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# Integrable representations of affine Lie-algebras

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## Introduction

A module $V$ for a Kac-Moody Lie-algebra $G$ is called integrable if (i) $V = \bigoplus_{\lambda \in H^*} V_\lambda$, (ii) the Chevalley generators $e_i, f_i$ act locally nilpotently on $G$. Let $\mathscr{I}_{\text{fin}}$ denote the category of integrable $G$-modules $V$ such that $\dim V_\lambda$ is finite for all $\lambda \in H^*$. In this article we classify the irreducible objects of the category $\mathscr{I}_{\text{fin}}$ for the non-twisted affine Lie-algebras.

Let $C$ denote the one-dimensional center of an affine Lie-algebra $\hat{L}(\mathfrak{g})$ and let $c \in C$ be the canonical central element [3]. If $V$ is an irreducible object of $\mathscr{I}_{\text{fin}}$ there exists an integer $n \equiv n(V)$ such that $cv = nv$ for all $v \in V$. If $n > 0$ (resp. $n < 0$) we prove that $V$ is an irreducible highest weight (resp. lowest weight) module in the category $\mathcal{O}$ (resp. $\mathcal{O}^-$) [3].

Let $(\alpha_0, \ldots, \alpha_n)$ be the simple roots of $\hat{L}(\mathfrak{g})$ and assume that $(\alpha_1, \ldots, \alpha_n)$ form a simple system for the underlying finite-dimensional simple Lie-algebra $\mathfrak{g}$. Let $\dot{\Gamma}_+$ denote the non-negative integral linear span of $\{\alpha_i : i = 1, \ldots, n\}$. Define a category $\tilde{\mathcal{O}}$ of $\hat{L}(\mathfrak{g})$ modules by $V \in \tilde{\mathcal{O}}$ if and only if (i) $cV = 0$, (ii) $V = \bigoplus_{\lambda \in H^*} V_\lambda$, (iii) the set $P(V) = \{\lambda \in H^* : V_\lambda \neq 0\}$ is contained in a finite union of cones $\tilde{D}(\lambda) = \{\lambda - \eta + n\delta : \eta \in \dot{\Gamma}_+,\ n \in \mathbb{Z}\}$. If $V \in \mathscr{I}_{\text{fin}}$ is irreducible and $cV = 0$ then we prove that $V \in \tilde{\mathcal{O}}$.

In section three we construct some examples of modules in $\tilde{\mathcal{O}}$. Let $T_0$ denote the homogeneous Heisenberg subalgebra of $\hat{L}(\mathfrak{g})$ and let $\mathfrak{S}$ denote the (graded) quotient of $U(T_0)$ by the ideal generated by the center of $T_0$. For every $\lambda \in H^*$ and every ideal $I$ of $\mathfrak{S}$ we construct modules $M(\lambda, I) \in \tilde{\mathcal{O}}$. The construction is analogous to the one for Verma modules. We prove that the irreducible objects of $\tilde{\mathcal{O}}$ are in bijective correspondence with the set $\{(\lambda, I) : \lambda \in H^*, I$ a maximal graded ideal in $\mathfrak{S}\}$ and determine the isomorphism classes of the irreducible modules.

In section four we classify the isomorphism classes of irreducible integrable modules in $\tilde{\mathcal{O}}$. Any such module has finite-dimensional weight spaces. For the affine Lie-algebra $A_1^{(1)}$ we see that for every $n > 0$ and every $a \in (\mathbb{C}^*)^n$ there exists

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a module $V(n,a)\in\tilde{\mathcal{O}}$ such that $V(n,a)$ is irreducible and integrable. Further if $V(n,a)$ and $V(m,b)$ are isomorphic then $n=m$ and $a=a'\sigma(b)$ for some element $\sigma$ of the permutation group $S_n$ and some $a'\in\mathbb{C}^*$.

## 1. Preliminaries

We recall the explicit realization of the non-twisted affine Lie-algebras (see [3], Chap. 7 for details).

Let $\mathfrak{g}$ denote a finite dimensional simple Lie-algebra, $\mathfrak{h}$ a Cartan subalgebra, $\dot{\Delta}$ the set of roots of $\mathfrak{g}$, $\dot{\pi}=\{\alpha_1,\dots,\alpha_n\}$ a simple system for $\dot{\Delta}$ and $\dot{\Delta}_+$ the corresponding set of positive roots. Let $\theta$ be the highest root of $\dot{\Delta}_+$.

Let $L=\mathbb{C}[t,t^{-1}]$ be the algebra of Laurent polynomials in the indeterminate $t$. The loop algebra

$$L(\mathfrak{g})=L\bigotimes_{\mathbb{C}}\mathfrak{g}$$

is an infinite-dimensional complex Lie-algebra with the bracket $[\ ]_0$ given by, $(P,Q\in L, x,y\in\mathfrak{g})$

$$[P\otimes x, Q\otimes y]_0=PQ\otimes[x,y].$$

Let $d$: $L(\mathfrak{g})\to L(\mathfrak{g})$ be the derivation of $L(\mathfrak{g})$ obtained by extending linearly the assignment

$$d(t^n\otimes x)=nt^n\otimes x.$$

The affine Kac-Moody Lie-algebra $\hat{L}(\mathfrak{g})$ associated to $\mathfrak{g}$ is obtained by adjoining to $L(\mathfrak{g})$ the derivation $d$ and a central element $c$. Explicitly,

$$\hat{L}(\mathfrak{g})=L(\mathfrak{g})\oplus\mathbb{C}c\oplus\mathbb{C}d$$

with the bracket given by $(x,y\in\mathfrak{g},\lambda,\mu,\lambda_1,\mu_1\in\mathbb{C})$

$$\begin{aligned}&[t^m\otimes x+\lambda c+\mu d, t^n\otimes y+\lambda_1 c+\mu_1 d]\\&\quad=t^{m+n}\otimes[x,y]+n\mu t^n\otimes y-m\mu_1 t^m\otimes x+m\delta_{m,-n}B(x,y)c\end{aligned}$$

where $B$: $\mathfrak{g}\times\mathfrak{g}\mapsto\mathbb{C}$ is a non-degenerate invariant form on $\mathfrak{g}$.

From now on we assume that $\mathfrak{g}$ is a fixed simple Lie-algebra and denote the algebra $\hat{L}(\mathfrak{g})$ by $G$. Let $H$ be the subalgebra

$$H=\mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}d$$

of $G$. Extend an element $\lambda\in\mathfrak{h}^*$ to an element of $H^*$ by setting $\lambda(c)=0=\lambda(d)$ so that $\mathfrak{h}^*$ is identified with a subspace of $H^*$. Define $\delta\in H^*$ by setting $\delta|_{\mathfrak{h}\oplus\mathbb{C}c}=0$, $\delta(d)=1$.

Let $\mathfrak{g}=\mathfrak{h}\bigoplus_{\alpha\in\dot{\Delta}}\mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$. For $\alpha\in\dot{\Delta}$, $n\in\mathbb{Z}$, set

$$G_{\alpha+n\delta}=t^n\otimes\mathfrak{g}_\alpha,$$

$$G_{n\delta}=t^n\otimes\mathfrak{h},\qquad n\neq 0.$$

Clearly $G_{\alpha+n\delta}$ and $G_{n\delta}$ are $H$-stable subspaces of $G$. Set $\Delta=\{\alpha+n\delta$: $\alpha\in\dot{\Delta}$, $n\in\mathbb{Z}\}\cup\{n\delta$: $n\in\mathbb{Z}-(0)\}$. One has the root space decomposition

$$G=H\oplus(\bigoplus_{\gamma\in\Delta}G_\gamma).$$

Let $\alpha_0$ denote the element $\delta-\theta$ of $\Delta$. The subset $\pi=\{\alpha_0,\ldots,\alpha_n\}$ forms a simple system for $\Delta$ and the corresponding positive system $\Delta_+$ is given by

$$\Delta_+=\{\alpha+n\delta:\alpha\in\dot{\Delta},n>0\}\cup\{n\delta:n>0\}\cup\dot{\Delta}_+.$$

Set $N_+=\bigoplus_{\alpha\in\Delta_+}G_\alpha$, $N_-=\bigoplus_{\alpha\in\Delta_+}G_{-\alpha}$. Clearly $N_+$ and $N_-$ are subalgebras of $G$ and one has

$$G=N_-\oplus H\oplus N_+.$$

The Lie-algebra $G$ admits a non-degenerate invariant bilinear form such that the restriction of the form to $H\times H$ is non-degenerate. Let $(\ ,\ )$ denote the form induced on $H^*$, $(\alpha,\alpha)\neq 0$ for all $\alpha\in\pi$. The Weyl group $W$ of $G$ is defined to be the subgroup of $\operatorname{Aut}H^*$ generated by the reflections $\{s_i: 0\leqq i\leqq n\}$, $s_i(\lambda)=\lambda-\frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i\forall\lambda\in H^*$. The Weyl group leaves $\Delta$ invariant. The subset $W\pi$ of $\Delta$ is called the set of real roots. A root $\alpha\in\Delta$ is imaginary if $\alpha\notin W\pi$. In fact,

$$W\pi=\{\alpha+n\delta:\ \alpha\in\dot{\Delta},\ n\in\mathbb{Z}\}.$$

Fix a Chevalley basis $\{e_\alpha:\alpha\in\dot{\Delta}\}\cup\{\check{\alpha}_i: i=1,\ldots,n\}$ for $\mathfrak{g}$. For $\alpha\in\dot{\Delta}$, $k\in\mathbb{Z}$ define elements $e_{\alpha,k}$, $e_k^{(i)}$ as follows:

$$e_{\alpha,k}=t^k\otimes e_\alpha,$$
$$e_k^{(i)}=t^k\otimes\check{\alpha}_i,\qquad i=1,\ldots,n,\ k\in\mathbb{Z}-(0).$$

For convenience, we set $e_{\alpha_i,0}=e_i$, $e_{-\alpha_i,0}=f_i$, $1\leqq i\leqq n$, $e_0=e_{-\theta,1}$, $f_0=e_{\theta,-1}$. The elements $e_i$, $f_i$, $i=0,\ldots,n$, are called the Chevalley generators of $G$. The subalgebra $H$ is spanned by the elements $\{\check{\alpha}_i: i=1,\ldots,n\}$ together with the central element $c$ and the derivation $d$. Set $\check{\alpha}_0=-\check{\theta}+\frac{2}{(\theta,\theta)}c$. For any $\gamma\in W\pi$, $\gamma=\alpha+n\delta$ the element $\check{\gamma}\in H$ is defined by $\check{\gamma}=[e_{\alpha,n},e_{-\alpha,-n}]=\check{\alpha}+\frac{2n}{(\alpha,\alpha)}c$.

