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Schwarz's lemma for circle packings

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1. Introduction

Conformal mappings can be approximated by circle packing isomorphisms; this is proved in [10]. Roughly speaking (see Sect. 5 for a more detailed description), a bounded region Ω is almost filled by ε-circles γ from the regular hexagonal ε-circle packing of the plane. Denote this approximate circle packing of Ω by Ω_ε. By results of Andreev and Thurston, there is a combinatorially isomorphic circle packing D_ε of the unit disk D. Denote this isomorphism, suitably normalized, by $\gamma \mapsto \gamma'$: $\Omega_\varepsilon \to D_\varepsilon$. Let f_ε be the piecewise linear quasiconformal mapping determined by the associated triangulations. As $\varepsilon \to 0$, f_ε converges uniformly on compacta to the Riemann mapping function f: $\Omega \to D$.

Let HCP_N be N-generations of the regular hexagonal circle packing, and let HCP'_N be a circle packing that is combinatorially isomorphic to HCP_N. Assume that D is the smallest disk containing HCP_N, and that HCP'_N is contained in D. In this context we prove the following analog of the classical lemma of Schwarz. The radii R_0, R'_0 of the generation zero circles of HCP_N and HCP'_N satisfy $R'_0 \leqq a R_0$, where a is an absolute constant.

This Schwarz lemma analog is used to prove Theorem 6.2: Given $K \Subset \Omega$, there is an absolute constant M_K such that as $\varepsilon \to 0$, $\operatorname{rad}\gamma'/\operatorname{rad}\gamma \leqq M_K$ for all ε-circles γ which meet K. This uniform boundness result is a key ingredient for proving Theorem 6.3 and its corollary. These results show that $\partial f_\varepsilon/\partial z$ is bounded on compacta (L_∞ norm) uniformly as $\varepsilon \to 0$. Furthermore, $\partial f_\varepsilon/\partial \bar{z} \to 0$ uniformly on compacta (L_∞ norm). The following open question was raised in [10]: does $\operatorname{rad}\gamma'/\operatorname{rad}\gamma$ converge to $|f'|$? Theorem 6.3 shows that $|\partial f_\varepsilon/\partial z| - (\operatorname{rad}\gamma'/\operatorname{rad}\gamma) \to 0$ as $\varepsilon \to 0$; hence the open question is equivalent to: does $|\partial f_\varepsilon/\partial z| - |f'| \to 0$?

The proof of Theorem 5.1, the Schwarz lemma analog, makes use of the fact that in any HCP'_N, the average of the radii of the circles of generation N is never smaller that the radius of the generation zero circle multiplied by a

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positive absolute constant (i.e., independent of N and the particular packing). This result, Theorem 2.2, is proved as a consequence of discrete potential theory for hexagonal lattices. In the course of developing the necessary fundamentals we prove the following result (Theorem 3.2) on the existence of a "fundamental potential": There is a discrete real valued function λ defined on the hexagonal lattice such that λ is harmonic (i.e., the value at a lattice point is the average of the values at the six neighbors) except at 0 and $\lambda(\alpha)=\frac{1}{2\pi}\log|\alpha| + \text{const.} + O\left(\frac{1}{|\alpha|}\right)$ for all lattice points α.

Acknowledgements. I would like to thank Stefan Warschawski for many interesting discussions on this material. While trying to prove Theorem 2.2 I had interesting discussions with my colleagues, Mike Fredman and Janos Komlos. They found a combinatorial proof [6]. I subsequently found the potential theoretic proof given here.

2. Subharmonicity of the radii

A *circle packing* is a collection of circles in the plane whose interiors are disjoint. The *nerve* of the packing is the imbedded graph whose vertices are the centers of the circles; the line segment joining two vertices is an edge if and only if the corresponding circles are tangent. Two circle packings are *combinatorially equivalent* if their nerves are isomorphic. The regular hexagonal packing of the plane by circles of equal radii is denoted by HCP; its nerve determines a paving of the plane by equilateral triangles. We let HCP_N ($N=0,1,2,\ldots$) denote N generations of HCP starting from some base circle. There are $6N$ circles of generation N if $N\geqq 1$. We let HCP'_N denote a packing that is combinatorially equivalent to HCP_N. (Clearly the circles in an HCP'_N need not be of equal radii. However, if a packing is combinatorially equivalent to all of HCP then it is, in fact, HCP [10].)

An HCP'_1 packing consists of an inner circle surrounded by six tangent circles (Fig. 2.1). The following result for such packings is stated in [2; p. 576] (the reference for a proof given there appears to be incorrect). For the convenience of the reader we give a self-contained proof here.

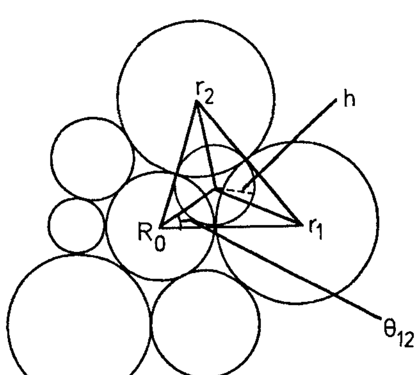

Fig. 2.1.

Lemma 2.1. *In any HCP_1' packing one has*

$$R_0 \leqq \tfrac{1}{6}(r_1 + r_2 + r_3 + r_4 + r_5 + r_6) \tag{2.1}$$

where R_0 is the radius of the inner circle and $r_1, r_2, \ldots, r_6$ are the radii of the outer circles.

Proof. Use the notation of Fig. 2.1 and consider three mutually tangent circles of radii R_0, r_1, r_2. The triangle of centers has area $[r_1 r_2 R_0(r_1 + r_2 + R_0)]^{\frac{1}{2}}$ by Heron's formula. This area can also be expressed as $h(r_1 + r_2 + R_0)$ where h is the radius of the inscribed circle. Thus $h = [r_1 r_2 R_0/(r_1 + r_2 + R_0)]^{\frac{1}{2}}$ and so we obtain

$$\tan \theta_{12} = \frac{1}{R_0}\sqrt{\frac{r_1 r_2 R_0}{r_1 + r_2 + R_0}} \tag{2.2}$$

for the vertex half angle θ_{12} (see Fig. 2.1). By the convexity of $\tan x$ on $0 < x < \pi/2$ we have (Jensen's Inequality; [7, p. 70ff.]).

$$\tan\left(\frac{\theta_{12} + \theta_{23} + \ldots + \theta_{61}}{6}\right) \leqq \tfrac{1}{6}(\tan \theta_{12} + \tan \theta_{23} + \ldots + \tan \theta_{61})$$

which, with the help of (2.2) and similar formulas, yields

$$\frac{1}{\sqrt{3}} \leqq \frac{1}{6R_0}\left(\sqrt{\frac{r_1 r_2 R_0}{r_1 + r_2 + R_0}} + \ldots + \sqrt{\frac{r_6 r_1 R_0}{r_6 + r_1 + R_0}}\right). \tag{2.3}$$

According to the inequality between geometric and arithmetic means we replace $r_1 r_2 R_0$ by $[(r_1 + r_2 + R_0)/3]^3$, and similarly for the remaining five terms in (2.3), and obtain the desired relation (2.1).

The next result extends the local property (2.1) to a global one.

