Invent. math. 77, 85–99 (1984)

Inventiones mathematicae
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An arithmetic characterisation of the symmetric monodromy groups of singularities

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Introduction

To each isolated complete intersection singularity is associated the monodromy group Γ. It is a subgroup of the automorphism group of the middle homology group of the Milnor fibre. This homology group is provided with a bilinear intersection form, which is symmetric or skew-symmetric according to the dimension. This group, which is free abelian, with this additional structure is called the Milnor lattice L. The form is left invariant by the monodromy group. Correspondingly the monodromy group is a subgroup of the orthogonal resp. symplectic group of the Milnor lattice, generated by reflections resp. symplectic transvections corresponding to certain geometrically defined elements in the Milnor lattice, the vanishing cycles. A natural question arising in this context is the question whether such a group is arithmetic or not and to describe this subgroup inside the automorphism group of the lattice.

There were different efforts concerning this question in the symmetric and the skew-symmetric case in the last years. In the symmetric case H. Pinkham proved in 1977 that the monodromy groups of the 14 exceptional unimodal hypersurface singularities and the 8 triangle complete intersection singularities can be identified with certain arithmetic subgroups $O^*(L)$ of $O(L)$ [21]. On the other hand it was noticed that Pinkham's characterisation is false for almost all hyperbolic singularities. But using a theorem of M. Kneser, we proved an arithmetic theorem, from which we could derive an extension of the characterisation $\Gamma = O^*(L)$ first to all singularities of Arnold's lists [7], and later to large classes of hypersurface singularities [8, 9], of course with the above exceptions. In the skew-symmetric case N. A'Campo proved in 1979 that the simple part (cf. Sect. 4) of the monodromy group is arithmetic for the A_μ-singularities [1]. His result was generalized by B. Wajnryb to the case of plane curve singularities [23], and later by S.V. Chmutov to all isolated hypersurface singularities [6]. Our proof and the proofs in the skew-symmetric case use the existence of bases of vanishing cycles with certain special properties. The difficulty of showing the existence of such a basis was the reason, that we could not extend our results further in the past.

Recently E. Looijenga and his student W.A.M. Janssen generalized the results in the skew-symmetric case in the following way [13]. They introduced the algebraic notion of a skew-symmetric vanishing lattice and its monodromy group and showed that the simple parts of these monodromy groups are arithmetic. The basic examples for these notions are the Milnor lattice together with the set of vanishing cycles and the monodromy group of any odd dimensional isolated complete intersection singularity.

The object of this note is to study the symmetric analogues of these notions and to extend in this way the results in the symmetric case. We prove that the characterisation $\Gamma = O^*(L)$, which implies in particular that the simple part of the monodromy group is arithmetic, is true for the monodromy groups of all symmetric vanishing lattices, which satisfy certain additional conditions, and which we call complete vanishing lattices (cf. Sect. 2). This theorem was already announced in [9, Note added in proof]. By this theorem we can on the one hand give a more satisfactory proof of our previous results and on the other hand extend these results and show $\Gamma = O^*(L)$ for all isolated hypersurface singularities and for all 2-dimensional isolated complete intersection singularities in $\mathbb{C}^4$, with the hyperbolic singularities excluded. As was known before this yields also a purely lattice theoretical description of the set of vanishing cycles and of the unipotent and simple part of the monodromy group.

The organisation of the paper is as follows. In § 1 we collect the basic notations and definitions we need from our previous papers. In § 2 we define the notion of a complete (symmetric) vanishing lattice and state our main algebraic Theorem (2.3), which is proven in § 3. The proof is based among other things on two lemmas, (3.4) and (3.5), which are entirely analogous to Lemmas (2.6) and (2.7) of [13]. In § 4 we derive a description of the unipotent and simple part of the monodromy group. In § 5 we apply our results to the monodromy groups of singularities and also discuss the cases in the beginning of the hierarchy of singularities, to which our theorem cannot be applied. These results are summarized in Theorem (5.5).

The author is grateful to E. Looijenga for valuable discussions. This work was done while the author stayed at the State University of Utrecht and was supported by the Netherlands Foundation of Mathematics S.M.C. with financial aid from the Netherlands Organisation for the Advancement of Pure Research (Z.W.O). The author takes the opportunity to thank these institutions for their hospitality. He also thanks the referee for pointing out that the condition about a sublattice of type A_6 in the announcement [9, Note added in proof] and in the earlier version was superfluous.

§ 1. Notations

(1.1) Let L be an even lattice, i.e. a free finitely generated $\mathbb{Z}$-module with a symmetric bilinear form $\langle\ ,\ \rangle$ satisfying $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$. We denote by $O(L)$ the group of units ($=$isometries) of L. A $\delta \in L$ with $\langle \delta, \delta\rangle = \pm 2$ is called a minimal vector of square length ± 2. The reflection s_δ corresponding to such a δ is defined by

$$s_\delta(x)=x-\frac{2\langle x,\delta\rangle}{\langle\delta,\delta\rangle}\delta.$$

Let $\Delta\subset L$ be a subset of L with $\langle\delta,\delta\rangle=\pm 2$ for all $\delta\in\Delta$. We denote by $\mathbb{Z}\cdot\Delta$ the sublattice generated by Δ, and by Γ_Δ the subgroup of $O(L)$ generated by all reflections s_δ for $\delta\in\Delta$. We define $R_-(L)$ to be the subgroup of $O(L)$ generated by all reflections s_v corresponding to vectors v with $\langle v,v\rangle=-2$. Similarly one can define $R_+(L)$. In what follows we restrict to sets of minimal vectors of square length -2, $R_-(L)$ etc. With a suitable modification in the definition of the real spinor norm σ below (reverse the $>$-sign), all the results are also true if one replaces minus signs by plus signs.

