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A Tauberian Theorem and Tangential Convergence for Bounded Harmonic Functions on Balls in $\mathbb{C}^n$

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If $u(x, y)$ is a bounded harmonic function on the upper half plane and $\lim_{x\to\infty} u(x, y_0)=a$ for some $y_0>0$, then $\lim_{x\to\infty} u(x, y)=a$ for any other positive y. This fact easy to prove by classical methods is also a consequence of the Wiener Tauberian theorem and the fact that the Fourier transform of the Poisson kernel never vanishes.

By studying the commutative algebra of radial integrable functions on the Heisenberg group, to which the Poisson kernel for the Siegel domain

$$D_r=\left\{(z_0, z_1, \ldots, z_r)\in\mathbb{C}^{r+1}: \operatorname{Im} z_0>\sum_{j=1}^{r}|z_j|^2\right\}$$

belongs, we obtain a similar result for bounded harmonic functions in D_r. Using the Cayley transform, this becomes a property of tangential convergence for bounded harmonic functions on the balls in $\mathbb{C}^{r+1}$ along appropriate surfaces.

Heisenberg Groups

Let H_r be the Heisenberg group, i.e. $H_r=\mathbb{C}^r\times\mathbb{R}$ with the multiplication given by

$$(z, t)(w, u)=(z+w, t+u+2\operatorname{Im}\langle z, w\rangle), \tag{1}$$

where $z=(z_1, \ldots, z_r)$, $w=(w_1, \ldots, w_r)$, $t, u\in\mathbb{R}$ and

$$\langle z, w\rangle=\sum_{j=1}^{r} z_j\bar{w}_j.$$

Abusing slightly the notation, we write $dz\,dt$ instead of $\frac{i}{2}(dz\wedge d\bar{z})\,dt$ for the differential of the Lebesgue measure on $\mathbb{C}^r\times\mathbb{R}$, which is the Haar measure on H_r.

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Bargmann Representations

For a real non-zero λ let $\mathscr{H}^\lambda$ be the Hilbert space of entire functions F on $\mathbb{C}^r$ such that

$$\int_{\mathbb{C}^r} e^{-2|\lambda||z|^2} |F(z)|^2 \, dz = |\lambda|^{-r} \|F\|_{\mathscr{H}^\lambda}$$

is finite. The monomials $z^n = z_1^{n_1} \dots z_r^{n_r}, n_j \in \mathbb{N}$, are orthogonal in $\mathscr{H}^\lambda$ and the functions

$$\phi_n^\lambda(z) = (n!)^{-1/2} \left(\frac{2}{\pi}\right)^{r/2} (2|\lambda| z)^n \qquad (n! = n_1! \dots n_r!) \tag{2}$$

form an orthonormal basis for $\mathscr{H}^\lambda$.

The group H_r acts on $\mathscr{H}^\lambda$ for positive λ by

$$U_{(z,t)}^\lambda F(w) = e^{-i\lambda t + \lambda(2\langle w, z\rangle - |z|^2)} F(w - z) \tag{3}$$

and for negative λ by

$$U_{(z,t)}^\lambda F(w) = e^{-i\lambda t - \lambda(2\langle w, \bar{z}\rangle - |z|^2)} F(w - \bar{z}). \tag{4}$$

For $\lambda = 0$ we get the 1 dimensional representations

$$\chi_w(z, t) = e^{-i\mathrm{Re}\langle z, w\rangle}. \tag{5}$$

These are the irreducible unitary representations of H_r up to equivalence.

For $f \in L^1(H_r)$ and $\lambda \neq 0$ we write U_f^λ for the operator $\int f(z,t) \, U_{(z,t)}^\lambda dz \, dt$.

