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Irreducible Characters of Semisimple Lie Groups III. Proof of Kazhdan-Lusztig Conjecture in the Integral Case

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1. Introduction

Let G be a connected linear semisimple Lie group. (Our precise hypotheses on G, which are slightly more general, are described in Sect. 2.) In [14], a conjecture describing the characters of the irreducible representations of G was formulated, generalizing that of Kazhdan-Lusztig for highest weight modules. That conjecture will be proved here, for those representations having the same infinitesimal character as some finite dimensional representation of G. Very roughly speaking, the idea of the proof is this. If $\pi \in \hat{G}$, known results of Hecht and Schmid (unpublished – see for example [14], Theorem 8.1) relate the character of π to its Lie algebra homology. On the other hand, (deep) techniques in algebraic geometry relate the conjectured character formula to the cohomology of certain geometric objects ([9, 10]). What is missing, therefore, is a connection between Lie algebra homology and the cohomology of the geometric objects. This connection is provided by the "localization" theory of Beilinson and Bernstein ([1]).

This sketch seems to leave nothing to do; and in fact the gaps to be filled are very technical. The main difficulty is that the Lie algebra homology groups studied by Beilinson and Bernstein are not the ones which are known to be related to character formulas. A more elementary way of formulating this difficulty is to say that the classification of irreducible Harish-Chandra modules given in [1] bears no obvious relation to that of Langlands. These matters are dealt with in Sect. 2–5; Theorem 1.13 below is the major new result. The last two sections reorganize the conjecture of [14] into a more convenient form, and prove some combinatorial results needed in [10].

Here is more detailed account of the proof of the character formulas. Fix a maximal compact subgroup K of G, and a complexification $K_{\mathbb{C}}$ of K. Let $\mathfrak{g}$ be the complexified Lie algebra of G, and $\mathscr{B}$ the variety of Borel subalgebras of $\mathfrak{g}$ (the usual flag manifold). The group $K_{\mathbb{C}}$ acts in a natural way on $\mathscr{B}$, and has finitely many orbits. Consider the set

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(1.1) $\mathscr{D}=\{$pairs $(Q, \mathscr{L}) \mid Q$ is a $K_{\mathbb{C}}$-orbit on $\mathscr{B}$, and $\mathscr{L}$ is a $K_{\mathbb{C}}$-homogeneous line bundle on Q with a flat connection$\}$.

If $\gamma=(Q, \mathscr{L}) \in \mathscr{D}$, we define $l(\gamma)$ (the *length* of γ) to be the dimension of Q. Fix once and for all an irreducible finite dimensional representation F of G.

Proposition 1.2 ([15], Theorem 2.2.4 and Proposition 2.3.8), *The set of infinitesimal equivalence classes of irreducible admissible representations of G having the same infinitesimal character as F is in a natural one-to-one correspondence with $\mathscr{D}$.*

This is just a reformulation of the Langlands classification. If $\gamma \in \mathscr{D}$, write

(1.3) $\bar{X}(\gamma)=$ irreducible $(\mathfrak{g}, K)$ module corresponding to γ

$X(\gamma)=$ standard induced-from-discrete series $(\mathfrak{g}, K)$ module containing $\bar{X}(\gamma)$ as a submodule

$\tilde{X}(\gamma)=$ standard module having $\bar{X}(\gamma)$ as a quotient

$\bar{\Theta}(\gamma)=$ character of $\bar{X}(\gamma)$

$\Theta(\gamma)=$ character of $X(\gamma)$ or $\tilde{X}(\gamma)$.

The problem we are interested in is expressing the irreducible characters in terms of the standard ones $\Theta(\gamma)$.

Suppose now that $\delta=(Q, \mathscr{L}) \in \mathscr{D}$. Since $\mathscr{L}$ is flat, there is a notion of locally constant section. Accordingly we may identify

(1.4) $\delta \leftrightarrow$ sheaf of germs of locally constant sections of $\mathscr{L}$;

this is a locally constant sheaf on Q, and carries a natural action of $K_{\mathbb{C}}$. We may also regard δ as a sheaf on all of $\mathscr{B}$, extending it to have all its stalks zero off of Q. This extension is a constructible sheaf on $\mathscr{B}$, and still carries a $K_{\mathbb{C}}$ action. This construction makes $\mathscr{D}$ into a natural basis for the Grothendieck group of constructible sheaves on $\mathscr{B}$ having a $K_{\mathbb{C}}$ action.

The closure $\bar{Q}$ of Q in $\mathscr{B}$ is an algebraic variety, often having rather complicated singularities. Because of these singularities, the line bundle $\mathscr{L}$ cannot in general be extended to $\bar{Q}$. However, Goresky and MacPherson in [6] (following a suggestion of Deligne to extend their own previous work) have shown how to attach to $\mathscr{L}$ an object which plays the role of such an extension. It is a complex $\tilde{\delta}$ of sheaves on $\bar{Q}$, defined up to quasi-isomorphism; in particular, the cohomology sheaves $\tilde{\delta}^i$ of the complex are well-defined. Often we will regard $\tilde{\delta}$ as a complex of sheaves on $\mathscr{B}$, extending it by zero off of Q. The precise characterization of $\tilde{\delta}$ will be recalled in Sect. 5 (see (5.13)). The group $K_{\mathbb{C}}$ acts on the quasi-isomorphism class of $\tilde{\delta}$, and therefore on the various $\tilde{\delta}^i$. These cohomology sheaves are constructible; so by the remarks at the end of the last paragraph, we can write

(1.5) $$\sum_i (-1)^i \tilde{\delta}^i = \sum_{\gamma \in \mathscr{D}} M(\gamma, \delta)\, \gamma$$

in the Grothendieck group of $K_{\mathbb{C}}$-equivariant constructible sheaves on $\mathscr{B}$. (Here $M(\gamma, \delta)$ is an integer.) We call $\tilde{\delta}$ *the DGM extension of* δ.

Theorem 1.6. *If the integers* $M(\gamma, \delta)$ *are defined by* (1.5), *then*

$$\bar{\Theta}(\delta) = \sum_{\gamma \in \mathscr{D}} (-1)^{l(\delta)-l(\gamma)} M(\gamma, \delta)\, \Theta(\gamma)$$

(*notation* (1.3)).

This theorem, due largely to Beilinson and Bernstein [1], reformulates the problem of computing characters in a geometric way. Its proof will be completed at the end of Sect. 5, but the idea is the following. We will show (following [1]) how to construct $\bar{\delta}$ from the irreducible representation $\bar{X}(\delta)$. Write $\mathfrak{Z}(\mathfrak{g})$ for the center of $U(\mathfrak{g})$, and $\mathfrak{I}_0 \subseteq \mathfrak{Z}(\mathfrak{g})$ for the maximal ideal which annihilates the fixed finite dimensional representation F. Put

$$\mathfrak{I} = U(\mathfrak{g})\, \mathfrak{I}_0 \qquad (1.7)$$

$$R = U(\mathfrak{g})/\mathfrak{I}.$$

By the infinitesimal character assumption, the various modules of (1.3) can be regarded as R modules. Fix a resolution

$$P^* \to \bar{X}(\delta) \to 0 \qquad (1.8)$$

of $\bar{X}(\delta)$ by finitely generated projective R-modules.

By the Borel-Weil theorem, there is a natural $K_{\mathbb{C}}$-homogeneous holomorphic line bundle on $\mathscr{B}$ of which F is the space of global sections. Write $\mathcal{O}_F$ for its sheaf of germs of holomorphic sections. Then $\mathcal{O}_F$ has a $K_{\mathbb{C}}$ action (by translation) and a $U(\mathfrak{g})$ action (by differentiation); and it is easy to check that the latter lifts to an R action (see Lemma 3.3 and Corollary 3.4). So we get a complex of sheaves

$$\mathscr{T} = \mathrm{Hom}_R(P^*, \mathcal{O}_F), \qquad (1.9)(a)$$

with cohomology sheaves

$$\mathscr{T}^i = \mathrm{Ext}^i_R(\bar{X}(\delta), \mathcal{O}_F). \qquad (1.9)(b)$$

The group $K_{\mathbb{C}}$ does not act on $\mathscr{T}$ itself, but it does act on the quasi-isomorphism class of $\mathscr{T}$. (The point is that the adjoint action of $K_{\mathbb{C}}$ on $\mathfrak{g}$ makes $K_{\mathbb{C}}$ act on equivalence classes of $U(\mathfrak{g})$ modules or R modules. P^* need not be preserved by this action, but it will obviously be moved into other projective resolutions of $\bar{X}(\delta)$; and all of these are quasi-isomorphic.) In particular, $K_{\mathbb{C}}$ acts on the cohomology sheaves $\mathscr{T}^i$.

On the level of stalks, this construction is an old friend. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, and let x be the corresponding point of $\mathscr{B}$. Then the stalk of $\mathscr{T}^i$ at x is

$$\mathscr{T}^i_x = \mathrm{Ext}^i_R(\bar{X}(\delta), (\mathcal{O}_F)_x). \qquad (1.10)(a)$$

Beilinson and Bernstein have shown that

$$\mathrm{Ext}^i_R(\bar{X}(\delta), (\mathcal{O}_F)_x) \cong \mathrm{Ext}^i_R(\bar{X}(\delta), (\hat{\mathcal{O}}_F)_x) \qquad (1.10)(b)$$

(see [1]; this is their assertion that the corresponding holonomic system has regular singularities). Choose a Levi decomposition $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, and recall our finite dimensional representation F. Computing the right side of (1.10)(b) (Proposition 4.1) leads to

(1.10)(c) $$\mathcal{T}_x^i \cong \operatorname{Hom}_{\mathfrak{h}}(H_i(\mathfrak{n}, \bar{X}(\delta)), H_0(\mathfrak{n}, F));$$

the objects $H_*(\ )$ are Lie algebra homology groups. The connection between Lie algebra homology and character theory ("the Osborne conjecture") gives a statement like Theorem 1.6, with $\tilde{\delta}$ replaced by $\mathcal{T}$. (Actually one needs a technical extension of the results of, say, Hecht and Schmid on the Osborne conjecture; this extension is the content of Sect. 4.) To prove Theorem 1.6, it remains to show that the complexes $\tilde{\delta}$ and $\mathcal{T}$ are quasi-isomorphic. Now $\tilde{\delta}$ is characterized by certain formal properties (see (5.13)), most of which were in some sense already known to hold for $\mathcal{T}$. The new result needed is

Theorem 1.11 (Beilinson-Bernstein; see [1]). *The complex $\mathcal{T}$ of* (1.9) *is self-dual (in the sense of Verdier) in the derived category of sheaves on $\mathcal{B}$ with constructible cohomology sheaves.*

The next problem is to compute the integers $M(\gamma, \delta)$ of (1.5) explicitly. Let u be an indeterminate, and $\mathcal{H}$ the Hecke algebra of the Weyl group of $\mathfrak{g}$. The free $\mathbb{Z}[u]$ module $\mathcal{M}$ with basis $\mathcal{D}$ can be made into an $\mathcal{H}$ module in a natural way (Definition 6.1). This gives enough structure to copy the definitions of [8], and define polynomials

$$P_{\gamma,\delta}(u) \qquad \gamma, \delta \in \mathcal{D}.$$

This procedure is described in Sect. 6, especially Proposition 6.11.

Theorem 1.12. *$M(\gamma, \delta) = P_{\gamma,\delta}(1)$. More precisely, $\tilde{\delta}^i$ (see (1.5)) is zero if i is odd, and (in a Grothendieck group)*

$$\tilde{\delta}^{2i} = \sum_{\gamma \in \mathcal{D}} (\text{coefficient of } u^i \text{ in } P_{\gamma,\delta})\, \gamma.$$

The first statement is proved in [10], using a generalization of the Weil conjectures proved by O. Gabber. The second (which has nothing to do with character theory in any case) is deduced from it in Sect. 7. At the same time, we prove that this algorithm for computing $M(\gamma, \delta)$ coincides with the one conjectured in [14] (Corollary 7.18).

This account deliberately minimizes the work of Beilinson and Bernstein. There are two reasons for this. First, no proofs of their results have appeared. Second, it would certainly be of interest to find direct representation-theoretic proofs of them. The reader who is willing to rely more heavily on [1] will be able to omit a little of Sect. 5 (as indicated there). From this point of view, the main result of the first five sections is

Theorem 1.13 (Corollary 4.8 below). *Suppose $\gamma = (Q, L) \in \mathcal{D}$. Write I_γ for the $(\mathfrak{g}, K)$ module attached to γ in* [1], §3, *and L_γ for its unique irreducible submodule. Then I_γ is isomorphic to the Langlands induced representation $X(\gamma)$ of* (1.3), *and L_γ is isomorphic to $\bar{X}(\gamma)$.*

Given this theorem, Theorem 1.6 follows from the parenthetical remark of [1], §4.

At about the time this paper was written, Beilinson, Bernstein, Deligne, and Gabber proved a "decomposition theorem" for the direct image of an intersection homology complex under a proper projective map of algebraic varieties; an account is to appear in the proceedings of the C.I.R.M. conference "Analyse et Topology sur les Espaces Singuliers", Marseille-Luminy, 1981. I have been informed that Beilinson and Bernstein have used this theorem to give a proof of Conjecture 3.15 of [13] in the integral case; that is, of (7.5) below. Granting this, one could simplify Sect. 7 substantially. In the absence of any written proof or announcement for their result, however, it seemed better to leave the present paper unchanged.

Many of these ideas are outside my own technical competence, and it is a pleasure to thank all of the people who offered help. G. Lusztig contributed directly to the formulation of Proposition 1.2 and Sects. 5 and 6. S. Kleiman and M. Artin provided additional guidance on geometry, and T. Kawai spent many hours explaining some of the mysteries of holonomic systems. O. Gabber, D. Kazhdan, and G. Zuckerman made several suggestions. Kazhdan was also kind enough to explain (and translate) the work of Bernstein and Beilinson before [1] appeared.

2. $K_{\mathbb{C}}$ Orbits on the Flag Manifold

The proper setting for the results of this paper is Harish-Chandra's category of reductive groups, and they can all be proved in that setting. However, this would lead to some minor technical problems which would obscure the main ideas. So we fix a reductive algebraic group $G_{\mathbb{C}}$ defined over $\mathbb{R}$, and assume that G has finite index in the set of real points of $G_{\mathbb{C}}$. (We do not even want to allow G to be little larger than the set of real points, as one often does. What we want, and what this definition assures, is that the component group of any Cartan subgroup of G should be finite and abelian.) Fix a maximal compact subgroup K of G, and write $K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ for its complexification. (Any finite dimensional representation of K extends uniquely to a holomorphic representation of $K_{\mathbb{C}}$. In particular, $K_{\mathbb{C}}$ acts on any $(\mathfrak{g}, K)$ module.)

Write G_0 for the identity component of G, $\mathfrak{g}_0$ for Lie(G), and $\mathfrak{g}$ for its complexification; similar notation is used for other groups. Fix a Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

$$G \cong K \cdot \exp(\mathfrak{p}_0)$$

and write θ for the corresponding involution (of $\mathfrak{g}_0$, G, $\mathfrak{g}$, or $G_{\mathbb{C}}$). Fix a non-degenerate symmetric invariant bilinear form $\langle\, ,\rangle$ on $\mathfrak{g}_0$, positive on $\mathfrak{p}_0$ and negative on $\mathfrak{k}_0$. We will use some other notation as in [14].

Proposition 2.1 (Matsuki [11]).

a) *Every θ-stable Cartan subalgebra of $\mathfrak{g}$ is conjugate by $K_{\mathbb{C}}$ to the complexification of a real θ-stable Cartan subalgebra.*

b) *If $\mathfrak{h}_0^1$ and $\mathfrak{h}_0^2$ are real θ-stable Cartan subalgebras and $k \in K_{\mathbb{C}}$ satisfies*

$$\mathrm{Ad}(k)(\mathfrak{h}^1) = \mathfrak{h}^2,$$

then there is an element k_0 of K such that

$$\mathrm{Ad}(k)|_{\mathfrak{h}^1} = \mathrm{Ad}(k_0)|_{\mathfrak{h}^1}.$$

c) *Every Borel subalgebra of $\mathfrak{g}$ contains a θ-stable Cartan subalgebra, whose $K_{\mathbb{C}}$-conjugacy class is unique.*

d) *Suppose B is a Borel subgroup of $G_{\mathbb{C}}$, and $H \subseteq B$ is a θ-stable Cartan subgroup. Then the natural map*

$$H \cap K_{\mathbb{C}}/(H \cap K_{\mathbb{C}})_0 \to B \cap K_{\mathbb{C}}/(B \cap K_{\mathbb{C}})_0$$

is an isomorphism. If in addition H is the complexification of a Cartan subgroup $H_{\mathbb{R}}$ of G, then

$$H_{\mathbb{R}} \cap K/(H_{\mathbb{R}} \cap K)_0 \to H \cap K_{\mathbb{C}}/(H \cap K_{\mathbb{C}})_0$$

is also an isomorphism.