The homogeneous Heisenberg subalgebra $T_0$ of $G$ is defined by

$$T_0=\mathbb{C}c\bigoplus_{k\in\mathbb{Z}-(0)}G_{k\delta}.$$

The elements $\{e_k^{(i)}: i=1,\ldots,n\}$ form a base for the space $G_{k\delta}$. Set $T=T_0+H=(L\otimes\mathfrak{h})\oplus\mathbb{C}c\oplus\mathbb{C}d$. If $\mathfrak{n}_\pm$ denote the subalgebras $\bigoplus_{\alpha\in\dot{\Delta}_+}\mathfrak{g}_{\pm\alpha}$, then one has the decomposition of $G$

$$G=L\otimes\mathfrak{n}_-\oplus T\oplus L\otimes\mathfrak{n}_+,$$

as $T$-stable subalgebras. For a subalgebra $B$ of $G$ let $U(B)$ denote the universal enveloping algebra of $B$. By the Poincaré-Birkhoff-Witt theorem one has (set $N_\pm^0=L\otimes\mathfrak{n}_\pm$)

$$U(G)=U(T)\oplus(N_+^0\,U(G)+U(G)N_-^0)$$

as $T$-stable subalgebras. Let $\beta'$: $U(G)\to U(T)$ denote the canonical projection onto $U(T)$. For $i\in(1,\ldots,n)$ let $L_i$: $U(G)\to U(G)$ denote left multiplication by $e_1^{(i)}$ and let $D_+$: $U(T_0)\to U(T_0)$ be the derivation extending $D_+(e_k^{(i)})=ke_{k+1}^{(i)}$. Set

$$Q_j^{(i)}=\frac{(D_++L_i)^j}{j!}\cdot 1,\qquad i\in(1,\ldots,n)\ j\geqq 0,\ \text{eg. } Q_0^{(i)}=1,\ Q_1^{(i)}=e_1^{(i)}.$$

In ([1], Lemma 7.5) H. Garland obtains the expression for the element $e^r_{-\alpha_i,1}\cdot e^s_i (r,s>0)$ in terms of the above decomposition. It is not hard to deduce from his formula that $(r>0, k\in\mathbb{Z})$

$$\beta'\left(\frac{e^r_{-\alpha_i,1}}{r!}\cdot\frac{e^r_i}{r!}\right)=(-1)^r\cdot Q^{(i)}_r,$$

$$\beta'\left(e_{-\alpha_i,k}\cdot\frac{e^r_{-\alpha_i,1}}{(r)!}\cdot\frac{e^{r+1}_i}{(r+1)!}\right)=(-1)^{r+1}\sum_{j=0}^{r} e^{(i)}_{j+k}\cdot Q^{(i)}_{r-j} \bmod U(T_0)c$$

(where $e^{(i)}_0=\check{\alpha}_i$). Let $\eta$: $G\to G$ be the automorphism of order two extending $\eta(e_{\alpha,n})=e_{-\alpha,n}$, $\eta(e^{(i)}_k)=-e^{(i)}_k$ $(\alpha\in\Delta, k, n\in\mathbb{Z})$. Clearly $\eta(N^0_+)=N^0_-$, $\eta(T)=T$. It is easy to check that the restriction of $\eta$ to $T$ commutes with $D_+$ and that $\eta\cdot L_i=-L_i\cdot\eta$. For $i\in(1,\dots,n)$ and $j>0$ set $P^{(i)}_j=\frac{(D_+-L_i)^j}{j!}\cdot 1=\eta(Q^{(i)}_j)$. Let $\bar\beta$: $U(G)\to U(T)$ be the projection onto $U(T)$ corresponding to the decomposition

$$U(G)=U(T)\oplus(N^0_- U(G)+U(G)N^0_+).$$

Then $\eta\cdot\beta'=\bar\beta\cdot\eta$ and we have:

(1.1) **Proposition.** *Let $i\in(1,\dots,n)$, $r,k\in\mathbb{Z}$, $r>0$. Then*

(i) $\bar\beta\left(e_{\alpha_i,k}\cdot\frac{e^r_{\alpha_i,1}}{r!}\cdot\frac{f^{r+1}_i}{(r+1)!}\right)=(-1)^r\sum_{j=0}^{r} e^{(i)}_{j+k}\cdot P^{(i)}_{r-j} \bmod U(T_0)c,$

(ii) $\bar\beta\left(\frac{e^{r+1}_{\alpha_i,1}}{(r+1)!}\cdot\frac{f^{r+1}_i}{(r+1)!}\right)=(-1)^{r+1}P^{(i)}_{r+1}.$

*Let $\Gamma_+$* (resp. $\dot\Gamma_+$) *denote the non-negative integral linear span of* $(\alpha_0,\dots,\alpha_n)$ (resp. $(\alpha_1,\dots,\alpha_n)$).

The category $\mathcal{O}$ of $G$-modules is defined as follows: a module $M\in\mathcal{O}$ if and only if:

(a) $M=\bigoplus_{\lambda\in H^*} M_\lambda$, where $M_\lambda=\{m\in M\colon hm=\lambda(h)m\,\forall h\in H\}$ and $\dim M_\lambda<\infty$,

(b) the set $P(M)=\{\lambda\in H^*\colon M_\lambda\neq 0\}$ is contained in a finite union of cones $D(\lambda)=\{\lambda-\eta\colon \eta\in\Gamma_+\}$.

For $\lambda\in H^*$ let $I_\lambda$ denote the left ideal in $U(G)$ generated by $N_+\cup\{h-\lambda(h)\colon h\in H\}$. The Verma-module $M(\lambda)$ is defined to be the quotient $U(G)/I_\lambda$. $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$ ([3], Chapt. 9).

(1.2) **Lemma.** *The set $\{L(\lambda)\colon \lambda\in H^*\}$ exhaust all the irreducible modules in $\mathcal{O}$. Further a module $L(\lambda)$ is integrable* (i.e. *the elements* $\{e_i, f_i\colon i=0,\dots,n\}$ *act locally nilpotently on* $L(\lambda)$) *if and only if* $(\lambda,\check{\alpha}_i)\in\mathbb{Z}_+$ *for all* $i=0,\dots,n$.

## 2. Integrable modules

(2.1) *Definition.* A module $V$ for the affine Lie-algebra $G$ is called integrable if:

(i) $V=\bigoplus_{\lambda\in H^*} V_\lambda$, where $V_\lambda=\{v\in V\colon hv=\lambda(h)v\,\forall h\in H\}$,

(ii) the elements $\{e_{\alpha,n}: \alpha\in\dot{\Delta}, n\in\mathbb{Z}\}$ act locally nilpotently on $V$, i.e. for every $v\in V$ there exists an integer $k=k(\alpha,n,v)$ such that $e^k_{\alpha,n}\cdot v=0$.

Let $\mathscr{I}$ denote the category of integrable $G$-modules and let $\mathscr{I}_{\text{fin}}$ be the subcategory of integrable modules with finite-dimensional weight spaces. For $V\in\mathscr{I}$ set

$$P(V)=\{\lambda\in H^*: V_\lambda\neq 0\}.$$

(2.2) **Lemma** ([3] Proposition 3.6). *Let $V\in\mathscr{I}$, $\lambda\in P(V)$. Then*

(a) $(\lambda,\check{\alpha}_i)\in\mathbb{Z}$ *for all* $i\in\{0,\dots,n\}$,

(b) $w\lambda\in P(V)$ *and* $\dim V_\lambda=\dim V_{w\lambda}$ *for all* $w\in W$,

(c) $\lambda+\alpha_i\notin P(V)$ (resp. $\lambda-\alpha_i\notin P(V)$) *implies* $(\lambda,\check{\alpha}_i)\geqq 0$ (resp. $(\lambda,\check{\alpha}_i)\leqq 0$).

(2.3) *Remark.* From the lemma it is clear that $V\in\mathscr{I}$ if and only if the elements $(e_i,f_i: i=0,\dots,n)$ act locally nilpotently on $V$. Further the statements (a) and (c) hold for all roots $\alpha\in W\pi$. The $\mathfrak{g}$-submodule generated by a vector $v\in V$ is finite-dimensional and hence $V$ breaks up as the direct sum of irreducible finite-dimensional $\mathfrak{g}$-modules.

For $\eta\in\dot{\Gamma}_+$, set

$$U(\mathfrak{n}_+)_\eta=\{x\in U(\mathfrak{n}_+): [h,x]=\eta(h)x\,\forall h\in H\}.$$

Define an ordering $\leqq$ on $H^*$ by: $\lambda\leqq\lambda'$ if and only if $\lambda'-\lambda|_d\in\dot{\Gamma}_+$. If $V\in\mathscr{I}$ and $0\neq v\in V_\lambda$, there exists $\eta\in\dot{\Gamma}_+$ such that $U(\mathfrak{n}_+)_\eta\cdot v\neq 0$ and $U(\mathfrak{n}_+)_{\eta'}\cdot v=0$ for all $\eta'\in\dot{\Gamma}_+$ such that $\eta'>\eta$. Further $(\lambda+\eta,\check{\alpha}_i)\in\mathbb{Z}_+$ for all $i=1,\dots,n$.