Theorem 2.2. *There is an absolute constant c such that for any $N = 1, 2, \ldots$ and any packing HCP_N' which is combinatorially equivalent to N generations of the regular hexagonal circle packing one has*

$$R_0 \leqq \frac{c}{6N}(r_1 + r_2 + \ldots + r_{6N}) \tag{2.4}$$

where R_0 is the radius of the generation zero circle of HCP_N' and $r_1, \ldots, r_{6N}$ are the radii of the generation N circles of HCP_N'.

The proof of Theorem 2.2 will be given in Sect. 4. It makes use of Theorem 3.5 which, in turn, requires an analysis of discrete potential theory on a hexagonal lattice. This analysis is accomplished in Sect. 3. It is possible for a reader to proceed quickly to Sect. 4; only the basic definitions at the beginning of Sect. 3 and the statement of Theorem 3.5 will be needed.

Remark. The motivation for Theorem 2 is the analogy between circle packings and conformal mappings (see Sect. 5). Specifically, the analog of (2.4) is the subharmonic mean value property

(2.5) $$|f'(0)| \leqq \frac{1}{2\pi} \int_0^{2\pi} |f'(e^{i\theta})| \, d\theta$$

which holds for a function f analytic in the closed unit disk D. As described in Sect. 5, if f is a Riemann mapping function we imagine f transforming an infinitesimal circle packing of D (this corresponds to HCP_N) onto an infinitesimal circle packing of $f(D)$ (this corresponds to HCP_N'). We interpret $|f'(z)|$ as the ratio of the radii of the image circle and the preimage circle. Thus (2.5) says that for the infinitesimal packing of $f(D)$, the average radius of the outer circles dominates the radius of the center circle. Since the harmonic mean value formula

(2.6) $$\log|f'(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f'(e^{i\theta})| \, d\theta$$

holds with equality, one might suspect that a more precise form of Theorem 2.2 could be obtained by changing (among other things) the arithmetic mean to the geometric mean.

3. Discrete potential theory

Let $\omega = e^{i\pi/3}$ and let $h > 0$. The set $HL(h) = \{hm + hn\omega \colon m, n \in \mathbb{Z}\}$ is called the *hexagonal lattice of mesh h*. The *neighbors* of $\alpha \in HL(h)$ are the six points $\alpha + h\omega^k$, $0 \leqq k \leqq 5$. The lattice point α is of generation $\leqq N$ if there is a sequence $0 = \alpha_0, \alpha_1, \ldots, \alpha_N = \alpha$ such that α_j is a neighbor of α_{j-1} for $1 \leqq j \leqq N$. We let $HL(h, N)$ denote the subset of $HL(h)$ consisting of lattice points of generation $\leqq N$. The set of lattice points of generation exactly N is denoted $\partial HL(h, N)$. If the value of h is understood we often simplify the symbols $HL(h)$, $HL(h, N)$, $\partial HL(h, N)$ to H, H_N, ∂H_N.

If u is a complex valued function defined at $\alpha \in H$ and its six neighbors, then the *discrete Laplacian* of u at α is defined by

(3.1) $$D_h u(\alpha) = \frac{2}{3h^2} \left\{ \sum_{k=0}^{5} u(\alpha + h\omega^k) - 6u(\alpha) \right\}.$$

To motivate (3.1), suppose $U(x, y)$ is class C^4 on the convex hull of the six neighbors of α. Apply Taylor's formula in the form ($\xi_k = \operatorname{Re} \omega^k$, $\eta_k = \operatorname{Im} \omega^k$):

(3.2) $$U(\alpha + h\omega^k) = \sum_{j=0}^{3} \frac{h^j}{j!} \left(\xi_k \frac{\partial}{\partial x} + \eta_k \frac{\partial}{\partial y} \right)^j U(\alpha) + \frac{h^4}{4!} \left(\xi_k \frac{\partial}{\partial x} + \eta_k \frac{\partial}{\partial y} \right)^4 U(\beta_k)$$

where β_k lies on the segment joining α and $\alpha + h\omega^k$. Add the six equations (3.2) for $k = 0, 1, \ldots, 5$. After simplifying by $\sum \xi_k = \sum \eta_k = 0$, $\sum \xi_k^2 = \sum \eta_k^2 = 3$, $\sum \xi_k \eta_k = \sum \xi_k^3 = \sum \xi_k^2 \eta_k = \sum \xi_k \eta_k^2 = \sum \eta_k^3 = 0$, $\sum \xi_k^4 = 9/4$, $\sum \eta_k^4 = 3/4$, $\sum \xi_k^3 \eta_k = \sum \xi_k \eta_k^3 = 0$ one obtains

(3.3) $$\sum_{k=0}^{5} U(\alpha + h\omega^k) = 6U(\alpha) + \frac{3h^2}{2} (U_{xx}(\alpha) + U_{yy}(\alpha)) + h^4 E(\alpha)$$

where $|E(\alpha)| \leqq 9M_4/4!$ if M_4 is an upper bound for the magnitude of the fourth order partial derivatives of U on this region. Thus the discrete Laplacian defined by (3.1) approximates the classical Laplacian to second order:

$$|D_h U(\alpha) - \Delta U(\alpha)| \leqq \frac{h^2 M_4}{4}. \tag{3.4}$$

A real-valued function u defined on H_N is said to be *subharmonic on* H_N if $D_h u(\alpha) \geqq 0$ for $\alpha \in H_N - \partial H_N$. Such functions obviously obey the

Maximum principle. *Suppose u is subharmonic on H_N. If u attains its maximum on $H_N - \partial H_N$ then u is constant on H_N.*

The maximum principle implies the existence, as well as uniqueness, of solutions to Dirichlet's problem for Poisson's equation.

Proposition 3.1. *Suppose $f\colon \partial H_N \to \mathbb{R}$ and $F\colon H_N - \partial H_N \to \mathbb{R}$ are given. Then there is a unique function $u\colon H_N \to \mathbb{R}$ such that $u = f$ on ∂H_N and $D_h u = F$ on $H_N - \partial H_N$.*

Proof. Consider D_h as a linear transformation of the following finite dimensional real vector spaces: the domain space is all real-valued functions on H_N which vanish on ∂H_N, and the range space is all real-valued function on H_{N-1}. The maximum principle shows that the null space of D_h consists of zero alone. Since the domain and range have the same dimension, D_h is surjective as well as injective.

Therefore there is a function u_0 on H_N which vanishes on ∂H_N and satisfies $D_h u_0 = F$. Let v be the function on H_N which is equal to f on ∂H_N and vanishes elsewhere. Then there is a function v_0 which vanishes on ∂H_N and satisfies $D_h v_0 = D_h v$. The function $u = u_0 + v - v_0$ is the desired solution. It is clearly unique.

Let $\alpha \in H_N$ and $\beta \in H_{N-1}$, $N \geqq 2$. We use Proposition 3.1 to define the discrete Green's function $g_N(\alpha, \beta)$ for H_N as follows:

$$g_N(\alpha, \beta) = 0 \quad \text{if } \alpha \in \partial H_N \tag{3.5}$$

$$D_h g_N(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha \in H_{N-1} - \{\beta\} \\ \dfrac{-2\sqrt{3}}{3h^2} & \text{if } \alpha = \beta \end{cases} \tag{3.6}$$

where the Laplacian, as in (3.6), is always taken with respect to the first variable of $g_N(\cdot, \cdot)$. The maximum principle shows that $g_N(\alpha, \beta) > 0$ if $\alpha \in H_{N-1}$. The symmetry of g_N may be deduced easily from Proposition 3.4.