Corollary 2.2. *The following sets are in one-to-one correspondence (under the obvious maps):*

a) *$K_{\mathbb{C}}$-conjugacy classes of pairs (B, χ), with B a Borel subgroup of $G_{\mathbb{C}}$, and χ a character of $(B \cap K_{\mathbb{C}})/(B \cap K_{\mathbb{C}})_0$.*

b) *$K_{\mathbb{C}}$-conjugacy classes of triples (H, Δ^+, χ), with H a θ-stable Cartan subgroup of $G_{\mathbb{C}}$, $\Delta^+ \subset \Delta(\mathfrak{g}, \mathfrak{h})$ a positive system, and χ a character of $(H \cap K_{\mathbb{C}})/(H \cap K_{\mathbb{C}})_0$.*

c) *K-conjugacy classes of triples $(H_{\mathbb{R}}, \Delta^+, \chi)$, with $H_{\mathbb{R}}$ a θ-stable Cartan subgroup of G, $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ a positive system, and χ a character of $(H_{\mathbb{R}} \cap K)/(H_{\mathbb{R}} \cap K)_0$.*

d) *G-conjugacy classes of triples $(H_{\mathbb{R}}, \Delta^+, \chi)$, with $H_{\mathbb{R}}$ a Cartan subgroup of G, $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ a positive system, and χ a character of $H_{\mathbb{R}}/(H_{\mathbb{R}})_0$.*

e) *G-conjugacy classes of pairs (B, χ) with B a Borel subgroup of $G_{\mathbb{C}}$, and χ a character of $B \cap G/(B \cap G)_0$.*

f) *The set $\mathscr{D}$ (cf. (1.1)) of $K_{\mathbb{C}}$-homogeneous flat line bundles on orbits of $K_{\mathbb{C}}$ on $\mathscr{B}$.*

g) *The set $\mathscr{E}$ of G-homogeneous flat line bundles on orbits of G on $\mathscr{B}$.*

Proof. The equivalence (a)–(c) is essentially Proposition 2.1, and that of (c)–(e) is known (cf. [17]). For (f) and (g), we only have to notice that homogeneous line bundles on a coset space are parameterized by characters of an isotropy group. It is easy to see that there is an invariant flat connection exactly when the character is trivial on the identity component. Q.E.D.

Our next goal is Proposition 1.2. We recall the version of the Langlands classification given in [12] or [15].

Definition 2.3. Suppose H is a Cartan subgroup of G. *A regular character* of H is an ordered pair $\lambda = (\Lambda, \bar{\lambda})$, with $\bar{\lambda} \in \mathfrak{h}^*$ and $\Lambda \in \hat{H}$, satisfying

a) If α is an imaginary root of $\mathfrak{h}$ in $\mathfrak{g}$, then $\langle \alpha, \bar{\lambda} \rangle$ is a non-zero real number. Set

$$(\Delta^+)^I = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \text{ is imaginary and } \langle \alpha, \bar{\lambda} \rangle > 0\}$$

$$(\Delta^+)^{I,c} = \{\alpha \in (\Delta^+)^I \mid \alpha \text{ is compact}\}$$

$$\rho^I = \rho((\Delta^+)^I), \quad \rho^{I,c} = \rho((\Delta^+)^{I,c}).$$

(In general we write $\rho(S)$ for half the sum of the elements of S, or half the sum of the weights of some abelian algebra on S.) Then

b) $d\Lambda = \bar{\lambda} + \rho^I - 2\rho^{I,c}$.

The set of regular characters of H is written $\hat{H}'$.

Theorem 2.4 (Langlands – see [12] or [15]). *There is a finite-to-one correspondence*

$$\{\bar{X}^i(\lambda)\} \leftrightarrow \lambda$$

between irreducible $(\mathfrak{g}, K)$ *modules and G-conjugacy classes of regular characters of Cartan subgroups. If* $\bar{\lambda}$ *is nonsingular, the correspondence is one-to-one. Each module* $\bar{X}^i(\lambda)$ *has infinitesimal character* $\bar{\lambda}$ *(with respect to the Harish-Chandra map from* $\mathfrak{Z}(\mathfrak{g})$ *to* $S(\mathfrak{h})$*)*.

Definition 2.5. (Suppose $H \subseteq G_{\mathbb{C}}$ is a θ-stable Cartan subgroup, and $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ is a positive root system. A root $\alpha \in \Delta$ is called

a) *real* if $\theta\alpha = -\alpha$,

b) *complex* if $\theta\alpha \neq \pm\alpha$,

c) *compact imaginary* if $\theta\alpha = \alpha$, and the corresponding root vector X_α lies in $\mathfrak{k}$,

d) *noncompact imaginary* if $\theta\alpha = \alpha$ and $X_\alpha \in \mathfrak{p}$.

We write $(\Delta^+)^I$, $(\Delta^+)^{I,c}$, ρ^I, $\rho^{I,c}$, $\rho^{I,n}$ as in Definition 2.3. Similarly, $(\Delta^+)^R$, etc., refers to the real roots.

Lemma 2.6. *Suppose* $H \subseteq G_{\mathbb{C}}$ *is a* θ*-stable Cartan subgroup, and* $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ *is a positive root system. Set*

$$\bar{B}_0 = \{\alpha \in \Delta^+ \mid \alpha \text{ is complex, and } \theta\alpha \in \Delta^+\}.$$

Choose a subset $B_0 \subseteq \bar{B}_0$ *such that*

$$\bar{B}_0 = B_0 \cup \theta B_0 \quad (\textit{disjoint}).$$

Put

$$B = B_0 \cup (\Delta^+)^{I,n},$$

and define

$$\phi = \phi(\Delta^+) = 2\rho(B),$$

regarded as a character of $H \cap K_{\mathbb{C}}$*. Then* ϕ *is independent of the choice of* B_0*, and satisfies*

$$d\phi = [\rho(\Delta^+) + \rho^I - 2\rho^{I,c}]|_{\mathfrak{h} \cap \mathfrak{k}}.$$

The (trivial) proof may be found in [15], Sect. 6.7.

The next result is a precise form of Proposition 1.2, in light of Theorem 2.4 and Corollary 2.2.

Proposition 2.7. *Suppose* $H \subseteq G$ *is a* θ*-stable Cartan subgroup, with Cartan decomposition*

$$H = TA = (H \cap K)(H \cap \exp \mathfrak{p}_0).$$

Fix a finite dimensional irreducible representation F of G. Put

$$S_1=\{\lambda\in\hat{H}'|\bar{X}(\lambda) \text{ has the same infinitesimal character as } F\}$$
$$S_2=\{(\Delta^+,\chi)|\Delta^+\subseteq\Delta(\mathfrak{g},\mathfrak{h}) \text{ is a positive system, and } \chi \text{ is a character of } T/T_0\}.$$

Then S_1 and S_2 are in one-to-one correspondence, as follows. Given (Δ^+,χ) in S_2, define $\lambda=(\Lambda,\bar{\lambda})$ by

a) $\Lambda|_T=[\mu|_T-\phi(\Delta^+)]\otimes\chi$

b) $\Lambda|_A=\mu|_A-\rho(\Delta^+)|_A$

c) $\bar{\lambda}=d\mu-\rho(\Delta^+)$.

Here $\mu\in\hat{H}$ is the Δ^+-lowest weight of F, and $\phi(\Delta^+)$ is defined as in Lemma 2.6. *In* (b), *$\rho(\Delta^+)$ makes sense as a group character since A is a vector group.*

This is a formal consequence of Lemma 2.6; we refer to Sect. 6.7 of [15] for details.

3. The Beilinson-Bernstein Theory

Recall from the introduction that we are fixing a finite dimensional irreducible representation F of G; and recall that $\mathscr{B}$ is the variety of Borel subalgebras of $\mathfrak{g}$.

Definition 3.1. Let $\mathscr{L}_F$ be the holomorphic line bundle on $\mathscr{B}$ associated to F by the Borel-Weil theorem. That is, the fiber $(\mathscr{L}_F)_x$ at a point x of $\mathscr{B}$ corresponding to a Borel subalgebra with nil radical $\mathfrak{n}$ is $F/\mathfrak{n}F$. Since $K_{\mathbb{C}}$ acts on $\mathscr{B}$ and on F, it acts on $\mathscr{L}_F$.

The group $G_{\mathbb{C}}$ need not act on $\mathscr{L}_F$ (although some covering group will). Even if it does, this action need not extend the action of $K_{\mathbb{C}}$ when G is disconnected. These technicalities serve to emphasize that it is the $K_{\mathbb{C}}$ action on $\mathscr{L}_F$ which is most important here. The Borel-Weil theorem asserts that the space of holomorphic sections of $\mathscr{L}_F$ is naturally isomorphic to F: if $v\in F$, the corresponding section is defined by

$$s_v(x)=v+\mathfrak{n}F\in F/\mathfrak{n}F\cong(\mathscr{L}_F)_x$$

(in the notation of Definition 3.1).

Definition 3.2. Let $\mathscr{O}_F$ denote the sheaf of germs of holomorphic sections of $\mathscr{L}_F$. The $\mathfrak{g}$ action on F makes $\mathfrak{g}$ act by first order differential operators on $\mathscr{O}_F$. Set

$$\mathscr{D}_F=\text{sheaf of germs of holomorphic differential operators on } \mathscr{L}_F,$$

a sheaf of non-commutative rings on $\mathscr{B}$. Then $\mathscr{O}_F$ is a sheaf of $\mathscr{D}_F$ modules – briefly, $\mathscr{O}_F$ is a $\mathscr{D}_F$ module. The $\mathfrak{g}$ action on $\mathscr{O}_F$ gives rise to a homomorphism

$$\pi_F\colon\ U(\mathfrak{g})\to\mathscr{D}_F(\mathscr{B});$$

the right side means the global sections of $\mathscr{D}_F$. Put

$$R=R_F=\text{image of } \pi_F.$$

The first problem is to determine the kernel of π_F. A holomorphic differential operator on $\mathscr{L}_F$ is zero if and only if its restriction to a single stalk of $\mathscr{O}_F$ (or to the completion of such a stalk) is zero. The completion of a stalk of a rank one locally free coherent sheaf is just a space of formal power series. This leads to the following general nonsense fact.

Lemma 3.3. *Suppose $x \in \mathscr{B}$ corresponds to a Borel subalgebra $\mathfrak{b}$ with nil radical $\mathfrak{n}$. Then, as a $U(\mathfrak{g})$ module (via π_F), the completion $(\hat{\mathscr{O}}_F)_x$ of $(\mathscr{O}_F)_x$ is isomorphic to*

$$\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), F/\mathfrak{n}F).$$

Here $U(\mathfrak{g})$ is regarded as a left $\mathfrak{b}$ module and a right $\mathfrak{g}$ module, making the Hom *a $\mathfrak{g}$ module.*

The object in the lemma is the full dual of a Verma module, so its annihilator can be computed as follows (see [5]).

Corollary 3.4. *Let $\mathfrak{I}_0 \subseteq \mathfrak{Z}(\mathfrak{g})$ (the center of $U(\mathfrak{g})$) be the maximal ideal annihilating F. Put*

$$\mathfrak{I} = U(\mathfrak{g})\,\mathfrak{I}_0.$$

Then (cf. Definition 3.2 and Lemma 3.3)

$$\ker \pi_F = \mathrm{Ann}(\hat{\mathscr{O}}_F)_x = \mathfrak{I}$$
$$R \cong U(\mathfrak{g})/\mathfrak{I}.$$

In particular, any $U(\mathfrak{g})$ module having the same infinitesimal character as F may be regarded as an R module.

It is shown in [1] that π_F is actually surjective. This is important for the proofs in [1], but not for stating the results. In any case, we may now speak of "$(R, K_{\mathbb{C}})$ modules" instead of "$(\mathfrak{g}, K)$ modules having the same infinitesimal character as F".

Theorem 3.5 (Beilinson-Bernstein [1]). *Let X be an $(R, K_{\mathbb{C}})$ module of finite length. Let $\mathscr{M} = \mathscr{M}_X$ be the $\mathscr{D}_F$ module obtained from X by extension of scalars: if $U \subseteq \mathscr{B}$ is open, then*

$$\mathscr{M}(U) = \mathscr{D}_F(U) \bigotimes_R X.$$

a) *$\mathscr{M}$ is a holonomic $\mathscr{D}_F$ module with regular singularities.*

b) $\mathrm{R\,Hom}_{\mathscr{D}_F}(\mathscr{M}, \mathscr{O}_F) \cong \mathrm{R\,Hom}_R(X, \mathscr{O}_F)$ *is an element of the derived category of sheaves on $\mathscr{B}$ with $K_{\mathbb{C}}$ action and constructible cohomology sheaves.*

c) *There is a contravariant functor $X \to \tilde{X}$ on $(R, K_{\mathbb{C}})$ modules which does not change composition series, and which maps to Verdier duality on the derived category of* (b). *In particular, if X is irreducible, then* $\mathrm{R\,Hom}_R(X, \mathscr{O}_F)$ *is self-dual in the derived category.*

d) *If $x \in \mathscr{B}$, then*

$$\mathrm{Ext}^i_R(X, \mathscr{O}_F)_x \cong \mathrm{Ext}^i_R(X, (\mathscr{O}_F)_x)$$
$$\cong \mathrm{Ext}^i_R(X, (\hat{\mathscr{O}}_R)_x).$$

Holonomic systems are defined are defined and discussed in [7]. "Regular singularities" can be defined by the second isomorphism in (d). (The first is obvious.) In (b), we can more or less think of $R\,\mathrm{Hom}$ as the set of all the Ext^i; the assertion is that $\mathrm{Ext}^i_R(X, \mathcal{O}_R)$ is a constructible sheaf with a $K_{\mathbb{C}}$ action. In particular, the stalks of these Ext sheaves are all finite dimensional vector spaces. The duality statements of (c) make sense only in the derived category, however. The hardest part of this theorem is formulating it; once formulated, it is a fairly easy consequence of the Kashiwara-Kawai theory of holonomic systems (see [7]), and the Bernstein-Gelfand-Gelfand translation principle. We will use only parts (c) and (d), and the reader is therefore invited to look for direct proofs of these.

For the benefit of the reader willing to rely more heavily on [1], we describe now some additional results stated there. They will not be used in the sequel, except to permit the omission of some references to [15]. In the setting of Theorem 3.5, fix a $K_{\mathbb{C}}$ orbit $Q \subseteq \mathscr{B}$. Then each of the sheaves $\mathrm{Ext}^i_R(X, \mathcal{O}_F)$ restricts to a locally constant sheaf on Q, which may be identified with a sum of elements of $\mathscr{D}$ supported on Q (see (1.1)):

$$\text{(3.6)(a)} \qquad \mathrm{Ext}^i_R(X, \mathcal{O}_F)|_Q \leftrightarrow \sum_{\substack{\gamma \in \mathscr{D} \\ \gamma \text{ supported on } Q}} n_i(\gamma, X).$$

This leads to an identity

$$\text{(3.6)(b)} \qquad \mathrm{Ext}^i_R(X, \mathcal{O}_F) = \sum_{\gamma \in \mathscr{D}} n_i(\gamma, X)\,\gamma$$

in the Grothendieck group of constructible sheaves with $K_{\mathbb{C}}$ action on $\mathscr{B}$.

Theorem 3.7 ([1], § 3–§ 4). *Suppose $\delta \in \mathscr{D}$, and i_0 = codimension of the underlying $K_{\mathbb{C}}$ orbit for δ. Then there is a unique $(R, K_{\mathbb{C}})$ module I_δ satisfying*

a) $n_i(\gamma, I_\delta) = 0$ if $i \neq i_0$ or $\gamma \neq \delta$
$n_{i_0}(\delta, I_\delta) = 1$.

I_δ has a unique irreducible submodule L_δ, which satisfies

b) $R\,\mathrm{Hom}_R(L_\delta, \mathcal{O}_F)\,[-i_0]$ *is quasi-isomorphic to the DGM extension $\tilde{\delta}$ of δ.*

c) *In the Grothendieck group of $(R, K_{\mathbb{C}})$-modules,*

$$L_\delta = \sum_{\gamma \in \mathscr{D}} (-1)^{l(\delta) - l(\gamma)} M(\gamma, \delta)\, I_\gamma$$

(*notation* (1.5), (1.1)).

Part (a) of this theorem is esentially a definition, and (c) follows from (a) and (3.6); the content of the result is (b).

4. Homology of Harish-Chandra Modules

In this section, we will study the spaces $\mathrm{Ext}^i_R(X, (\hat{\mathcal{O}}_F)_x)$ when X is an $(R, K_{\mathbb{C}})$ module, and $x \in \mathscr{B}$. Let $\mathfrak{b}$ be the Borel subalgebra corresponding to x, $\mathfrak{n}$ its nil radical, and $\mathfrak{h}$ a θ-stable Cartan subalgebra of $\mathfrak{b}$ (Proposition 2.1(c)). Since $H \cap K_{\mathbb{C}}$ acts on X and $\mathcal{O}_F$, and fixes x, $\mathrm{Ext}^i_R(X, (\hat{\mathcal{O}}_F)_x)$ is an $H \cap K_{\mathbb{C}}$-module. We will state all of the main results before beginning the proofs.

Proposition 4.1. *If X is any R module, then (with notation as above) there is a natural isomorphism*

$$\mathrm{Ext}^i_R(X, (\hat{\mathcal{O}}_F)_x) \cong \mathrm{Hom}_{\mathfrak{h}}(H_i(\mathfrak{n}, X), F/\mathfrak{n} F).$$

If X is actually an $(R, K_{\mathbb{C}})$ module, this isomorphism respects the action of $H \cap K_{\mathbb{C}}$. In particular, $(H \cap K_{\mathbb{C}})_0$ acts trivially.

Because of this proposition, we must study the $\mathfrak{n}$ homology of Harish-Chandra modules. For various special $\mathfrak{n}$, this has been done by Hecht and Schmid and in [14]; but it is important not to restrict $\mathfrak{n}$. The idea is to combine those special results with information about how homology varies with $\mathfrak{n}$.