Algebra of Radial Functions

This algebra, which is crucial to all what follows, has been investigated before and various facts about it and its relation to the Szegö kernel can be found in [3, 4]. For $\theta = (\theta_1, \dots, \theta_r)$, $\theta_j \in \mathbb{R}$ we define an automorphism α_θ of H_r by

$$\alpha_\theta : (z, t) \to (e^{i\theta} z, t),$$

where

$$e^{i\theta} z = (e^{i\theta_1} z_1, \dots e^{i\theta_r} z_r).$$

We have

$$\begin{aligned} U_{(e^{i\theta}z,t)}^\lambda &= A_\theta^{-1} \, U_{(z,t)}^\lambda A_\theta \qquad \text{for } \lambda > 0, \\ U_{(e^{i\theta}z,t)}^\lambda &= A_\theta \, U_{(z,t)}^\lambda A_\theta^{-1} \qquad \text{for } \lambda < 0, \end{aligned} \tag{6}$$

where $A_\theta F(z) = F(e^{i\theta} z)$. Also, for $\lambda = 0$,

$$\chi_w(e^{i\theta} z, t) = \chi_{e^{-i\theta} w}(z, t).$$

We say that a function f on H_r is radial, if

$$f(z, t) = f(e^{i\theta} z, t) \qquad \text{for all } \theta.$$

In virtue of (6), if f is radial, the operators U_f^λ, $\lambda \neq 0$, and A_θ commute and, since

$$A_\theta \phi_n^\lambda = e^{i\langle \theta, n\rangle} \phi_n^\lambda,$$

we have

$$U_f^\lambda \phi_n^\lambda = \hat{f}(\lambda, n)\, \phi_n^\lambda, \qquad \text{where } \hat{f}(\lambda, n) \in \mathbb{C}. \tag{7}$$

Also, for $\lambda = 0$, we write

$$\hat{f}(0, \rho) = \int f(z,t)\, \chi_w(z,t)\, dz\, dt, \qquad \text{where } \rho = (|w_1|, \ldots, |w_r|). \tag{8}$$

Let $\mathscr{A}$ denote the space of radial functions in $L^1(H_r)$. Since α_θ are automorphisms of H_r, $\mathscr{A}$ is a closed $*$-subalgebra of $L^1(H_r)$. Formula (7) shows that the algebra $\mathscr{A}$ is commutative. Since $L^1(H_r)$ is symmetric [7], the $*$-subalgebra $\mathscr{A}$ is also symmetric.

Proposition 1. *All non-zero multiplicative functionals on $\mathscr{A}$ are either of the form*

$$(a)\ f \to \hat{f}(\lambda, n) \qquad \textit{as in } (7)$$

or of the form

$$(b)\ f \to \hat{f}(0, \rho) \qquad \textit{as in } (8).$$

Proof. Let ψ be a non-zero multiplicative linear functional on $\mathscr{A}$. Since $\mathscr{A}$ is a symmetric $*$-subalgebra of $L^1(H_r)$ there exists an irreducible $*$-representation π of $L^1(H_r)$ and a unit vector ξ in the Hilbert space $\mathscr{H}^\pi$ such that $\pi_f \xi = \psi(f)\xi$ for f in $\mathscr{A}$. If $\mathscr{H}^\pi$ is one-dimensional, then ψ has the form (b). Otherwise $\pi = U^\lambda$ for some $\lambda \neq 0$ and $\mathscr{H}^\pi = \mathscr{H}^\lambda$. Since $\{U_f^\lambda : f \in \mathscr{A}\}$ is a $*$-algebra of operators which are diagonal on the basis ϕ_n^λ, we have $\xi = \phi_n^\lambda$ for some n and (a) follows.