We want to prove the existence of a function on $HL(1)$ which is harmonic away from the origin and which grows like $(1/2\pi)\log|z|$ up to an error term of order $O(1/|z|)$. We have been unable to find this result in the literature. In the case of the square lattice the existence of such a fundamental potential was first proved by McCrea-Whipple [9]. A different proof, but without the $O(1/|z|)$ error estimate, appears in Spitzer [11] and Van der Pol [12]. Other references are: Wasow [14] and Forsythe-Wasow [5] for comments on McCrea-Whipple [9], in particular, that [9] actually shows that the error estimate is $O(1/|z|^2)$; Duffin [4] for the higher dimensional case which is considerably simpler.

Theorem 3.2. *There is a real-valued function λ on the hexagonal lattice $HL(1)$ of mesh 1 such that*

(i) $D_1\lambda(0)=\frac{2\sqrt{3}}{3}$,

(ii) $D_1\lambda(\alpha)=0$ *if* $\alpha\in HL(1)-\{0\}$,

(iii) $\lambda(\alpha)=\frac{1}{2\pi}\log|\alpha|+\text{const.}+O\left(\frac{1}{|\alpha|}\right)$ *as* $\alpha\to\infty$, $\alpha\in HL(1)$.

The proof of Theorem 3.2[1] will refer to the following Lemmas 3.3 and 3.4:

Lemma 3.3. *The discrete Laplacian of the function* $\alpha\mapsto e^{i\operatorname{Re}\alpha\bar{z}}$: $HL(1)\to\mathbb{C}$ *is*

$$D_1 e^{i\operatorname{Re}\alpha\bar{z}}=-\tfrac{8}{3}e^{i\operatorname{Re}\alpha\bar{z}}S(z) \tag{3.7}$$

where $S(z)$ is defined by

$$S(z)=\sin^2\frac{x}{2}+\sin^2\frac{x+y\sqrt{3}}{4}+\sin^2\frac{x-y\sqrt{3}}{4}, \qquad (z=x+iy). \tag{3.8}$$

The function S can be written

$$S(z)=\tfrac{3}{8}|z|^2-\tfrac{3}{128}|z|^4+E(z) \tag{3.9}$$

where $E(z)/|z|^4$ and its first order partial derivatives are uniformly bounded on compact subsets of $\mathbb{C}$.

Proof of Lemma 3.3. Direct application of Definition 3.1 gives

$$\begin{aligned} D_1 e^{i\operatorname{Re}\alpha\bar{z}} &=\tfrac{2}{3}e^{i\operatorname{Re}\alpha\bar{z}}\left[\sum_{k=0}^{2}\left(e^{i\operatorname{Re}\omega^k\bar{z}}+e^{-i\operatorname{Re}\omega^k\bar{z}}\right)-6\right] \\ &=-\tfrac{8}{3}e^{i\operatorname{Re}\alpha\bar{z}}\left[\sin^2\frac{x}{2}+\sin^2\frac{x+y\sqrt{3}}{4}+\sin^2\frac{x-y\sqrt{3}}{4}\right] \end{aligned} \tag{3.10}$$

which proves (3.7). Another computation shows that

$$\begin{aligned} 2S(z) &=1-\cos x+1-\cos\frac{x+y\sqrt{3}}{2}+1-\cos\frac{x-y\sqrt{3}}{2} \\ &=\frac{x^2}{2}+\tfrac{1}{8}(x+y\sqrt{3})^2+\tfrac{1}{8}(x-y\sqrt{3})^2-\frac{x^4}{4!}-\frac{(x+y\sqrt{3})^4}{4!\,16}-\frac{(x-y\sqrt{3})^4}{4!\,16}+\dots \\ &=\tfrac{3}{4}|z|^2-\tfrac{3}{64}|z|^4+2E(z) \end{aligned} \tag{3.11}$$

where

$$2E(z)=x^6\phi(x)+(x+y\sqrt{3})^6\phi(x+y\sqrt{3})+(x-y\sqrt{3})^6\phi(x-y\sqrt{3}) \tag{3.12}$$

[1] It should be remarked, although this fact will not be used, that conditions (ii) and (iii) of Theorem 3.2 determine λ uniquely up to an additive constant

and ϕ is the entire function

$$\phi(t)=\frac{1}{6!}-\frac{t^2}{8!}+\frac{t^4}{10!}-\dots.$$

One can check that the functions

$$\frac{x^6}{(x^2+y^2)^2},\quad \frac{(x+y\sqrt{3})^6}{(x^2+y^2)^2},\quad \frac{(x-y\sqrt{3})^6}{(x^2+y^2)^2}$$

and their first order partial derivatives are uniformly bounded on compact subsets of $\mathbb{C}$. By (3.12), the same is therefore true for $E(z)/|z|^4$.

Lemma 3.4. *Let Ω be a closed parallelogram in the plane. Let $f\colon \Omega\to\mathbb{R}$ be of uniformly bounded variation on Ω; that is, there is an M such that the variation of f, as a function of one variable, along any vertical or horizontal line in Ω is less than M. Then*

$$\iint_{\Omega} e^{i\operatorname{Re}\alpha\bar z} f(x,y)\,dx\,dy=O\left(\frac{1}{|\alpha|}\right) \tag{3.13}$$

as $\alpha\to\infty$, $\alpha\in HL(1)$.

Proof of Lemma 3.4. Let $\alpha=m+n\,e^{i\pi/3}=\mu+i\,\nu$ where $\mu=m+(n/2)$, $\nu=n\sqrt{3}/2$. Because of bounded variation we can integrate by parts and obtain

$$\int_{x_1(y)}^{x_2(y)} e^{i\operatorname{Re}\alpha\bar z} f(x,y)\,dx=\frac{e^{i\nu y}}{i\mu}\left(f(z)\,e^{i\mu x}\big|_{x_1(y)}^{x_2(y)}-\int_{x_1(y)}^{x_2(y)} e^{i\mu x}\,d_x f\right)$$
$$=O\left(\frac{1}{\mu}\right),$$

and therefore the integral in (3.13) is $O(1/\mu)$. Similarly, it is $O(1/\nu)$. Therefore it is $O(1/(|\mu|+|\nu|))=O(1/\sqrt{\mu^2+\nu^2})=O(1/|\alpha|)$.

Proof of Theorem 3.2. The potential λ for the unit hexagonal lattice $HL(1)$ is defined by

$$\lambda(\alpha)=\frac{\sqrt{3}}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{(1-e^{i(mu+nv)})\,du\,dv}{4\left(\sin^2\frac{u}{2}+\sin^2\frac{v}{2}+\sin^2\frac{u-v}{2}\right)},\qquad (\alpha=m+n\,e^{i\pi/3}). \tag{3.14}$$

Note that the linear change of variables

$$u=x,\qquad v=\tfrac{1}{2}x+\frac{\sqrt{3}}{2}y \tag{3.15}$$

transforms (3.14) into

$$\lambda(\alpha)=\frac{3}{8\pi^2}\iint_{\Omega}\frac{(1-e^{i\operatorname{Re}\alpha\bar z})\,dx\,dy}{4S(z)} \tag{3.16}$$

where $\Omega=\{z=x+iy\colon -\pi\leqq x\leqq\pi,\ -2\pi-x\leqq y\sqrt{3}\leqq 2\pi-x\}$ and S is defined by (3.8).