(2.4) **Theorem.** *Let $V\in\mathscr{I}_{\text{fin}}$ be irreducible and let $k$ be the integer such that $cv=kv$ for all $v\in V$. Then*

(i) *if* $k>0$ (resp. $k<0$) *there exists an element* $0\neq v\in V$ (resp. $0\neq w\in V$) *such that* $N_+v=0$ (resp. $N_-w=0$),

(ii) *if* $k=0$ *there exist nonzero elements* $v_0, w_0\in V$ *such that* $N^0_+v_0=0$, $N^0_-w_0=0$.

(2.5) *Remark.* Observe that if $k>0$ then $V$ is an object of the intersection $\mathscr{I}\cap\mathscr{O}$ and hence by Lemma (1.2) $V$ is isomorphic to $L(\lambda)$ for some $\lambda\in H^*$ with $(\lambda,\check{\alpha}_i)\in\mathbb{Z}_+$, for all $i=0,\dots,n$. If $k<0$ then $V$ is isomorphic to an integrable irreducible lowest weight module.

We need the following Lemma:

(2.6) **Lemma.** *Let $V\in\mathscr{I}_{\text{fin}}$. The subsets $P_+(V)$, $P_-(V)$ of $P(V)$ defined by*

$$P_+(V)=\{\lambda\in P(V): V_{\lambda+\eta}=0\ \forall\eta\in\dot{\Gamma}_+-(0)\}=\{\lambda\in P(V): \mathfrak{n}_+V_\lambda=0),$$
$$P_-(V)=\{\lambda\in P(V): V_{\lambda-\eta}=0\ \forall\eta\in\dot{\Gamma}_+-(0)\}=\{\lambda\in P(V);\ \mathfrak{n}_-V_\lambda=0\}$$

*are non-empty.*

We recall the following fact about finite-dimensional irreducible modules for $\mathfrak{g}$ ([2], Chap. 6, Proposition 21.3).

(2.7) **Lemma.** *Let $F=\bigoplus_{\lambda\in\mathfrak{h}^*}F_\lambda$ be an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight $\mu$. Let $\nu\in\mathfrak{h}^*$ be such that $(\nu,\check{\alpha}_i)\in\mathbb{Z}_+$ for all $i=1,\dots,n$ and $\mu-\nu\in\dot{\Gamma}_+$. Then $F_\nu\neq\{0\}$.*

*Proof of Lemma* (2.6). We prove the Lemma for $P_+(V)$, the proof for $P_-(V)$ is similar. By Remark (2.3) we can choose $\lambda \in P(V)$ such that $(\lambda, \check{\alpha}_i) \in \mathbb{Z}_+$ for all $i = 1, \dots, n$. Since $\dim V_\lambda$ is finite, it follows that the subspace $U(\mathfrak{n}_+) V_\lambda$ is finite-dimensional. Hence there exists an element $\eta \in \dot{\Gamma}_+$ such that

$$U(\mathfrak{n}_+)_\eta V_\lambda \neq 0 \quad \text{and} \quad U(\mathfrak{n}_+)_{\eta'} V_\lambda = 0 \quad \text{if } \eta' > \eta.$$

This proves that $\mathfrak{n}_+ V_{\lambda+\eta} = 0$. We now establish the equivalence of the two definitions. Let $\mu \in P(V)$ be such that $\mathfrak{n}_+ V_\mu = 0$. If $V_{\mu+\eta} \neq 0$ for some $\eta \in \dot{\Gamma}_+$ we choose $\eta' \in \dot{\Gamma}_+$ such that there exists $0 \neq v \in V_{\mu+\eta+\eta'}$ with $\mathfrak{n}_+ v = 0$. Then $F = U(\mathfrak{g}) v$ is a finite-dimensional irreducible module with highest weight $\mu + \eta + \eta'$. By Lemma (2.7) it follows that $F_\mu = F \cap V_\mu$ is non-zero. This contradicts the fact that $\mathfrak{n}_+ V_\mu = 0$.

*Proof of Theorem* (2.4)(i). Assume $k > 0$. Let $\lambda \in P_+(V)$, $(\lambda, \check{\alpha}_i) \in \mathbb{Z}_+ \, \forall i = 1, \dots, n$. The set

$$\Delta(\lambda) = \{\gamma \in W\pi \cap \Delta_+ : (\lambda, \check{\gamma}) \leqq 0\}$$

is a finite (possibly empty) subset of $\Delta_+$. Fix a positive integer $r$ such that $\alpha + s\delta \in \Delta_+ - \Delta(\lambda)$ for all $\alpha \in \dot{\Delta}$, $s \geqq r$.

**Claim 1.** $V_{\lambda + s\delta} = 0$ *for all* $s \geqq r$. *Assume that the claim is false. For some* $\alpha \in \dot{\Delta}_+$, *set* $\gamma = -\alpha + s\delta$. *Then* $(\lambda, \check{\gamma}) > 0$ *and hence by Lemma* (2.7) *it follows that* $V_{\lambda - \gamma + s\delta}$ $(= V_{\lambda+\alpha})$ *is non-zero contradicting the choice of* $\lambda$.

*Fix an integer* $p \geqq 0$ *such that* $V_{\lambda + p\delta} \neq 0$ *and* $V_{\lambda + s\delta} = 0$ *for all* $s > p$.

**Claim 2.** *For all* $m > 0$ *and* $\alpha \in \dot{\Delta}_+$ *we have* $V_{\lambda + \alpha + (m+p)\delta} = 0$. *Assume that the claim is false. Since* $(\lambda + \alpha, \check{\alpha}) > 0$ *if* $\alpha \in \dot{\Delta}_+$ *it follows from Lemma* (2.2) *that* $V_{\lambda + (m+p)\delta} \neq 0$ *contradicting the choice of* $p$.

**Claim 3.** *For all* $\alpha \in \dot{\Delta}_+$ *and all integers* $m > r$ *we have*

$$V_{\lambda - \alpha + (m+p)\delta} = 0.$$

*The proof is similar to the proof of the Claim* 2. *Observe that* $(\lambda - \alpha, \check{\gamma}) > 0$ *if* $\gamma = -\alpha + (m-1)\delta$.

Let $0 \neq v \in V_{\lambda + p\delta}$. From claims 1–3 it follows that

$$G_{r\delta} \cdot v = 0 \quad \text{for all } r > 0$$

and

$$G_{\alpha + s\delta} \cdot v = 0$$

for all but finitely many values of $s$. Since $V$ is integrable the elements $\{e_{\alpha,k} : \alpha \in \dot{\Delta}, k \in \mathbb{Z}\}$ act locally nilpotently on $V$ and hence the subspace $U(N_+)v$ is finite-dimensional. Let $v_1, \dots, v_q$ be a basis for $U(N_+)v$ with weights $\mu_1, \dots, \mu_q$. As a $U(N_-)$-module $V$ is generated by the elements $(v_1, \dots, v_q)$ and hence the set

$$P(V) \subseteq \bigcup_{i=1}^{q} D(\mu_i).$$

This implies that $V \in \mathcal{O}$ and hence by ([3], Proposition 9.3, Lemma 10.1) it follows that $V$ is isomorphic to $L(\lambda_0)$ for some $\lambda_0 \in H^*$ with $(\lambda_0, \check{\alpha}_i) \in \mathbb{Z}_+$ for all $i = 0, \ldots, n$. This completes the proof of Theorem (2.4)(i) in the case $k > 0$. For $k < 0$ the proof is similar. We work with $P_-(V)$ rather than $P_+(V)$.

(ii) Assume $k=0$. Let $\lambda \in P_+(V)$. If $V_{\lambda+\alpha+n\delta} = 0$ for all $\alpha \in \dot{\Delta}_+$ and all $n \in \mathbb{Z}$ the theorem follows. If $V_{\lambda+\alpha+r\delta} \neq 0$ for some $\alpha \in \dot{\Delta}_+$ and $r \in \mathbb{Z}$ set $\mu = \lambda + \alpha + r\delta$.

**Claim.** $V_{\mu+\beta+s\delta} = 0$ *for all* $\beta \in \dot{\Delta}_+$ *and all* $s \in \mathbb{Z}$. *Suppose the claim is false. Since* $\alpha, \beta \in \dot{\Delta}_+$ *it follows that either* $(\alpha+\beta, \check{\alpha})$ *or* $(\alpha+\beta, \check{\beta})$ *is positive, say* $(\alpha+\beta, \check{\alpha}) > 0$. *Set* $\gamma = \alpha + (s+r)\delta$. *Then* $(\mu+\beta, \check{\gamma}) > 0$ *and hence by Lemma* (2.2), $V_{\mu+\beta+s\delta-\gamma} = V_{\lambda+\beta} \neq 0$ *contradicting* $\lambda \in P_+(V)$. *The claim follows and hence* $N_+^0 V_\mu = 0$.

This completes the proof of the theorem.

## 3. The category $\tilde{\mathcal{O}}$

Throughout this section and the next we shall deal only with elements $\lambda \in H^*$ such that $(\lambda, c) = 0$, i.e. $\lambda \in (\mathfrak{h} \oplus \mathbb{C}d)^*$. The category $\tilde{\mathcal{O}}$ of $G$-modules is defined as follows: a module $M$ is an object of $\tilde{\mathcal{O}}$ if and only if:

(i) $M = \bigoplus_{\lambda \in H^*} M_\lambda$,

(ii) there exist finitely many elements $\lambda_1, \ldots, \lambda_r \in H^*$ such that the set $P(M) = \{\lambda \in H^* : M_\lambda \neq 0\}$ is contained in a union

$$P(M) \subseteq \bigcup_{i=1}^{r} \tilde{D}(\lambda_i)$$

where $\tilde{D}(\lambda_i) = \{\lambda_i - \eta + n\delta : \eta \in \dot{\Gamma}_+, n \in \mathbb{Z}\}$.

Observe that the center of $G$ acts trivially on all objects in $\tilde{\mathcal{O}}$. The morphism in $\tilde{\mathcal{O}}$ are the $G$-module maps. If $M \in \tilde{\mathcal{O}}$ then any submodule or quotient module is also in $\tilde{\mathcal{O}}$. Also finite direct sums and tensor products of modules in $\tilde{\mathcal{O}}$ are in $\tilde{\mathcal{O}}$.