(1.2) Let $\ker L$ denote the kernel of L and define $\bar{L}=L/\ker L$. Moreover let $\bar{L}_{\mathbb{R}}=\bar{L}\otimes\mathbb{R}=\mathbb{R}\cdot\bar{L}$. We define a homomorphism

$$\sigma\colon\ O(L)\to\{+1,-1\}$$

(real spinor norm) as follows. Let $g\in O(L)$. Let $\bar{g}$ be the induced element in $O(\bar{L})\subset O(\bar{L}_{\mathbb{R}})$. Then $\bar{g}$ can be written $\bar{g}=s_{v_1}\circ\ldots\circ s_{v_r}$ in $O(\bar{L}_{\mathbb{R}})$. Define

$$\sigma(g)=\begin{cases}+1 & \text{if } \langle v_i,v_i\rangle>0 \text{ for an even number of indices,}\\ -1 & \text{otherwise.}\end{cases}$$

Denote by $L^\#$ the dual lattice $\mathrm{Hom}(L,\mathbb{Z})$ and let $j\colon L\to L^\#$ be the natural homomorphism. Let τ be the canonical homomorphism

$$\tau\colon\ O(L)\to\mathrm{Aut}(L^\#/jL).$$

We define

$$O^*(L)=\ker\sigma\cap\ker\tau.$$

(1.3) Let $f\in L$ be an isotropic vector (i.e. $\langle f,f\rangle=0$) and $w\in L$ a vector orthogonal to f. Then the Eichler-Siegel-transformation [11] $\psi_{f,w}\in O(L)$ is defined by

$$\psi_{f,w}(x)=x+\langle x,f\rangle w-\langle x,w\rangle f-\tfrac{1}{2}\langle w,w\rangle\langle x,f\rangle f$$

for $x\in L$. Let U be a unimodular hyperbolic plane with a basis of isotropic vectors $\{f_1,f_2\}$ satisfying $\langle f_1,f_2\rangle=1$. Suppose that L is the orthogonal direct sum of U and an even lattice M, $L=M\perp U$. We define $\Psi_U(L)$ to be the subgroup of $O(L)$ generated by the transformations $\psi_{f_1,w},\psi_{f_2,w}$ for arbitrary $w\in M$. Some properties of these transformations are listed in [8]. We recall in particular one property, which we shall use frequently.

(1.4) **Lemma.** *Let $L=U'\perp U$ for another unimodular hyperbolic plane U'. Then for each vector $x\in U'\perp U$ there exists a $\varphi\in\Psi_U(U'\perp U)$, such that $\varphi(x)=\alpha f_1+\beta f_2$ with $\alpha|\beta$.*

A proof can be found in [8].

One has the following inclusions between the various groups:

$$\Gamma_\Delta\subset R_-(L)\subset O^*(L)\subset O(L),$$

$$\Psi_U(\mathrm{L})\subset O^*(L),$$

and if M is generated by minimal vectors of square length -2

$$\Psi_U(L) \subset R_-(L).$$

The last inclusion follows from the fact that the mapping $M \to \Psi_U(L)$, $w \mapsto \psi_{f_1,w}$ is a homomorphism, and the identity

$$\psi_{f_1,w} = s_w \circ s_{w-f_1}$$

for $w \in M$ with $\langle w, w \rangle = -2$.

(1.5) Let finally Δ be a subset of L. We define an equivalence relation on Δ, denoted by $\sim_\Delta$, as follows: for $\delta, \delta' \in \Delta$ $\delta \sim_\Delta \delta'$ if and only if there exists a sequence $\delta = \delta_0, \delta_1, \ldots, \delta_k = \delta'$ with $\langle \delta_{i-1}, \delta_i \rangle = 1$, $\delta_i \in \Delta$, for $1 \leqq i \leqq k$, or $\delta = \delta'$.

§ 2. Complete vanishing lattices

We introduce here the symmetric analogue of the basic notion of [13].

(2.1) **Definition.** Let L be an even lattice and $\Delta \in L$ a subset of L. A pair (L, Δ) is called a *vanishing lattice*, if Δ satisfies the following conditions:

(i) Δ consists of minimal vectors of square length -2.
(ii) Δ generates L.
(iii) Δ is a Γ_Δ-orbit.
(iv) Unless rank $L = 1$, there exist δ_1, $\delta_2 \in \Delta$ with $\langle \delta_1, \delta_2 \rangle = 1$.

(2.2) **Definition.** A vanishing lattice (L, Δ) is called *complete*, if there exists an orthogonal splitting $L = L'' \perp U' \perp U$ with unimodular hyperbolic planes U, U', such that the following conditions are satisfied:

(i) There exist $\omega_1, \omega_2 \in \Delta \cap L''$ with $\langle \omega_1, \omega_2 \rangle = 1$.
(ii) Let $\{f_1, f_2\}$ resp. $\{f_1', f_2'\}$ be bases of isotropic vectors of U resp. U' with $\langle f_1, f_2 \rangle = \langle f_1', f_2' \rangle = 1$. Let

$$\Omega = \{\omega_1, \omega_2, \omega_1 - f_1, f_1 - f_2, \omega_1 - f_1', f_1' - f_2'\}.$$

Then $\Omega \subset \Delta$.

We are ready to state our main theorem.

(2.3) **Theorem.** *Let (L, Δ) be a complete vanishing lattice. Then*

$$\Gamma_\Delta = R_-(L).$$

The proof of Theorem (2.3) will be given in Sect. 3.

Since a complete vanishing lattice contains a sublattice $A_2 \perp U' \perp U$, where A_2 denotes a root lattice of type A_2 [4] (with bilinear form multiplied by -1), the assumptions of Kneser's Theorem [15, Satz 4] are fulfilled. This theorem is formulated for nondegenerate lattices and uses the rational spinor norm, but one can easily generalize it to degenerate lattices and derive the identification $R_-(L) = O^*(L)$ under the same assumptions (cf. [7, Proof of Theorem 3.1c)]). Combining these results, we get

(2.4) **Theorem.** *Let (L, Δ) be a complete vanishing lattice. Then*

$$\Gamma_\Delta = O^*(L).$$

One can also derive a purely lattice theoretical description of the set Δ for a complete vanishing lattice.