In order to discuss the convergence of the integral in (3.14) we note that

$$|1-e^{i(mu+mv)}| \leqq |m\,u+m\,v| \leqq \sqrt{m^2+n^2}\,\sqrt{u^2+v^2}$$

and that $\sin(t/2) \geqq t/\pi$ for $0 \leqq t \leqq \pi$. These estimates give

$$\frac{|1-e^{i(mu+nv)}|}{\sin^2\dfrac{u}{2}+\sin^2\dfrac{v}{2}+\sin^2\dfrac{u-v}{2}} \leqq \frac{\pi^2\sqrt{m^2+n^2}}{\sqrt{u^2+v^2}}$$

and the singularity $(u^2+v^2)^{-\frac{1}{2}}$ is integrable over the square.

Next we compute $D_1\lambda(\alpha)$ for λ in the form (3.16). First Lemma 3.1 and then a change of variables according to (3.15) gives

$$D_1\lambda(\alpha) = \frac{1}{4\pi^2}\iint_{\Omega} e^{i\operatorname{Re}\alpha\bar{z}}\,dx\,dy = \frac{\sqrt{3}}{6\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(mu+mv)}\,du\,dv$$

$$= \frac{\sqrt{3}}{6\pi^2}\left(\int_{-\pi}^{\pi} e^{imu}\,du\right)\left(\int_{-\pi}^{\pi} e^{inv}\,dv\right).$$

The final expression vanishes unless $m=n=0$, in which case it gives $D_1\lambda(0) = \dfrac{2\sqrt{3}}{3}$.

To prove part (iii) of the theorem we begin with (3.16) and decompose the integral as follows:

$$(3.17)\qquad \frac{32\pi^2}{3}\lambda(\alpha) = \tfrac{8}{3}\iint_{\Omega}(1-e^{i\operatorname{Re}\alpha\bar{z}})\frac{dx\,dy}{|z|^2} + \iint_{\Omega}(1-e^{i\operatorname{Re}\alpha\bar{z}})\frac{|z|^2-(8/3)S(z)}{S(z)\,|z|^2}\,dx\,dy.$$

In the second integral on the right of (3.17) replace $S(z)$ in the numerator by its expression in (3.9). We write

$$\frac{32\pi^2}{3}\lambda(\alpha) = I_1+I_2-I_3$$

where

$$(3.18)\qquad I_1 = \tfrac{8}{3}\iint_{\Omega}(1-e^{i\operatorname{Re}\alpha\bar{z}})\frac{dx\,dy}{|z|^2},$$

$$(3.19)\qquad I_2 = \iint_{\Omega}\frac{(1/16)\,|z|^4-(8/3)\,E(z)}{S(z)\,|z|^2}\,dx\,dy,$$

$$(3.20)\qquad I_3 = \iint_{\Omega} e^{i\operatorname{Re}\bar{z}}\,\frac{(1/16)\,|z|^4-(8/3)\,E(z)}{S(z)\,|z|^2}\,dx\,dy.$$

Since $\sin^2(t/2) \geqq t^2/\pi^2$ for $-\pi \leqq t \leqq \pi$ one has

$$(3.21)\qquad S(z) = \sin^2\frac{x}{2}+\sin^2\frac{x+y\sqrt{3}}{4}+\sin^2\frac{x-y\sqrt{3}}{4}$$

$$\geqq \frac{1}{\pi^2}\left(x^2+\left(\frac{x+y\sqrt{3}}{2}\right)^2+\left(\frac{x-y\sqrt{3}}{2}\right)^2\right) = \frac{3\,|z|^2}{2\pi^2}.$$

This estimate, together with the fact that $E(z)/|z|^4$ is bounded (Lemma 3.3), shows that I_2 converges.

Next we show that $I_3 = O(1/|\alpha|)$ by appealing to Lemma 3.4. In order to verify that the hypotheses of this lemma are satisfied it suffices to show that the function

$$\frac{(1/16)\,|z|^4 - (8/3)\,E(z)}{S(z)\,|z|^2} = \frac{|z|^2}{S(z)}\left(\frac{1}{16} - \frac{8}{3}\,\frac{E(z)}{|z|^4}\right) \tag{3.22}$$

is bounded and has bounded partial derivatives on Ω. Since $E(z)/|z|^4$ has these properties, it suffices to show that $|z|^2/S(z)$ has them. Inequality (3.21) shows that $|z|^2/S(z)$ is bounded. If we take partial derivatives of

$$\frac{|z|^2}{S(z)} = \frac{1}{\dfrac{3}{8} - \dfrac{3\,|z|^2}{128} + \dfrac{E(z)}{|z|^2}}$$

we find that they will be bounded provided $E(z)/|z|^2$ has bounded partial derivatives, which is true because it holds for each factor of $|z|^2(E(z)/|z|^4)$.

We now show that

$$I_1 = \frac{16\pi}{3} \log|\alpha| + \text{const.} + O\left(\frac{1}{|\alpha|}\right). \tag{3.23}$$

Let $\Delta \subset \Omega$ be the disk $\{|z| \leqq \pi\}$. Write

$$I_1 = \tfrac{8}{3} \iint_{\Omega-\Delta} \frac{dx\,dy}{|z|^2} - \tfrac{8}{3} \iint_{\Omega-\Delta} e^{i\,\mathrm{Re}\,\alpha\bar{z}} \frac{dx\,dy}{|z|^2} + \tfrac{8}{3} \iint_{\Omega} (1 - e^{i\,\mathrm{Re}\,\alpha\bar{z}}) \frac{dx\,dy}{|z|^2}. \tag{3.24}$$

The first integral on the right side of (3.24) is a constant. The second integral is $O(1/|\alpha|)$ by Lemma 3.4, which applies to this situation because the function which vanishes on Δ and is equal to $|z|^{-2}$ on $\Omega - \Delta$ is of uniformly bounded variation.

The third integral in (3.24) can be transformed as follows:

$$\begin{aligned} \tfrac{8}{3} \iint_{\Delta} (1 - e^{i\,\mathrm{Re}\,\alpha\bar{z}}) \frac{dx\,dy}{|z|^2} &= \tfrac{8}{3} \iint_{\Delta} (1 - \cos \mathrm{Re}\,\alpha\bar{z}) \frac{dx\,dy}{|z|^2} \\ &= \frac{32}{3} \int_0^{\pi/2} \int_0^{\pi} \frac{1 - \cos(|\alpha|\,r\cos\theta)}{r}\,dr\,d\theta. \end{aligned} \tag{3.25}$$

The inner integral above can be written

$$\begin{aligned} \int_0^{\pi} \frac{1 - \cos(|\alpha|\,r\cos\theta)}{r}\,dr &= \int_0^{|\alpha|\pi\cos\theta} \frac{1 - \cos u}{u}\,du \\ &= \int_1^{|\alpha|\pi\cos\theta} \frac{du}{u} + \int_{|\alpha|\pi\cos\theta}^{\infty} \frac{\cos u}{u}\,du + \gamma \end{aligned} \tag{3.26}$$

where $\gamma = \int_0^1 (1 - \cos u)\,(du/u) - \int_1^\infty \cos u\,(du/u)$ is Euler's constant. We now have

