (5.10.2). Finally condition (c) is the equality.

(5.10.3) $$\sum_{\Lambda' \in \mathcal{S}_Z} \langle \rho(\Lambda'), R[\Lambda] \rangle_{G_n^F} = \begin{cases} 2^d & \text{if } \Lambda = Z \\ 0 & \text{if } \Lambda \in \mathcal{S}_Z, \Lambda \neq Z, \text{ of defect one.} \end{cases}$$

The left hand side of this equality can be written

$$\Big\langle \sum_{\Lambda' \in \mathcal{S}_Z} \rho(\Lambda'), R[\Lambda] \Big\rangle_{G_n^F} = \sum_{\hat{\Phi} \subset \Phi} \langle \underline{c}(Z, \Phi, \hat{\Phi}), R[\Lambda] \rangle_{G_n^F} \quad \text{(for a fixed } \Phi\text{)}.$$
$$= \langle 2^d R[Z], R[\Lambda] \rangle_{G_n^F}$$

which is the right hand side of (5.10.3). We may therefore apply Proposition 3.8 and get the formula $[\Lambda', \Lambda] = (\Lambda', \Lambda)$. The Theorem is proved.

6. An Application

6.1. We preserve the notations in 5.7. In addition, we shall assume, as we may, that the special symbol Z (of rank n) has $2m+1$ entries where $m \equiv n \pmod 2$. To the symbol

$$\Lambda_M = \begin{pmatrix} Z_2 \amalg (Z_1 - M) \\ Z_2 \amalg M \end{pmatrix} \in \mathcal{S}_Z$$

($M \subset Z_1$, $|M| \equiv d \pmod 2$) we associate the symbol $\Lambda_{M'} \in \mathcal{S}_Z$ where $M' = M_{\text{ev}} \amalg (Z_1 - M)_{\text{odd}}$ (i.e. the set of even entries in M union odd entries in $Z_1 - M$). Note that $M' \subset Z_1$ satisfies again $|M'| \equiv d \pmod 2$ since

$$|M| + |M'| \equiv |(Z_1)_{\text{odd}}| \equiv \sum_{z \in Z_1} z \equiv (\text{sum of all entries in } Z) = n + m^2 \equiv 0 \pmod 2.$$

6.2. **Lemma.** *With the notations of* 6.1 *and the assumptions of* 5.9, *we have*

$$\langle\rho(\Lambda_M), R_w\rangle_{G_n^F}=(-1)^{a[Z]}\langle\rho(\Lambda_{M'}), R_{w_0 w}\rangle_{G_n^F}$$

for all $w\in W_n$.

Proof. Using 5.9 and the formula $\mathrm{Tr}(w_0 w, [\Lambda_P])=\varepsilon_{[\Lambda_P]}\,\mathrm{Tr}(w, [\Lambda_P])$, $(P\subset Z_1, |P|=d)$ we see that it is enough to show that

$$(-1)^{(\Lambda_P, \Lambda_M)}=(-1)^{a[Z]}(-1)^{(\Lambda_P, \Lambda_{M'})}\varepsilon_{[\Lambda_P]}$$

for any $P\subset Z_1$, $|P|=d$.

But this follows immediately from definitions and from 2.2, 2.10.

6.3. **Theorem.** *We preserve the assumptions of* 5.8. *Let h be an integer, and let* $\Lambda_M\in\mathscr{S}_Z$ $(M\subset Z_1, |M|\equiv d\ (\mathrm{mod}\ 2))$ *be as in* 6.1. *Then the multiplicity of* $\rho(\Lambda_M)$ *in the virtual* G_n^F*-module* $\sum(-1)^i H_c^i(X_{w_0})^{(h)}$ (=*part of weight* h) *is equal to* $(-1)^{a[Z]}\dim[\Lambda_{M'}]$ *if* $h=\nu-\frac{a[Z]+A[Z]}{2}$ (*where Z is as in* 6.1) *and* $|M'|=d$ (*see* 6.1); *otherwise, it is zero.*

Proof. By 4.7, this multiplicity is given by

$$\sum_E \mathrm{Tr}(T_{w_0}, E; h/2)\langle\rho(\Lambda_M), R(E)\rangle_{G_n^F}$$