(2.5) **Proposition.** *Let (L, Δ) be a vanishing lattice with $L = L'' \perp U' \perp U$. Assume $\Psi_U(L) \subset \Gamma_\Delta$. (This condition is in particular satisfied, if $\Gamma_\Delta = O^*(L)$). Then*

$$\Delta = \{v \in L \mid \langle v, v\rangle = -2 \text{ and } \langle v, L\rangle = \mathbb{Z}\}.$$

Proof. This proposition follows from a more general result due to E. Looijenga (cf. [5, 4.2]).

For the convenience of the reader we give a direct proof following the same lines as [19, Proof of Theorem (2.4)], cf. also [22, App. to §6]. That Δ is contained in the set on the righthand side is clear by the definition of a vanishing lattice. Therefore let v be an element of the set on the righthand side. We write $v = v' + v''$ with $v' \in U' \perp U$ and $v'' \in L''$. Let $\{f_1, f_2\}$ be a basis of U as above. By (1.4) there exists a $\varphi \in \Psi_U(U' \perp U)$ such that $\tilde{v}' = \varphi(v')$ satisfies $\langle \tilde{v}', f_2\rangle \mid \langle \tilde{v}', f_1\rangle$. But since $\varphi(v'') = v''$, it follows that also $\tilde{v} = \varphi(v)$ satisfies $\langle \tilde{v}, f_2\rangle \mid \langle \tilde{v}, f_1\rangle$. We can therefore assume that this is already true for v. Since $\langle v, L\rangle = \mathbb{Z}$, there exists a $y \in L$ with $\langle v, y\rangle = -1$. Write $y = y_1 + y_2$, where $y_1 \in L'' \perp U'$ and $y_2 \in U$. Then

$$\psi_{f_2, y_1}(v) = \alpha f_1 + \beta f_2 + v_1$$

with $v_1 \in L'' \perp U'$. Then $\gcd(\alpha, \beta) = 1$, since $\alpha = \langle v, f_2\rangle$ and

$$\begin{aligned}\beta &= \langle v, f_1\rangle - \langle v, y_1\rangle - \tfrac{1}{2}\langle y_1, y_1\rangle \langle v, f_2\rangle \\ &\equiv -\langle v, y_1\rangle \bmod \alpha \equiv -\langle v, y\rangle \bmod \alpha \equiv 1 \bmod \alpha.\end{aligned}$$

Again by (1.4), after an application of an element of $\Psi_U(U' \perp U)$ we can assume that $\alpha = 1$. Then $\psi_{f_2, -v_1}$ maps this vector to a vector of the form $f_1 + \varepsilon f_2$. But

$$\langle f_1 + \varepsilon f_2, f_1 + \varepsilon f_2\rangle = \langle v, v\rangle = -2$$

implies $\varepsilon = -1$. So each minimal vector v of square length -2 with $\langle v, L\rangle = \mathbb{Z}$ can be mapped to the vector $f_1 - f_2$ by an element of $\Psi_U(L) \subset \Gamma_\Delta$. This proves the proposition.

From Proposition (2.5) we immediately get the following corollary which can be regarded as a sort of converse of Theorem (2.4).

(2.6) **Corollary.** *Let (L, Δ) be a vanishing lattice. Assume that there exists an orthogonal splitting $L = L'' \perp U' \perp U$, such that the following conditions are satisfied:*

(i) *L'' contains a sublattice of type A_2.*
(ii) *$\Psi_U(L) \subset \Gamma_\Delta$.*

Then (L, Δ) is complete.

(2.7) *Remark.* Let (L, Δ_L) and (M, Δ_M) be vanishing lattices. We say that (L, Δ_L) *contains* (M, Δ_M) if M is a primitive sublattice of L and $\Delta_M \subset \Delta_L$. Note that if (L, Δ_L) contains a complete vanishing lattice, then it is itself complete.

§ 3. Special subsets

The proof of Theorem (2.3) will be reduced to a slight generalization of the key theorem [8, Theorem 3]. This theorem involved the notion of a special basis. We generalize this notion to more general subsets. As above L always denotes an even lattice.

(3.1) **Definition.** A subset $\Lambda \subset L$ is called *special*, if the following conditions are satisfied:

(i) Λ consists of minimal vectors of square length -2.
(ii) There exist $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ such that

$$\langle \lambda_1, \lambda_2 \rangle = 1,$$
$$\langle \lambda_1, \lambda \rangle = 0, \qquad \langle \lambda_2, \lambda \rangle = \langle \lambda_3, \lambda \rangle \text{ for all } \lambda \in \Lambda \text{ with } \lambda \neq \lambda_1, \lambda_2.$$

(iii) Let $\Lambda' = \Lambda - \{\lambda_1, \lambda_2\}$. Then $\lambda \sim_{\Lambda'} \lambda_3$ for all $\lambda \in \Lambda'$.

Let Λ be a special subset of L. Let $\Lambda' = \Lambda - \{\lambda_1, \lambda_2\}$ as above, and let

$$f_1 = -\lambda_2 + \lambda_3$$
$$f_2 = -\lambda_1 - \lambda_2 + \lambda_3$$
$$U = \mathbb{Z} f_1 + \mathbb{Z} f_2.$$

Then U is a unimodular hyperbolic plane and

$$\mathbb{Z} \cdot \Lambda = U \perp \mathbb{Z} \cdot \Lambda'.$$

(3.2) **Theorem.** *Let $\Lambda \subset L$ be a special subset. Then*

$$\Psi_U(\mathbb{Z} \cdot \Lambda) \subset \Gamma_\Lambda.$$

If moreover another unimodular hyperbolic plane U' is contained in $\mathbb{Z} \cdot \Lambda'$, then

$$R_-(\mathbb{Z} \cdot \Lambda) = \Gamma_\Lambda.$$

The proof of Theorem (3.2) is the same as the proof of [8, Theorem 3] applied to the lattice $K = \mathbb{Z} \cdot \Lambda$, except that we do not have a basis B' of $K' = \mathbb{Z} \cdot \Lambda'$, but only a generating system Λ'. But the linear independence of the elements of B' is not used.