(3.27)
$$\frac{32\pi^2}{3}\lambda(\alpha)=I_1+\text{const.}+O\left(\frac{1}{|\alpha|}\right)$$
$$=\text{const.}+O\left(\frac{1}{|\alpha|}\right)+\frac{32}{3}\int_0^{\pi/2}\int_1^{|\alpha|\pi\cos\theta}\frac{du}{u}\,d\theta+\int_0^{\pi/2}\int_{|\alpha|\pi\cos\theta}^{\infty}\frac{\cos u}{u}\,du\,d\theta$$
$$=\text{const.}+O\left(\frac{1}{|\alpha|}\right)+\frac{16\pi}{3}\log|\alpha|+\frac{32}{3}\int_0^{\pi/2}\int_{|\alpha|\pi\cos\theta}^{\infty}\frac{\cos u}{u}\,du\,d\theta.$$

The proof will be complete if we show that

$$\int_0^{\pi/2}\int_{\rho\cos\theta}^{\infty}\frac{\cos u}{u}\,du\,d\theta=O\left(\frac{1}{\rho}\right)\qquad\text{as } \rho\to\infty$$

or equivalently, by the substitution $t=\rho\cos\theta$, that

(3.28)
$$\int_0^{\rho}\int_t^{\infty}\cos u\frac{du}{u}\frac{dt}{\sqrt{\rho^2-t^2}}=O\left(\frac{1}{\rho}\right).$$

Split the above integral into a sum $\int_0^1\int_t^\infty+\int_1^\rho\int_t^\infty$. In the first integral use the obvious estimate

$$\left|\int_t^{\infty}\cos u\frac{du}{u}\right|\leqq\log\frac{1}{t}+k,\qquad 0<t<1$$

where $k=\int_1^\infty\cos u(du/u)$, and obtain

(3.29)
$$\left|\int_0^1\int_t^{\infty}\cos u\frac{du}{u}\frac{dt}{\sqrt{\rho^2-t^2}}\right|\leqq\int_0^1\log\frac{1}{t}\frac{dt}{\sqrt{\rho^2-t^2}}+k\int_0^1 d\left(\sin^{-1}\frac{t}{\rho}\right)=O\left(\frac{1}{\rho}\right).$$

In the second integral we use the estimate

(3.30)
$$\int_t^{\infty}\cos u\frac{du}{u}=\frac{-\sin t}{t}+\frac{O(1)}{t^2}$$

which follows from integration-by-parts:

$$\int_t^{\infty}\cos u\frac{du}{u}=\frac{\sin u}{u}\bigg|_t^{\infty}+\int_t^{\infty}\sin u\frac{du}{u^2}=\frac{-\sin t}{t}+\frac{O(1)}{t^2}.$$

Thus

(3.31)
$$\int_1^{\rho}\int_t^{\infty}\cos u\frac{du}{u}\frac{dt}{\sqrt{\rho^2-t^2}}=\int_1^{\rho}\frac{-\sin t\,dt}{t\sqrt{\rho^2-t^2}}+\int_1^{\rho}\frac{O(1)\,dt}{t^2\sqrt{\rho^2-t^2}}.$$

The second integral is clearly finite and the first can be estimated by

$$(3.32)\quad \int_1^\rho \frac{\sin t\,dt}{t\sqrt{\rho^2-t^2}} = \frac{1}{\rho}\int_{1/\rho}^1 \frac{\sin(\rho s)}{s}\frac{ds}{\sqrt{1-s^2}}$$
$$= \frac{1}{\rho}\int_{1/\rho}^1 \left(\frac{\sin(\rho s)}{s\sqrt{1-s^2}} - \frac{\sin(\rho s)}{s}\right) ds + \frac{1}{\rho}\int_{1/\rho}^1 \frac{\sin(\rho s)}{s}\,ds$$
$$= \frac{1}{\rho}\int_{1/\rho}^1 \frac{\sin(\rho s)\,s\,ds}{\sqrt{1-s^2}(1+\sqrt{1-s^2})} + \frac{1}{\rho}\int_1^\rho \frac{\sin t}{t}\,dt = O\left(\frac{1}{\rho}\right).$$

This completes the proof of (3.28) and, consequently, Theorem 3.2.

The Green's function $g_N(\alpha,\beta)$ for $HL(h,N)$ was defined by (3.5) and (3.6). We shall need the following upper estimate for $g_N(\alpha,0)$ as α varies over the lattice points of generation $N-1$.

Proposition 3.3. *There is an absolute constant b such that for all $N\geqq 2$, if $\alpha\in\partial HL(h,N-1)$ then*

$$(3.33)\qquad g_N(\alpha,0)\leqq\frac{b}{N}.$$

Proof. We may assume unit mesh size $h=1$. In proving (3.33) we may also assume that α lies in the upper horizontal edge of $HL(1,N-1)$, that is, among the points (recall $\omega=e^{i\pi/3}$)

$$(3.34)\qquad E_N=\{(N-1)\,\omega,(N-1)\,\omega-1,(N-1)\,\omega-2,\dots,(N-1)\,\omega-(N-1)\}.$$

The method we are going to use requires special care at corner points. For that reason we replace $HL(1,N)$ by a larger configuration $\tilde H_N$ obtained from $HL(1,2N)$ by removing the upper N rows of lattice points and the lower N rows of lattice points (see Fig. 3.1). We define and construct the discrete Green's function $\tilde g_N(\alpha,0)$ for $\tilde H_N$, just as was done in the case of H_N. The maximum Principle implies that $g_N(\alpha,0)\leqq\tilde g_N(\alpha,0)$ for $\alpha\in H_N$. Therefore it suffices to prove

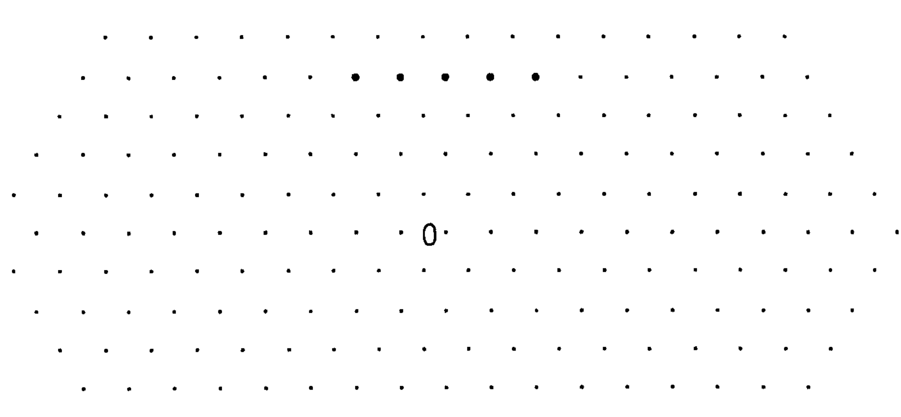

Fig. 3.1.

(3.35)
$$\tilde{g}_N(\alpha, 0) \leqq \frac{\text{const.}}{N}$$

for $\alpha \in E_N$.

Let us show that inequality (3.35) holds if the discrete objects $\tilde{H}_N, \tilde{g}_N$ are replaced by analogous nondiscrete objects Ω, G. That is, let Ω be the convex hull of $\tilde{H}_N$, a region with polygonal border. Let $G(z,0)$ be the classical Green's function for Ω with pole at 0. We wish to show that

(3.36)
$$G(a, 0) \leqq \frac{\text{const.}}{N}$$

if a lies on the convex hull of E_N.

Let $z \mapsto \phi(z/N)$ be the Riemann mapping function of Ω onto the unit disk, normalized to send the origin to the origin. That is, ϕ is the Riemann map of the region Ω_1 obtained from Ω by homothetically shrinking it to diameter 4. Let M be the maximum of $|\phi'(\zeta)|$ as $N\zeta$ varies over the convex hull of $HL(1, N)$. The reflection principle shows that M is finite.