(sum over all irreducible $Q[W_n]$-modules E). If the term corresponding to E is non-zero, then, using 1.11 and 5.8, we have $h=\nu-\frac{a(E)+A(E)}{2}$ (since $\mathrm{Tr}(T_{w_0}, E; h/2)\neq 0$) and $a[Z]=a(E)$, $A[Z]=A(E)$ (since $\langle\rho(\Lambda_M), R(E)\rangle_{G_n^F}\neq 0$). Hence the multiplicity is zero unless $h=\nu-\frac{a[Z]+A[Z]}{2}$. Hence the multiplicity for $h=\nu-\frac{a[Z]+A[Z]}{2}$ is equal to the sum of multiplicities over all h, i.e. to $\langle\rho(\Lambda_M), R_{w_0}\rangle_{G_n^F}$. By Lemma 6.2, the last inner product is equal to $(-1)^{a[Z]}\langle\rho(\Lambda_{M'}), R_1\rangle_{G_n^F}$. It remains to use the formula

$$\langle\rho(\Lambda_{M'}), R_1\rangle_{G_n^F}=\begin{cases}\dim[\Lambda_{M'}] & \text{if } |M'|=d\\ 0, & \text{otherwise}\end{cases}.$$

6.4. *Remark.* It seems likely that for Λ_M as above such that $|M'|=d$, we have $\langle\rho(\Lambda_M), H_c^i(X_{w_0})\rangle_{G_n^F}\neq 0$ if and only if $i=2\nu-A[Z]$.

6.5. According to [11, 3.9], to each unipotent representation ρ of G_n^F (as in 5.8) one can associate a sign $\lambda_\rho=\pm 1$ such that, whenever ρ is contained in a generalized eigenspace of Frobenius F: $H_c^i(X_w)\to H_c^i(X_w)$, the corresponding eigenvalue of F is of the form $\lambda_\rho\cdot q^k$, where k is an integer. It also follows from [11, 3.33] that, if $\rho=\rho(\Lambda_M)$ $(M\subset Z_1, |M|\equiv d \bmod 2)$, as in 6.1, then λ_ρ depends only on the integer $\frac{1}{2}(|M|-d)$. We shall prove the following result (which was proved in a different way in [2, 2.5.3] assuming the conjecture [11, 4.3].)

6.6. **Proposition.** *With the previous notations, and assumptions of* 5.8, *we have* $\lambda_\rho=(-1)^{\frac{1}{2}(|M|-d)}$.

Proof. The Frobenius map $F\colon X_{w_0}\to X_{w_0}$ has no fixed points. Therefore, the fixed point formula, together with 6.3 shows that

$$\sum_Z\sum_M(-1)^{a[Z]}\dim[\Lambda_{M'}]\,\lambda_{\rho(\Lambda_M)}\,q^{\nu-\frac{a[Z]+A[Z]}{2}}\dim\rho(\Lambda_M)=0$$

(the first sum is over all special symbols Z of rank n, up to equivalence, the second sum is over all subsets $M\subset Z_1$, $|M|\equiv d\,(\mathrm{mod}\,2)$ such that $|M'|=d$, see 6.1.)

It is enough to show that the same identity holds with $\lambda_{\rho(\Lambda_M)}$ replaced by $(-1)^{\frac{1}{2}(|M|-d)}$, for then the desired formula would follow by induction on $\frac{1}{2}(|M|-d)$. Moreover, it is enough to prove this identity with q replaced by $(-q)$. Under this change, $(-1)^{a[Z]}\dim\rho(\Lambda_M)$ becomes $\dim\rho(\Lambda_{M'})$. Thus, we must prove

$$\sum_Z\sum_M(-1)^{\frac{1}{2}(|M|-d)}\dim[\Lambda_{M'}]\,(-q)^{\nu-\frac{a[Z]+A[Z]}{2}}\dim\rho(\Lambda_{M'})=0$$

(the summation is as before).

A direct computation shows that

$$(-1)^{\frac{1}{2}(|M|-d)+\nu-\frac{a[Z]+A[Z]}{2}}=\varepsilon_{\Lambda_{M'}}\cdot(-1)^n.$$

Hence the identity to be proved is

$$(-1)^n\sum_E\varepsilon_E\dim(E)\,q^{\nu-\frac{a(E)+A(E)}{2}}\dim(\rho_E)=0$$

(summation is over all irreducible $Q[W]$-modules E) where ρ_E is the irreducible principal series representation of G_n^F corresponding to $\tilde{E}$.

But this identity simply expresses the fact that, in the standard representation of the Hecke algebra H on the space of functions on complete flags, the trace of T_{w_0} is equal to zero.

This completes the proof.

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