(3.3) *Example.* Let (L, Δ) be a complete vanishing lattice and Ω be the set of Definition (2.2). Then Ω is special: Let $\lambda_1 = f_1 - f_2$, $\lambda_2 = \omega_1 - f_1$, $\lambda_3 = \omega_1$. One easily checks that all elements of Ω are equivalent with respect to $\sim_\Omega$. For $\omega \in \Omega' = \Omega - \{\lambda_1, \lambda_2\}$ one even has $\omega \sim_{\Omega'} \lambda_3$. Moreover $U' \subset \mathbb{Z} \cdot \Omega'$. Theorem (3.2) implies

$$\Psi_U(U' \perp U) \subset \Psi_U(\mathbb{Z} \cdot \Omega) \subset \Gamma_\Omega.$$

(3.4) **Lemma.** *Let (L, Δ) be a complete vanishing lattice. Then Δ is an equivalence class with respect to $\sim_\Delta$.*

Proof. Each equivalence class $\tilde{\Delta}$ is a $\Gamma_{\tilde{\Delta}}$-orbit. For $\langle\delta,\delta'\rangle=1$ implies $s_{\delta'}s_\delta(\delta')=\delta$. On the other hand $\Gamma_{\tilde{\Delta}}(\tilde{\Delta})\subset\tilde{\Delta}$. To prove this, we remark first, that if $\delta\in\tilde{\Delta}$, then also $s_\delta(\delta)=-\delta$ is in $\tilde{\Delta}$. For there is a $\gamma\in\Gamma_\Delta$ with $\gamma(\omega_2)=\delta$ and

$$\{\gamma(\omega_2), \gamma(\omega_1-f_1), \gamma(f_1-f_2)\}$$

is the basis of a root system of type A_3, which is contained in $\tilde{\Delta}$. Now let $\delta,\delta'\in\tilde{\Delta}$. Then one can easily show by induction on the minimal length of a sequence $\delta=\delta_0,\delta_1,\ldots,\delta_k=\delta'$, $\langle\delta_{i-1},\delta_i\rangle=1$ for $1\leqq i\leqq k$, $\delta_i\in\tilde{\Delta}$, that $s_\delta(\delta')\in\tilde{\Delta}$.

Since Γ_Δ preserves the symmetric bilinear form, Γ_Δ permutes equivalence classes. Now let $\tilde{\Delta}$ be an equivalence class and $\delta\in\Delta$. It suffices to show, that there exists a $\tilde{\delta}\in\tilde{\Delta}$, such that $s_\delta(\tilde{\delta})=\tilde{\delta}$, i.e. $\langle\delta,\tilde{\delta}\rangle=0$. For this implies that Γ_Δ leaves $\tilde{\Delta}$ invariant, hence $\tilde{\Delta}=\Gamma_\Delta\cdot\tilde{\Delta}=\Delta$.

Let $\delta_1\in\tilde{\Delta}$. Then there is a $\gamma\in\Gamma_\Delta$ with $\gamma(\omega_1)=\delta_1$. Then $\gamma(\Omega)\subset\tilde{\Delta}$. We may assume without loss of generality that $\delta_1=\omega_1$, since we can replace $U'\perp U$ by $\gamma^{-1}(U'\perp U)$ and ω_1,ω_2 by $\gamma^{-1}(\omega_1)$, $\gamma^{-1}(\omega_2)$. Then $\Omega\subset\tilde{\Delta}$. Since the lattice $U'\perp U\subset\mathbb{Z}\cdot\Omega$ is unimodular, we can write $L=L''\perp U'\perp U$. Write δ as $\delta=\delta''+\delta'$ with $\delta''\in L''$, $\delta'\in U'\perp U$. By (1.4) there is a $\varphi\in\Psi_U(U'\perp U)\subset\Gamma_\Omega$ with $\varphi(\delta')\in U'$. But $\varphi(\delta'')=\delta''$, hence $\varphi(\delta)\in U^\perp$, where $U^\perp$ denotes the orthogonal complement of U. But

$$\langle\delta,\varphi^{-1}(f_1-f_2)\rangle=\langle\varphi(\delta),f_1-f_2\rangle=0,$$

and $f_1-f_2\in\tilde{\Delta}$, and $\varphi\in\Gamma_{\tilde{\Delta}}$. Since each equivalence class $\tilde{\Delta}$ is a $\Gamma_{\tilde{\Delta}}$-orbit, the lemma is proven.

(3.5) **Lemma.** *Let (L, Δ) be a complete vanishing lattice. Let*

$$\Delta_0:=\{\delta\in\Delta\,|\,\langle\omega_1,\delta\rangle=1 \text{ or } \delta=\omega_1\}.$$

Then $\Gamma_{\Delta_0}=\Gamma_\Delta$ and $L=\mathbb{Z}\cdot\Delta_0$.

Proof. The proof is the same as that of Lemma (2.7) in [13]. For the convenience of the reader we repeat it here. Let $\delta\in\Delta$. Let $l(\delta)$ denote the minimal length of a sequence $\omega_1=\delta_0,\delta_1,\ldots,\delta_k=\delta$ such that $\langle\delta_{i-1},\delta_i\rangle=1$, $\delta_i\in\Delta$ for $1\leqq i\leqq k$, which exists according to Lemma (3.4). We prove by induction on $l(\delta)$: $\delta\in\Gamma_{\Delta_0}\cdot\omega_1$. If $l(\delta)=0$ then $\delta=\omega_1$. Now let $k=l(\delta)>0$ and a sequence as above be given. By the induction hypothesis there exists a $\gamma\in\Gamma_{\Delta_0}$ such that $\gamma(\delta_{k-1})=\omega_1$. But

$$\langle\omega_1,\gamma(\delta_k)\rangle=\langle\gamma(\delta_{k-1}),\gamma(\delta_k)\rangle=\langle\delta_{k-1},\delta_k\rangle=1,$$

hence $s_{\gamma(\delta_k)}\in\Gamma_{\Delta_0}$. Then

$$s_{\delta_k}=\gamma^{-1}\,s_{\gamma(\delta_k)}\,\gamma\in\Gamma_{\Delta_0}.$$

Therefore

$$\delta_k=s_{\delta_{k-1}}s_{\delta_k}(\delta_{k-1})\in\Gamma_{\Delta_0}\cdot\omega_1.$$

But $\Delta=\Gamma_{\Delta_0}\cdot\Delta_0$ implies $\Gamma_{\Delta_0}=\Gamma_\Delta$ and $L=\mathbb{Z}\cdot\Delta=\mathbb{Z}\cdot(\Gamma_{\Delta_0}\cdot\Delta_0)\subset\mathbb{Z}\cdot\Delta_0$.