Let a be a point on the convex hull of E_N. Let z_0 be a point on the upper edge of the convex hull of $HL(1, N)$ such that $|z_0 - a| \leqq 1$. Then

(3.37)
$$\left|\phi\left(\frac{z_0}{N}\right) - \phi\left(\frac{a}{N}\right)\right| \leqq \left|\int_{a/N}^{z_0/N} \phi'(\zeta)\, d\zeta\right| \leqq \frac{M}{N}.$$

With the help of (3.37) we obtain, for N sufficiently large,

$$G(a, 0) = -\log\left|\phi\left(\frac{a}{N}\right)\right| \leqq -\log\left|1 - \frac{M}{N}\right| \leqq \frac{2M}{N}.$$

This proves (3.36).

To complete the proof we show that (3.36) implies (3.35) by proving

(3.38)
$$|\tilde{g}_N(\alpha, 0) - G(\alpha, 0)| \leqq \frac{\text{const.}}{N}, \qquad \alpha \in E_N.$$

Introduce the following functions: λ_0 is a discrete harmonic function on $HL(1) - \{0\}$ which satisfies

(3.39)
$$\lambda_0(\alpha) = \frac{1}{2\pi} \log|\alpha| + O\left(\frac{1}{|\alpha|}\right)$$

(the existence of λ_0 is assured by Theorem 3.2 which shows $\lambda_0 = \lambda + \text{const.}$); $P_d[f]$ is the solution of the discrete Dirichlet problem for $\tilde{H}_N$ with boundary values f; and $P_c[f]$ is the solution of the classical Dirichlet problem for Ω with boundary values f. Since

$$G(z, 0) = \frac{1}{2\pi} \log \frac{1}{|z|} - P_c\left[\frac{1}{2\pi} \log \frac{1}{|z|}\right](z),$$

(3.38) will follow from the three inequalities (α varies over E_N):

(3.40)
$$\left|\tilde{g}_N(\alpha, 0) + \lambda_0(\alpha) - P_d\left[\frac{1}{2\pi} \log|z|\right](\alpha)\right| \leqq \frac{\text{const.}}{N},$$

(3.41) $$\left|P_c\left[\frac{1}{2\pi}\log\frac{1}{|z|}\right](\alpha)-P_d\left[\frac{1}{2\pi}\log\frac{1}{|z|}\right](\alpha)\right|\leqq\frac{\text{const.}}{N},$$

(3.42) $$\left|\lambda_0(\alpha)-\frac{1}{2\pi}\log|\alpha|\right|<\frac{\text{const.}}{N}.$$

Estimate (3.40) follows from the Maximum Principle since the function on the left is harmonic for $\alpha\in\tilde{H}_N-\partial\tilde{H}_N$ and is equal to $\lambda_0(\beta)-\frac{1}{2\pi}\log|\beta|$ for $\beta\in\partial\tilde{H}_N$. These boundary values are $O(1/|\beta|)$ by (3.39), and $|\beta|=O(N)$ for $\beta\in\partial\tilde{H}_N$.

Estimate (3.42) follows from (3.39) since $|\alpha|=O(N)$ for $\alpha\in E_N$.

Estimate (3.41) is, from a certain viewpoint, a fundamental problem of numerical analysis: to estimate the error between a classical solution to Dirichlet's problem and its finite difference analogue. Such error estimates are easy to derive if the boundary value function and the boundary of the region are both sufficiently smooth. The case of nonsmooth boundary values and boundaries was investigated in Laasonen [8]. The following is a consequence of Laasonen's discussion. *If a region Ω' has a piecewise smooth boundary, and if the interior angles at all corners of Ω' are less than π, then the error $\varepsilon_h(z)$ between the solution to the classical Dirichlet problem with analytic boundary values and its finite difference analogue for the square grid lattice of mesh h satisfies $\varepsilon_h(z)=O(h)$ as $h\to 0$, uniformly on compact subsets of $\Omega'\cup\partial\Omega'-\{$corner points of $\partial\Omega'\}$.*

Only two modifications to Laasonen's argument [8] are needed to make the above statement apply to the hexagonal grid of mesh h; both modifications are completely trivial. The first one is to replace Eq. (1) of [8] with Eqs. (3.3) and (3.4) of the present paper. The second modification is to verify that $D_h|z-z_0|^2=4$, whether this discrete Laplacian refers to the hexagonal or the square grid.

To prove (3.41) we replace Ω and $\tilde{H}_N$ by their images Ω' and $\tilde{H}'_N$ under the homothetic contraction $z\mapsto z/N$. Let P'_d, P'_c be the operators which assign to a function on $\partial\Omega'$ the solution to the discrete (respectively classical) Dirichlet problem for Ω' (respectively $\tilde{H}'_N$). Laasonen's result gives

(3.43) $$\left\|P'_c\left[\frac{1}{2\pi}\log\frac{1}{|z|}\right]-P'_d\left[\frac{1}{2\pi}\log\frac{1}{|z|}\right]\right\|_K=O\left(\frac{1}{N}\right)$$

where the supremum norm is over the subset $K\subset\Omega'$ consisting of all z/N such that $z\in HL(1,N)$; (3.43) is equivalent to (3.41). This completes the proof of Proposition 3.3.

The next result is an analog of Green's second identity. We follow the treatment in Duffin [4] which requires only superficial modifications to apply to the hexagonal lattice.

Proposition 3.4. *Let u, v be functions defined on H_N, the first N generations of the hexagonal lattice of mesh h. Then*

(3.44) $$\sum_{\alpha\in H_{N-1}} v(\alpha)D_h u(\alpha)-u(\alpha)D_h v(\alpha)=\frac{2}{3h^2}\sum_{(\alpha,\beta)} v(\alpha)u(\beta)-u(\alpha)v(\beta)$$

where the second sum is over all pairs (α,β) *such that* $\alpha\in\partial H_{N-1}$, $\beta\in\partial H_N$ *and* α *is a neighbor of* β.

Proof. Set $u'(\alpha)=u(\alpha)$ for $\alpha\in H_{N-1}$ and $u'(\alpha)=0$ for all other points α in H, the infinite hexagonal lattice of mesh h. Define v' similarly. Then

$$\begin{aligned}\sum_{\alpha\in H} v'(\alpha)D_h u'(\alpha) &= \sum_{\alpha\in H} v'(\alpha)\frac{2}{3h^2}\left(\sum_{k=0}^{5} u'(\alpha+h\omega^k)-6u'(\alpha)\right)\\ &= \sum_{\gamma\in H} u'(\gamma)\frac{2}{3h^2}\left(\sum_{k=0}^{5} v'(\gamma-h\omega^k)-6v'(\gamma)\right)\\ &= \sum_{\gamma\in H} u'(\gamma)D_h v'(\gamma).\end{aligned}$$

Thus

(3.45) $$\sum_{\alpha\in H} v'(\alpha)D_h u(\alpha)-u'(\alpha)D_h v'(\alpha)=0.$$

Group the nonzero terms of (3.45) into two parts:

(3.46) $$\sum_{\alpha\in H_{N-1}} v'(\alpha)D_h u'(\alpha)-u'(\alpha)D_h v'(\alpha)+\sum_{\alpha\in H_{N-2}} v'(\alpha)D_h u'(\alpha)-u'(\alpha)D_h v'(\alpha)=0.$$

For $\alpha\in\partial H_{N-1}$ we have $D_h u'(\alpha)=D_h u(\alpha)-\frac{2}{3h^2}\sum u(\beta)$ where the sum is over the points $\beta\in\partial H_N$ which are neighbors of α. A similar formula holds for v'. The second summation in (3.46) can be rewritten with u, v instead of u', v'. These observations transform (3.46) into (3.44) as desired.