(3.1) **Lemma.** *Any module* $M \in \tilde{\mathcal{O}}$ *contains elements* $0 \neq m \in M_\lambda$ *such that* $N_+^0 m = 0$.

*Proof.* Recall the ordering $\leq$ on $H^*$ defined by $\lambda \leq \mu$ if and only if $\mu - \lambda|_{\mathfrak{d}} \in \dot{\Delta}_+$. The condition (ii) in the definition of $\tilde{\mathcal{O}}$ implies that there exists $\lambda \in P(M)$, which is maximal with respect to $\leq$. Clearly $N_+^0 M_\lambda = 0$.

We construct examples of modules in $\tilde{\mathcal{O}}$. Thus we say that a module $M \in \tilde{\mathcal{O}}$ is a highest weight module of weight $\lambda$ if there exists $0 \neq m \in M$ such that

$$N_+^0 m = 0, \quad hm = \lambda(h)m, \quad M = U(G)m.$$

For $\lambda \in H^*$, $(\lambda, c) = 0$, let $\mathbb{C}_\lambda$ denote the one-dimensional $B^0 = H \oplus N_+^0$ module defined by

$$h \cdot 1 = \lambda(h)1, \quad N_+^0 \cdot 1 = 0.$$

Set $\tilde{M}(\lambda)=U(G)\underset{U(B_0)}{\otimes}\mathbb{C}_\lambda$, $v_\lambda=1\otimes 1$. Define an action of $U(G)$ on $\tilde{M}(\lambda)$ by left multiplication. Let $\mathfrak{S}$ denote the quotient $U(T_0)/U(T_0)c$ and let $p$: $U(T_0)\to\mathfrak{S}$ be the canonical homomorphism. Set $\phi(e_k^{(i)})=x_k^{(i)}$. It is easy to see that $\mathfrak{S}$ is in fact the polynomial algebra in the infinitely many variables $\{x_k^{(i)}: k\in\mathbb{Z}, i=1,\dots,n\}$. For $k\in\mathbb{Z}$, $\eta\in\dot{\Gamma}_+$, set

$$U(N_-^0)_{\eta+k\delta}=\{x\in U(N_-^0): [h,x]=-(\eta+k\delta)(h)x\,\forall h\in H\},$$

$$U(T_0)_{k\delta}=\{x\in U(T_0): [h,x]=k\delta(h)x\,\forall h\in H\}$$

$$\mathfrak{S}_k=p(U(T_0)_{k\delta}).$$

Since $T_0$ is an $H$-stable subalgebra of $G$ we have a $\mathbb{Z}$-grading on the rings $U(T_0)$ and $\mathfrak{S}$,

$$U(T_0)=\bigoplus_{k\in\mathbb{Z}}U(T_0)_{k\delta},$$

$$\mathfrak{S}=\bigoplus_{k\in\mathbb{Z}}\mathfrak{S}_k.$$

Further there exists a bijective correspondence between $H$-stable left ideals in $U(T_0)$ containing $c$ and graded ideals in $\mathfrak{S}$.

(3.2) **Lemma.** (i) *As an $N_-^0$-module $\tilde{M}(\lambda)$ is free and*

$$\tilde{M}(\lambda)\simeq U(N_-^0)\otimes\mathfrak{S}\otimes\mathbb{C}_\lambda.$$

(ii) $P(\tilde{M}(\lambda))=\tilde{D}(\lambda)$. *If $\eta\in\dot{\Gamma}_+$, $k\in\mathbb{Z}$ then*

$$\tilde{M}(\lambda)_{\lambda-\eta+k\delta}=\bigoplus_{q-p=k}(U(N_-^0)_{\eta+p\delta}\otimes\mathfrak{S}_q\otimes\mathbb{C}_\lambda)$$

(iii) *Let $I$ be any proper graded ideal in $\mathfrak{S}$. The $G$-submodule of $\tilde{M}(\lambda)$ generated by the elements $\{xv_\lambda: x\in p^{-1}(I)\}$ is proper and the quotient $M(\lambda,I)$ satisfies*

$$M(\lambda,I)\simeq U(N_-^0)\otimes(\mathfrak{S}/I)\otimes\mathbb{C}_\lambda,$$

$$M(\lambda,I)_{\eta+k\delta}=\bigoplus_{q-p=k}U(N_-^0)_{\eta+p\delta}\otimes(\mathfrak{S}/I)_q\otimes\mathbb{C}_\lambda.$$

*Proof.* Parts (i) and (ii) of the lemma are clear. For part (iii) set

$$M'=U(G)Jv_\lambda\qquad(v_\lambda=1\otimes 1\otimes 1\in\tilde{M}(\lambda)),$$

where $J=p^{-1}(I)$ is a proper ideal in $U(T_0)$. If $v_\lambda\in M'$ then

$$v_\lambda=gxv_\lambda$$

for some $g\in U(G)$, $x\in J$. Since $N_+^0xv_\lambda=0$ for all $x\in U(T_0)$ it follows that

$$v_\lambda=\bar{\beta}(g)xv_\lambda$$

where $\bar{\beta}$: $U(G)\mapsto U(T)$ was defined in Sect. 1. Part (i) of the lemma implies that $p(\bar{\beta}(g)x)=1$ and hence $1\in I$ contradicting the fact that $I$ is a proper ideal.

Let $\hat{\mathfrak{S}}$ denote the set of maximal graded ideals in $\mathfrak{S}$.

(3.3) **Corollary.** *Let $M\in\tilde{\mathcal{O}}$ be irreducible. There exists $\lambda\in H^*$ and $I\in\hat{\mathfrak{S}}$ such that $M$ is a quotient of $M(\lambda, I)$.*

*Proof.* By Lemma (3.1) there exists $0\neq m\in M_\lambda$ such that $N_+^0 m=0$, $M=U(G)m$. From the definition of $\tilde{M}(\lambda)$ it is clear that there exists a morphism $f$: $\tilde{M}(\lambda)\to M\to 0$ with $f(v_\lambda)=m$. Set $J=\{x\in U(T_0): xm=0\}$, $I=p(J)$. By Lemma (3.2) the map $f$ factors through to a morphism $\bar{f}$: $M(\lambda, I)\to M\to 0$.

Let $I'(\neq I)$ be any graded ideal containing $I$ and let $J'=p^{-1}(I)$. Then $M=U(G)J'm$ (since $M$ is irreducible) and as in the proof of Lemma (3.2) there exists $x\in J'$, $y\in U(T_0)$ with $(yx-1)m=0$ i.e. $(yx-1)\in J\subseteq J'$. This proves that $1\in J'$ and hence $I'=\mathfrak{S}$.

(3.5) **Theorem.** *Let $\lambda\in(\mathfrak{h}\oplus\mathbb{C}d)^*$, $I\in\hat{\mathfrak{S}}$.*

(i) *As a $U(N_-^0)$-module $M(\lambda, I)$ is free and $\dim M(\lambda, I)_{k\delta}\leq 1$ for all $k\in\mathbb{Z}$.*

(ii) *$M(\lambda, I)$ has a unique irreducible quotient $V(\lambda, I)$.*

(iii) *The set $\{V(\lambda, I)$: $\lambda\in(\mathfrak{h}\oplus\mathbb{C}d)^*$, $I\in\hat{\mathfrak{S}}\}$ exhaust the irreducible modules in $\tilde{\mathfrak{O}}$.*

(iv) *The modules $M(\lambda, I)$ and $M(\lambda', I')$ are isomorphic if and only if $I=I'$ and $\lambda=\lambda'+n\delta$ for some $n\in\mathbb{Z}$ with $(\mathfrak{S}/I)_n\neq 0$.*

(3.6) **Lemma.** *Let $I\in\hat{\mathfrak{S}}$. As a graded ring $\mathfrak{S}/I$ is isomorphic either to $\mathbb{C}$ (in which case $x_k^{(i)}\in I$ for all $i=1,\dots,n$ and all $k\in\mathbb{Z}$) or to a Laurent subring $\mathbb{C}[t^r, t^{-r}]$ of $\mathbb{C}[t, t^{-1}]$*, grad $t=1$.

*Proof.* Set $A=\mathfrak{S}/I$. Clearly $A$ is a simple $\mathbb{Z}$-graded algebra over $\mathbb{C}$ and hence every non-zero graded element is invertible. The Lemma is a consequence of the following fact: Let $B=\bigoplus_{n\in\mathbb{Z}} B_n$ be a $\mathbb{Z}$-graded commutative algebra over $\mathbb{C}$, $B\neq\mathbb{C}$. Let $r$ be the least positive, integer such that $B_r\neq 0$. Assume there exists an invertible element $t^r\in B_r$. Then $B_p=0$ if $p\not\equiv 0(r)$ and the homomorphism $B\to B_0\otimes\mathbb{C}[t^{-r}, t^r]$ defined by $b_n\to(b_n t^{-n})\otimes t^n$ $(n\equiv 0(r))$ is an isomorphism of graded rings.

*Proof of Theorem* (3.5). (i) This is immediate from Lemma (3.2)(iii) and Lemma (3.6).

(ii) If $M'$ is any proper submodule of $M(\lambda, I)$ then $v_\lambda\notin M'$ since $M(\lambda, I)=U(G)v_\lambda$. By part (i) we have $M'\cap M(\lambda, I)_\lambda=\{0\}$ and hence the sum $M_I$ of all proper submodules of $M(\lambda, I)$ is again proper. The quotient $V(\lambda, I)=M(\lambda, I)/M_I$ is thus the unique irreducible one.

(iii) This is immediate from Corollary (3.3) and part (ii) above.