Definition 4.2. Suppose $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are Borel subalgebras of $\mathfrak{g}$, and $\mathfrak{h} \subseteq \mathfrak{b}_1 \cap \mathfrak{b}_2$ is a Cartan subalgebra. Define

$$\Delta_{12} = \Delta(\mathfrak{b}_1/(\mathfrak{b}_1 \cap \mathfrak{b}_2), \mathfrak{h}) \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$$
$$d_{12} = \text{cardinality of } \Delta_{12} = \dim(\mathfrak{b}_1/(\mathfrak{b}_1 \cap \mathfrak{b}_2))$$
$$\delta_{12} = \text{weight of } \mathfrak{h} \text{ on } \Lambda^{d_{12}}(\mathfrak{b}_1/(\mathfrak{b}_1 \cap \mathfrak{b}_2)) \in \mathfrak{h}^*.$$

We also may regard δ_{12} as a character of the Cartan subgroup with Lie algebra $\mathfrak{h}$, or any subgroup of it.

Proposition 4.3. *Suppose $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are two Borel subalgebras of $\mathfrak{g}$ containing the θ-stable Cartan subalgebra $\mathfrak{h}$, with nil radicals $\mathfrak{n}_1$ and $\mathfrak{n}_2$. Let X be a $(\mathfrak{g}, K_{\mathbb{C}})$ module of finite length, and $\lambda \in \mathfrak{h}^*$. Assume that for every root $\alpha \in \Delta_{12}$ (Definition 4.2), we have*

a) *if $\alpha = \theta\alpha$, then α is compact;*
b) *if $\alpha \neq \theta\alpha$, then $\theta\alpha \in \Delta(\mathfrak{b}_1 \cap \mathfrak{b}_2)$; and*
c) *$\langle \check{\alpha}, \lambda \rangle$ is not a negative integer.*

Write $E(\mu)$ for the μ generalized weight space of an $\mathfrak{h}$ module E. Then there is a natural isomorphism of $(\mathfrak{h}, H \cap K_{\mathbb{C}})$-modules

$$H_p(\mathfrak{n}_2, X)(\lambda + \rho(\mathfrak{n}_2)) \cong H_{p+d_{12}}(\mathfrak{n}_1, X)(\lambda + \rho(\mathfrak{n}_1)) \otimes \Lambda^{d_{12}}(\mathfrak{b}_1/\mathfrak{b}_1 \cap \mathfrak{b}_2).$$

At this point we can proceed in two different ways. The Langlands standard representations may be characterized in terms of homology groups related to real parabolic subalgebras (Hecht and Schmid, unpublished), or in terms of homology groups related to θ-stable parabolic subalgebras ([15]). The former approach is more natural when the Langlands representations are regarded as induced from discrete series. The latter is more accessible in print, however, so we will adopt it.

Definition 4.4. Suppose $\gamma = (Q, \mathcal{L}) \in \mathcal{D}$ (see (1.1)); fix a Borel subalgebra $\mathfrak{b}_1 = \mathfrak{h} + \mathfrak{n}_1$ corresponding to a point x of Q, (with $\mathfrak{h}$ θ-stable), and write χ for the character of $(H\cap K_{\mathbb{C}})/(H\cap K_{\mathbb{C}})_0$ defined by $\mathscr{L}$ (Corollary 2.2). A *Z-good Borel subalgebra for* γ is a Borel $\mathfrak{b}_2=\mathfrak{h}+\mathfrak{n}_2$, containing $\mathfrak{h}$, with the following properties:

a) If $\alpha\in\Delta(\mathfrak{b}_2,\mathfrak{h})$, then either $\theta\alpha=-\alpha$, or $\theta\alpha\in\Delta(\mathfrak{b}_2,\mathfrak{h})$.

b) If $\alpha\in\Delta(\mathfrak{b}_1\cap\theta\mathfrak{b}_1,\mathfrak{h})$, then $\alpha\in\Delta(\mathfrak{b}_2,\mathfrak{h})$.

c) If $\alpha\in\Delta(\mathfrak{b}_1,\mathfrak{h})$ and $\theta\alpha=-\alpha$, then $\alpha\in\Delta(\mathfrak{b}_2,\mathfrak{h})$.

The terminology "Z-good" refers to the fact that $\mathfrak{b}_2$ is well suited to the Zuckerman construction of the standard representation $X(\gamma)$ (see (1.3)); $\mathfrak{b}_2$ is as close to being θ-stable as possible.

Lemma 4.5. *Z-good Borel subalgebras exist for every* γ. *If* $\mathfrak{b}_2$ *is one, then*

$$\operatorname{codim} Q=\dim(\mathfrak{n}_2\cap\mathfrak{p})-d_{12}$$

(*notation* (4.2)).

The next result is the promised relationship between standard representations and Lie algebra homology.

Theorem 4.6. *Suppose* $\gamma=(Q,\mathscr{L})\in\mathscr{D}$. *Fix a Z-good Borel subalgebra* $\mathfrak{b}=\mathfrak{h}+\mathfrak{n}_2$ *for* γ, *and use the notation of Definition* 4.4. *Put*

$$\mathbb{C}_\mu=(F/\mathfrak{n}_1 F)\otimes\delta_{12}^*\otimes\chi,$$

a one-dimensional $(\mathfrak{h},H\cap K_{\mathbb{C}})$ *module.* (*Here F is our fixed finite dimensional representation,* δ_{12} *is given by Definition* 4.2, *and* χ *by Definition* 4.4). *Recall the standard modules of* (1.3). *If* $\gamma'\in\mathscr{D}$, *then*

$$\begin{aligned}\dim\operatorname{Hom}_{\mathfrak{h},H\cap K_{\mathbb{C}}}(H_i(\mathfrak{n}_2,X(\gamma')),\mathbb{C}_\mu)&=0 \quad \text{if } \gamma\neq\gamma' \text{ or } i\neq\dim\mathfrak{n}_2\cap\mathfrak{p}\\ &=1 \quad \text{if } \gamma=\gamma' \text{ and } i=\dim\mathfrak{n}_2\cap\mathfrak{p}.\end{aligned}$$

This is distilled from [15]; the proof (by concatenation of square brackets) is in an appendix. Using Proposition 4.3 and Lemma 4.5, we can rewrite it as

Corollary 4.7. *Suppose* $\gamma=(Q,\mathscr{L})$, $\gamma'=(Q',\mathscr{L}')$ *are in* $\mathscr{D}$. *Fix a Borel subalgebra* $\mathfrak{b}=\mathfrak{h}+\mathfrak{n}$ *corresponding to a point of Q (with* $\mathfrak{h}$ θ*-stable), and let* χ *be the character of* $H\cap K_{\mathbb{C}}$ *corresponding to* $\mathscr{L}$ (*Corollary* 2.2). *Then*

$$\begin{aligned}\dim\operatorname{Hom}_{\mathfrak{h},H\cap K_{\mathbb{C}}}(H_i(\mathfrak{n},X(\gamma')),(F/\mathfrak{n}F)\otimes\chi)&=0 \quad \text{if } \gamma\neq\gamma' \text{ or } i\neq\operatorname{codim} Q\\ &=1 \quad \text{if } \gamma=\gamma' \text{ and } i=\operatorname{codim} Q.\end{aligned}$$

Using Proposition 4.1 and Theorem 3.5(d), we deduce

Corollary 4.8. *Suppose* $\gamma=(Q,\mathscr{L})\in\mathscr{D}$. *Then* $\operatorname{Ext}^i_R(X(\gamma),\mathcal{O}_F)$ *is zero unless* $i=\operatorname{codim} Q$. *The stalks of* $\operatorname{Ext}^{\operatorname{codim} Q}_R(X(\gamma),\mathcal{O}_F)$ *are zero except on* Q; *and* $\operatorname{Ext}^{\operatorname{codim} Q}_R(X(\gamma),\mathcal{O}_F)|_Q=$ *sheaf of locally constant sections of* $\mathscr{L}$.

In light of the definition of I_γ in Theorem 3.7, this proves Theorem 1.13. Another consequence of Corollary 4.7 is this.

Corollary 4.9. *Suppose X is an* $(R,K_{\mathbb{C}})$ *module of finite length, with character* $\Theta(X)$. *Write*

$$\Theta(X)=\sum_{\gamma\in\mathscr{D}} M(\gamma,X)\,\Theta(\gamma)$$

(*notation* (1.3)); *here* $M(\gamma, X)$ *is an integer. Fix* $\gamma \in \mathscr{D}$, *and use the notation of Corollary* 4.7. *Then*

$$M(\gamma, X) = (-1)^{\operatorname{codim} Q} \sum_i (-1)^i \operatorname{mult}(\chi, \operatorname{Ext}^i_R(X, (\hat{\mathcal{O}}_F)_x)).$$

For the reader willing to accept Theorem 3.7, this gives Theorem 1.6. We will give another proof of Theorem 1.6, using only Theorem 3.5, in Sect. 5.

We turn now to the proofs, beginning with Proposition 4.1. Suppose first that $i=0$. By Lemma 3.3,

$$(\hat{\mathcal{O}}_F)_x \cong \operatorname{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), F/\mathfrak{n} F).$$

So

$$\begin{aligned}\operatorname{Ext}^0_R(X, (\hat{\mathcal{O}}_F)_x) &= \operatorname{Hom}_R(X, \operatorname{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), F/\mathfrak{n} F)) \\ &= \operatorname{Hom}_{U(\mathfrak{g})}(X, \operatorname{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), F/\mathfrak{n} F)) \\ &\cong \operatorname{Hom}(X, F/\mathfrak{n} F),\end{aligned}$$

the last isomorphism being Frobenius reciprocity. Since $\mathfrak{h} \cong \mathfrak{b}/\mathfrak{n}$, this last Hom is isomorphic to

$$\operatorname{Hom}_{\mathfrak{h}}(X/\mathfrak{n} X, F/\mathfrak{n} F) = \operatorname{Hom}_{\mathfrak{h}}(H_0(\mathfrak{n}, X), F/\mathfrak{n} F),$$

as we wished to show. For $i>0$, we need a technical fact.

Lemma 4.10 (Casselman-Osborne [4]). *If* M *is any* R *module, then* $H_i(\mathfrak{n}, M)$ *is a semisimple* $\mathfrak{h}$ *module. If* M *is a projective* R *module, then* $H_i(\mathfrak{n}, M)$ *is zero for* $i>0$.

Suppose now that $i>0$, and that Proposition 4.1 has been proved for $i-1$. Choose a projective R-module P mapping onto X:

$$0 \to K \to P \to X \to 0. \tag{4.11}$$

Then $\operatorname{Ext}^i_R(P, (\hat{\mathcal{O}}_F)_x) = 0$, and $H_i(\mathfrak{n}, P)$ is zero by the lemma. Now write the long exact sequences for $\operatorname{Ext}_R(*, (\hat{\mathcal{O}}_F)_x)$ and $H_i(\mathfrak{n}, *)$ associated to (4.11). By the lemma, we may apply $\operatorname{Hom}_{\mathfrak{h}}(*, F/\mathfrak{n} F)$ to the second without destroying its exactness. Comparing the two resulting complexes, and using the induction hypothesis, we get the proposition. To study the $H \cap K_{\mathbb{C}}$ action, one makes this group operate on the set of all projective resolutions of X; we leave the details to the reader. Q.E.D.

Proposition 4.3 requires a little preliminary work.

Lemma 4.11 (Hochschild-Serre spectral sequence – see for example [3], p. 351). *Suppose* $\mathfrak{n}$ *is a Lie algebra,* $\mathfrak{u} \subseteq \mathfrak{n}$ *is an ideal, and* M *is any* $\mathfrak{n}$ *module. Then there is a spectral sequence*

$$H_p(\mathfrak{n}/\mathfrak{u}, H_q(\mathfrak{u}, M)) \Rightarrow H_{p+q}(\mathfrak{n}, M).$$

The differential d_N *has bidegree* $(N, 1-N)$ $(N=2,3,4,\ldots)$. *If* $\mathfrak{u}$ *has codimension one in* $\mathfrak{n}$, *this degenerates to a family of short exact sequences*

$$0 \to H_0(\mathfrak{n}/\mathfrak{u}, H_p(\mathfrak{u}, M) \to H_p(\mathfrak{n}, M) \to H_1(\mathfrak{n}/\mathfrak{u}, H_{p-1}(\mathfrak{u}, M)) \to 0.$$

Since $H_p(\mathfrak{h}, *)$ is zero if $p > \dim \mathfrak{h}$, the last assertion follows from the first.

Here is the strategy of the proof of Proposition 4.3. By induction on d_{12}, we may reduce to the case $d_{12}=1$. In this case, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are contained in a common parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$, with $[\mathfrak{l},\mathfrak{l}]\cong\mathfrak{sl}(2)$. Using Lemma 4.11, Proposition 4.3 can be reduced to a fact about the representations $H_*(\mathfrak{u},X)$ of $\mathfrak{sl}(2)$. The first problem, therefore, is to get some control on these representations.

Lemma 4.12. *Suppose X is a $(\mathfrak{g},K)$ module of finite length, and $\mathfrak{h}\subseteq\mathfrak{g}$ is a θ-stable Cartan subalgebra. Fix a Borel subalgebra $\mathfrak{b}\supseteq\mathfrak{h}$, and a parabolic $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$, with $\mathfrak{q}\supseteq\mathfrak{b}$, $\mathfrak{l}\supseteq\mathfrak{h}$. Assume that*

$$[\mathfrak{l},\mathfrak{l}]=\mathfrak{sl}(2)$$
$$\theta\mathfrak{l}\neq\mathfrak{l}.$$

Let α be the unique root of $\mathfrak{h}$ in $\mathfrak{l}\cap\mathfrak{b}$, and x_α a root vector.

a) *$\mathfrak{h}$ acts in a locally finite way on $H_*(\mathfrak{u},X)$; that is, $H_*(\mathfrak{u},X)$ is a direct sum of generalized weight spaces of $\mathfrak{h}$.*

b) *If $\theta\alpha\in\Delta(\mathfrak{u},\mathfrak{h})$, x_α acts locally nilpotently on $H_*(\mathfrak{u},X)$.*

Proof. Since $\theta\mathfrak{l}\neq\mathfrak{l}$, $\theta\alpha\neq\pm\alpha$. Put $\mathfrak{z}$ = center of $\mathfrak{l}$; then it follows that $\mathfrak{z}$ and $\mathfrak{h}\cap\mathfrak{k}$ span $\mathfrak{h}$. Since $\mathfrak{h}\cap\mathfrak{k}$ acts semisimply on X and on $\mathfrak{u}$, it does on $H_*(\mathfrak{u},X)$ as well. By the Casselman-Osborne theorem in [4], $\mathfrak{z}$ acts in a locally finite way on $H_*(\mathfrak{u},X)$. This proves (a). For (b), $\theta x_\alpha\in\mathfrak{u}$, so θx_α acts by zero on $H_*(\mathfrak{u},X)$. So it is enough to show that $y=x_\alpha+\theta x_\alpha$ acts nilpotently on $H_*(\mathfrak{u},X)$. But $y\in\mathfrak{k}$, and y is nilpotent (since it lies in the nil radical of $\mathfrak{b}$); so y acts nilpotently on X and on $\mathfrak{u}$. Therefore it acts nilpotently on the standard complex for computing $H_*(\mathfrak{u},X)$. Q.E.D.

If X has regular infinitesimal character, then [4] implies that $\mathfrak{z}$ acts semisimply on $H_*(\mathfrak{u},X)$, so $\mathfrak{h}$ does as well.

Lemma 4.13. *Suppose Y is a module for the reductive Lie algebra $\mathfrak{l}$, and $[\mathfrak{l},\mathfrak{l}]\cong\mathfrak{sl}(2)$. Choose a Cartan subalgebra $\mathfrak{h}\subseteq\mathfrak{l}$, and a Chevalley basis $\{h_\alpha, x_\alpha, x_{-\alpha}\}$ of $[\mathfrak{l},\mathfrak{l}]$ (with $h_\alpha\in\mathfrak{h}$; here $\alpha\in\mathfrak{h}^*$ denotes the weight of $\mathfrak{h}$ on x_α). For $\mu\in\mathfrak{h}^*$, write $Y(\mu)$ for the μ generalized weight space of Y. Fix μ, and assume that*

$$h_\alpha \text{ acts locally finitely on } Y$$
$$x_\alpha \text{ acts nilpotently on } Y$$
$$\langle\check{\alpha},\mu\rangle+1 \text{ is not a negative integer.}$$

Then

a) *x_α: $Y(\mu)\to Y(\mu+\alpha)$ is surjective, and $x_{-\alpha}$: $Y(\mu+\alpha)$ is injective.*

b) *There are direct sum decompositions*

$$Y(\mu)=Y(\mu)^{x_\alpha}\oplus(x_{-\alpha}Y(\mu+\alpha))$$

c) $H_0(\mathbb{C}x_\alpha, Y)(\mu+\alpha)=0$

$H_1(\mathbb{C}x_{-\alpha}, Y)(\mu)=0$

d) *There is a natural isomorphism*

$$[H_0(\mathbb{C}x_{-\alpha}, Y)(\mu)]\otimes(\mathbb{C}x_\alpha)\cong H_1(\mathbb{C}x_\alpha, Y)(\mu+\alpha).$$

Proof. The assumptions on Y imply that the Casimir operator

$$\Omega = x_{-\alpha} x_\alpha + \tfrac{1}{4} h_\alpha (h_\alpha + 2) = x_\alpha x_{-\alpha} + \tfrac{1}{4} h_\alpha (h_\alpha - 2) \tag{4.14}$$

acts in a locally finite way on Y. After passing to a direct summand of Y, we may therefore assume that there is a constant c such that $\Omega - c$ acts nilpotently on Y. Set

$$\mu_\alpha = \mu(h_\alpha) = \langle \check{\alpha}, \mu \rangle.$$

Then (4.14) gives

$$x_\alpha x_{-\alpha}|_{Y(\mu+\alpha)} = c - \tfrac{1}{4}\mu_\alpha(\mu_\alpha + 2) + M; \tag{4.15(a)}$$

here M is a locally nilpotent operator on $Y(\mu+\alpha)$. Similarly, for any integer $k \geqq 0$, we find

$$x_{-\alpha} x_\alpha|_{Y(\mu+k\alpha)} = c - \tfrac{1}{4}\mu_\alpha(\mu_\alpha + 2) + k(\mu_\alpha + k + 1) + N_k, \tag{4.15(b)}$$

with N_k locally nilpotent.