(3.6) **Proposition.** *Let (L, Δ) be a complete vanishing lattice. Let*

$$\Lambda_0 := \{\delta \in \Delta \mid \langle f_1, \delta\rangle = \langle f_2, \delta\rangle = 0\}$$
$$\Lambda' := \{\delta \in \Lambda_0 \mid \delta \sim_{\Lambda_0} \omega_1\}$$
$$\Lambda := \Lambda' \cup \{\omega_1 - f_1, f_1 - f_2\}.$$

Then Λ is a special subset of L with $\Omega \subset \Lambda \subset \Delta$, $U' \subset \mathbb{Z}\cdot\Lambda'$ and $L = \mathbb{Z}\cdot\Lambda$.

Proof. It is clear that Λ is special and that $\Omega \subset \Lambda$. Since $U' \subset \mathbb{Z}\cdot\Omega$, also $U' \subset \mathbb{Z}\cdot\Lambda'$. Thus we have only to show that $L = \mathbb{Z}\cdot\Lambda$. By the previous lemma it suffices to show that $\Delta_0 \subset \mathbb{Z}\cdot\Lambda$.

Let $\delta \in \Delta_0$, $\delta \neq \omega_1$. We have $L = L'' \perp U' \perp U$. As above, applying (1.4) shows that there exists a $\varphi \in \Psi_U(U' \perp U)$ with $\varphi(\delta) \in U^\perp$ and

$$\langle \varphi(\delta), \omega_1\rangle = \langle \varphi(\delta), \varphi(\omega_1)\rangle = \langle \delta, \omega_1\rangle = 1$$

since $\omega_1 \in L''$ and $\delta \in \Delta_0$. Thus $\tilde{\delta} = \varphi(\delta) \in \Lambda'$. But $\varphi \in \Gamma_\Omega \subset \Gamma_\Lambda$. Hence

$$\delta = \varphi^{-1}(\tilde{\delta}) \in \Gamma_\Lambda(\Lambda') \subset \mathbb{Z}\cdot\Lambda.$$

This proves the proposition.

Theorem (2.3) now follows from Proposition (3.6) and Theorem (3.2).

§ 4. The unipotent and simple part of the monodromy group

(4.1) Let (L, Δ) be a vanishing lattice. The group $\Gamma = \Gamma_\Delta$ acts on $\bar{L} = L/\ker L$ with its induced symmetric bilinear form. The image of Γ in $O(\bar{L})$ is denoted by Γ_s, the kernel of this representation by $\Gamma_u \subset \Gamma$. One refers to $\Gamma_u^{(\sim)}$ resp. Γ_s as the *unipotent* resp. *simple* part of the monodromy group. We have $\Gamma_s = \Gamma/\Gamma_u$, but a priori it is not clear whether Γ is the semidirect product of Γ_s and Γ_u.

We let $\ker(L) \otimes \bar{L}$ act on L by

$$v \otimes w(x) = x + \langle x, w\rangle\, v.$$

Note that $v \otimes w$ corresponds to the Eichler-Siegel-transformation $\psi_{-v,w}$ defined in Sect. 1.

(4.2) **Proposition.** *Let (L, Δ) be a complete vanishing lattice. Then*

$$\Gamma_u = \ker(L) \otimes \bar{L}.$$

In particular Γ_u is abelian. Moreover $\Gamma_s = O^*(\bar{L})$ and $\Gamma = \Gamma_u \Gamma_s$. In particular Γ_s is of finite index in $O(\bar{L})$ and hence arithmetic.

Proof. Let $e_1, \ldots, e_m$ be a basis of $\ker L$. One can easily show that for each $\gamma \in \Gamma_u$ there exist unique $v_1, \ldots, v_m \in \bar{L}$ such that

$$\gamma(x) = x + \sum_{i=1}^{m} \langle x, v_i\rangle\, e_i.$$

On the other hand, $\ker(L) \otimes \bar{L}$ is generated by the transformations $\psi_{-v,w}$, where $v \in \ker L$ and $w \in \bar{L}$. By [8, Property (a)] $\sigma(\psi_{-v,w})=1$, and obviously $\psi_{-v,w} \in \ker \tau$. Here σ, τ are the homomorphisms defined in § 1. Hence

$$\psi_{-v,w} \in O^*(L).$$

By Theorem (2.4) $\Gamma = O^*(L)$, and therefore $\psi_{-v,w} \in \Gamma_u$ (cf. also [7, Proof of Theorem 3.1 c)]).

Finally a given splitting

$$0 \to \ker L \to L \rightleftarrows \bar{L} \to 0, \qquad \text{i.e. } L = \bar{L} \oplus \ker L,$$

leads to a split short exact sequence

$$1 \to \ker(L) \otimes \bar{L} \to O^*(L) \rightleftarrows O^*(\bar{L}) \to 1$$

(cf. [7, Sect. 3]).