(iv) Let $f$: $M(\lambda, I)\to M(\lambda', I')$ be a $G$-module isomorphism. Then $\lambda\in\tilde{D}(\lambda')$, $\lambda'\in\tilde{D}(\lambda)$ and hence $\lambda=\lambda'+n\delta$ for some $n\in\mathbb{Z}$ with $(\mathfrak{S}/I)_{-n}\neq 0$. Further there exists $g\in U(T_0)$ with $v'_\lambda=gf(v_\lambda)$. Let $x\in I$ and $y\in U(T_0)$ with $p(y)=x$. Then the equation

$$0=f(ygv_\lambda)=ygf(v_\lambda)=yv'_\lambda$$

proves that $x\in I'$. Similarly we can prove that $I'\subseteq I$ and hence $I=I'$. For the converse, observe that for any $x\in\mathfrak{S}_n$, $x\notin I$ the map $yv_\lambda\to yxv'_\lambda$ $(y\in U(N_-^0))$ is a $G$-module isomorphism.

Let $L_r$ $(r>0)$ denote the subring $\mathbb{C}[t^r, t^{-r}]$ of $L$ and let $\mathfrak{H}'$ denote the set of graded ring homomorphisms $\Lambda$: $\mathfrak{S} \mapsto L$ with $\Lambda(1)=1$ and such that $\operatorname{im}(\Lambda)=L_r$ for some $r>0$. If $r=0$ then $L_0=\mathbb{C}$ and $\Lambda_0$ is the trivial homomorphism $\Lambda_0(1)=1$, $\Lambda_0(x_k^{(i)})=0$ for all $i=1,\dots,n$, $k\in\mathbb{Z}-(0)$. Set $\mathfrak{H}=\mathfrak{H}'\cup\{\Lambda_0\}$.

Given $\Lambda\in\mathfrak{H}$, $\operatorname{im}\Lambda=L_r$ and $\lambda\in(\mathfrak{h}\oplus\mathbb{C}d)^*$ define a $T_0$-module structure on $L_r$ by:

$$x t^{rs}=\Lambda(p(x))t^{rs}, \qquad x\in U(T_0),$$

$$N_+^0 L_r=0, \qquad h t^{rs}=(\lambda+rs\delta)(h)t^{rs}.$$

Denote the corresponding module by $L_{\Lambda,\lambda}$. It is clear that $L_{\Lambda,\lambda}$ is an irreducible $T_0+B^0$ module and that $L_{\Lambda,\lambda}=L_{\Lambda,\lambda+rs\delta}$ for all $s\in\mathbb{Z}$. Let $M(\lambda,\Lambda)$ denote the induced module $U(G)\underset{U(B^0+T_0)}{\otimes} L_{\Lambda,\lambda}$. If $I\in\mathfrak{S}$ then by Lemma (3.6) it follows that $I=\operatorname{kernel}\Lambda$ for some $\Lambda\in\mathfrak{H}$. It is now not hard to see that $M(\lambda,I)$ is isomorphic to $M(\lambda,\Lambda)$.

(3.8) **Proposition.** *The modules $M(\lambda,\Lambda)$ and $M(\lambda',\Lambda')$ are isomorphic if and only if* (i) $\lambda=\lambda'+n\delta$ *for some* $n\in\mathbb{Z}$ *with* $\Lambda'(\mathfrak{S}_n)\neq0$ (ii) *there exists* $0\neq a\in\mathbb{C}$ *such that for all* $k\in\mathbb{Z}$, *and all* $x\in\mathfrak{S}_k$

$$\Lambda(x)=a^k\Lambda'(x).$$

*Proof.* Set $I=\operatorname{kernel}\Lambda$, $I'=\operatorname{kernel}\Lambda'$. If the pairs $(\lambda,\Lambda)$ and $(\lambda',\Lambda')$ satisfy conditions (i) and (ii) then $I=I'$ and hence $M(\lambda,\Lambda)$ and $M(\lambda',\Lambda')$ are isomorphic by Theorem (3.5).

Conversely if $M(\lambda,\Lambda)$ and $M'(\lambda',\Lambda')$ are isomorphic then $\lambda=\lambda'+n\delta$ for some $n\in\mathbb{Z}$ and kernel $\Lambda=\operatorname{kernel}\Lambda'$. Hence there exists $r\geqq0$ such that $\operatorname{im}(\Lambda)=\operatorname{im}(\Lambda')=L_r$. If $r=0$ then $\Lambda=\Lambda'$. If $r>0$, then there exists $x\in\mathfrak{S}_r$ with $\Lambda(x)\neq0$, $\Lambda'(x)\neq0$. Set

$$\Lambda(x)=at^r, \qquad \Lambda'(x)=bt^r, \qquad a,b\in\mathbb{C}-(0).$$

The result is immediate from Lemma (3.6) since $(\mathfrak{S}/I)$ is spanned by the elements $x^s$, and we have

$$\Lambda(x^s)=(ab^{-1})^s\Lambda'(x^s).$$

The modules $M(\lambda,\Lambda)$ have a unique irreducible quotient which we denote by $V(\lambda,\Lambda)$. Clearly an isomorphism of $M(\lambda,\Lambda)$ and $M(\lambda',\Lambda')$ induces an isomorphism of the quotients. One can imitate the proof of Proposition (3.8) to obtain the following parametrization of the isomorphism classes of irreducible modules in $\tilde{\mathcal{O}}$.

(3.9) **Proposition.** *The modules $V(\lambda,\Lambda)$ and $V(\lambda',\Lambda')$ are isomorphic if and only if*

(i) $\lambda=\lambda'+n\delta$ *for some* $n\in\mathbb{Z}$, $\Lambda'(\mathfrak{S}_n)\neq0$,

(ii) *there exists* $0\neq a\in\mathbb{C}$ *such that for all* $k\in\mathbb{Z}$ *and all* $x\in\mathfrak{S}_k$

$$\Lambda(x)=a^k\Lambda'(x).$$

(3.10) *Remark.* If $\operatorname{im}\Lambda=L_r$ then $M(\lambda,\Lambda)$ is generated as a $U(G)$-module by $t^{rs}$ for any $s\in\mathbb{Z}$ and hence

$$\dim V(\lambda,\Lambda)_{rs\delta}=\dim M(\lambda,\Lambda)_{rs\delta} \quad \text{for all } s\in\mathbb{Z}.$$

## 4. Integrable modules in $\tilde{\mathcal{O}}$

In this section we obtain the necessary and sufficient condition for the modules $V(\lambda, \Lambda)$ to be integrable. We use the notation of Section 3. Let $\mathbb{C}^*$ denote the set of non-zero complex numbers.

Set $P_+ = \{\lambda \in (\mathfrak{h} \oplus \mathbb{C} d)^* : (\lambda, \check{\alpha}_i) \in \mathbb{Z}_+ \ \forall i \in (1, \ldots, n)\}$. For $\lambda \in P_+$ set

$$J_\lambda = \{i \in (1, \ldots, n) : (\lambda, \check{\alpha}_i) > 0\},$$

$$r_\lambda = \sum_{i \in J_\lambda} (\lambda, \check{\alpha}_i).$$

If $r_\lambda > 0$ identify the set $(\mathbb{C}^*)^{r_\lambda}$ with the product $(\mathbb{C}^*)^{(\lambda, \check{\alpha}_{i_1})} \times \ldots \times (\mathbb{C}^*)^{(\lambda, \check{\alpha}_{i_k})}$ where $i_1 < \ldots < i_k$ are the elements of $J_\lambda$. For every $a \in (\mathbb{C}^*)^{r_\lambda}$, $a = (a_{ij})$, $i \in J_\lambda$, $1 \leqq j \leqq (\lambda, \check{\alpha}_i)$, define a graded homomorphism $\Lambda_a \colon \mathfrak{S} \to L$ by extending

$$\Lambda_a(x_k^{(i)}) = 0 \qquad \forall k \in \mathbb{Z},\ i \notin J_\lambda,$$

$$\Lambda_a(x_k^{(i)}) = \left( \sum_{j=1}^{(\lambda, \alpha_i)} a_{ij}^k \right) t^k \qquad \forall k \in \mathbb{Z},\ i \in J_\lambda.$$

Set $\mathfrak{H}_\lambda = \{\Lambda_a : a \in (\mathbb{C}^*)^{r_\lambda}\}$. If $r_\lambda = 0$ then set $\mathfrak{H}_\lambda = \{\Lambda_0\}$, (where $\Lambda_0$ was defined in Section 3). Observe that $\mathfrak{H}_\lambda = \mathfrak{H}_\mu$ if $\lambda = \mu + s\delta$, $s \in \mathbb{Z}$.

(4.1) **Lemma.** *For all $\lambda \in P_+$, $a \in (\mathbb{C}^*)^{r_\lambda}$ the image of $\Lambda_a$ is a Laurent ring* i.e. $\mathfrak{H}_\lambda \subseteq \mathfrak{H}$.

(4.2) **Theorem.** *$V(\lambda, \Lambda)$ is integrable if and only if $\lambda \in P_+$ and $\Lambda \in \mathfrak{H}_\lambda$.*

Define an equivalence relation on $\mathfrak{H}_\lambda$ by: $\Lambda_a \sim \Lambda_{a'}$, if and only if there exists $b \in \mathbb{C}^*$ and permutations $\sigma_i$ of $(1, \ldots, (\lambda, \check{\alpha}_i))$, $i \in J_\lambda$ such that

$$a_{ij} = b a'_{i, \sigma_i(j)}.$$

The following Corollary which is now a trivial consequence of Proposition (3.9) gives the parametrization of the isomorphism classes of irreducible integrable $G$-modules in $\tilde{\mathcal{O}}$.

(4.3) **Corollary.** *The integrable modules $V(\lambda, \Lambda_a)$ and $V(\mu, \Lambda_b)$ are isomorphic if and only if: $\lambda = \mu + k\delta$ for some $k \in \mathbb{Z}$ with $\Lambda_b(\mathfrak{S}_k) \neq 0$ and $\Lambda_a \sim \Lambda_b$.*

To simplify the notation we prove the results for the affine Lie-algebra $A_1^{(1)}$ and sketch a proof of the general case at the end of the section. We recall the following well-known results.