Suppose first that $c \neq \tfrac{1}{4}\mu_\alpha(\mu_\alpha + 2)$. Then $x_\alpha x_{-\alpha}|_{Y(\mu+\alpha)}$ and $x_{-\alpha} x_\alpha|_{Y(\mu)}$ are invertible; so by (4.15) (with $k=0$),

$$x_\alpha\colon\ Y(\mu) \to Y(\mu+\alpha)$$
$$x_{-\alpha}\colon\ Y(\mu+\alpha) \to Y(\mu)$$

are necessarily isomorphisms; so (a) and (b) are clear. Next, suppose that $c = \tfrac{1}{4}\mu_\alpha(\mu_\alpha + 2)$. Since $\mu_\alpha + 1$ is not a negative integer, 4.15(b) says that $x_{-\alpha} x_\alpha|_{Y(\mu+k\alpha)}$ is invertible, for all positive integers k. In particular,

$$x_\alpha\colon\ Y(\mu + k\alpha) \to Y(\mu + (k+1)\alpha)$$

is injective for $k \geqq 1$. Since x_α is locally nilpotent on Y, this forces $Y(\mu+\alpha) = 0$. Now (a) and (b) are clear in this case as well. Parts (c) and (d) are immediate consequences of (a) and (b), respectively. Q.E.D.

When $d_{12} = 1$, Proposition 4.3 is an immediate consequence of Lemmas 4.11 (with $\mathfrak{u} = \mathfrak{n}_1 \cap \mathfrak{n}_2$), 4.12, and 4.13 (with $Y = H_p(\mathfrak{n}_1 \cap \mathfrak{n}_2, X)$). (Actually, we also need the analogue of Lemma 4.12 when α is a compact root, but this is a triviality: obviously $H_*(\mathfrak{u}, X)$ is a locally finite representation of $\mathfrak{l}$ in that case.) The general case follows by induction on d_{12}; we leave the straightforward details to the reader.

Proof of Lemma 4.5. In the notation of Definition 4.4, let ρ_1 denote half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{n}_1$. Choose a maximally regular element $x \in \mathfrak{h} \cap \mathfrak{k}$, in the real span of the roots. Define $\mathfrak{b}_2$ by

$$\Delta(\mathfrak{b}_2, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \langle \alpha, \rho_1|_{\mathfrak{h}\cap\mathfrak{k}} \rangle > 0;\ \text{or}\ \langle \alpha, \rho_1|_{\mathfrak{h}\cap\mathfrak{k}} \rangle = 0,\ \text{and}\ \langle \alpha, x \rangle > 0;\ \text{or}\ \alpha\ \text{is real, and}\ \alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h})\}.$$

Clearly, $\mathfrak{b}_2$ is Z-good. For the codimension statement, suppose $\mathfrak{b}_2$ is any Z-good Borel subalgebra. Now

$$\begin{aligned}\operatorname{codim} Q &= \dim \mathscr{B} - \dim K_{\mathbb{C}} \cdot x\\ &= (\dim \mathfrak{g} - \dim \mathfrak{b}_1) - (\dim \mathfrak{k} - \dim \mathfrak{b}_1 \cap \mathfrak{k})\\ &= (\dim \mathfrak{g} - \dim \mathfrak{k}) - (\dim \mathfrak{b}_1 - \dim \mathfrak{b}_1 \cap \mathfrak{k})\\ \operatorname{codim} Q &= \dim \mathfrak{p} - \dim (\mathfrak{b}_1/(\mathfrak{b}_1 \cap \mathfrak{k})).\end{aligned}$$

We have

$$\begin{aligned}\dim \mathfrak{b}_1 \cap \mathfrak{k} = \dim (\mathfrak{h} \cap \mathfrak{k}) &+ |\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is compact imaginary}\}|\\ &+ \tfrac{1}{2}\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is complex, and } \theta\alpha \text{ is positive}\}|;\end{aligned}$$

the last set corresponds to elements $X_\alpha + \theta X_\alpha$ of $\mathfrak{b}_1 \cap \mathfrak{k}$. Therefore

$$\begin{aligned}\dim (\mathfrak{b}_1/\mathfrak{b}_1 \cap \mathfrak{k}) = \dim (\mathfrak{h} \cap \mathfrak{p}) &+ |\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is noncompact}\}|\\ &+ \tfrac{1}{2}|\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is complex, } \theta\alpha \text{ is positive}\}\\ &+ |\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \theta\alpha \text{ is negative}\}|.\end{aligned}$$

Similarly,

$$\begin{aligned}\dim \mathfrak{p} = \dim \mathfrak{h} \cap \mathfrak{p} &+ 2|\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is noncompact}\}|\\ &+ |\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is complex or real}\}|.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{codim} Q = &|\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is noncompact imaginary}\}|\\ &+ \tfrac{1}{2}|\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is complex and } \theta\alpha \text{ is positive}\}|.\end{aligned}$$

On the other hand

$$\begin{aligned}\dim \mathfrak{n}_2 \cap \mathfrak{p} = &|\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is noncompact imaginary}\}\\ &+ \tfrac{1}{2}|\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is complex}\}\end{aligned}$$

$$d_{12} = \tfrac{1}{2}\{\alpha \in \Delta(\mathfrak{b}_1, \mathfrak{h}) \mid \alpha \text{ is complex, and } \theta\alpha \text{ is negative}\}.$$

These last three formulas give

$$\operatorname{codim} Q = \dim \mathfrak{n}_2 \cap \mathfrak{p} - d_{12},$$

as we wished to prove. Q.E.D.

5. More about $K_{\mathbb{C}}$ Orbits

The proof of the Kazhdan-Lusztig conjecture in [2] and [9] relies heavily on a knowledge of the Bruhat order: that is, a combinatorial description of when one Bruhat cell is contained in the closure of another. In this section, we consider the analogous problem for $\mathscr{D}$, which could be formulated as follows: if γ and δ are in $\mathscr{D}$, when does γ occur in one of the DGM cohomology sheaves $\tilde{\delta}^i$ (see (1.5))? (Here "occurs" must be understood in the sense of the Grothendieck group of constructible sheaves with a $K_{\mathbb{C}}$ action, for which $\mathscr{D}$ is a basis.) We would like to say that $\gamma \leqq \delta$ if γ occurs in some $\tilde{\delta}^i$. Unfortunately, this relation is not transitive (in contrast to the situation for Bruhat cells). Of course we can make it transitive by fiat. The resulting relation (which should be called the Bruhat order) is worthy of study - it is the smallest order relation with the property that

$$\bar{X}(\gamma) \text{ occurs in } X(\delta) \Rightarrow \gamma \leqq \delta$$

(notation (1.3)), as is proved in [16]. However, I know of no simple combinatorial description of it: it is sometimes very difficult to distinguish two elements of $\mathscr{D}$ supported on the same $K_{\mathbb{C}}$ orbit. We sweep this problem under the rug (temporarily) by not trying to do so. The order relation to be defined (Definition 5.8) will be intermediate between the one given by containment of closures of underlying orbits and the Bruhat order. The geometry involved in the definition was suggested by Lusztig.

Suppose $x \in \mathscr{B}$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is the Borel subalgebra corresponding to x, and $s \in W(\mathfrak{g}, \mathfrak{h})$ is a reflection through a simple root. Write L^s_x for the projective line of type s through x. This is the set of all Borel subalgebras $\mathfrak{b}'$ such that $\mathfrak{b}' = \mathfrak{b}$, or $\mathfrak{b}$ and $\mathfrak{b}'$ are in relative position s. If $\mathfrak{p}_s = \mathfrak{l}_s + \mathfrak{u}_s$ is the parabolic subalgebra of type s containing $\mathfrak{b}$, with $[\mathfrak{l}_s, \mathfrak{l}_s] \cong \mathfrak{sl}(2)$, then these are exactly the Borel subalgebras contained in $\mathfrak{p}_s$.

Lemma 5.1. *Suppose $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$, with $\mathfrak{h}$ θ-stable; write $x \in \mathscr{B}$ for the corresponding point. Suppose s is a simple reflection in the Weyl group, and α is the corresponding simple root of $\mathfrak{h}$ in $\mathfrak{n}$.*

a) *If α is compact imaginary, then $L^s_x \subseteq K_{\mathbb{C}} \cdot x$.*

b) *If α is noncompact imaginary, then $L^s_x \cap (K_{\mathbb{C}} \cdot x)$ consists of exactly one or two points. If there is one, we say α is type I. If there are two, we say α is type II.*

c) *If α is real, then*

$$L^s_x \cap (K_{\mathbb{C}} \cdot x) = L^s_x - \{y^+, y^-\}.$$

We say that α is type I (respectively, type II) if $y^\pm$ lie in different (respectively, the same) $K_{\mathbb{C}}$-orbits.

d) *If α is complex, and $\theta\alpha \in \Delta^+$, then $L^s_x \cap (K_{\mathbb{C}} \cdot x) = \{x\}$.*

e) *If α is complex, and $\theta\alpha \notin \Delta^+$, then*

$$L^s_x \cap (K_{\mathbb{C}} \cdot x) = L^s_x - \{y\}.$$

In each case, if $L^s_x \cap (K_{\mathbb{C}} \cdot x)$ is finite, there is a unique $K_{\mathbb{C}}$-orbit $K_{\mathbb{C}} \cdot y$ such that

$$L^s_x \cap (K_{\mathbb{C}} \cdot y) \text{ is open.}$$

Proof. Since $K_{\mathbb{C}}$ has finitely many orbits on $\mathscr{B}$, the last assertion is clear. Let P_s be the parabolic subgroup of type s containing B; then

$$L^s_x = P_s \cdot x.$$

To compute $L^s_x \cap (K_{\mathbb{C}} \cdot x)$, we must therefore find the image of

$$(K_{\mathbb{C}} \cap P_s)/(K_{\mathbb{C}} \cap B) \to P_s/B.$$

So we must determine the group $(K_{\mathbb{C}}\cap P_s)/(K_{\mathbb{C}}\cap U_s H_s)$, with U_s the nil radical of P_s, and H_s the kernel of the root α. This may be regarded as a subgroup of $PSL(2,\mathbb{C})$. Its Lie algebra is easy to find; and we deduce the following structure for it (with cases labelled as in the lemma).

a) $PSL(2,\mathbb{C})$,

b) I) = c) I) $\quad T=\left\{\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix} \middle| x\in\mathbb{C}-\{0\}\right\}$

b) II) = c) II) $\quad N(T)=T\cup\left\{\begin{pmatrix} 0 & -x \\ x^{-1} & 0\end{pmatrix} \middle| x\in\mathbb{C}-\{0\}\right\}$

d) = e) $\quad TN=\left\{\begin{pmatrix} x & a \\ 0 & x^{-1}\end{pmatrix} \middle| x\in\mathbb{C}-\{0\},\ a\in\mathbb{C}\right\}.$

So we must determine the orbits of these groups on $\mathbb{P}^1$; but this is trivial. Q.E.D.

Definition 5.2. Suppose $\gamma\in\mathcal{D}$ corresponds to (H,Δ^+,χ) (Corollary 2.2); let $x\in\mathcal{B}$ be the corresponding Borel subgroup. Define the *weak τ-invariant of γ* (or of the underlying orbit) by

$$\tau_w(K_{\mathbb{C}}\cdot x)=\tau_w(\gamma)=\{s\in S \mid L_x^s\cap(K_{\mathbb{C}}\cdot x) \text{ is infinite}\}.$$

If $s\in\tau_w(\gamma)$, we define $s\circ\gamma$ to be the empty set. If $s\notin\tau_w(\gamma)$, we define $s\circ(K_{\mathbb{C}}\cdot x)$ to be the orbit $K_{\mathbb{C}}\cdot y$ described at the end of Lemma 5.1. Then let $s\circ\gamma$ be the subset of $\mathcal{D}$ consisting of flat $K_{\mathbb{C}}$-homogeneous line bundles on $s\circ(K_{\mathbb{C}}\cdot x)$, which extend γ to $(K_{\mathbb{C}}\cdot x)\cup(K_{\mathbb{C}}\cdot y)$. Finally, recall from (1.1) the *length* of γ, $l(\gamma)$, which is the dimension of $K_{\mathbb{C}}\cdot x$.

We need to compute $s\circ\gamma$ explicitly. The proof of Lemma 5.1 allows us to reduce this problem to $SL(2)$. The most complicated case is (b). In that case, the Levi factor $\mathfrak{l}_s$ of $\mathfrak{p}_s$ is θ-stable, and $[\mathfrak{l}_s,\mathfrak{l}_s]$ is the complexification of $\mathfrak{sl}(2,\mathbb{R})$. Therefore, it contains a unique $(L_s\cap K_{\mathbb{C}})$-conjugacy class of θ-stable Cartan subalgebras, distinct from the class of $\mathfrak{h}$. Write $\mathfrak{h}^\alpha$ for a representative of this class, and $(\Delta^+)^\alpha$ for a positive root system for $\mathfrak{h}^\alpha$ compatible with $\mathfrak{p}_s$. In case b) I), we have

$$H^\alpha\cap K_{\mathbb{C}}\subseteq H\cap K_{\mathbb{C}},$$

and we define

$$\chi^\alpha=\chi|_{H^\alpha\cap K_{\mathbb{C}}};$$

recall that γ corresponds to (H,Δ^+,χ). Set

(5.3)(I) $\quad\gamma^\alpha$ = element of $\mathcal{D}$ corresponding to $(H^\alpha,(\Delta^+)^\alpha,\chi^\alpha)$.

In the correspondence with regular characters given by Proposition 2.7, this is consistent with the notation of [15], Definition 8.3.6 (to which we refer the reader for further details). In case b) II) $H^\alpha\cap H\cap K_{\mathbb{C}}$ has index two in $H^\alpha\cap K_{\mathbb{C}}$; so we may define $\chi_\pm^\alpha$ by

$$\chi_+^\alpha\oplus\chi_-^\alpha=\operatorname{Ind}_{H^\alpha\cap H\cap K_{\mathbb{C}}}^{H^\alpha\cap K_{\mathbb{C}}}(\chi|_{H^\alpha\cap H\cap K_{\mathbb{C}}}).$$

Set

(5.3)(II) $\gamma^\alpha_\pm$ = element of $\mathscr{D}$ corresponding to $(H^\alpha, (\Delta^+)^\alpha, \chi^\alpha)$.

The two elements γ^α_+ and γ^α_- are not canonically labelled: there are two of them, but there is no reason to prefer one over the other (as the notation might seem to indicate). Once these definitions are understood, it is a simple matter to compute $s \circ \gamma$; and we find

Lemma 5.4. *In the setting of Definition* 5.2, *suppose* $s \notin \tau_w(\gamma)$. *Then* $s \circ \gamma$ *consists of elements of length* $l(\gamma)+1$. *Let* α *be the corresponding simple root.*

a) *If* α *is complex, then* $s \circ \gamma = \{(H, s\Delta^+, \chi)\}$.

b) *If* α *is type I noncompact imaginary, then* $s \circ \gamma = \{\gamma^\alpha\}$.

c) *If* α *is type II noncompact imaginary, then* $s \circ \gamma = \{\gamma^\alpha_+, \gamma^\alpha_-\}$.

Lemma 5.5. *Suppose* $x, y \in \mathscr{B}$, *and*

$$(K_{\mathbb{C}} \cdot y) \subseteq \overline{(K_{\mathbb{C}} \cdot x)}$$
$$s \notin \tau_w(K_{\mathbb{C}} \cdot y) \cup \tau_w(K_{\mathbb{C}} \cdot x).$$

Then

$$s \circ (K_{\mathbb{C}} \cdot y) \subseteq s \circ (K_{\mathbb{C}} \cdot x).$$

(*Definition* 5.2).

(Here and elsewhere, we tacitly identify simple reflections for different Borel subgroups in the canonical way.)

Proof. The set

$$\bar{A} = \bigcup_{z \in \overline{(K_{\mathbb{C}} \cdot x)}} L^s_z$$

is obviously closed, and contains $s \circ (K_{\mathbb{C}} \cdot y)$. But

$$A = \bigcup_{z \in K_{\mathbb{C}} \cdot x} L^s_z$$

is dense in $\bar{A}$, and $s \circ (K_{\mathbb{C}} \cdot x)$ is dense in A. The lemma follows. Q.E.D.

Write S for the set of reflections around simple roots (with respect to any pair $\mathfrak{h} \subseteq \mathfrak{b}$ of Cartan and Borel subalgebras).