We are now able to prove the main result we need for this section.

Theorem 3.5. *There is an absolute constant* c *with the following property. Let* u *be a positive subharmonic function on* H_N, *the first* N *generations of the hexagonal lattice of mesh* h. *Let* $\partial H_N=\{\beta_1,\beta_2,\dots,\beta_{6N}\}$. *Then*

(3.47) $$u(0)\leq\frac{c}{6N}(u(\beta_1)+u(\beta_2)+\dots+u(\beta_{6N})).$$

Proof. Apply Proposition 3.4 with $v(\alpha)=g_N(\alpha,0)$ where g_N is the discrete Green's function for H_N defined in (3.5) and (3.6); one obtains

(3.48) $$(\text{something positive})+\frac{2\sqrt{3}}{3h^2}u(0)=\frac{2}{3h^2}\sum_{(\alpha,\beta)} g_N(\alpha,0)u(\beta).$$

In the summation above there are at most two α's for each $\beta\in\partial H_N$, and Proposition 3.3 gives an estimate b/N for $g_N(\alpha,0)$ since $\alpha\in\partial H_{N-1}$. In this way we are led to (3.47).

4. Proof of Theorem 2.2

We are considering a circle packing HCP'_N which is combinatorially equivalent to N generations of the regular hexagonal packing of circles of equal radii. After associating the lattice point $0 \in HL(1,N)$ with the generation 0 circle of HCP'_N and arbitrarily assigning the lattice point $1 \in HL(1,N)$ to a first generation circle of HCP'_N we obtain a natural bijection of $HL(1,N)$ with HCP'_N. Let $r(\alpha)$ be the radius of the circle in HCP'_N associated to the lattice point $\alpha \in HL(1,N)$. By Lemma 2.1, $r\colon HL(1,N) \to \mathbb{R}$ is a discrete subharmonic function. Now apply Theorem 3.5 and obtain the desired inequality (2.4).

5. Schwarz's lemma for circle packings

A conformal mapping of a disk into itself has a derivative at the center which is no greater than one in modulus. Theorem 2.2 allows us to prove the following analogous result for circle packing isomorphisms.

Theorem 5.1. *There is an absolute constant a with the following property. Let HCP_N be N generations of the regular hexagonal circle packing. Let D be the smallest disk which contains HCP_N. Let HCP'_N be any circle packing combinatorially equivalent to HCP_N and also contained in D. Then*

$$R'_0 \leqq a R_0 \tag{5.1}$$

where R_0 and R'_0 are the generation zero circles in HCP_N and HCP'_N.

Proof. Let r_{kj}, $1 \leqq j \leqq 6k$, be the radii of the generation k circles of HCP'_N. Then

$$\left(\sum_{j=1}^{6k} r_{kj}\right)^2 \leqq 6k \sum_{j=1}^{6k} r_{kj}^2.$$

Use Theorem 2.2 to estimate the left-hand side and obtain

$$6k c^{-2} R_0'^2 \leqq \sum_{j=1}^{6k} r_{kj}^2.$$

Sum from $k=1$ to N to get

$$3c^{-2} R_0'^2 N(N+1) \leqq \sum_{k=1}^{n} \sum_{j=1}^{6k} r_{kj}^2 \leqq (\operatorname{rad} D)^2 = (2N+1)^2 R_0^2.$$

Thus (5.1) holds with

$$a^2 = \frac{c^2}{3} \max_{1 \leqq N} \left(4 + \frac{1}{N(N+1)}\right) = \frac{3c^2}{2}.$$

6. Applications to conformal mapping

We now consider the situation discussed in Rodin-Sullivan [10], namely, the approximation of the Riemann mapping function by circle packing isomorphisms. We are given a bounded region Ω in the plane and points z_0, z_1 in Ω. For each $\varepsilon>0$, form the circle packing Ω_ε defined as follows. Consider the regular hexagonal circle packing of the plane by circles of radius ε and let H_ε consist of those circles contained in Ω. Let I_ε, the set of inner circles, consist of all circles γ such that γ is the last term in a finite sequence of circles with the properties: (i) each circle in the sequence belongs to H_ε, (ii) the six H_ε-neighbors of each circle in the sequence also belongs to H_ε, (iii) each circle in the sequence is tangent to the preceding circle, and (iv) the flower of the initial circle in the sequence contains z_0 (the *flower* of a circle in H_ε or a combinatorially equivalent circle packing is the closed connected set consisting of that circle and its interior, the six neighboring circles and their interiors, and the six triangular interstices so formed).

Let B_ε, the set of *border circles*, consist of all circles from H_ε which are not inner circles but which are tangent to inner circles. The border circles are contained in Ω and can be arranged in a sequence such that each one is tangent to its predessesor, and the first is tangent to the last. The polygonal line joining successive centers of the circles in this *border cycle* is a Jordan curve which surrounds all the inner circles I_ε. Define Ω_ε to be the union of I_ε and B_ε. We shall refer to Ω_ε as the ε-circle packing approximation to Ω_ε (with distinguished point z_0).

By Andreev's theorem [1] (see also [13]) there is a circle packing D_ε contained in the unit disk D which is combinatorially equivalent to Ω_ε and such that all the circles of D_ε which correspond to border circles of Ω_ε are tangent to the circumference of D. W. Thurston conjectured that this circle correspondence $\Omega_\varepsilon \to D_\varepsilon$, suitably normalized, converges to the Riemann map of Ω onto D as $\varepsilon \to 0$. By *suitably normalized* we shall always mean the following. Perform a Mobius transformation fixing D so that a circle of Ω_ε whose flower contains z_0 corresponds to a circle of D_ε whose flower contains the origin. Then perform a rotation of D so that a circle of Ω_ε whose flower contains a prescribed point z_1 of Ω will correspond to a circle of D_ε which lies on the positive real radius of D.

This conjecture is proved in Rodin-Sullivan [10]. It is shown there that as $\varepsilon \to 0$, the piecewise linear mappings f_ε converge to the Riemann map $f: \Omega \to D$, where f_ε is the simplicial map of the triangulated regions determined by the canonical imbeddings of the nerves of Ω_ε and D_ε. The normalization of D_ε described above implies that f is normalized by $f(z_0)=0$ and $\operatorname{Re} f(z_1)>0$.

Consider again the circle packing isomorphism $\gamma \mapsto \gamma'$ of $\Omega_\varepsilon \to D_\varepsilon$; here as elsewhere we intend that D_ε is to be suitably normalized. It will be very useful to know that the map $\gamma \mapsto \operatorname{rad} \gamma'/\operatorname{rad} \gamma$ is uniformly bounded above (independently of ε) on compact subsets of Ω.