**Lemma A.** *If $a_1, \ldots, a_k \in \mathbb{C}^*$ are distinct, then, the matrix $(a_i^j)$ $1 \leqq i, j \leqq k$ is non-singular.*

**Lemma B.** *Let $(a_n)_{n \in \mathbb{Z}}$ be elements of $\mathbb{C}$ satisfying a recurrence relation of type*

$$a_n = \sum_{j=1}^{r} A_j a_{n-j}$$

*where, $A_j \in \mathbb{C}$ $(1 \leqq j \leqq r)$ is independent of n. Assume that $A_r \neq 0$. Let $a_{(1)}, \ldots, a_{(r)}$ be the (non-zero) roots of the polynomial $X^r - \sum_{j=1}^{r} A_j X^{r-j}$. Then $a_n = \sum_{j=1}^{r} B_j a_{(j)}^n \forall n \in \mathbb{Z}$, where $B_1, \ldots, B_r \in \mathbb{C}$ depend on $a_1, \ldots, a_r$.*

**Lemma C.** *Let I be the ideal in a polynomial algebra $\mathbb{C}[X_1, \ldots, X_n]$ generated by elements of the form*

$$Y_i = \left(\sum_{j=1}^{n} b_{ij} X_j\right)^p, \quad i \in (1, \ldots, n),$$

*where $b_{ij} \in \mathbb{C}$, $1 \leqq i, j \leqq n$ and p is some positive integer. If the matrix $(b_{ij})$ is non-singular then I is of finite co-dimension* i.e. *there exists an integer $q > 0$ such that $X_j^q \in I$ for all $j \in (1, \ldots, n)$.*

*From now on G denotes the affine Lie-algebra of type $A_1^{(1)}$. Let $y, h, x$ be the standard basis for $sl(2, \mathbb{C})$ and let $y_n, h_n, x_n$ denote the elements $t^n \otimes y$, $t^n \otimes h$, $t^n \otimes x$ of G. Set $y_0 = y$, $h_0 = h$, $x_0 = x$. Clearly,*

$$N_-^0 = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} y_n, \quad T_0 = \mathbb{C}c \bigoplus_{n \in \mathbb{Z} - (0)} \mathbb{C} h_n, \quad N_+^0 = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} x_n.$$

*The algebra $\mathfrak{S}$ is the polynomial algebra $\mathbb{C}[\bar{h}_n \colon n \in \mathbb{Z} - (0)]$ and the map $p \colon U(T_0) \to \mathfrak{S}$ satisfies $p(h_n) = \bar{h}_n$.*

*Proof of Lemma* (4.1). Let $\lambda \in P_+$, $(\lambda, h) = n > 0$ and let $a = (a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$. Assume that $a_1, \ldots, a_k$ are the distinct elements in $(a_1, \ldots, a_n)$ and that $a_i$ occurs with multiplicity $p_i$, $1 \leqq i \leqq k$. Then

$$\Lambda_a(\bar{h}_r) = \left(\sum_{i=1}^{k} p_i a_i^r\right) t^r \quad \forall r \in \mathbb{Z}.$$

By Lemma A it follows that there exists $i, j \in (1, \ldots, k)$ such that $\Lambda_a(\bar{h}_i) \neq 0$, $\Lambda_a(\bar{h}_{-j}) \neq 0$. Let $r, s \in (1, \ldots, k)$ be the smallest integers such that there exists $Q \in \mathfrak{S}_r$, $Q_* \in \mathfrak{S}_{-s}$ with $\Lambda_a(Q) \neq 0$ and $\Lambda_a(Q_*) \neq 0$. If $r \geqq s$ write $r = sp + q$, $0 \leqq q < s$. Since $QQ_*^p \in \mathfrak{S}_q$ and $\Lambda_a(QQ_*^p) \neq 0$, the minimality of $r$ forces $q = 0$. If $p > 1$ then $\Lambda_a(QQ_*^{p-1}) \neq 0$ and $QQ_*^{p-1} \in \mathfrak{S}_s$ $(s < r)$. Hence $p = 1$ and $r = s$. A similar argument proves that $\Lambda_a(\mathfrak{S}_q) = 0$ if $q \neq o(r)$ and so $\Lambda_a$ maps onto the ring $L_r = \mathbb{C}[t^r, t^{-r}]$. If $r \leqq s$ the proof is similar.

We now prove Theorem (4.2). We need the following consequence of Lemma (2.2) and Theorem (3.5). Let $v_\lambda$ denote the element $1 \otimes 1$ of $M(\lambda, \Lambda)$ and $\bar{v}_\lambda$ the image of $v_\lambda$ in $V(\lambda, \Lambda)$. Let $\bar{\beta} \colon U(G) \to U(T)$ be the canonical map defined in § 1.

Given any $n$-tuple of integers $r = (r_1, \ldots, r_n)$ set $x_{(r)} = x_{r_1} \ldots x_{r_n}$, $y_{(r)} = y_{r_1} \ldots y_{r_n}$.

(4.4) **Lemma.** *Let $\lambda \in P_+$, $(\lambda, h) = n$. The following are equivalent:*

(i) *$V(\lambda, \Lambda)$ is integrable,*

(ii) $y_{(r)} \cdot \bar{v}_\lambda = 0$ $\forall r \in \mathbb{Z}^{n+1}$,

(iii) $\Lambda(\bar{\beta}(x_{(r)} y_{(s)})) = 0$ $\forall r, s \in \mathbb{Z}^{n+1}$.

*Proof.* (i)⇒(ii). This is clear from Lemma (2.2)(b) and the fact that $y_{(r)} \cdot \bar{v}_\lambda$ has weight $\lambda - (n+1)\alpha$ if $r \in \mathbb{Z}^{n+1}$.

(ii)⇒(i). This is by Definition (2.1).

(ii)⇔(iii). By Theorem (3.5)(ii) it follows that $y_{(r)}\cdot \bar{v}_\lambda=0\,\forall r\in\mathbb{Z}^{n+1}$ if and only if the set $\{y_{(r)}\cdot v_\lambda\colon r\in\mathbb{Z}^{n+1}\}$ generates a proper submodule of $M(\lambda, \Lambda)$. Equivalently, (by Remark (3.10))

$$x_{(s)}\cdot y_{(r)}\cdot v_\lambda=0\ \forall r, s\in\mathbb{Z}^{n+1}$$

i.e. $\Lambda(\bar{\beta}(x_{(s)}y_{(r)})=0\ \forall r,\ s\in\mathbb{Z}^{n+1}$.

Recall the elements $P_j\in U(T_0)$ defined in §1. Thus if $D_+\colon U(T_0)\to U(T_0)$ is the derivation obtained by extending $D_+(h_n)=nh_{n+1}$, and $L_1\colon U(T_0)\to U(T_0)$ is left multiplication by $h_1$, then

$$P_j=\frac{(D_+-L_1)^j}{j!}\cdot 1,\qquad j\geqq 0.$$

By Proposition (1.1)(ii) we have

$$P_j=(-1)^j\bar{\beta}\left(\frac{x_1^j}{j!}\frac{y^j}{j!}\right).$$

Set $\bar{P}_j=p(P_j)$. Proposition (1.1)(i) gives us the following recursive formula for $\bar{P}_j$,

$$\bar{P}_j=-\frac{1}{j}\sum_{i=0}^{j-1}\bar{h}_{i+1}\bar{P}_{j-i-1}. \tag{4.5}$$

Let $\lambda\in P_+$, $(\lambda,h)=n$ and $\Lambda_a\in\mathfrak{H}_\lambda$. For $0\leqq j\leqq n$ we have

$$\Lambda_a(\bar{P}_j)=((-1)^j\textstyle\sum a_{i_1}\dots a_{i_j})t^j \tag{4.6}$$

where the sum is over $j$-tuples $i_1<\dots<i_j$, $i_k\in(1,\dots,n)$. If $j=1$ then $\bar{P}_1=-h_1$ and hence $\Lambda_a(\bar{P}_1)=\left(-\sum_{i=1}^n a_i\right)t$. Assume that (4.6) holds for all $j\leqq i$. Substituting the values of $\Lambda_a(\bar{h}_{i+1})=\left(\sum_{j=1}^n a_j^{i+1}\right)t^{i+1}$ and $\Lambda_a(P_j)(0\leqq j\leqq i)$ in (4.5) gives (4.6) for $i+1$. Conversely if for some $\Lambda\in\mathfrak{H}_\lambda$ there exists $a_1,\dots,a_n\in\mathbb{C}^*$ such that (4.6) holds, then,

$$\Lambda(\bar{h}_j)=\left(\sum_{i=1}^n a_i^j\right)t^j \tag{4.7}$$

for $0\leqq j\leqq n$.

*Proof of Theorem* (4.2). Assume that $V(\lambda,\Lambda)$ is integrable. By Lemma (2.2) we know that $\lambda\in P_+$ i.e. $(\lambda,h)=n\in\mathbb{Z}_+$. Define scalars $a_r\in\mathbb{C}$, $r\in\mathbb{Z}$ by,

$$\Lambda(\bar{h}_r)=a_r t^r,\ r\neq 0,\qquad a_0=(\lambda,h)=n.$$

For $r\in\mathbb{Z}$ let $(r)$ denote the element $(r,1,\dots,1)$ of $\mathbb{Z}^{n+1}$. By Lemma (4.4)(iii) and Proposition (1.1)(i) we have

$$0=\Lambda(\bar{\beta}(x_{(r)}y^{n+1}))=\sum_{j=0}^n a_{j+r}\Lambda(\bar{P}_{n-j})t^{j+r}.$$

**Claim.** $\Lambda(\bar{P}_n)\neq 0$.

*If* $\Lambda(\bar{P}_n)=0$ *then the preceding equality together with Proposition* (1.1)(i) *implies that*

$$0=\Lambda\left(\sum_{j=0}^{n-1}\bar{h}_{j+r+1}\bar{P}_{n-j-1}\right)=\Lambda(\bar{\beta}(x_{r+1}x_1^{n-1}y^n)).$$

*Equivalently,*

$$x_{r+1}\cdot x_1^{n-1}\cdot y^n\cdot\bar{v}_\lambda=0\ \forall r\in\mathbb{Z}.$$