Definition 5.6. Suppose $\gamma \in \mathscr{D}$ corresponds to (H, Δ^+, χ). The *strong τ-invariant of* γ is defined by

$$\begin{aligned}\tau_s(\gamma) &= \{s \in S \mid \gamma \in s \circ (\gamma'), \text{ for some } \gamma' \in \mathscr{D}\}\\ &= \{s \in S \mid L^s_x \cap (K_{\mathbb{C}} \cdot x) \text{ is a proper open subset of } L^s_x,\\ &\qquad \text{and the flat bundle on it extends to all of } L^s_x\}.\end{aligned}$$

Set

$$\tau(\gamma) = \text{Borho-Jantzen-Duflo } \tau\text{-invariant of } \bar{X}(\gamma) \subseteq S.$$

Proposition 5.7 ([15], Theorem 8.5.18). *We have*

$$\tau_s(\gamma) \subseteq \tau(\gamma) \subseteq \tau_w(\gamma).$$

More precisely,

$$\tau(\gamma)=\tau_s(\gamma)\cup\{s\in S\,|\,\text{corresponding root } \alpha \text{ is compact imaginary}\}$$
$$\tau_w(\gamma)=\tau(\gamma)\cup\{s\in S\,|\,\text{corresponding root } \alpha \text{ is real}\}.$$

Definition 5.8. The *Bruhat $\mathscr{G}$-order* on $\mathscr{D}$ is defined to be the smallest order relation with the following properties.

a) If $\gamma'<\delta'$, $\gamma\in s\circ\gamma'$, and $\delta\in s\circ\delta'$, then $\gamma<\delta$.

b) If $\gamma<\delta'$, and $\delta\in s\circ\gamma'$, then $\gamma<\delta$.

Here γ, δ, γ', and δ' are in $\mathscr{D}$, $s\in S$, and we use the notation of Definition 5.2.

The difference between this order and the Bruhat order (described at the beginning of the section) is this. In (a), suppose that $s\circ\delta'=\{\delta,\delta''\}$. If $\gamma'<\delta'$ in the Bruhat order, then it can be shown that $\gamma<\delta$ or $\gamma<\delta''$; but it is not easy to tell which of these relations holds. We therefore weaken the Bruhat order by including both.

Lemma 5.9. *Suppose $\gamma=(Q_1,\mathscr{L}_1)$, $\delta=(Q_2,\mathscr{L}_2)$ are in $\mathscr{D}$, and $\gamma<\delta$ (in the Bruhat $\mathscr{G}$-order). Then*

a) $l(\gamma)<l(\delta)$,

b) $Q_1\subseteq\overline{Q_2}$.

Proof. This follows from Lemma 5.5, just as one proves the relation between the Bruhat order and containments of Schubert varieties. Q.E.D.

Theorem 5.10. *Suppose $\gamma,\delta\in\mathscr{D}$, and $\bar{X}(\gamma)$ occurs in $X(\delta)$ (see (1.3)). Then $\gamma\leqq\delta$.*

This is a consequence of the proof of Theorem 8.6.6 of [15]; in fact those familiar with that argument will see that Definition 5.8 is tailored exactly to fit it.

The reader who believes Theorem 3.7 may now proceed to Sect. 6. The rest of this section is devoted to proving a version of Theorem 3.7 from Theorem 3.5. As in Sect. 4, the main step is taken from [15], and we relegate the proof to an appendix.

Theorem 5.11. *In the setting and notation of Theorem 4.6, let X be an $(R,K_{\mathbb{C}})$ module of finite length; and suppose that*

$$\operatorname{Hom}_{\mathfrak{h},H\cap K_{\mathbb{C}}}(H_i(\mathfrak{n}_2,X),\mathbb{C}_\mu)\neq 0.$$

Then there is an irreducible subquotient $\bar{X}(\delta)$ of X (some $\delta\in\mathscr{D}$), with

a) $\gamma\leqq\delta$,

b) $\dim\mathfrak{n}_2\cap\mathfrak{p}-(l(\delta)-l(\gamma))\leqq i\leqq\dim\mathfrak{n}_2\cap\mathfrak{p}$.

If actually $i=\dim\mathfrak{n}_2\cap\mathfrak{p}$, then $\bar{X}(\gamma)$ is an irreducible subquotient of X. Conversely, if $i=\dim\mathfrak{n}_2\cap\mathfrak{p}$, then

$$\dim\operatorname{Hom}_{\mathfrak{h},H\cap K_{\mathbb{C}}}(H_i(\mathfrak{n}_2,\bar{X}(\gamma)),\mathbb{C}_\mu)=1.$$

Corollary 5.12. *Suppose $\gamma=(Q,\mathscr{L})$ and $\delta=(Q',\mathscr{L}')$ are in $\mathscr{D}$. Fix a Borel subalgebra $\mathfrak{b}=\mathfrak{h}+\mathfrak{n}$ corresponding to a point of Q (with $\mathfrak{h}$ θ-stable), and let χ be the character of $H\cap K_{\mathbb{C}}$ corresponding to $\mathscr{L}$ (Corollary 2.2). Suppose that*

$$\operatorname{Hom}_{\mathfrak{h},H\cap K_{\mathbb{C}}}(H_i(\mathfrak{n},\bar{X}(\delta)),(F/\mathfrak{n}F)\otimes\chi)\neq 0.$$

Then

a) $\gamma\leqq\delta$.
b) $Q\subseteq\bar{Q}'$.
c) $\operatorname{codim}Q'\leqq i\leqq\operatorname{codim}Q$.
d) *If* $i=\operatorname{codim}Q$, *then* $\gamma=\delta$.
e) *If* $i=\operatorname{codim}Q$, *and* $\gamma=\delta$, *then*

$$\dim\operatorname{Hom}_{\mathfrak{h},H\cap K_{\mathbb{C}}}(H_i(\mathfrak{n},\bar{X}(\delta)),(F/\mathfrak{n}F)\otimes\chi)=1.$$

Proof. Using Proposition 4.3 and Lemma 4.5, we find that everything but (b) is contained in Theorem 5.11. Part (b) follows from (a) and Lemma 5.9. Q.E.D.

Fix $\delta=(Q',\mathscr{L}')\in\mathscr{D}$, and write $\tilde{\delta}$ for the DGM extension of δ to $\bar{Q}'$ (discussed before (1.5)). We extend $\tilde{\delta}$ by zero off of $\bar{Q}'$, to regard it as a complex of sheaves on $\mathscr{B}$, with a $K_{\mathbb{C}}$ action defined up to quasi-isomorphism. As is shown in [6], $\tilde{\delta}$ is characterized by the following properties:

(5.13) i) $\tilde{\delta}$ is self-dual in the sense of Verdier, up to a Tate twist.
ii) $\tilde{\delta}^i$ is supported on $\bar{Q}'$.
iii) $\tilde{\delta}^0|_{Q'}=\delta$.
iv) $\operatorname{codim}\operatorname{supp}\tilde{\delta}^i\geqq\operatorname{codim}Q'+i+1$ $(i>0)$.
v) $\tilde{\delta}^i=0,\ i<0$.

Corollary 5.14. *Suppose $\delta=(Q',\mathscr{L}')\in\mathscr{D}$. Then*

$$R\operatorname{Hom}_R(\bar{X}(\delta),\mathscr{O}_F)[-\operatorname{codim}Q']$$

is quasi-isomorphic to the DGM extension $\tilde{\delta}$ of δ.

Proof. We have to verify the properties (5.13)(i)–(v). The first is Theorem 3.5(b). The rest are statements about the cohomology sheaves $\operatorname{Ext}^i_R(\bar{X}(\delta),\mathscr{O}_F)$. By Theorem 3.5(d) and Proposition 4.1, they amount to statements about $\operatorname{Hom}_{\mathfrak{h},H\cap K_{\mathbb{C}}}(H_i(\mathfrak{n},\bar{X}(\delta)),(F/\mathfrak{n}F)\otimes\chi)$ (as $\mathfrak{h}+\mathfrak{n}$ and χ vary). As such, they are exactly the conclusions of Corollary 5.12: (5.13)(ii) is (5.12)(b), (iii) is (e), and (iv) and (v) are (c) and (d). Q.E.D.

Theorem 1.6 is a consequence of Corollary 5.14 and Corollary 4.9.

6. The Polynomials $P_{\gamma,\delta}$

In this section we will set up the combinatorial formalism underlying the computation of the DGM extension of δ. In [8], Kazhdan and Lusztig did this for the Weyl group (corresponding to the case of Schubert varieties) in a selfcontained way. Here we are not so successful. We will prove the uniqueness of our proposed $P_{\gamma,\delta}$; but the proof of existence is postponed to [10], and uses algebraic geometry. To understand the definition in detail, we need a slight extension of Lemma 5.4.

Lemma 6.1 ([15], Proposition 8.3.18). *Suppose $\gamma \in \mathscr{D}$ corresponds to (H, Δ^+, χ) (Corollary 2.2). Fix a simple reflection $s \in S$, and let $\alpha \in \Delta^+$ be the corresponding simple root. Assume that $s \in \tau_s(\gamma)$; that is, that $s \in \tau(\gamma)$, but α is not compact imaginary.*

a) If α is complex, then there is a unique element $\gamma' \in \mathscr{D}$ such that $\gamma \in s \circ \gamma'$. The root for γ' corresponding to α is also complex.

b) If α is type II real (Lemma 5.1(c)), there is a unique element $\gamma_\alpha \in \mathscr{D}$ such that $\gamma \in s \circ \gamma_\alpha$. The root for γ_α corresponding to α is type II noncompact imaginary.

c) If α is type I real (Lemma 5.1(c)), there are exactly two elements $\gamma_\alpha^\pm$ of $\mathscr{D}$ such that $\gamma \in s \circ \gamma_\alpha^\pm$. The roots for $\gamma_\alpha^\pm$ corresponding to α are type I noncompact imaginary.

The elements $\gamma_\alpha^\pm$ and $\gamma_\alpha^\pm$ may be constructed by Cayley transforms, in analogy with the discussion before (5.3). Write $x \in \mathscr{B}$ for the point corresponding to (H, Δ^+), and $Q = K_{\mathbb{C}} \cdot x$ for the underlying orbit of γ. Let Q', $Q\alpha$, $Q_\alpha^\pm$ be the underlying orbits of γ', γ_α, $\gamma_\alpha^\pm$. The $Q \cap L_x^s$ is open (by Definition 5.6); and

$$L_x^s = (Q \cap L_x^s) \cup (Q' \cup L_x^s) \qquad \text{(case (a) of Lemma 6.1)}$$
$$(6.2) \qquad L_x^s = (Q \cap L_x^s) \cup (Q_\alpha \cap L_x^s) \qquad \text{(case (b))}$$
$$L_x^s = (Q \cap L_x^s) \cup (Q_\alpha^\pm \cap L_x^s) \qquad \text{(case (c))}.$$

(This is clear from Lemma 5.1).

Definition 6.3. Suppose $\gamma \in \mathscr{D}$ corresponds to (H, Δ^+, χ), and $s \in S$ is a simple reflection. Define $s \times \gamma$ to be the element of $\mathscr{D}$ corresponding to $(H, s\Delta^+, \chi)$. In terms of notation already developed, this can be described as follows. Write $\alpha \in \Delta^+$ for the corresponding simple root.

a) Suppose α is complex. If $\theta\alpha \in \Delta^+$, then $\{s \times \gamma\} = s \circ \gamma$. If $\theta\alpha \notin \Delta^+$, then $\{\gamma\} = s \circ (s \times \gamma)$ (Lemma 5.4(a)).

b) If α is type I noncompact imaginary, write $\{\delta\} = s \circ \gamma$. Then $\{\delta_\alpha^\pm\} = \{\gamma, s \times \gamma\}$; that is, $s \times \gamma$ is the unique element distinct from γ such that $s \circ (s \times \gamma) = s \circ \gamma$ (Lemma 6.1(c)).

c) If α is type II real, write $\delta = \gamma_\alpha$ (Lemma 6.1(b)). Then $s \circ \delta = \{\gamma, s \times \gamma\}$.

d) In all other cases, $s \times \gamma = \gamma$.

The assertions in (b)–(d) are verified in Proposition 8.3.18 of [15]. We are now in a position to formulate the first main definition: that of an analogue of the Hecke algebra.

Definition 6.4. Let $\mathscr{M}$ be the free $\mathbb{Z}[u, u^{-1}]$ module with basis $\mathscr{D}$; here u is an indeterminate. More generally, fix an abelian group B, and an element $u \in B$ of infinite order; and let $\mathscr{M}'$ be the free $\mathbb{Z}[B]$ module with basis $\mathscr{D}$. We identify $\mathscr{M}$ as a $\mathbb{Z}[u, u^{-1}]$ submodule of $\mathscr{M}'$ by identifying the indeterminate u with

$u \in B$. If $s \in S$ is a simple reflection, we define a $\mathbb{Z}[u, u^{-1}]$-linear map T_s on $\mathcal{M}$ (or a $\mathbb{Z}[B]$-linear map on $\mathcal{M}'$) on basis elements γ, as follows. We use the notation of Lemma 5.4 and 6.1.

a) If α is compact imaginary, $T_s \gamma = u\gamma$.

b1) If α is complex and $\theta\alpha \in \Delta^+$, then $T_s \gamma = s \circ \gamma = s \times \gamma$.

c1) If α is type II noncompact imaginary, then

$$\begin{aligned} T_s \gamma &= \gamma + \sum_{\gamma' \in s \circ \gamma} \gamma' \\ &= \gamma + \gamma_+^\alpha + \gamma_-^\alpha . \end{aligned}$$

d1) If α is type I noncompact imaginary (so that $s \circ \gamma = \{\gamma^\alpha\}$),

$$T_s \gamma = \gamma^\alpha + s \times \gamma.$$

e) If α is real, but $s \notin \tau(\gamma)$,

$$T_s \gamma = -\gamma.$$

b2) If α is complex and $\theta\alpha \notin \Delta^+$,

$$T_s \gamma = u(s \times \gamma) + (u-1)\gamma.$$

c2) If α is type II real, and $s \in \tau(\gamma)$,

$$T_s \gamma = (u-1)\gamma - s \times \gamma + (u-1)\gamma_\alpha.$$

d2) If α is type I real, and $s \in \tau(\gamma)$,

$$\begin{aligned} T_s \gamma &= (u-2)\gamma + (u-1) \sum_{\gamma \in s \circ \gamma'} \gamma' \\ &= (u-2)\gamma + (u-1)(\gamma_\alpha^+ + \gamma_\alpha^-). \end{aligned}$$

For the moment, this definition cannot be very well motivated. A geometric explanation of it is given in [10]. The point of Lemma 7.8 is that this definition captures the inductive formulas for computing homology given in Theorem 7.2 of [14].

Lemma 6.5. *The operators* T_s *satisfy* $(T_s+1)(T_s-u)=0$.

Proof. By Definition 6.4, the matrix of T_s is a direct sum of matrices of the form

$$(u), \quad (-1), \quad \begin{pmatrix} 0 & 1 \\ u & u-1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ u-1 & u-1 & u-2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ u-1 & u-1 & -1 \\ u-1 & -1 & u-1 \end{pmatrix}.$$

All of these satisfy the desired relation. Q.E.D.

It is also true that the operators $\{T_s | s \in S\}$ satisfy the braid relations, so that we have defined a Hecke algebra action. This can be checked by hand - with some effort because of the large number of cases - but a conceptual geometric proof is given in [10]. In any case we do not need this fact.

Definition 6.6 ([15], Definition 8.3.1). Let $\sim$ denote the equivalence relation on $\mathscr{D}$ generated by relations $\gamma \sim s \times \gamma$ whenever s corresponds to a simple real root for γ (Definition 6.3); and let $\mathscr{G}(\gamma)$ be the equivalence class of γ.

The next lemma is used to guarantee that inductive arguments on the Bruhat $\mathscr{G}$-order of Definition 5.8 can be kept going. The analogous statement for the Bruhat order on W is that if $w \in W$, $w \neq 1$, then there is a simple reflection s with $l(sw) < l(w)$.

Lemma 6.7 ([15], Chapter 9). *Suppose $\gamma, \delta \in \mathscr{D}$, and $\gamma < \delta$ (in the Bruhat $\mathscr{G}$-order). Then at least one of the following conditions holds.*

a) *For some $s \in \tau(\delta)$, the corresponding simple root is complex.*

b) *For some $s \notin \tau(\gamma)$, the corresponding simple roots for γ and δ are complex and real, respectively.*

c) *There is an $s \notin \tau(\gamma)$ such that the corresponding simple root is noncompact imaginary, and a $\delta' \in \mathscr{G}(\delta)$ such that $s \in \tau(\delta')$, and the corresponding simple root is real.*

Lemma 6.8. *There is at most one $\mathbb{Z}$-linear map $D: \mathscr{M}' \to \mathscr{M}'$ with the following properties. Write (that is, define $R_{\gamma,\delta}$ by)*

$$D(\delta) = u^{-l(\delta)} \sum_{\gamma} R_{\gamma,\delta}(u)\, \gamma.$$

a) $D(bm) = b^{-1} D(m)$ $(m \in \mathscr{M}',\ b \in B)$.

b) $D((T_s+1)m) = u^{-1}(T_s+1) D(m)$ $(m \in \mathscr{M}',\ s \in S)$.

c) $R_{\delta,\delta} = 1$.

d) $R_{\gamma,\delta} \neq 0$ *only if* $\gamma \leqq \delta$.