§ 5. Applications

(5.1) Now let X be an n-dimensional complete intersection in $\mathbb{C}^k$ with an isolated singularity $x \in X$. Let F: $\mathscr{X} \to S$ be a suitable representative of the semiuniversal deformation of the germ (X, x) and denote by $D \subset S$ the corresponding discriminant. Put $S' = S - D$, $\mathscr{X}' = F^{-1}(S')$ and $F' = F|_{\mathscr{X}'}$. Then F': $\mathscr{X}' \to S'$ is a C^∞-fibre bundle, where each fibre is a smooth manifold diffeomorphic to the Milnor fibre, which is defined by [20], [12]. Fix a point $t \in S'$. The corresponding fibre X_t of F' has the homotopy type of a bouquet of μ n-spheres. The only interesting homology group is therefore $H_n(X_t, \mathbb{Z})$. This is a free $\mathbb{Z}$-module of rank μ, endowed with a bilinear form $\langle\ ,\ \rangle$, given by the intersection product. This form is symmetric resp. skew-symmetric, if n is even resp. odd. We consider here the symmetric case, for the skew-symmetric case see [13]. Then the above group is an even lattice, which we denote by L and call the *Milnor lattice*. It is generated by the set Δ of *vanishing cycles*. These are defined as follows: Choose a regular point c on the discriminant D and a path η: $[0,1] \to S$ from c to t, such that $\eta((0,1]) \subset S'$. The fibre over c has only one singular point, which is an ordinary double point. The fibre over $\eta(r)$, r small, contains a distinguished n-sphere, which vanishes, as r approaches 0. After the choice of an orientation and transport along η to t, one gets an element of Δ. For a vanishing cycle $\delta \in \Delta$

$$\langle \delta, \delta \rangle = (-1)^{n/2}\, 2.$$

If (X, x) is not an ordinary double point, then there exist $\delta_1, \delta_2 \in \Delta$ with $\langle \delta_1, \delta_2 \rangle = 1$. The image of the natural representation

$$\rho\colon\ \pi_1(S', t) \to \operatorname{Aut}(L)$$

is called the *monodromy group* Γ. It is a subgroup of $O(L)$ generated by the reflections s_δ corresponding to the vanishing cycles $\delta \in \Delta$. The set of vanishing cycles Δ forms one orbit under Γ. For more details see e.g. the forthcoming book of Looijenga [18].

It follows that (L, Δ) is a vanishing lattice and Γ its monodromy group Γ_Δ.

(5.2) Now assume that X is an n-dimensional hypersurface, with $n \equiv 2 \pmod 4$, or a 2-dimensional complete intersection in $\mathbb{C}^4$. In the hypersurface case the assumption that $n \equiv 2 \pmod 4$ is no restriction, because a singularity (X, x) of arbitrary dimension n is stably equivalent to one with $n \equiv 2 \pmod 4$. By the remarks of (1.1) the case $n \equiv 0 \pmod 4$ is also covered. According to Arnold's and Wall's classification theorems [3], [25] we have to distinguish between the following classes of singularities. Here the first name refers to the type of the Milnor lattice, in brackets are given other usual names.

1) Elliptic (simple) singularities
2) Parabolic (simply elliptic) singularities
3) Hyperbolic (cusp) singularities
4) Others.

The Milnor lattices, sets of vanishing cycles and monodromy groups of the elliptic singularities are the root lattices, root systems and Weyl groups of type A_μ, D_μ, E_6, E_7 and E_8. One can easily check that in these cases $\Gamma = O^*(L)$.

The Milnor lattices of the parabolic singularities are of the type $L = Q' \perp (0) \perp (0)$ where (0) denotes a 1-dimensional zero form and Q' is a root lattice of type E_8, E_7, E_6 or D_5. The monodromy groups are semidirect products of the corresponding Weyl groups and $\ker(L) \otimes \bar{L}$. Thus also $\Gamma = O^*(L)$ (cf. Sect. 4).

For the hyperbolic singularities, Gabrielov [14] in the hypersurface case, and Looijenga [16, III. 3.7] in the complete intersection case have shown that the monodromy group Γ is the semi-direct product of W by Q, where W is the Coxeter group corresponding to the Coxeter graphs $T_{p,q,r}$ (Fig. 1) resp. $\Pi_{p,q,r,s}$ (Fig. 2), and Q is the lattice corresponding to one of these graphs. Here $L = Q \perp (0)$. Note that one gets the parabolic case for $1/p + 1/q + 1/r = 1$ and $(p, q, r, s) = (2, 2, 2, 2)$. The equality $\Gamma = O^*(L)$ would imply $W = O^*(Q)$, in particular W would be of finite index in $O(Q)$, since Q is nondegenerate. But if W is of finite index in $O(Q)$, then according to [4, Ch. V, § 4, Exercice 12] $T_{p,q,r}$ resp. $\Pi_{p,q,r,s}$ have to define a Coxeter system of hyperbolic type. This is only the case for $(p, q, r) = (2, 3, 7)$, $(2, 4, 5)$, $(3, 3, 4)$ and $(p, q, r, s) = (2, 2, 2, 3)$. Therefore except for these cases the characterisation $\Gamma = O^*(L)$ is false.

For the four remaining cases one can show that $\Gamma = O^*(L)$. We sketch here the proof, which is due to Brieskorn (unpublished) in the hypersurface case. By Sect. 4 it suffices to show that $W = O^*(Q)$. Here $Q = Q' \perp U$, where Q' is a root lattice of type E_8, E_7, E_6 or D_5. One can easily show that Q' has the property