Proposition 2 (cf. [8]). *If $f \in \mathscr{A}$, then*

$$\hat{f}(\lambda, n) = \int f(z,t)\, e^{-i\lambda t} e^{-|\lambda||z|^2} \prod_{j=1}^{r} L_{n_j}(2|\lambda|\, |z_j|^2)\, dz\, dt, \tag{9}$$

where L_k is the Laguerre polynomial of degree k, that is

$$L_k(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{(-x)^j}{j!}.$$

Proof. By (7), for $\lambda > 0$

$$\hat{f}(\lambda, n) = \lambda^r \int f(z,t)\, e^{-i\lambda t + \lambda(2\langle w, z\rangle - |z|^2)}\, \phi_n^\lambda(w - z)\, \overline{\phi_n^\lambda(w)}\, e^{-2\lambda |w|^2}\, dz\, dt\, dw.$$

Thus it suffices to evaluate

$$\begin{aligned} &\lambda^r \int e^{2\lambda\langle w, z\rangle} \phi_n^\lambda(w - z)\, \overline{\phi_n^\lambda(w)}\, e^{-2\lambda|w|^2}\, dw \qquad\qquad (10)\\ &= \frac{(2\lambda)^{|n|+r}}{n!\, \pi^r} \int \left(\sum_p \frac{(2\lambda)^{|p|}}{p!} w^p \bar{z}^p \right) \left(\sum_q \binom{n}{q} (-1)^{|q|} w^{n-q} z^q \right) \bar{w}^n e^{-2\lambda|w|^2}\, dw \\ &= \frac{(2\lambda)^{|n|+r}}{n!\, \pi^r} \sum_p \frac{(-2\lambda)^{|p|}}{p!} \binom{n}{p} \prod_{j=1}^{r} |z_j|^{2p_j} \int e^{-2\lambda|w|^2} \prod_{j=1}^{r} |w_j|^{2n_j}\, dw, \end{aligned}$$

since for $m \neq n$ $\int w^m \bar{w}^n e^{-2\lambda|w|^2} dw = 0$. Because

$$\frac{\pi n_j!}{(2\lambda)^{n_j+1}} = \int |w_j|^{2n_j} e^{-2\lambda|w_j|^2} dw_j,$$

(10) is equal to

$$\sum_p \binom{n}{p} \frac{1}{p!} \prod_{j=1}^{r} (-2\lambda|z_j|^2)^{p_j} = \prod_{j=1}^{r} L_{n_j}(2\lambda|z_j|^2).$$

For $\lambda < 0$ the calculation is the same, and thus Proposition 2 follows.

For $s > 0$ a dilation of H_r is defined by

$$a_s(z, t) = (s^{-1/2} z, s^{-1} t).$$

a_s is an automorphism of H_r and so $(a_s f)(z, t) = s^{-r-1} f(a_s(z, t))$ defines an automorphism of $L^1(H_r)$ which preserves $\mathscr{A}$. For a functional ψ on $\mathscr{A}$, let $\langle f, a_s^* \psi \rangle = \langle a_s f, \psi \rangle$. a_s^* maps the Gelfand space $\mathscr{M}(\mathscr{A})$ of non-zero multiplicative functionals on $\mathscr{A}$ homeomorphically onto itself. On the other hand, if $f \in L^1(H_r)$ and $\int f(z, t)\, dz\, dt = 1$, $\{a_s f\}$ is an approximate identity in $L^1(H_r)$ as $s \to 0$.

Proposition 3. *$\mathscr{A}$ is a (commutative) regular algebra and the set of functions f in $\mathscr{A}$ whose Gelfand transform $\hat{f}$ has compact support in $\mathscr{M}(\mathscr{A})$ is dense in $\mathscr{A}$.*

Proof. We apply Dixmier's functional calculus. By [1] there exists a k such that for $f = f^*$ in $\mathscr{A}$ with compact support in H_r, the functions $F \in C^k(\mathbb{R})$ which vanish at 0 operate on $\hat{f}$ into $\hat{\mathscr{A}}$. This proves that $\mathscr{A}$ is regular.