*Thus the element* $x_1^{n-1}\cdot y^n\cdot\bar{v}_\lambda$ *generates a proper submodule of* $V(\lambda,\Lambda)$ *and hence* $x_1^{n-1}\cdot y^n\cdot\bar{v}_\lambda=0$. *Since* $y_{-1}\cdot y^n\cdot\bar{v}_\lambda=0$ *it follows from the standard representation theory of* $sl(2,\mathbb{C})$ *that* $y^n\cdot\bar{v}_\lambda=0$ *contradicting* $(\lambda,h)=n$. *This proves the claim.*

Let $(A_j)_{0\leq j\leq n}$ be such that $\Lambda(\bar{P}_j)=A_jt^j$. The scalars $(a_r)_{r\in\mathbb{Z}}$ satisfy

$$a_{r+n}=-\sum_{j=0}^{n-1}a_{j+r}A_{n-j}$$

and hence by Lemma B it follows that

$$a_r=\sum_{i=1}^{n}B_ia_{(i)}^r\qquad\forall r\in\mathbb{Z},$$

where $B_1,\ldots,B_n$ are determined by $a_1,\ldots,a_n$ and $a_{(1)},\ldots,a_{(n)}$ are the roots of the polynomial $\left(X^n+\sum_{j=0}^{n-1}A_{n-j}X^j\right)$. Further for $0\leq j\leq n$,

$$A_j=(-1)^j\sum_{i_1<\ldots<i_j}a_{(i_1)}\ldots a_{(i_j)}.$$

By (4.7) we have $B_i=1$ for all $i\in(1,\ldots,n)$ and hence $\Lambda=\Lambda_a$ where $a=(a_{(1)},\ldots,a_{(n)})\in(\mathbb{C}^*)^n$.

We now prove the converse. Thus let $\lambda\in P_+$, $(\lambda,h)=n\geq 0$ and $\Lambda=\Lambda_a\in\mathfrak{H}_\lambda$. If $n=0$ then $\Lambda=\Lambda_0$. For all $p,q\in\mathbb{Z}$ we have,

$$x_p\cdot y_q\cdot v_\lambda=h_{p+q}\cdot v_\lambda=0$$

and hence the elements $\{y_q\cdot v_\lambda\colon q\in\mathbb{Z}\}$ generate a proper submodule of $M(\lambda,\Lambda_0)$. Thus by Theorem (3.5)(ii)

$$\bar{y}_p\cdot\bar{v}_\lambda=0$$

for all $p\in\mathbb{Z}$ and $V(\lambda,\Lambda_0)$ is the trivial $G$-module.

Assume now that $n>0$ and $\Lambda=\Lambda_a$ for some $a=(a_1,\ldots,a_n)\in(\mathbb{C}^*)^n$. Let $r>0$ be such that $\Lambda_a$ maps onto $L_r=\mathbb{C}[t^r,t^{-r}]$. Let $\mathbb{Z}_r$ denote the set of residues modulo $r$. For every $i\in\mathbb{Z}_r$ define a linear map $\phi_i\colon M(\lambda+i\delta,\Lambda)\to U(N_-^0)\otimes L$ by extending

$$\phi_i(g\otimes t^{qr})=g\otimes t^{p+qr+i}$$

for all $p,q\in\mathbb{Z}$, $g\in U(N_-^0)_p$, where $U(N_-^0)_p$ is the subspace $\{x\in U(N_-^0)\colon [d,x]=px\}$. Clearly $\phi_i$ is injective and $\phi_i(M(\lambda+i\delta,\Lambda))$ acquires a natural $G$-module structure so that $\phi_i$ is a $G$-module map. Denote this module by $M_i$. Set $v_i=\phi_i(\bar{v}_{\lambda+i\delta})$; notice that $v_i=1\otimes t^i$, $i\in\mathbb{Z}_r$. Let $M$ denote the $G$-module $\bigoplus_{i\in\mathbb{Z}_r}M_i$; the underlying vector space of $M$ is $U(N_-^0)\otimes L$. We shall prove that $M$ has an integrable quotient $\bar{M}$ such that if $\eta\colon M\to\bar{M}\to 0$ denotes the canonical map, then $\eta(M_i)\neq 0$ for all $i\in\mathbb{Z}_r$. Thus $\bar{M}_i=\eta(M_i)$ is an integrable quotient of $M_i$ and the theorem follows since $V(\lambda+i\delta,\Lambda)$ is a further quotient of $\bar{M}_i$.

We have the following explicit formulae for the action of $G$ on $M$. Let $g\in U(N_-^0)_k, m, p, q\in\mathbb{Z}$, then,

$$
\begin{aligned}
& d(g\otimes t^m)=(\lambda+m\delta)(d)g\otimes t^m\\
(4.8)\qquad & y_q(g\otimes t^m)=y_q g\otimes t^{m+q}\\
& h_q(g\otimes t^m)=g\otimes\Lambda(\bar{h}_q)t^m+[h_q,g]\otimes t^{m+q}\\
& x_q(y_p g\otimes t^m)=(y_p x_q+h_{p+q})(g\otimes t^{m-p}).
\end{aligned}
$$

(4.9) **Lemma.** *Let $g\in U(N_-^0)$ be an element of degree one* i.e. $g=\sum_{i\in F} c_i y_i$ *where $F$ is some finite subset of the integers. For any positive integer $p$ and any $q\in\mathbb{Z}$, the action of $x_q$ and $h_q$ on the element $g^{p+1}\otimes 1$ of $M$ is given by:*

$$x_q(g^{p+1}\otimes 1)=(p+1)[g^{p-1}\sum_{i,j\in F} c_i c_j(y_i\lambda_{j+q}-p\,y_{i+j+q})]\otimes t^q,$$

$$h_q(g^{p+1}\otimes 1)=(g^p\sum_{i\in F} c_i(y_i\lambda_q-2(p+1)\,y_{i+q}))\otimes t^q,$$

*where $\lambda_q\in\mathbb{C}$ is defined by $\Lambda(\bar{h}_q)=\lambda_q t^q$.*

The proof of the Lemma is an easy induction on $p>0$.

(4.10) **Proposition.** *There exists a proper ideal $I\subseteq U(N_-^0)$ such that*

(a) *the quotient $R=U(N_-^0)/I$ is finitely generated,*

(b) *$I\otimes L$ is a $G$-stable subspace of $M$.*

Let $M'$ denote the G-module $R\otimes L$ and $\eta'\colon M\to M'$ be the natural map, Then $\eta'(v_i)\neq 0$ and hence $\eta'(M_i)(=M_i')$ is non-zero for all $i\in\mathbb{Z}_r$.

(4.11) **Corollary.** *$M_i'$ has finite dimensional weight spaces for all $i\in\mathbb{Z}_r$.*

(4.12) **Proposition.** *There exists a proper ideal $J\subseteq R$ such that*

(a) *the quotient $F=R/J$ is finite-dimensional,*

(b) *$J\otimes L$ is a $G$-stable subspace of $R\otimes L$.*

Let $\bar{M}$ denote the $G$-module $F\otimes L$ and let $\bar{\eta}\colon M'\to\bar{M}$ be the natural map. As before $\bar{\eta}(M_i')=\bar{M}_i$ is non-zero for all $i\in\mathbb{Z}_r$. Set $v_i'=\eta'(v_i)$, $\bar{v}_i=\bar{\eta}(v_i')$. For all $p,q\in\mathbb{Z}$, $p>0$, we have from (4.8) that

$$y_q^p\cdot v_i'=(y_q')^p\otimes t^{pq+i},$$

$$y_q^p\cdot\bar{v}_i=(\bar{y}_q)^p\otimes t^{pq+i}$$

where $y_q'=\eta'(y_q)$, $\bar{y}_q=\bar{\eta}(y_q')$. By Proposition (4.12) there exists an integer $>0$ such that $\bar{y}_p^{+1}=0$ for all $p\in\mathbb{Z}$ and hence

$$y_p^{+1}\cdot\bar{v}_i=0\qquad\forall p\in\mathbb{Z},\ i\in\mathbb{Z}_r.$$

Since the adjoint representation of $G$ on $U(G)$ is integrable and $\bar{M}=\bigoplus_{i\in\mathbb{Z}_r} U(G)\bar{v}_i$ it follows that the elements $(y_p)_{p\in\mathbb{Z}}$ act locally nilpotently on $\bar{M}$ and hence $\bar{M}$ is integrable. This proves the theorem modulo Proposition (4.10)–(4.12).

Recall that $\Lambda=\Lambda_a$ for some $a=(a_1,\ldots,a_n)\in(\mathbb{C}^*)^n$. Let $(a_1,\ldots,a_k)$ be the distinct elements in $(a_1,\ldots,a_n)$ and assume that $a_i$ occurs with multiplicity $p_i$, $1\leqq i\leqq k$. Then for all $q\in\mathbb{Z}$ we have,

$$\Lambda(\bar{h}_q)=\left(\sum_{i=1}^{k} p_i a_i^q\right)t^q.$$

Set $\lambda_q=\sum_{i=1}^{k} p_i a_i^q$. Let $Q$ denote the polynomial $(X-a_1)\ldots(X-a_k)$ and $(Q_i)$ be polynomials such that $Q=(X-a_i)Q_i$. For $j\in(1,\ldots,k)$ let $A_j$ (resp. $B_{ij}$) denote the coefficient of $X^{k-j}$ in $Q$ (resp. $Q_i$). Set $A_0=1$, $B_{i,k+1}=0$ for all $i\in(1,\ldots,k)$. Then

$$A_j=B_{i,j+1}-a_iB_{i,j}$$

for all $i,j\in(1,\ldots,k)$.

By definition

$$\sum_{j=0}^{k} A_{k-j}a_p^j=0 \qquad \forall p\in(1,\ldots,k),$$

$$\sum_{j=0}^{k-1} B_{i,k-j}a_p^j=0 \qquad \forall p\in(1,\ldots,k),\ p\neq i$$

and hence, we have

$$(*) \qquad \sum_{j=0}^{k} A_{k-j}\lambda_{j+p}=0$$

$$(**) \qquad \sum_{j=0}^{k-1} B_{i,k-j}\lambda_{j+p}=p_i\sum_{j=0}^{k-1} B_{i,k-j}a_i^{j+p}$$

for all $p\in\mathbb{Z}$, $i\in(1,\ldots,k)$.