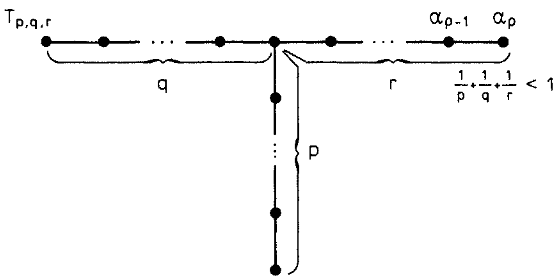

Fig. 1

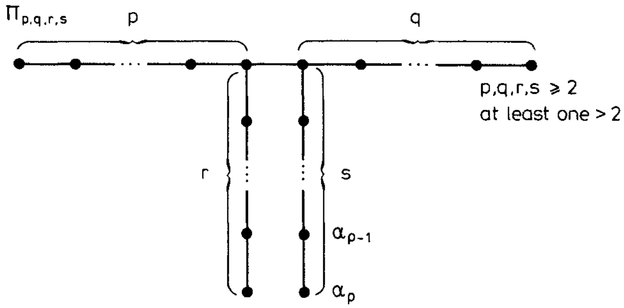

Fig. 2

(R) of [24, § 5]. By [24, 5.2]

$$O(Q)=O(Q')\cdot O(U)\cdot \Psi_U(Q).$$

(This result is only formulated for unimodular lattices, but the proof goes also through for Q.) This implies

$$O^*(Q)=O^*(Q')\cdot O^*(U)\cdot \Psi_U(Q). \tag{1}$$

But $O^*(Q')=W'$, where W' is the Weyl group of Q', and $O^*(U)=\{1, s_{f_1-f_2}\}$, where f_1, f_2 is a basis of isotropic vectors of U. Here f_1, f_2 can be taken as $f_1 =\tilde{\alpha}+\alpha_{p-1}$, $f_2=\alpha_p+f_1$, where $\tilde{\alpha}$ is the longest root of Q' [4]. The group $\Psi_U(Q)$ is generated by transformations

$$\psi_{f_i,\alpha}=s_\alpha s_{\alpha-f_i}$$

where α is a root of Q'. Using these facts, one can easily show that all groups on the right hand side of (1) are contained in W.

In the hypersurface case the results up to now are already stated in [7, 8].

(5.3) We consider now the remaining singularities, namely class 4). Assume first that X is a hypersurface. Then it follows from Arnold's classification theorems [2], that (X, x) deforms into a unimodal exceptional singularity (Y, y). This was shown by D. Siersma (personal communication to E. Brieskorn). As each unimodal exceptional singularity deforms into one of the unimodal exceptional singularities E_{12}, Z_{11} and Q_{10}, the statement above can even be strengthened. But this implies that the vanishing lattice (L_X, Δ_X) of (X, x) contains the vanishing lattice (L_Y, Δ_Y) of one of these singularities. But $L_Y=L'_Y \perp U' \perp U$, $A_2 \subset L'_Y$ and $\Gamma_Y=O^*(L_Y)$ by our earlier results [7, Theorem 3.1] (cf. also [8, 9]). By Corollary (2.6) and Remark (2.7) it follows that (L_X, Δ_X) is a complete vanishing lattice. Therefore Theorem (2.4) implies $\Gamma_X=O^*(L_X)$.

Now let X be a 2-dimensional complete intersection in $\mathbb{C}^4$ with a singularity x of class 4). Then it follows easily from Wall's classification results, that (X, x) deforms into the triangle complete intersection singularity J'_9 in Wall's notation [25], given by a mapping

$$f\colon \mathbb{C}^4 \to \mathbb{C}^2, \quad f(w,x,y,z)=(x\,y+z^2, w^2+x\,z+y^3),$$

which is $D_{2,3,10}$ in Looijenga's notation [17]. We are grateful to C.T.C. Wall for pointing this out to us. (The main point here is that the property of an isolated singularity defined by a map-germ f: $(\mathbb{C}^4, 0) \to (\mathbb{C}^2, 0)$ being hyperbolic is characterised at the 2-jet level [25]. So one only has to show that the 2-jet of the map-germ defining (X, x) can be simplified to that defining the J'-series, and then apply the classification of the J'-series [loc. cit.].) This again implies (see e.g. [18]) that the vanishing lattice (L_X, Δ_X) of (X, x) contains the vanishing lattice (L_Y, Δ_Y) of J'_9. The following Proposition (5.4) shows that (L_Y, Δ_Y) is complete. Therefore we get as above by Remark (2.7) and Theorem (2.4) $\Gamma_X = O^*(L_X)$.

We remark that for the 22 triangle complete intersection singularities (for the definition see [17]), which include the 14 exceptional unimodal hypersurface singularities, this was already proven by Pinkham [21] using algebraic-geometric methods.

(5.4) **Proposition.** *Let (L, Δ) be the vanishing lattice of the singularity J'_9. Then (L, Δ) is complete.*

Proof. In order to show that (L, Δ) is complete, we want to apply Corollary (2.6). Let (Z, z) be any of the 8 triangle complete intersection singularities, which are not hypersurface singularities, and L_Z be the corresponding Milnor lattice. As already noted by Pinkham [21], there exists a basis of minimal vectors of square length -2 of L_Z, such that the matrix of the bilinear form with respect to this basis is described by the following "Dynkin diagram" (as

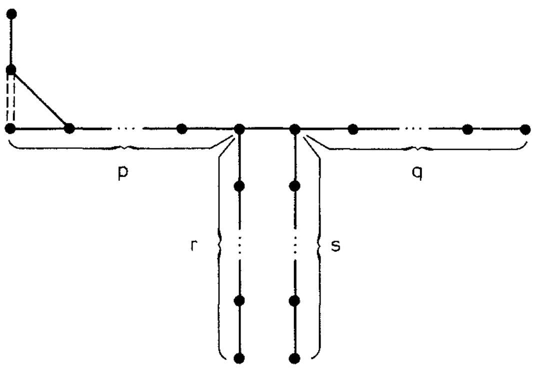

Fig. 3

usual the sign (dotted = minus) and number of edges between vertices i, j is given by the corresponding entry of the matrix) for a certain quadruple (p, q, r, s) with $p, q, r, s \geqq 2$ and at least one >2 (cf. also [10]). In particular this basis is a special subset of L_Z in the sense of Definition (3.1). For $(Z, z) = J'_9$ one has $(p, q, r, s) = (2, 2, 2, 3)$. From the remarks following Definition (3.1) and from (5.2) it follows that

$$L = L_Z = D_5 \perp U' \perp U.$$

In particular $A_2 \subset L' = D_5$.