Suppose D exists. Then there is an algorithm for computing the various $R_{\gamma,\delta}$; they are polynomials in u, of degree at most $l(\delta) - l(\gamma)$. Furthermore,

e) D^2 *is the identity.*

f) D *preserves* $\mathscr{M} \subseteq \mathscr{M}'$.

g) *On $\mathscr{M}$, the specialization of D to $u = 1$ is the identity.*

Proof. Assuming that D exists, we will show how to compute $R_{\gamma,\delta}$ from (a)-(d). The algorithm will show that $R_{\gamma,\delta}$ is a polynomial in u of degree at most $l(\delta) - l(\gamma)$, and that

(6.9)
$$R_{\gamma,\delta}(1) = \begin{cases} 1 & \gamma = \delta \\ 0 & \gamma \neq \delta. \end{cases}$$

Then (f) and (g) follow. By (c) and (d), D is invertible. Its inverse will have the same properties, and so must coincide with D by the uniqueness. So the main problem is to find the algorithm. It proceeds by induction on $l(\delta)$; and then for fixed δ, by downward induction on $l(\gamma)$. By (c) and (d), we may assume that $\gamma < \delta$. We will use Lemma 6.7. Suppose first that $\tau(\delta)$ contains a reflection s such that the corresponding simple root for δ is complex. Then

$$l(s \times \delta) = l(\delta) - 1$$
$$T_s(s \times \delta) = \delta.$$

The identity (b) for $m=s\times\delta$ can be written as

$$D(\delta)=u^{-1}(T_s+1-u)\,D(s\times\delta)$$

or

$$\sum_{\gamma\in\mathscr{D}}(-1)^{l(\gamma)}R_{\gamma,\delta}\gamma=-\sum_{\gamma\in\mathscr{D}}(-1)^{l(\gamma)}R_{\gamma,s\times\delta}[(T_s+1-u)\gamma].$$

Equating the coefficients of γ gives formulas for $R_{\gamma,\delta}$. Next, if s corresponds to a real root in $\tau(\delta)$, then the argument above gives

(6.10) $$R_{\gamma,\delta}+R_{\gamma,s\times\delta}=\text{known polynomial in } u.$$

Suppose there is a reflection $s\notin\tau(\gamma)$ as in Lemma 6.7(b). If we apply (b) with $m=\delta$, and equate coefficients of $s\times\gamma=T_s\gamma$ on both sides, we get a formula for $R_{\gamma,\delta}$ in terms of known $R_{\gamma',\delta'}$. Finally, suppose s and δ' are as in Lemma 6.7(c). By (6.10), it is enough to compute $R_{\gamma,\delta'}$. For concreteness, we suppose s is type II for δ' and type I for γ; the other cases are easier. Recall that

$$T_s\delta'=(u-1)\,\delta'-s\times\delta'+(u-1)\,\delta'_\alpha$$
$$T_s\gamma=s\times\gamma+\gamma^\alpha.$$

Apply (b) to $m=s\times\delta'$. Equating coefficients of $s\times\gamma$ gives

(6.11)(a) $R_{\gamma,s\times\delta'}+R_{s\times\gamma,s\times\delta'}=\text{known polynomial in } u.$

Equating coefficients of γ^α gives

(6.11)(b) $u\,R_{\gamma,\delta'}+R_{s\times\gamma,s\times\delta'}=\text{known polynomial in } u.$

Then combining (6.10) and (6.11) gives

$$(u+1)\,R_{\gamma,\delta'}=\text{known polynomial in } u.$$

Such an equation forces $(u+1)$ to divide the right side. The degree estimate and (f) follow by writing this out more explicitly. Q.E.D.

From now on, we assume that our abelian group B (Definition 6.4) has a multiplicative norm, written $|\ |$; and that $|u|>1$. In the application in [10], B is the group of non-zero algebraic numbers modulo roots of unity; u is the image of a prime power q; and the norm is the geometric mean of the absolute values of the conjugates in $\mathbb{C}$.

Corollary 6.12. *Suppose D exists; and suppose that for some $\delta\in\mathscr{D}$ there is an element*

$$C_\delta=\sum_{\gamma\leqq\delta}P_{\gamma,\delta}\in\mathscr{M}'$$

with the following properties.

a) $D(C_\delta)=u^{-l(\delta)}\,C_\delta$.

b) $P_{\gamma,\gamma}=1$.

c) *If $\gamma\neq\delta$, then $P_{\gamma,\delta}$ is a $\mathbb{Z}$-linear combination of elements of B of norm between 1 and $|u|^{1/2(l(\delta)-l(\gamma)-1)}$.*

Then $P_{\gamma,\delta}$ is a computable polynomial in u. (In particular, C_δ is unique.)

The proof, by downward induction on $l(\delta)$, is almost word-for-word the same as that of Theorem 1.1 in [8], and we leave it to the reader. A different computation of P is contained in Proposition 6.14.

Definition 6.13. Suppose $\gamma \leqq \delta$, and C_δ exists. Let $\mu(\gamma, \delta)$ be the coefficient of $u^{1/2(l(\delta)-l(\gamma)-1)}$ in $P_{\gamma,\delta}$.

Proposition 6.14. *Suppose that D and all the C_δ exist. Fix γ and δ in $\mathcal{D}$. Then the polynomials of Corollary* 6.12 *satisfy the following formulas, which (together with $P_{\gamma,\gamma}=1$, and $P_{\gamma,\delta}\neq 0$ only if $\gamma\leqq\delta$) characterize them completely. More precisely, if $P_{\gamma',\delta'}$ is known whenever $l(\delta')<l(\delta)$, or $l(\delta')=l(\delta)$ and $l(\gamma')>l(\gamma)$, then the formulas determine $P_{\gamma,\delta}$. Fix s, and define α_γ to be the simple root corresponding to s for γ.*

Case I. $s\notin\tau(\delta)$. *Write*

$$P_{\gamma, s\circ\delta} = \sum_{\gamma'\in s\circ\delta} P_{\gamma,\delta}.$$

$$U^s_{\gamma,\delta} = \sum_{\substack{z\in\mathcal{D}\\ s\in\tau(z)\\ \gamma\leqq z\leqq\delta}} \mu(z,\delta)\, u^{1/2(l(\delta)-l(z)+1)} P_{\gamma,z}.$$

a) α_γ *is complex, and* $s\notin\tau(\gamma)$.

$$P_{\gamma,s\circ\delta} = P_{s\circ\gamma,s\circ\delta} = u P_{s\circ\gamma,\delta} + P_{\gamma,\delta} - U^s_{\gamma,\delta}.$$

b) α_γ *is compact imaginary.*

$$P_{\gamma,s\circ\delta} = (u+1) P_{\gamma,\delta} - U^s_{\gamma,\delta}.$$

c) α_γ *is type I noncompact. In the notation of Definition* 6.4(d1),

$$P_{\gamma,s\circ\delta} = P_{\gamma^\alpha,s\circ\delta} = P_{s\times\gamma,s\circ\delta} = (u-1) P_{\gamma^\alpha,\delta} + P_{\gamma,\delta} + P_{s\times\gamma,\delta} - U^s_{\gamma,\delta}.$$

d) α_γ *is type II noncompact. Write* $\{\gamma^\alpha_\pm\}=\{\gamma',\gamma''\}$. *Then*

$$P_{\gamma,s\circ\delta} = P_{\gamma',s\circ\delta} + P_{\gamma'',s\circ\delta} = (u-1)(P_{\gamma',\delta}+P_{\gamma'',\delta}) + 2P_{\gamma,\delta} - U^s_{\gamma,\delta}$$
$$P_{\gamma',s\circ\delta} = u P_{\gamma',\delta} - P_{\gamma'',\delta} + P_{\gamma,\delta} - U^s_{\gamma',\delta}.$$

e) α_γ *real,* $s\notin\tau(\gamma)$. *Then* $P_{\gamma,s\circ\delta}=0$.

Case II. $s\in\tau(\delta)$. *We use notation for γ as in Case I, and letter the cases in the same way.*

a) $P_{\gamma,\delta}=P_{s\circ\gamma,\delta}$.
c) $P_{\gamma,\delta}=P_{\gamma^\alpha,\delta}=P_{s\times\gamma,\delta}$.
d) $P_{\gamma,\delta}=P_{\gamma',\delta}+P_{\gamma'',\delta}(\{\gamma^\alpha_\pm\}=\{\gamma',\gamma''\})$.
e) $P_{\gamma,\delta}=0$.

Proof. That the formulas characterize P follows from Lemma 6.7, by an argument formally similar to (but in detail simpler than) that given for Lemma 6.8. To prove them, we recall from [10], Sect. 5, the formulas

(6.15) $(T_s+1)\,C_\delta=\sum_{\delta'\in s\circ\delta} C_{\delta'}+\sum_{\substack{z\leq\delta\\ s\in\tau(z)}}\mu(z,\delta)\,u^{1/2(l(\delta)-l(z)+1)}\,C_z\quad(s\notin\tau(\delta)).$

(6.16) $(T_s-u)\,C_\delta=0\quad(s\in\tau(\delta)).$

(To avoid circular reasoning, it is important to recall that the proofs of these formulas did not rely on Theorem 1.12 of [10], whose proof is to be given below.) Most of the formulas of the proposition follow from Definition 6.4 by comparing coefficients of γ on the two sides of these identities (compare [8]), and we leave them to the reader. Those of Case II(c) are slightly more subtle, however. By (6.16) and Definition 6.4(d), comparing coefficients of γ, $s\times\gamma$, and γ^α gives

$$\begin{aligned}-u\,P_{\gamma,\delta}+P_{s\times\gamma,\delta}+(u-1)\,P_{\gamma^\alpha,\delta}&=0\\ -u\,P_{s\times\gamma,\delta}+P_{\gamma,\delta}+(u-1)\,P_{\gamma^\alpha,\delta}&=0\\ P_{\gamma,\delta}+P_{s\times\gamma,\delta}-2\,P_{\gamma^\alpha,\delta}&=0.\end{aligned}$$

Subtracting the first two of these gives

$$(u+1)\,P_{\gamma,\delta}=(u+1)\,P_{s\times\gamma,\delta}.$$

It follows that $P_{\gamma,\delta}=P_{s\times\gamma,\delta}$. Inserting this in the last of the three formulas above gives II(c). Q.E.D.

7. Proof of Theorem 1.12

Since the material in this section is only of very technical interest, we have essentially assumed that the reader is quite familiar with [14] or Chapter 9 of [15]. The proofs really require this. We first recall the main results of [10].

Theorem 7.1 ([10]). *The map D of Lemma* 6.8, *and the elements C_δ of Corollary* 6.12 *both exist. Write $\tilde{\delta}$ for the DGM extension of δ to the closure of the underlying orbit* (*see* (5.13)). *Then*

$$\sum_i(-1)^i\,(\text{multiplicity of }\gamma\text{ in }\tilde{\delta}^i)=P_{\gamma,\delta}(1).$$

Because of Corollary 5.14 and Corollary 4.9, this is equivalent to

Corollary 7.2. $\bar{\Theta}(\delta)=\sum_\gamma(-1)^{l(\delta)-l(\gamma)}\,P_{\gamma,\delta}(1)\,\gamma.$

Our goal is to deduce Theorem 1.12, in the following equivalent form (compare Proposition 4.1 and Corollary 5.14).

Theorem 7.3. *Fix $\delta,\gamma\in\mathscr{D}$, $x\in\mathscr{B}$ in the underlying orbit of γ, and a Levi decomposition*

$$\mathfrak{b}=\mathfrak{h}+\mathfrak{n}$$

of the Borel subalgebra corresponding to x, with $\mathfrak{h}$ θ-stable. Write d for the codimension of the support of δ in $\mathscr{B}$. Let χ be the character of

$(H \cap K_{\mathbb{C}})/(H \cap K_{\mathbb{C}})_0$ *corresponding to* γ. *Then*

$$\begin{aligned}&\dim \operatorname{Hom}_{\mathfrak{h}, H \cap K_{\mathbb{C}}}(H_q(\mathfrak{n}, \bar{X}(\delta)), (F/\mathfrak{n}F) \otimes \chi)\\ &\quad = \textit{coefficient of } u^{1/2(q-d)} \textit{ in } P_{\gamma,\delta}.\end{aligned}$$

For convenience, we define (with d the codimension of $\operatorname{supp}(\delta)$)

(7.4)(a) $$Q_{\gamma,\delta} = \sum_q u^{1/2(q-d)}(\dim \operatorname{Hom}_{\mathfrak{h}, H \cap K}(H_q(\mathfrak{n}, \bar{X}(\delta)), (F/\mathfrak{n}F) \otimes \chi),$$

an element of $\mathbb{Z}[u^{1/2}, u^{-1/2}]$. Then we want to show that $P = Q$. Put

(7.4)(b) $\mu'(\gamma, \delta) = \text{coefficient of } u^{1/2(l(\delta) - l(\gamma) - 1)}$ in $Q_{\gamma,\delta}$.

To clarify the argument, we separate two inductive hypotheses.

(7.5) $A(m)$ If $l(\delta) < m$, $s \notin \tau(\delta)$, and α is the corresponding simple root, then $U_\alpha(\bar{X}(\delta))$ is completely reducible.

Here $U_\alpha(X)$ is the $(\mathfrak{g}, K)$ module constructed in [13] or [15] using tensor products with finite dimensional representations. It is defined whenever X is irreducible with regular infinitesimal character, and α is a simple integral root not in the τ invariant of X.

(7.6) $B(m)$ If $l(\delta) \leq m$, then $P_{\gamma,\delta} = Q_{\gamma,\delta}$.

The idea of the proof of Theorem 7.3 is this. In [14], there is an algorithm for computing the polynomials $Q_{\gamma,\delta}$ (or something equivalent) under the assumption (7.5) (for all m). We will show that this algorithm corresponds exactly to the recursion formulas for $P_{\gamma,\delta}$ given in Proposition 6.14. On the other hand, we already know (by Theorem 7.1) that $P_{\gamma,\delta}(1) = Q_{\gamma,\delta}(1)$. We will show roughly that if (7.5) fails, then the algorithm for computing $Q_{\gamma,\delta}$ is so seriously affected that even the value at 1 must change; this was stated without proof in [14]. So (7.5) cannot fail, and we will be done.

There is a problem of exposition here. The arguments of [14] are stated in terms of θ-stable parabolic subalgebras exclusively. The Beilinson-Bernstein theory shows that this is not very clever; and when rephrased in terms of the homology groups appearing in Theorem 7.3, the proofs often become simpler and clearer. They do not become trivial, however, and we do not propose to rewrite [14] here: it is much faster to make do with the results proved in [14] than to start from scratch. This has the effect of obscuring the ideas almost completely. For example, Lemma 7.8 could be proved by reduction to $SL(2)$, if [14] had been written from the Beilinson-Bernstein point of view. Apologies are tendered to the reader for this situation.

Lemma 7.7. *Assume* $A(m)$. *Then the vanishing results and induction formulas of Theorem* 7.2 *in* [14] *hold for* $l(\gamma^1) \leq m$.

These results state that the various cohomology groups are completely reducible and vanish in every other degree; and then show how to compute the cohomology of $U_\alpha(\bar{X}(\gamma^1))$ from that of $\bar{X}(\gamma^1)$. (*A* typical example is written before (7.11) below.) They are based on the hypothesis that certain $U_\alpha(*)$ be completely reducible: $A(m)$ is precisely what is used, as the reader can check.

Lemma 7.8. *Assume* $A(m)$. *Then the* $Q_{\gamma,\delta}$ *satisfy all the formulas of Proposition* 6.14 *when* $l(\delta)<m$ *(in Case I) or* $l(\delta)\leq m$ *(in Case II), with* μ *replaced by* μ'.