(4.13) *Remark.* It is not hard to see that the matrix $B=(B_{ij})$, $1\leqq i,j\leqq k$ is non-singular. In fact if $A'=(a_{ij})$ is defined by $a_{ij}=a_j^{k-i}$ $1\leqq i$, $j\leqq k$, then $BA'$ is a diagonal non-singular matrix.

*Proof of Proposition* (4.10). For $p\in\mathbb{Z}$ let $g_p$ denote the element

$$g_p=\sum_{j=0}^{k} A_{k-j}y_{j+p}$$

of $U(N_-^0)$. For $q\in\mathbb{Z}$ observe from (4.8) and $(*)$ that

$$x_q(g_p\otimes 1)=1\otimes\left(\sum_{j=0}^{k} A_{k-j}\lambda_{j+p+q}\right)t^q=0,$$

$$h_q(g_p\otimes 1)=(\lambda_q g_p-2g_{p+q})\otimes t^q.$$

Let $I$ be the ideal in $U(N_-^0)$ generated by the elements $(g_p)_{p\in\mathbb{Z}}$. Clearly $I$ is proper ($I\subseteq U(N_-^0)N_-^0$) and the preceding equalities prove that $I\otimes C[t,t^{-1}]$ is $G$-stable. If $R$ denotes the quotient $U(N_-^0)I$ and $y'_p$ the image of $y_p$ in $R$, then

$$y'_{p+k}=-\sum_{j=0}^{k-1} A_{k-j}y'_{j+p}.$$

Since $A_k=(-1)^k(a_1\dots a_k)$ is non-zero it follows that the set $\{y'_p\colon p\in\mathbb{Z}\}$ is spanned by any $k$-consecutive elements $\{y'_{q+1},\dots,y'_{q+k}\}$. Hence $R$ is finitely generated; in fact $R$ is isomorphic to the polynomial algebra in $k$-variables.

The proof of Corollary (4.11) is immediate since $R$ is finitely generated and

$$(M)_{-p\alpha+q\delta}=U(N_-^0)^p\otimes\mathbb{C}t^q$$

where $U(N_-^0)^p=\{x\in U(N_-^0)\colon [h,x]=-p\alpha(h)\}$.

*Proof of Proposition* (4.12). For $q\in\mathbb{Z}$, $i\in(1,\dots,k)$, define elements $v_{q,i}\in R$ by:

$$v_{q,i}=\sum_{j=0}^{k-1} B_{i,k-j}y'_{j+q}.$$

Observe that: $v_{q,i}-a_iv_{q-1,i}=\sum_{j=0}^{k} A_{k-j}y'_{j+q}=0$ $\left(\text{recall that } g_q=\sum_{j=0}^{k} A_{k-j}y'_{j+q}\in I\right)$.
Hence for all $q>0$ we have

$$v_{q,i}=a_i^q v_i,\qquad v_{-q,i}=a_i^{-q}v_i$$

where $v_i=v_{0,i}$ for $i\in(1,\dots,k)$.

Let $J$ be the ideal of $R$ generated by the elements $\{v_i^{p_i+1}\colon i\in 1,\dots,k\}$. We show that $J$ satisfies the conditions of Proposition (4.12). Let $q\in\mathbb{Z}$. The following equalities prove that $J\otimes C[t,t^{-1}]$ is $G$-stable.

$$(h_q v_i^{p_i+1}\otimes 1)=v_i^{p_i+1}(\lambda_q-2(p_i+1)a_i^q)\otimes t^q,$$
$$(x_q v_i^{p_i+1}\otimes 1)=0.$$

The first equality follows immediately from Lemma (4.10) and the definition of the elements $v_{q,i}$. For the second observe (from Lemma (4.9)) that:

$$(\dagger')\qquad x_q v_i^{p_i+1}=(p_i+1)v_i^{p_i-1}\left(v_i\sum_{j=0}^{k-1} B_{i,k-j}\lambda_{j+q}-p_i\sum_{j,l=0}^{k-1} B_{i,k-l}B_{i,k-j}y'_{j+q+l}\right)\otimes t^q.$$

By the equality (**) we have

$$\sum_{j=0}^{k-1} B_{i,k-j}\lambda_{j+q}=p_i\sum_{j=0}^{k-1} B_{i,k-j}a_i^{j+q}.$$

Also, by the definition of $v_{q,i}$ we have

$$\sum_{j,l=0}^{k-1} B_{i,k-j}B_{i,k-l}y'_{j+q+l}=\sum_{j=0}^{k-1} B_{i,k-j}v_{j+q,i}=\left(\sum_{j=0}^{k-1} B_{i,k-j}a_i^{j+q}\right)v_i.$$

This proves that the expression on the right hand side of ($\dagger'$) is zero.

The matrix $(B_{ij})1 \leqq i, j \leqq k$ is non-singular (see Remark (4.13)) and hence by Lemma C we conclude that the quotient $R/J$ is finite dimensional. This proves the proposition. We have proved Theorem (4.2) in the case when $G$ is of type $A_1^{(1)}$.

Let $G$ now denote an arbitrary non-twisted affine Lie-algebra, we use the notation of Sect. 1. It is clear from the proof given for $A_1^{(1)}$ that in general $\mathfrak{H}_\lambda \subseteq \mathfrak{H}$ and that if $V(\lambda, \Lambda)$ is integrable then $\lambda \in P_+$ and $\Lambda \in \mathfrak{H}_\lambda$. We deduce the converse from the $A_1^{(1)}$ case. For $i \in (1, \ldots, n)$ let $G_i$ denote the subalgebra of $G$ spanned by the elements $\{\otimes e_i t^k, \otimes f_i t^k, e_k^{(i)}: k \in \mathbb{Z}\}$ together with $c$ and $d$. Then $G_i$ is isomorphic to an affine Lie-algebra of type $A_1^{(1)}$. Set

$$H_i = \mathbb{C}c \oplus \mathbb{C}d \oplus \mathbb{C}\check{\alpha}_i,$$

$$T_i = \mathbb{C}c \bigoplus_{k \in \mathbb{Z}} \mathbb{C}e_k^{(i)}.$$

Let $\lambda \in P_+$, $\Lambda \in \mathfrak{H}_\lambda$ and let $\lambda_i$(resp. $\Lambda_i$) denote the restriction of $\lambda$ (resp. $\Lambda$) to $H_i \left(\text{resp. } \frac{U(T_i)}{U(T_i)c}\right)$. Then the $G_i$-module $M(\lambda_i, \Lambda_i)$ is a $G_i$-sub-module of $M(\lambda, \Lambda)$. Since $\Lambda_i \in \mathfrak{H}_{\lambda_i}$ we know that $M(\lambda_i, \Lambda_i)$ has an integrable quotient. Set $(\lambda, \check{\alpha}_i) = r_i$. Equivalently,

(†) the submodule generated by the elements $\{f_{i,k}^{r_i+1} v_\lambda: k \in \mathbb{Z}\}$ intersects the weight spaces $M(\lambda_i, \Lambda_i)_{\lambda_i + s\delta}$ trivially for all $s \in \mathbb{Z}$, where $f_{i,k} = t^k \otimes f_i$.

If we prove that in fact the elements $\{f_{i,k}^{r_i+1}: i \in (1, \ldots, n), k \in \mathbb{Z}\}$ generate a proper $G$-submodule of $M(\lambda, \Lambda)$ then it follows that $V(\lambda, \Lambda)$ is integrable.

For simplicity we take $i=1$, $k=0$ and set $f_{i,0} = f$, $r_i + 1 = r$. The proof for any $i, k$ is similar. Suppose that there exists $g \in U(G)$ with

$$g f^r v_\lambda = v_\lambda.$$

Write $g$ as a sum

$$g = \sum_j y_j x_j$$

where $y_j \in U(N_-^0)$, $x_j \in U(T \oplus N_+^0)_{\eta] + p]\delta}$, $\eta_j \in \dot{\Gamma}_+$, $p_j \in \mathbb{Z}$.

Since the weights of $M(\lambda, \Lambda)$ are in $\tilde{D}(\lambda)$ it follows that

$$x_j f^r v_\lambda = 0$$

if $\eta_j \neq m\alpha_1$ for some $0 \leqq m \leqq r$; in fact, we have,

$$v_\lambda = g f^r v_\lambda = x f^r v_\lambda$$

for some $x \in U(T \oplus N_+^0)_{r\alpha_1}$. [Note that if $\eta_j = m\alpha_1$, $m < r$ then $y_j x_j f^r v_\lambda$ has weight less than $\lambda$]. Write $x$ as a sum

$$x = \sum_{q \in \mathbb{Z}} P_{-q} x_q$$

corresponding to the decomposition

$$U(T \oplus N_+^0)_{r\alpha_1} = \bigoplus_{q \in \mathbb{Z}} (U(T)_{-q\delta} \otimes U(N_+^0)_{r\alpha_1 + q\delta}).$$

Choose $q \in \mathbb{Z}$ such that $x_q f^r v_\lambda \neq 0$. Since $U(N_+^0)_{r\alpha_1+q\delta} \subseteq U(G_1)$ it follows that the $G_1$-module generated by $f^r v_\lambda$ intersects the weight space $M(\lambda_1, \Lambda_1)_{\lambda_1+q\delta}$ contradicting (†). This proves the theorem.

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**Note added in proof**

(i) In [4] we obtain explicit realizations of the modules $V(\lambda, \Lambda_a)$, $\lambda \in P_+$, $a \in (\mathbb{C}^*)^{r_\lambda}$. The modules are unitary for a compact form of $G$ if and only if $|a_i| = |a_j|\ \forall i, j$, where $a = (a_1, \ldots, a_{r_\lambda})$.
(ii) In [5] we prove analogous results for the twisted affine Lie-algebras.