Thus it remains to show that there is an orthogonal splitting $L = D_5 \perp U' \perp U$ such that $\Psi_U(L) \subset \Gamma = \Gamma_\Delta$. Now J_9' deforms into the hyperbolic singularity $\Pi_{2,2,2,3}$. Let $F: \mathscr{Y} \to S$ be a suitable representative of the semiuniversal deformation of a germ (Y, y) of type J_9', and assume $F(y) = 0 \in S$. Let s be a point of the discriminant D close to 0, such that the fibre of F over s has exactly one singular point of type $\Pi_{2,2,2,3}$. The multiplicity of D at 0 resp. s is 10 resp. 9 (1 added to the corresponding Milnor number). Then there exists a primitive embedding $i_*: L' \to L$ of the Milnor lattice L' of $\Pi_{2,2,2,3}$ into the Milnor lattice $L = D_5 \perp U' \perp U$ of J_9'. We may assume that L' is embedded as $K := i_*(L') = D_5 \perp U' \perp \mathbb{Z} f_1$, where $\{f_1, f_2\}$ is a basis of isotropic vectors of U. Moreover there exist vanishing cycles $\delta_1', \ldots, \delta_9' \in \Delta' \subset L'$ and $\delta_1, \ldots, \delta_{10} \in \Delta \subset L$ which generate L' resp. L, such that $i_*(\delta_i') = \delta_i$ for $i = 1, \ldots, 9$, and such that the corresponding reflections $s_{\delta_1'}, \ldots, s_{\delta_9'}$ resp. $s_{\delta_1}, \ldots, s_{\delta_{10}}$ generate the monodromy group Γ' resp. Γ of $\Pi_{2,2,2,3}$ resp. J_9' (see e.g. [18, (7.13)]). Let G be the subgroup of Γ generated by the reflections $s_{\delta_1}, \ldots, s_{\delta_9}$. Then K is G-invariant and one has $s_{\delta_j}(i_* v) = i_*(s_{\delta_j'} v)$ for $v \in L'$ and $1 \leqq j \leqq 9$. Now let $w \in M := D_5 \perp U'$ be arbitrary. Then $\psi_{f_1, w}|_K \in O^*(K)$. But we have seen in (5.2) that $\Gamma' = O^*(L')$. Therefore there exist $r \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_r \in \{\delta_1, \ldots, \delta_9\}$, such that

$$\psi_{f_1, w}|_K = \prod_{j=1}^{r} s_{\alpha_j}|_K.$$

But since $M \subset K = (\mathbb{Z} f_1)^\perp$ is nondegenerate, there is only one way to extend $\psi_{f_1, w}|_K$ to an element of $O(L)$, and this is $\psi_{f_1, w}$. Thus

$$\psi_{f_1, w} = \prod_{j=1}^{r} s_{\alpha_j} \in \Gamma.$$

Since $f_2 \notin \mathbb{Z} \cdot \{\delta_1, \ldots, \delta_9\}$, $\delta_{10} = \varepsilon f_2 + a f_1 + v$ with $\varepsilon \in \{+1, -1\}$, $a \in \mathbb{Z}$ and $v \in M$. Then

$$\psi_{f_1, -\varepsilon v}(\delta_{10}) = \varepsilon f_2 - \varepsilon f_1 \in \Delta.$$

Hence $s_{f_1 - f_2} \in \Gamma$. Now let $w \in M$ with $\langle w, w \rangle = -2$. Then by [8, Formula (c2)]

$$\psi_{f_2, w} = \psi_{f_1, w} \circ s_w \circ s_{f_1 - f_2} \circ \psi_{f_1, w}.$$

Since $s_w|_K \in O^*(K)$, $s_w \in \Gamma$ as above, and hence $\psi_{f_2, w} \in \Gamma$. Since M is generated by such vectors w, this shows $\Psi_U(L) \subset \Gamma$ (cf. § 1). The assertion now follows from Corollary (2.6).

We summarise the above results in the following theorem

(5.5) **Theorem.** *Let Γ be the monodromy group of an evendimensional isolated hypersurface singularity (X, x) or a 2-dimensional isolated complete intersection singularity (X, x) in $\mathbb{C}^4$. Assume that (X, x) is not hyperbolic or hyperbolic of type $T_{2,3,7}$, $T_{2,4,5}$, $T_{3,3,4}$ or $\Pi_{2,2,2,3}$. Then*

$$\Gamma = O^*(L).$$

In particular the simple part Γ_s is arithmetic. For the hyperbolic singularities which are not of the above types the conclusion is false: In these cases Γ_s is not of finite index in $O^*(\bar{L})$.

From Propositions (2.5) and (4.2) we get descriptions of the set of vanishing cycles and of the unipotent and simple part of the monodromy group for all singularities satisfying the conditions of the theorem and belonging to class 4). One can show that these descriptions are also true for the other singularities, for which the conclusion of the theorem holds, except for the set of vanishing cycles of an ordinary double point, i.e. a singularity of type A_1. There is an interesting consequence concerning the question, whether there exists a basis of vanishing cycles of the Milnor lattice of a complete intersection singularity, such that the monodromy group is generated by the corresponding reflections (which is true for hypersurface, but in general unknown for complete intersection singularities). One particular consequence of Theorem (5.5), of the description of the set of vanishing cycles (Proposition (2.5)), and of Theorem (3.2) is that the bases given above for the 8 triangle complete intersection singularities are bases of vanishing cycles satisfying this property. In the same way one can show the existence of such bases for other singularities (cf. [10]).

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Oblatum 24-VIII-1983 & 21-XII-1983

Note added in proof

By Proposition (2.5) a vanishing lattice (L, Δ) is complete if and only if the following two conditions are satisfied:

(i) L contains a sublattice $A_2 \perp U' \perp U$.

(ii) Δ contains all minimal vectors of square length -2 of this sublattice.

(In Definition (2.2) we only claimed that Δ contains a basis of this sublattice consisting of such vectors.) Note that condition (i) alone is not sufficient: One can construct examples of nondegenerate vanishing lattices (L, Δ) satisfying condition (i), where the group Γ_Δ is *not* of finite index in $O(L)$.