Now let $f \in \mathscr{A}$ have compact support and $\hat{f}(0, 0) = 1$. Choose F in $C^k(\mathbb{R})$ such that $F(1) = 1$ and $F(x) = 0$ for $|x| \leq \frac{1}{2}$. Then $F \circ \hat{f} = \hat{g}$, where $g \in \mathscr{A}$ and $\int g(z, t)\, dz\, dt = F \circ \hat{f}(0, 0) = 1$; also $\operatorname{supp} \hat{g}$ is compact in $\mathscr{M}(\mathscr{A})$. Thus $(a_s g)^\wedge$ has also compact support in $\mathscr{M}(\mathscr{A})$ and since $\{a_s g\}$ is an approximate identity as $s \to 0$, Proposition 3 follows.

Corollary 4. *If I is a proper closed right ideal in $L^1(H_r)$, then there is a ψ in $\mathscr{M}(\mathscr{A})$ such that $\hat{f}(\psi) = 0$ for every $f \in I \cap \mathscr{A}$.*

The proof follows from the fact that $\mathscr{A}$ contains an approximate identity for $L^1(H_r)$, so $I \cap \mathscr{A}$ is a proper ideal in $\mathscr{A}$; also $\mathscr{A}$ has the Wiener property, as Proposition 3 shows.

Harmonic Functions and the Poisson Kernel

The following situation has been extensively studied. We specifically refer here to [5] and [6].

Let

$$D_r = \{(z, z_0) \in \mathbb{C}^r \times \mathbb{C} : \operatorname{Im} z_0 > |z|^2\}$$

on which the Heisenberg group H_r acts by translations

$$H_r \times D_r \ni ((w, u), (z, z_0)) \to (w, u) \cdot (z, z_0) = (w + z, z_0 + u + i|w|^2 + 2i\langle z, w\rangle) \in D_r.$$

Introducing new coordinates t, ε, z

$$z_0 = t + i(\varepsilon + |z|^2)$$
$$z = z$$

$D_r \cong H_r \times \mathbb{R}_+$ and the level surfaces for the variable ε are the orbits of H_r in D_r. Also H_r is identified with the boundary ∂D_r of D_r.

Let Δ be the Laplace-Beltrami operator for the Bergman metric on D_r. The bounded harmonic functions u on D_r, i.e. $\Delta u = 0$, have boundary values a.e. on ∂D_r, i.e.

$$\lim_{\varepsilon \to 0} u(z, t, \varepsilon) = \varphi(z, t) \text{ a.e.,} \tag{11}$$

where $\varphi \in L^\infty(H_r)$. Moreover

$$u(z, t, \varepsilon) = (\varphi * P_\varepsilon)(z, t), \tag{12}$$

where

$$P_\varepsilon(z, t) = c_r \varepsilon^{r+1}((|z|^2 + \varepsilon)^2 + t^2)^{-r-1}$$

$c_r = \dfrac{2^{r-1} r!}{\pi^{r+1}}$ and the convolution is on H_r.

We notice that $P_\varepsilon \in L^1(H_r)$ and is radial, i.e. $P_\varepsilon \in \mathscr{A}$. P_ε can be expressed as

$$P_\varepsilon = c_r^{-1} \varepsilon^{r+1} |S_\varepsilon|^2 \tag{13}$$

where

$$S_\varepsilon(z, t) = c_r(\varepsilon + |z|^2 - i t)^{-r-1}$$

is the Szegö kernel for D_r, which determines the orthogonal projection of $L^2(H_r)$ onto the Hardy space $H^2(D_r)$. Since

$$S_\varepsilon(z, t) = \frac{c_r}{r!} \int_0^\infty e^{i\mu t} e^{-\mu(\varepsilon + |z|^2)} \mu^r \, d\mu \tag{14}$$

we have

$$P_\varepsilon(z, t) = c_r \varepsilon^{r+1} \int_0^\infty \int_0^\infty e^{i(\mu - \eta)t} e^{-(\mu + \eta)(\varepsilon + |z|^2)} \mu^r \eta^r \, d\mu \, d\eta \tag{15}$$

Proposition 5. *For every $\varepsilon > 0$ $\hat{P}_\varepsilon$ does not vanish at any point in $\mathscr{M}(\mathscr{A})$.*