Proof. The idea is that the formulas of Proposition 6.14 correspond exactly to the induction formulas mentioned in the previous lemma. Those induction formulas are for θ-stable parabolic subalgebras, however, so we have to use the ideas of Sect. 4 and the appendix to transfer the homology appearing in (7.4) to a θ-stable setting. To do this for (6.14)(I)(a), (b), (e) and (6.14)(II) is rather easy; so we will concentrate on (for definiteness) (6.14)(I)(c). Fix γ, $\delta\in\mathscr{D}$. Choose x, $\mathfrak{b}$, $\mathfrak{h}+\mathfrak{n}$ attached to γ as in Theorem 7.3. Let $\alpha\in\Delta(\mathfrak{h},\mathfrak{n})$ be the simple root corresponding to s; to be in the setting of (6.14)(I)(c), we assume α is type I noncompact imaginary. Let $G^\alpha\supseteq H$ be generated by the $SL(2)$ through this root α, and let H^α be a θ-stable Cartan in G^α-obtained by Cayley transforms through α. Fix

$$\mathfrak{b}^\alpha=\mathfrak{h}^\alpha+\mathfrak{n}^\alpha$$

so that $\mathfrak{n}^\alpha\cap\mathfrak{n}$ has codimension one in $\mathfrak{n}$, and define γ^α by (5.3). Choose $\mathfrak{q}^\alpha=\mathfrak{l}^\alpha+\mathfrak{u}^\alpha$ for γ^α as in (A2) and Corollary 4.10, and choose

$$\mathfrak{q}^0=\mathfrak{l}+\mathfrak{u}^0\subseteq\mathfrak{l}^\alpha$$

for γ in L^α in the same way. Then (since the Cartan involutions for $\mathfrak{h}$ and $\mathfrak{h}^\alpha$ differ only by s), the parabolic

$$\mathfrak{q}=\mathfrak{l}+(\mathfrak{u}^0+\mathfrak{u}^\alpha)=\mathfrak{l}+\mathfrak{u}$$

has the corresponding properties for γ. By Corollary A10(c) and Proposition 4.3, if we write

$$r=l(\delta)-l(\gamma)-\dim(\mathfrak{u}\cap\mathfrak{p}),$$

then

$$Q_{\gamma,\delta}=\sum u^{1/2(q+r)}\,(\text{multiplicity of } \bar{X}^L(\gamma_{\mathfrak{q}}) \text{ in } H_q(\mathfrak{u},\bar{X}(\delta));$$

here $\gamma_{\mathfrak{q}}$ is defined by in Corollary (A4). Similarly, if

$$r^\alpha=l(\delta)-l(\gamma^\alpha)-\dim(\mathfrak{u}^\alpha\cap\mathfrak{p})$$

$$(7.9)\quad Q_{\gamma^\alpha,\delta}=\sum_q u^{1/2(q+r^\alpha)}\,(\text{multiplicity of } \bar{X}^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha}) \text{ in } H_q(\mathfrak{u}^\alpha,\bar{X}(\delta)).$$

We need to relate $Q_{\gamma,\delta}$ to $\mathfrak{q}^\alpha$ as well. We use the Hochschild-Serre spectral sequence (Lemma 4.11); so we have to compute

$$\text{multiplicity of } \bar{X}^L(\gamma_{\mathfrak{q}}) \text{ in } H_p(\mathfrak{u}^0,Y)=m_p(Y),$$

for any irreducible $(\mathfrak{l}^\alpha, L^\alpha\cap K)$ module Y. We will use Theorem 5.11 for the group L^α, with $\gamma_{\mathfrak{q}^\alpha}$ playing the role of γ. Since H^α is split in L^α, $\gamma^\alpha_{\mathfrak{q}^\alpha}$ is maximal for the Bruhat $\mathscr{G}$-order for L^α. Since

$$l^\alpha(\gamma^\alpha_{\mathfrak{q}^\alpha})=l(\gamma_{\mathfrak{q}^\alpha})+1,$$

it is easy to check that $\gamma^{\alpha}_{\mathfrak{q}^\alpha}$ is the *only* regular character for L^α (up to conjugacy) which follows $\gamma_{\mathfrak{q}^\alpha}$ in this order. By Theorem 5.11 and Corollary A10(c), we conclude that

$$m_{\dim(\mathfrak{u}^0\cap\mathfrak{p})}(\bar{X}^{L^\alpha}(\gamma_{\mathfrak{q}^\alpha}))=1$$

$$m_i(\bar{X}^{L^\alpha}(\gamma_{\mathfrak{q}^\alpha}))=0, \quad i \neq \dim \mathfrak{u}^0\cap\mathfrak{p}$$

$$m_i(\bar{X}^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha}))=0, \quad i \neq \dim \mathfrak{u}^0\cap\mathfrak{p}-1$$

$$m_i(Y)=0, \quad Y \text{ irreducible}, \quad Y \neq \bar{X}^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha}), \quad \bar{X}^{L^\alpha}(\gamma_{\mathfrak{q}^\alpha}).$$

Finally, notice that $\bar{X}^{L^\alpha}(\gamma_{\mathfrak{q}^\alpha})$ occurs once as a composition factor of $X^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha})$ (by induction by stages from $SL(2,\mathbb{R})$). Now $\bar{X}_L(\gamma_{\mathfrak{q}})$ cannot occur in $H_*(\mathfrak{u}^0, X^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha}))$ (Corollary A10(b) and Theorem 4.6). The formulas above therefore force

$$m_{\dim(\mathfrak{u}^0\cap\mathfrak{p})-1}\bar{X}^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha})=1.$$

Now Lemma 4.11 gives

multiplicity of $\bar{X}^L(\gamma_{\mathfrak{q}})$ in $H_q(\mathfrak{u}, \bar{X}(\delta))$
$=$multiplicity of $\bar{X}^{L^\alpha}(\gamma_{\mathfrak{q}^\alpha})$ in $H_{q\text{-}\dim(\mathfrak{u}^0\cap\mathfrak{p})}(\mathfrak{u}^\alpha, \bar{X}(\delta))$
$+$multiplicity of $\bar{X}^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha})$ in $H_{q\text{-}\dim(\mathfrak{u}^0\cap\mathfrak{p})+1}(\mathfrak{u}^\alpha, \bar{X}(\delta))$.

In terms of the Q polynomials, this is

(7.10) $$Q_{\gamma,\delta}=\sum_q u^{1/2(q-r^\alpha+1)} (\text{multiplicity of } \bar{X}^{L^\alpha}(\gamma_{\mathfrak{q}^\alpha}) \text{ in } H_q(\mathfrak{u}^\alpha, \bar{X}(\delta)))+Q_{\gamma^\alpha,\delta}.$$

Now suppose $s\notin\tau(\delta)$, and $l(\delta)<m$. Set

$$Q^s_{\gamma,\delta}=\sum_q u^{1/2(q-d)} \dim(\operatorname{Hom}_{\mathfrak{h},T}(H_q(\mathfrak{n}, U_\alpha(\bar{X}(\delta))), (F/\mathfrak{n}F)\otimes\chi))$$

in analogy with (7.4); here α is the simple root corresponding to s, d is the codimension of the support of δ minus one, and other notation follows (7.4). Suppose γ is of the type considered above, so that we are in Case I(c) of Proposition 6.14. Then Theorem 7.2 of [14] shows how to compute $Q^s_{\gamma^\alpha,\delta}$. The formula in question is

$$\begin{aligned} m[\bar{X}^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha}), H_q(\mathfrak{u}^\alpha, U_\alpha(\bar{X}(\delta)))] &= m[\bar{X}^{L^\alpha}(\gamma^\alpha_{\mathfrak{q}^\alpha}), (H_{q+1}\oplus H_{q-1})(\mathfrak{u}^\alpha, \bar{X}(\delta))] \\ &+ m[\bar{X}^{L^\alpha}(\gamma_{\mathfrak{q}^\alpha}), H_q(\mathfrak{u}^\alpha, \bar{X}(\delta))] \\ &+ m[\bar{X}^{L^\alpha}(s\times\gamma_{\mathfrak{q}^\alpha}), H_q(\mathfrak{u}^\alpha, \bar{X}(\delta))]. \end{aligned}$$

Now (7.9) and (7.10) give

$$Q^s_{\gamma^\alpha,\delta}=(u+1)Q_{\gamma^\alpha,\delta}+(Q_{\gamma,\delta}-Q_{\gamma^\alpha,\delta})+(Q_{s\times\gamma,\delta}-Q_{\gamma^\alpha,\delta}),$$

or

$$Q^s_{\gamma^\alpha,\delta} = (u-1)\,Q_{\gamma^\alpha,\delta} + Q_{\gamma,\delta} + Q_{s\times\gamma,\delta}. \tag{7.11}$$

By $A(m)$, $U_\alpha(\bar{X}(\delta))$ is completely reducible; so

$$m[\bar{X}(\gamma^\alpha), U_\alpha(\bar{X}(\delta))] = \dim \operatorname{Hom}_{\mathfrak{g},K}(\tilde{X}(\gamma^\alpha), U_\alpha(\bar{X}(\delta)).$$

By Corollary A18, this is

$$= \text{coefficient of } u^{1/2(l(\delta)-l(\gamma^\alpha)+1)} \text{ in } Q^s_{\gamma^\alpha,\delta}.$$

By (7.11), this is

$$= \text{coefficient of } u^{1/2(l(\delta)-l(\gamma^\alpha)-1)} \text{ in } Q_{\gamma^\alpha,\delta}$$

$$m[\bar{X}(\gamma^\alpha), U_\alpha(\bar{X}(\delta))] = \mu'(\gamma^\alpha,\delta).$$

Combined with analogous results for other γ, this gives

$$U_\alpha(\bar{X}(\delta)) = \sum_{\delta' \in s \circ \delta} \bar{X}(\delta') \oplus \sum_{\substack{z \in \mathscr{D} \\ s \in \tau(z) \\ z \leqq \delta}} \mu'(z,\delta)\,\bar{X}(\gamma). \tag{7.12}$$

Now (7.11) and (7.12) imply the analogue of Proposition 6.14(I)(c), with (P,μ) replaced by (Q,μ'). Q.E.D.

Lemma 7.13. *$A(m)$ and $B(m-1)$ imply $B(m)$.*

Proof. Fix $\gamma' < \delta'$, with $l(\delta') \leqq m$; we want to show $P_{\gamma',\delta'} = Q_{\gamma',\delta'}$. By $B(m-1)$, we may assume $l(\delta') = m$; and we may assume the result is known for (γ'',δ'') whenever $l(\gamma'') > l(\gamma')$. By Lemma 7.8, we are done if $P_{\gamma',\delta'}$ can be computed using only the formulas in Case I of Proposition 6.14, with $l(\delta) < m$; or those in Case II, with $l(\delta) \leqq m$. Using Lemma 6.7, we may therefore assume that there is an $s \notin \tau(\delta') \cup \tau(\gamma')$, with the corresponding simple root α for δ' real; the corresponding root for γ' is complex or noncompact imaginary; and there is a second simple reflection $s' \in \tau(\delta')$, with the corresponding simple root type II real. By Lemma 7.8 and Proposition 6.14(I),

$$P_{\gamma',\delta'} + P_{\gamma',s'\times\delta'} = Q_{\gamma',\delta'} + Q_{\gamma',s'\times\delta'}$$

(notation 6.3). If $s \in \tau(s' \times \delta')$, then Lemma 7.8 and Proposition 6.14(II) give $P_{\gamma',s'\times\delta'} = Q_{\gamma',s'\times\delta'}$, and we are done; so we may suppose $s \notin \tau(s' \times \delta')$. To compute $P_{\gamma',\delta'}$, we would use Proposition 6.14(I)(a) or (c). To prove the analogous formulas for Q, we need to know that $U_\alpha(\bar{X}(\delta'))$ is completely reducible. This is part of $A(m+1)$, which is not yet available. However, since we know the homology of $\bar{X}(\delta') \oplus \bar{X}(s' \times \delta')$, we can argue as in the proof of (7.14) below to deduce that $U_\alpha(\bar{X}(\delta') \oplus \bar{X}(s' \times \delta'))$ is completely reducible. The desired formula for Q follows. Q.E.D.

Proof of Theorem 7.3. We will show that

$$A(m) + B(m) \Rightarrow A(m+1), \tag{7.14}$$

using Corollary 7.2. Since $A(0)$ is empty, Lemma 7.13 implies that $B(m)$ is true for all m; and this is the conclusion of the theorem. The proof of (7.14) is based on the following fact.

Lemma 7.15. *Let X be an $(R, K_{\mathbb{C}})$ module such that*

a) $\dim \operatorname{Hom}_{(\mathfrak{g},K)}(I, X) = \dim \operatorname{Hom}_{(\mathfrak{g},K)}(X, I)$

for any irreducible I. Suppose further that for any $\gamma \in \mathscr{D}$,

b) $m(\bar{X}(\gamma), X) = \dim \operatorname{Hom}_{(\mathfrak{g},K)}(X, \bar{X}(\gamma))$.

Then X is completely reducible.

We leave the (formal) proof to the reader. So suppose $\delta \in \mathscr{D}$, $l(\delta) = m$. Fix $s \notin \tau(\delta)$, and let α be a corresponding simple root. We will apply Lemma 7.15 to $U_\alpha(\bar{X}(\delta))$; hypothesis (a) follows from Proposition 3.17 of [13]. By Lemma 7.7, we can compute the homology of $U_\alpha(\bar{X}(\delta))$ by the formulas of Theorem 7.2 in [14]. Since we already know the homology of $\bar{X}(\delta)$ is given by P (this is $B(m)$), the proof of Lemma 7.8 shows that, if q_0 is the codimension of supp(γ),

$$\dim \operatorname{Hom}_{\mathfrak{h}, H \cap K}(H_{q_0}(\mathfrak{n}, U_\alpha(\bar{\pi}(\delta))), F/\mathfrak{n}F \otimes \chi) = \mu(\gamma, \delta)$$

(notation as in Theorem 7.3). By Corollary A18 and Proposition 4.3,

$$\dim \operatorname{Hom}_{\mathfrak{g}, K}(U_\alpha(\bar{\pi}(\delta)), \pi(\gamma))) = \mu(\gamma, \delta). \tag{7.16}$$

Write

$$\tilde{\mu}(\gamma, \delta) = m(\bar{\pi}(\gamma), U_\alpha(\bar{\pi}(\delta))). \tag{7.17}$$

By Lemma 7.15, we only have to show that $\mu(\gamma, \delta) = \tilde{\mu}(\gamma, \delta)$. In the Grothendieck group of $(R, K_{\mathbb{C}})$ modules,

$$\sum_{\delta' \in s \circ \delta} \bar{X}(\delta') = U_\alpha(\bar{X}(\delta)) - \sum_{\substack{z \leq \delta \\ s \in \tau(z)}} \tilde{\mu}(z, \delta) \bar{X}(z).$$

But we know also the cohomology of $U_\alpha(\bar{X}(\delta))$ and of all the $\bar{X}(\gamma)$ with $\gamma \leqq \delta$; so with notation as in the proof of Lemma 7.8,

$$\sum_{\delta' \in s \circ \delta} \bar{X}(\delta') = \sum_{\gamma \in \mathscr{D}} [P^s_{\gamma,\delta}(1) - \sum_{\substack{z \leq \delta \\ s \in \tau(z)}} \tilde{\mu}(z, \delta) P_{\gamma,z}(1)](-1)^{l(\gamma) - l(\delta) - 1} X(\gamma);$$

here $P^s_{\gamma,\delta}$ is computed from P as $Q^s_{\gamma,\delta}$ is from Q. In light of the recursion formulas (6.14) for P, and Corollary 7.2,

$$\sum_{\delta' \in s \circ \delta} \bar{X}(\delta') = \sum_{\gamma \in \mathscr{D}} [P^s_{\gamma,\delta}(1) - \sum_{\substack{z \leq \delta \\ s \in \tau(z)}} \mu(z, \delta) P_{\gamma,z}(1)](-1)^{l(\gamma) - l(\delta) - 1} X(\gamma).$$

So

$$\sum_{\substack{z \leq \delta \\ s \in \tau(z)}} \mu(z, \delta) P_{\gamma,z}(1) = \sum_{\substack{z \leq \delta \\ s \in \tau(z)}} \tilde{\mu}(z, \delta) P_{\gamma,z}(1)$$

for all γ. Now (7.16) and (7.17) imply that $\tilde{\mu} \geqq \mu$; so we deduce that

$$[\tilde{\mu}(z,\delta)-\mu(z,\delta)]\,P_{\gamma,z}(1)=0$$

for all z and γ. Taking $z=\gamma$ gives $\mu=\tilde{\mu}$ as desired. Q.E.D.

As a consequence of the proof, we obtain

Corollary 7.18. *If X is an irreducible $(\mathfrak{g},K)$ module having the same infinitesimal character as F, and $\alpha\notin\tau(X)$, then $U_\alpha(X)$ is completely reducible; that is, Conjecture* 3.15 *of* [13] *is true for integral infinitesimal character.*

Appendix: An Introduction to [15]

We give here proofs of Theorems 4.6 and 5.11. Fix once and for all $\gamma\in\mathscr{D}$. Choose a θ-stable Cartan subgroup $H=TA$ of G, a positive root system $\Delta^+\subseteq\Delta(\mathfrak{g},\mathfrak{h})$; write

(A1)(a) $$\mathfrak{b}_1=\mathfrak{h}+\mathfrak{n}_1$$

for the corresponding Borel subalgebra. Fix a character

(A1)(b) $$\chi\in(H/H_0)^\wedge\cong(T/T_0)^\wedge$$

and assume that (H,Δ^+,χ) corresponds to γ under the equivalences of Corollary 2.2. Finally, fix a Z-good Borel subalgebra

(A1)(c) $$\mathfrak{b}_2=\mathfrak{h}+\mathfrak{n}_2$$

for γ (Definition 4.4). The first point is to construct an intermediate parabolic subalgebra for induction by stages. Set

(A2) $\mathfrak{l}$ = centralizer of $\mathfrak{t}$ in $\mathfrak{g}$, L = centralizer of $\mathfrak{t}$ in G
$\mathfrak{u}=\mathfrak{n}_2\cap\theta\mathfrak{n}_2$
$\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$.

Lemma A3. *In the notation just defined, $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ is a θ-stable Levi decomposition of a θ-stable parabolic subalgebra of $\mathfrak{g}$. The group L is split, with maximally split Cartan subgroup $H=TA$. The intersection $\mathfrak{b}_2\cap\mathfrak{l}_0=\mathfrak{p}_0^L$ is the Lie algebra of a Borel subgroup $P^L=TAN^L$ of L.*

This is elementary, and we leave the argument to the reader. Next, we define the character of H which will define the representation of L we wish to consider. Recall the one-dimensional representation $\mathbb{C}_\mu$ of H defined in Theorem 4.6. Write $\rho(\mathfrak{l})$ for half the sum of the roots of $\mathfrak{h}$ in $\bar{\mathfrak{n}}^L$; this is zero on $\mathfrak{t}$, and so can be regarded as a character of H trivial on T. Set

(A4)(a) $$\gamma_\mathfrak{q}=\text{character of } H \text{ on } \mathbb{C}_\mu\otimes\mathbb{C}_{-\rho(\mathfrak{l})}\in\hat{H}.$$

Since H is split in L, a character of H may be regarded naturally as a regular character with respect to L (Definition 2.3); so we may sometimes write

(A4)(b) $$\gamma_\mathfrak{q}=(\Gamma_\mathfrak{q},\bar{\gamma}_\mathfrak{q}).$$

Here $\Gamma_\mathfrak{q}$ is just the character $\gamma_\mathfrak{q}$, and $\bar{\gamma}_\mathfrak{q}$ is its differential. Put

(A5) ρ_i = half sum of roots of $\mathfrak{h}$ in $\mathfrak{n}_i$ $(i=1,2)$
$\rho(\mathfrak{u})$ = half sum of roots of $\mathfrak{h}$ in $\mathfrak{u}$
$=\rho(\mathfrak{n}_2)-\rho(\mathfrak{l})$
λ = weight of $\mathfrak{h}$ or H on $F/\mathfrak{n}_1F$.