Theorem 6.2. *Let K be a compact subset of Ω. There is a constant M_K with the following property. Let $\varepsilon>0$ be sufficiently small and let $\gamma \mapsto \gamma'$ be the circle packing isomorphism of an ε-circle packing approximation Ω_ε of Ω onto a suitably normalized circle packing D_ε of the unit disk D. Then* $\operatorname{rad} \gamma'/\operatorname{rad} \gamma \leq M_K$ *for all circles γ of Ω_ε which intersect K.*

Proof. For sufficiently small $\varepsilon>0$, consider a circle γ in Ω_ε such that $\gamma\cap K\neq\emptyset$. Let N be maximal with respect to the property that Ω_ε contains an HCP_N configuration centered at γ. Let Δ be the smallest disk containing this HCP_N. The isomorphism $\Omega_\varepsilon\to D_\varepsilon$ makes this HCP_N correspond to an HCP_N' in the unit disk D. If we rescale the unit disk to a disk the size of Δ we may apply Theorem 6.1, the Schwarz lemma analog, to obtain

$$\lambda \operatorname{rad}\gamma' \leqq a \operatorname{rad}\gamma \tag{6.2}$$

where λ is the scaling factor, $\lambda=\operatorname{rad}\Delta$. We have

$$\tfrac{1}{2}\operatorname{dist}(K,\mathbb{C}-\Omega)\leqq\lambda\leqq\operatorname{dist}(K,\mathbb{C}-\Omega)$$

and therefore (6.2) gives an upper bound independent of ε of the desired type:

$$\frac{\operatorname{rad}\gamma'}{\operatorname{rad}\gamma}\leqq\frac{2a}{\operatorname{dist}(K,\mathbb{C}-\Omega)}.$$

This completes the proof of Theorem 6.2.

To a bounded region Ω and sufficiently small $\varepsilon>0$ we have associated a normalized circle packing isomorphism $\gamma\mapsto\gamma'$ of $\Omega_\varepsilon\to D_\varepsilon$ and a corresponding piecewise linear quasi-conformal mapping f_ε of the triangulated regions formed by the canonical imbedding of the nerves of Ω_ε and D_ε. As $\varepsilon\to0$ the mappings f_ε converge uniformly on compact subsets of Ω to the similarly normalized Riemann mapping function f of Ω onto D. It is not yet known if the map $z\mapsto\operatorname{rad}\gamma'/\operatorname{rad}\gamma$, where γ is some circle in Ω_ε whose flower contains z, converges to $|f'(z)|$ as $\varepsilon\to0$. Theorem 6.2 allows us to prove that this problem is equivalent to a problem on the convergence of the partial derivatives of f_ε.

Theorem 6.3. *As $\varepsilon\to0$ the maps $z\mapsto\operatorname{rad}\gamma'/\operatorname{rad}\gamma$ described above converge to the map $z\mapsto|f'(z)|$ uniformly on compacta of Ω if and only if for each compact K in Ω*

$$\left\|\left|\frac{\partial f_\varepsilon}{\partial z}\right|-|f'|\right\|_{K,\infty}\to0$$

where the norm is the essential supremum on K.

Proof. Consider a circle γ in Ω_ε and two tangent circles neighboring it. Let Δ denote the triangle formed by their centers. Suppose the isomorphism $\Omega_\varepsilon\to D_\varepsilon$ maps these circles to three mutually tangent circles of radii r_1, r_2, and r_3 with r_1 the radius of γ', the image of γ. A calculation, the details of which are given in Lemma 6.4 below, shows that the linear transformation f_ε restricted to Δ satisfies

$$2\varepsilon^2\left|\frac{\partial f_\varepsilon}{\partial z}\right|^2=\tfrac{1}{12}[(r_1+r_2)^2+(r_1+r_3)^2+(r_2+r_3)^2]+\frac{1}{\sqrt{3}}[(r_1r_2r_3(r_1+r_2+r_3)]^{\frac{1}{2}}, \tag{6.3}$$

$$2\varepsilon^2\left|\frac{\partial f_\varepsilon}{\partial \bar z}\right|^2=\tfrac{1}{12}[(r_1+r_2)^2+(r_1+r_3)^2+(r_2+r_3)^2]-\frac{1}{\sqrt{3}}[r_1r_2r_3(r_1+r_2+r_3)]^{\frac{1}{2}}. \tag{6.4}$$

Choose N to be maximal with respect to the property that there is an HCP_N configuration contained in Ω_ε and centered at γ. If we restrict γ to intersect a fixed compact subset K of Ω then $N\to\infty$ as $\varepsilon\to 0$. By the Hexagonal Packing Lemma of [9] we have $r_j/r_1=1+O(s_N)$, $(j=2,3)$, where $s_N\to 0$ as $N\to\infty$. If we use this $O(s_N)$ notation and the fact that $r_1/\varepsilon\leqq M_K$ by Theorem 6.2, then (6.3) and (6.4) can be written

$$\left|\frac{\partial f_\varepsilon}{\partial z}\right|^2=\left(\frac{r_1}{\varepsilon}\right)^2+O(s_N),\qquad \left|\frac{\partial f_\varepsilon}{\partial \bar z}\right|^2=O(s_n). \tag{6.5}$$

Theorem 6.3 follows from the first Eq. (6.5). The proof has also shown:

Corollary. *The quasi-conformal mappings f_ε have the following properties: $\partial f_\varepsilon/\partial z$ is uniformly bounded on compact subsets in the L_∞ norm, and $\partial f_\varepsilon/\partial \bar z$ converges to zero on compact subsets in the L_∞ norm.*

We now give the lemma used in the proof of Theorem 6.3.

Lemma 6.4. *Let Δ be the triangle formed by the centers of three mutually tangent unit circles. Let Δ' be the triangle formed by the centers of three mutually tangent circles of radii r_1, r_2, r_3. Let T be a linear transformation which maps Δ onto Δ'. Then*

$$2\left|\frac{\partial T}{\partial z}\right|^2=\tfrac{1}{12}\left[(r_1+r_2)^2+(r_1+r_3)^2+(r_2+r_3)^2\right]+\frac{1}{\sqrt{3}}\left[r_1 r_2 r_3(r_1+r_2+r_3)\right]^{\frac{1}{2}}, \tag{6.6}$$

$$2\left|\frac{\partial T}{\partial \bar z}\right|^2=\tfrac{1}{12}\left[(r_1+r_2)^2+(r_1+r_3)^2+(r_2+r_3)^2\right]-\frac{1}{\sqrt{3}}\left[r_1 r_2 r_3(r_1+r_2+r_3)\right]^{\frac{1}{2}}. \tag{6.7}$$

Proof. Place the vertices of Δ at q, $q\omega$, $q\omega^2$ where $q=2/\sqrt{3}$ and $\omega=e^{2\pi i/3}$. Write T in the form $T(z)=az+b\bar z$. Add the three equations:

$$(r_1+r_2)^2=|T(q)-T(q\omega)|^2$$
$$(r_1+r_3)^2=|T(q)-T(q\omega^2)|^2$$
$$(r_2+r_3)^2=|T(q\omega)-T(q\omega^2)|^2$$

and write the result in the form

$$|a|^2+|b|^2=\tfrac{1}{12}\left[(r_1+r_2)^2+(r_1+r_3)^2+(r_2+r_3)^2\right]. \tag{6.8}$$

The area of Δ' is $[r_1 r_1 r_3(r_1+r_2+r_3)]^{\frac{1}{2}}$ and hence the determinant of T must be

$$|a|^2-|b|^2=\frac{1}{\sqrt{3}}\left[r_1 r_2 r_3(r_1+r_2+r_3)\right]^{\frac{1}{2}}. \tag{6.9}$$

Equations (6.6) and (6.7) follow immediately from (6.8) and (6.9).

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