Proof. By Proposition 1, we check $\hat{P}_\varepsilon(0, \rho)$ and $\hat{P}_\varepsilon(\lambda, n)$. In view of (8) and (9), for both cases we compute

$$\int_{-\infty}^{+\infty} P_\varepsilon(z, t) e^{-i\lambda t} \, dt$$

which equals, by (15)

$$c_r \varepsilon^{r+1} \int_{\max(0, -\lambda)}^{\infty} e^{-(2\eta + \lambda)(\varepsilon + |z|^2)} (\eta + \lambda)^r \eta^r \, d\eta \tag{16}$$

If we take $\lambda = 0$ and evaluate the Fourier transform in $\mathbb{C}^r$ at ρ, we obtain $\hat{P}_\varepsilon(0, \rho)$.

Since the Fourier transform of $e^{-2\eta|z|^2}$ never vanishes in $\mathbb{C}^r$, it is immediate to see that $\hat{P}_\varepsilon(0,\rho)\neq 0$ for every ε and ρ.

For $\lambda\neq 0$ we proceed similarly and, using (9) and (16), we reduce ourselves to prove that for any $\gamma>|\lambda|$ and any $n=(n_1,\dots n_r)$, $n_j\in\mathbb{N}$, the integral

$$\int_{\mathbb{C}^r} e^{-\gamma|z|^2} e^{-|\lambda||z|^2} \prod_{j=1}^{r} L_{n_j}(2|\lambda|\ |z_j|^2)\,dz \tag{17}$$

is different from 0. By change of variables, (17) becomes

$$\left(\frac{\pi}{|\lambda|}\right)^r \int_0^\infty \dots \int_0^\infty e^{-\frac{\gamma+|\lambda|}{2|\lambda|}(t_1+\dots t_r)} \prod_{j=1}^{n} L_{n_j}(t_j)\,dt_1\dots dt_r. \tag{18}$$

As one can easily verify,

$$L_m(x)=\frac{1}{m!}e^x D^m(x^m e^{-x}),$$

(cf. [2] vol. 2, p. 188). Thus, (18) turns into

$$\begin{aligned}
&\left(\frac{\pi}{|\lambda|}\right)^r \frac{1}{n!}\int_0^\infty \dots \int_0^\infty e^{-\frac{\gamma-|\lambda|}{2|\lambda|}(t_1+\dots+t_r)} \prod_{j=1}^{r} D_j^{n_j}(t^{n_j}e^{-t_j})\,dt_1\dots dt_r\\
&=\left(\frac{\pi}{|\lambda|}\right)^r \frac{1}{n!}\frac{(\gamma-|\lambda|)^{|n|}}{(2|\lambda|)^{|n|}} \int_0^\infty \dots \int_0^\infty e^{-\frac{\gamma-|\lambda|}{2|\lambda|}(t_1+\dots+t_r)} t^n e^{-(t_1+\dots+t_r)}\,dt_1\dots dt_r\\
&=\pi^r\frac{(\gamma-|\lambda|)^{|n|}}{(\gamma+|\lambda|)^{|n|+r}}
\end{aligned}$$

which is positive.

Theorem. *Let u be a bounded harmonic function on D_r. Assume that for an $\varepsilon_0>0$*

$$\lim_{(z,t)\to\infty} u(z,t,\varepsilon_0)=a.$$

Then for any $\varepsilon>0$

$$\lim_{(z,t)\to\infty} (u(z,t,\varepsilon)=a.$$

Proof. Let φ be as in (11). Consider the right ideal in $L^1(H_r)$

$$I=\{f\in L^1(H_r)\colon \lim_{(z,t)\to\infty} \varphi * f(z,t)=a\int f(z,t)\,dz\,dt\}.$$

By assumption, (12) shows that $P_{\varepsilon_0}\in I\cap A$. By Corollary 4 and Proposition 5, $I=L^1(H_r)$ and the theorem follows.

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