The differential of the weight δ_{12} of Definition 4.2 is easily seen to be $\rho_1-\rho_2$. Using this, we calculate

(A6) $$\begin{aligned} d\mu&=\lambda-\rho_1+\rho_2\\ \bar{\gamma}_\mathfrak{q}&=(\lambda-\rho_1)+(\rho_2-\rho(l))\\ &=(\lambda-\rho_1)+\rho(\mathfrak{u}).\end{aligned}$$

Lemma A 7.

a) *If* $\alpha \in \Delta^+$, $\langle \alpha, \lambda - \rho_1 \rangle < 0$

b) *If* $\alpha \in \Delta^+ \cap \Delta(\mathfrak{l})$, $\langle \alpha, \bar{\gamma}_{\mathfrak{q}} \rangle < 0$

c) *If* $\alpha \in \Delta(\mathfrak{u})$, *then either* $\langle \alpha, \lambda - \rho_1 \rangle < 0$, *or* $\langle \theta\alpha, \lambda - \rho_1 \rangle < 0$.

Proof. Part (a) follows from the Cartan-Weyl theory: since λ is the lowest weight of a finite dimensional representation $-\lambda$ is dominant. Parts (b) and (c) then follow from Definition 4.4. Q.E.D.

Define (using normalized, $L \cap K$-finite induction)

(A 8) $X^L(\gamma_{\mathfrak{q}}) = \mathrm{Ind}_{P_L}^{G_L}(\gamma_{\mathfrak{q}})$
$\bar{X}^L(\gamma_{\mathfrak{q}})$ = Langlands subrepresentation of $X^L(\gamma_{\mathfrak{q}})$.

By Lemma A 7(b), $\bar{X}^L(\gamma_{\mathfrak{q}})$ really has a unique Langlands subrepresentation.

Theorem A 9 (Schmid, Casselman - see [15], Sect. 6.4). *With notation as above, suppose* Y *is an* $\mathfrak{l}$, $L \cap K$ *module of finite length. Define* $\mathbb{C}_\mu$ *by Theorem* 4.6

a) $\mathrm{Hom}_{\mathfrak{h},T}(H_p(\mathfrak{n}^L, Y), \mathbb{C}_\mu) = 0$, $p > 0$.

b) $\mathrm{Hom}_{\mathfrak{h},T}(H_0(\mathfrak{n}^L, Y), \mathbb{C}_\mu) \cong \mathrm{Hom}_{\mathfrak{l}, L\cap K}(Y, X^L(\gamma_{\mathfrak{q}}))$.

c) *The dimension of the* μ*-isotypic subspace of* $H_0(\mathfrak{n}^L, Y)$ *is the multiplicity of* $\bar{X}^L(\gamma_{\mathfrak{q}})$ *in* Y.

d) *If* $\mathrm{Ext}^*_{\mathfrak{l}, L\cap K}(Y, X^L(\gamma_{\mathfrak{q}})) \neq 0$, *then* $\bar{X}^L(\gamma_{\mathfrak{q}})$ *occurs in* Y.

Part (b) is an observation of Jacquet, and is easy to prove; it is included mainly to motivate (c).

Corollary A 10. *In the setting of Theorems* 4.6 *and* A9, *let* X *be any* $(\mathfrak{g}, K)$ *module of finite length. Then*

a) $H_p(\mathfrak{n}_2, X) \cong H_0(\mathfrak{n}^L, H_p(\mathfrak{u}, X))$

b) $\mathrm{Hom}_{\mathfrak{h},T}(H_p(\mathfrak{n}_2, X), \mathbb{C}_\mu) \cong \mathrm{Hom}_{\mathfrak{l}, L\cap K}(H_p(\mathfrak{u}, X), X^L(\gamma_{\mathfrak{q}}))$.

c) *If* X *is actually an* $(R, K_{\mathbb{C}})$ *module, then*

$$\dim \mathrm{Hom}_{\mathfrak{h},T}(H_p(\mathfrak{n}_2, X), \mathbb{C}_\mu) = \text{multiplicity of } \bar{X}^L(\gamma_{\mathfrak{q}}) \text{ in } H_p(\mathfrak{u}, X).$$

Proof. Part (a) follows from Theorem A9(a), the Hochschild-Serre spectral sequence (Lemma 4.11) and [15], Lemma 6.3.33 (which guarantees that $H_p(\mathfrak{u}, X)$ has finite length as an $(\mathfrak{l}, L\cap K)$ module). Part (b) follows from (a) and Theorem A9(b). Part (c) follows from (a), Lemma 4.10 (semisimplicity of homology for R modules), and Theorem A9(c). Q.E.D.

Next, we need the results of [15] about constructing the standard representations by cohomological parabolic induction. Proposition 2.7 associates to γ (and our fixed $\mathfrak{b}_1 \supseteq \mathfrak{h}$) a regular character in $\hat{H}'$. In the long run, confusion seems to be minimized by a rather serious abuse of notation at this point. We will write γ for this regular character, as well as the element of $\mathscr{D}$:

(A 11)(a) $$\gamma = (\Gamma, \bar{\gamma}) \in \hat{H}'.$$

For technical reasons, we also need to consider

(A 11)(b) $$\gamma^c = (\Gamma^{-1}, -\bar{\gamma}) \in \hat{H}'$$

(A 11)(c) $$\gamma^c_{\mathfrak{q}} = (\Gamma_{\mathfrak{q}}^{-1}, -\bar{\gamma}_{\mathfrak{q}})$$

(cf. (A 4)), the latter being a regular character for H with respect to L.

Lemma A 12. *The definitions* (A 4) *and* (A 11) *are related by*

$$\Gamma|_T = \Gamma_{\mathfrak{q}}|_T \otimes \{\Lambda^{\dim \mathfrak{u}\cap\mathfrak{p}}(\mathfrak{u}\cap\mathfrak{p})^*\}$$

$$\bar{\gamma} = \bar{\gamma}_{\mathfrak{q}} - \rho(\mathfrak{u}) = \lambda - \rho_1.$$

The second assertion is immediate from Proposition 2.7 and (A 6). The first is elementary (compare [15], proof of Lemma 9.4.9). We leave the details to the reader.

Theorem A 13 ([15], Theorem 8.2.4). *The standard representation $X(\gamma^c)$ for G is obtained from $X^L(\gamma^c_{\mathfrak{q}})$ by cohomological parabolic induction:*

$$X(\gamma^c) = \mathscr{R}^S_{\mathfrak{q}}(X^L(\gamma^c_{\mathfrak{q}})),$$

and all other $\mathscr{R}^i$ are zero on $X^L(\gamma^c_{\mathfrak{q}})$.

Proof. Lemma A 12 guarantees that $\gamma^c_{\mathfrak{q}}$ is (as the notation suggests) the character for L constructed in Lemma 8.1.2 of [15]; and Lemma A 7(c) is exactly the positivity hypothesis needed in Theorem 8.2.4 of [15]. Q.E.D.

Theorem A 14 (Zuckerman's spectral sequence – [15], Proposition 6.3.2). *Suppose X is any $(\mathfrak{g}, K)$ module. Then there is a first quadrant spectral sequence*

$$\begin{aligned}&\mathrm{Ext}^p_{\mathfrak{l}, L\cap K}(H^{\dim(\mathfrak{u}\cap\mathfrak{p})-q}(\mathfrak{u}, X), X^L(\gamma^c_{\mathfrak{q}}))\\ &\quad\Rightarrow\ \mathrm{Ext}^{p+q}_{\mathfrak{g},K}(X, X(\gamma^c)).\end{aligned}$$

Proposition A 15 ([15], Proposition 8.5.6). *Write Y^c for the K-finite dual of the $(\mathfrak{g}, K)$ module Y. Then the standard module $\tilde{X}(\gamma)$ containing $\bar{X}(\gamma)$ as a quotient (notation (1.3)) is*

$$\tilde{X}(\gamma) \cong X(\gamma^c)^c.$$

We need two more formulas about X^c.

Lemma A 16. *Suppose X and Y are $(\mathfrak{g}, K)$ modules.*

a) $\mathrm{Ext}^i_{\mathfrak{g},K}(X, Y^c) \cong \mathrm{Ext}^i_{\mathfrak{g},K}(Y, X^c)$

b) $H_i(\mathfrak{u}, X^c) \cong H^i(\mathfrak{u}, X)^c$.

Proof. Part (a) is [15], Corollary 9.2.5. Part (b) may be proved by a similar argument, or simply by inspecting the standard complexes (as $H_i(\mathfrak{u}, X^*) \cong H^i(\mathfrak{u}, X)^*$ is proved). We leave this to the reader. Q.E.D.

If we use Proposition A 15 and Lemma A 16 to rewrite Theorem A 14, we obtain

Corollary A 17. *Suppose X is any $(\mathfrak{g}, K)$ module of finite length. There is a first quadrant spectral sequence*

$$\mathrm{Ext}^p_{\mathfrak{l}, L\cap K}(\tilde{X}^L(\gamma_{\mathfrak{q}}), H_{\dim(\mathfrak{u}\cap\mathfrak{p})-q}(\mathfrak{u}, X)) \ \Rightarrow\ \mathrm{Ext}^{p+q}_{\mathfrak{g},K}(\tilde{X}(\gamma), X).$$

Corollary A 18. *Suppose X is an $(R, K_{\mathbb{C}})$ module of finite length. Then $\mathrm{Ext}^*_{\mathfrak{g},K}(\tilde{X}(\gamma), X) \neq 0$ if and only if*

$$\mathrm{Hom}_{\mathfrak{h},T}(H_*(\mathfrak{n}_2, X), \mathbb{C}_\mu) \neq 0.$$

Suppose both are non-zero; let n be the lowest degree in which the first does not vanish, and m in the highest degree in which the second does not. Then

$$n + m = \dim \mathfrak{u} \cap \mathfrak{p},$$

and the two spaces have the same dimensions in this degree.

Proof. Suppose $\mathrm{Ext}^*_{\mathfrak{g},K}(\tilde{X}(\gamma), X) \neq 0$. By Corollary A 17, we find p and q with

$$\mathrm{Ext}^p_{\mathfrak{l}, L\cap K}(\tilde{X}^L(\gamma_{\mathfrak{q}}), H_{\dim(\mathfrak{u}\cap\mathfrak{p})-q}(\mathfrak{u}, X)) \neq 0.$$

By Lemma A 16 and Proposition A 15,

$$\mathrm{Ext}^p_{\mathfrak{l}, L\cap K}(H_{\dim(\mathfrak{u}\cap\mathfrak{p})-q}(\mathfrak{u}, X)^c, X^L(\gamma^c_{\mathfrak{q}})) \neq 0.$$

By Theorem A 9(d), $\bar{X}^L(\gamma^c_{\mathfrak{q}})$ occurs in $H_{\dim(\mathfrak{u}\cap\mathfrak{p})-q}(\mathfrak{u}, X)^c$. By Proposition A 15, $\bar{X}^L(\gamma_{\mathfrak{q}})$ occurs in $H_{\dim(\mathfrak{u}\cap\mathfrak{p})-q}(\mathfrak{u}, X)$. By Corollary A 10(c),

$$\mathrm{Hom}_{\mathfrak{h},T}(H_{\dim \mathfrak{u}\cap\mathfrak{p}-q}(\mathfrak{n}_2, X), \mathbb{C}_\mu) \neq 0.$$

In particular,

$$\mathrm{Hom}_{\mathfrak{h},T}(H_*(\mathfrak{n}_2, X), \mathbb{C}_\mu) \neq 0,$$

proving half of the first assertion. Conversely, suppose this Hom is non-zero in degree m, and m is as large as possible with this property. Set

$$d = \dim \operatorname{Hom}_{\mathfrak{h},T}(H_m(\mathfrak{n}_2, X), \mathbb{C}_\mu).$$

By Corollary A10(c), $\bar{X}^L(\gamma_\mathfrak{q})$ occurs in $H_m(\mathfrak{u}, X)$ with multiplicity d, and not at all in higher degrees. By Proposition A15 and Theorem A9, it follows that

$$\dim \operatorname{Hom}_{\mathfrak{l}, L\cap K}(H_m(\mathfrak{u}, X)^c, X^L(\gamma_\mathfrak{q}^c)) = d$$

$$\operatorname{Ext}^p_{\mathfrak{l}, L\cap K}(H_r(\mathfrak{u}, X)^c, X^L(\gamma_\mathfrak{q}^c)) = 0, \qquad r > m.$$

Using Lemma A16 to rewrite this, we get

$$\dim \operatorname{Hom}_{\mathfrak{l}, L\cap K}(\tilde{X}^L(\gamma_\mathfrak{q}), H_m(\mathfrak{u}, X)) = d$$

$$\operatorname{Ext}^p_{\mathfrak{l}, L\cap K}(\tilde{X}^L(\gamma_\mathfrak{q}), H_r(\mathfrak{u}, X)) = 0, \qquad r > m.$$

By Corollary A17,

(**)
$$\dim \operatorname{Ext}^{\dim(\mathfrak{u}\cap\mathfrak{p}) - m}(\tilde{X}(\gamma), X) = d$$

$$\operatorname{Ext}^s(\tilde{X}(\gamma), X) = 0, \qquad s < \dim \mathfrak{u}\cap\mathfrak{p} - m.$$

So $\operatorname{Ext}^*(\tilde{X}(\gamma), X) \neq 0$, completing the proof of the first claim; and (**) is the rest. Q.E.D.

Theorem A19 ([15], Lemma 9.2.18). *Suppose $X(\psi)$ is a standard $(\mathfrak{g}, K)$ module distinct from $X(\gamma)$. Then*

$$\operatorname{Ext}^*_{\mathfrak{g},K}(\tilde{X}(\gamma), X(\psi)) = 0.$$

Theorem A20 ([15], Corollary 8.1.21). *Suppose $Y = X(\gamma)$, $\bar{X}(\gamma)$, or $\tilde{X}(\gamma)$. Then $\bar{X}^L(\gamma_\mathfrak{q})$ occurs in $H_*(\mathfrak{u}, Y)$ exactly once, in degree* $\dim \mathfrak{u}\cap\mathfrak{p}$.

Actually, the result cited concerns the occurrence of $\bar{X}^L(\gamma_\mathfrak{q}^c)$ in $H^*(\mathfrak{u}, Y^c)$; but by Lemma A16, the results are equivalent. Theorem 4.6 is a consequence of Corollary A18, Theorem A19, Theorem A20, and Corollary A10(b).

Proof of Theorem 5.11. Under the hypothesis of the theorem, Corollary A18 shows that (since $\mathfrak{u}\cap\mathfrak{p} = \mathfrak{n}_2\cap\mathfrak{p}$)

$$\operatorname{Ext}^{\dim(\mathfrak{u}\cap\mathfrak{p}) - i_0}_{\mathfrak{g},K}(\tilde{X}(\gamma), X) \neq 0$$

for some $i_0 \geqq i$. In particular, it follows that $i \leqq \dim \mathfrak{u}\cap\mathfrak{p}$; and the statements about the case $i = \dim \mathfrak{u}\cap\mathfrak{p}$ are clear. For the rest, we proceed by induction on the maximum $n(X)$ of the numbers $l(\delta)$, for those $\delta \in \mathscr{D}$ such that $\bar{X}(\delta)$ occurs in X. Using the long exact sequence in homology, we reduce to the case $X = \bar{X}(\delta)$. If $\delta = \gamma$, Theorem A20 and Corollary A10(b) show that $i = \dim \mathfrak{u}\cap\mathfrak{p}$ (as desired). So suppose $\delta \neq \gamma$. Consider the long exact sequence

$$0 \to \bar{X}(\delta) \to X(\delta) \to Y \to 0.$$

By [15], Proposition 8.6.19, $n(Y) \leqq l(\delta) - 1$. By Theorem 4.6,

$$\operatorname{Hom}_{\mathfrak{h},T}(H_*(\mathfrak{n}_2, X(\delta)), \mathbb{C}_\mu) = 0.$$

By the long exact sequence in homology,

$$\operatorname{Hom}_{\mathfrak{h},T}(H_{i+1}(\mathfrak{n}_2, Y), \mathbb{C}_\mu) \neq 0.$$

By inductive hypothesis,

$$\dim(\mathfrak{u}\cap\mathfrak{p}) - (n(Y) - l(\gamma)) \leqq i + 1,$$

or

$$\dim(\mathfrak{u}\cap\mathfrak{p}) - (l(\delta) - l(\gamma)) \leqq i + (n(Y) - l(\delta)) + 1.$$

The inequality on $n(Y)$ mentioned above gives

$$\dim(\mathfrak{u}\cap\mathfrak{p}) - (l(\delta) - l(\gamma)) \leqq i$$

as desired. The inductive hypothesis also provides a constituent $\bar{X}(\psi)$ of Y such that $\gamma \leqq \psi$. Since Y is a quotient of $\bar{X}(\delta)$, Theorem 5.10 says that $\psi \leqq \delta$. Since $\leqq$ is defined to be transitive, $\gamma \leqq \delta$. Q.E.D.

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