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On Tits' conjecture and other questions concerning Artin and generalized Artin groups

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§ 1. Introduction

Let Γ be a graph with vertex set $\mathbf{x}$. By this we mean the elements of $\mathbf{x}$ (the *vertices* of Γ), together with certain two element subsets of $\mathbf{x}$ (the *edges* of Γ). Let ϕ be a function which assigns to each edge $\{x, y\}$ of Γ the symmetrized closure[1] of a word of the form

$$(x y x \ldots)(y x y \ldots)^{-1}$$

where the two bracketed words have the same length >1. Let $\mathscr{A}(\Gamma, \phi)$ be the presentation with generating set $\mathbf{x}$ and defining relators

$$\bigcup_{\{x, y\}} \phi\{x, y\} \qquad (\{x, y\} \text{ an edge in } \Gamma).$$

The group $A(\Gamma, \phi)$ defined by $\mathscr{A}(\Gamma, \phi)$ is called an *Artin group*. Such groups have been widely studied [1–8, 10].

Our aim is to investigate certain Artin groups (and generalizations of these groups) using small cancellation theory. For the reader's convenience, we briefly review the main definitions (see [9], Chap. V, for further details). Let $\mathscr{P}$ be a group presentation, say $\mathscr{P} = \langle \mathbf{y}; \mathbf{r} \rangle$, where $\mathbf{r}$ is symmetrized. A word P on $\mathbf{y}$ is a *piece* (relative to $\mathscr{P}$) if there are distinct elements PU, PV of $\mathbf{r}$. We say that $\mathscr{P}$ satisfies the small cancellation condition $C(p)$ (p a positive integer) if no element of $\mathbf{r}$ is a product of less than p pieces. In this paper we will be interested in the $C(4)$ condition. This condition is only useful when combined with a second condition, $T(4)$, defined as follows: $\mathscr{P}$ satisfies $T(4)$ if and only if, whenever R_1, R_2, R_3 are elements of $\mathbf{r}$ with $R_1 \neq R_2^{-1}$, $R_2 \neq R_3^{-1}$, $R_3 \neq R_1^{-1}$, then one of the products $R_1 R_2$, $R_2 R_3$, $R_3 R_1$ is freely reduced.

It is not difficult to show that $\mathscr{A}(\Gamma, \phi)$ satisfies the small cancellation condition $C(4)$. It is thus of interest to know when the presentation also satisfies

[1] The *symmetrized closure* $\mathbf{s}^*$ of a set $\mathbf{s}$ of words consists of all cyclic permutations of non-empty elements of $\mathbf{s}$, together with the inverses of these elements. A set $\mathbf{s}$ is *symmetrized* if $\mathbf{s}^* = \mathbf{s}$

$T(4)$. We will see below, as a special case of a more general result (Theorem 1), that *$\mathcal{A}(\Gamma, \phi)$ satisfies $T(4)$ if and only if Γ contains no triangles.* Several results follow from this using small cancellation theory. (In each of the statements below it is assumed that Γ does not contain a triangle.)

1. If Γ is finite then the word and conjugacy problems are solvable for the presentation $\mathcal{A}(\Gamma, \phi)$.

2. If $\mathbf{x}_0$ is a subset of $\mathbf{x}$ and Γ_0 is the induced subgraph on $\mathbf{x}_0$, then the natural mapping of $A(\Gamma_0, \phi)$ into $A(\Gamma, \phi)$ is an embedding.

3. If $\mathbf{x}_1$ is a subset of $\mathbf{x}$ and no two elements of $\mathbf{x}_1$ are adjacent in Γ, then the subgroup of $A(\Gamma, \phi)$ generated by $\mathbf{x}_1$ is free, with free basis $\mathbf{x}_1$.

4. (Tits' conjecture [2] for "triangle-free" Artin groups.) Any relation between the elements x^2 ($x \in \mathbf{x}$) is a consequence of the relations

$$s^2 t^2 = t^2 s^2,$$

where $\{s, t\}$ ranges over all edges of Γ for which $\phi\{s, t\}$ is the symmetrized closure of $\{sts^{-1}t^{-1}\}$.

The first of these results follows from general theorems concerning the word and conjugacy problems for $C(4)$, $T(4)$ presentations [9, pp. 259–267]. The other results will be proved later in a wider context (see Theorem 2).

We can generalize the concept of an Artin group by allowing a larger class of functions ϕ than that specified above. In this paper we consider functions ϕ which assign to each edge $\{x, y\}$ of Γ a non-empty symmetrized set of words of the form

$$x^{\varepsilon_1} y^{\delta_1} \dots x^{\varepsilon_n} y^{\delta_n}$$

where $n \geq 2$, $|\varepsilon_i| = |\delta_i| = 1$ for $i = 1, 2, \dots, n$. (Words of the above form are said to be *cyclically square-free.*) For simplicity, we write $\mathbf{r}_{xy}$ instead of $\phi\{x, y\}$ (note then that $\mathbf{r}_{xy} = \mathbf{r}_{yx}$), and we let

$$\mathbf{r} = \bigcup_{\{x,y\}} \mathbf{r}_{xy} \qquad (\{x, y\} \text{ an edge of } \Gamma).$$

We denote the presentation $\langle \mathbf{x}; \mathbf{r} \rangle$ by $\mathcal{G}(\Gamma, \phi)$, and we write $G(\Gamma, \phi)$ for the group defined by this presentation. It will be convenient to assume that $\mathbf{r}$ satisfies the condition: $U^k, U^l \in \mathbf{r}$ implies $|k| = |l|$. This is no great restriction; if it does not hold then $\mathcal{G}(\Gamma, \phi)$ will not even satisfy $C(2)$.

Our first concern is to establish necessary and sufficient conditions for $\mathcal{G}(\Gamma, \phi)$ to satisfy $C(4)$ and $T(4)$. To state our results we introduce some terminology. We say that an edge $\{x, y\}$ of Γ is *degenerate* if $\mathbf{r}_{xy}$ is the symmetrized closure of a single word of the form $(xy^{\varepsilon})^k$ ($k \geq 2$, $\varepsilon = \pm 1$). We say that a triangle $\{x, y\}$, $\{y, z\}$, $\{z, x\}$ in Γ is *coherently degenerate* if there are integers $k, l, m \geq 2$ and integers $\varepsilon, \delta = \pm 1$ such that $\mathbf{r}_{xy}, \mathbf{r}_{yz}, \mathbf{r}_{zx}$ are the symmetrized closures of $(xy^{\varepsilon})^k$, $(y^{\varepsilon} z^{\delta})^l$, $(z^{\delta} x)^m$, respectively.

Theorem 1. (i) *$\mathcal{G}(\Gamma, \phi)$ satisfies $T(4)$ if and only if every triangle in Γ is coherently degenerate.*

(ii) *$\mathcal{G}(\Gamma, \phi)$ satisfies $C(4)$ if and only if each of the presentations $\langle x, y; \mathbf{r}_{xy} \rangle$ ($\{x, y\}$ a non-degenerate edge of Γ) satisfies $C(4)$.*

Our second theorem concerns properties of $G(\Gamma, \phi)$.

Theorem 2. *Suppose $\mathscr{G}(\Gamma, \phi)$ satisfies $C(4)$ and $T(4)$.*

(i) *If $\mathscr{G}(\Gamma, \phi)$ is finite then the word and conjugacy problems for $\mathscr{G}(\Gamma, \phi)$ are solvable.*

(ii) *If $\mathbf{x}_0$ is a subset of $\mathbf{x}$ and Γ_0 is the induced subgraph on $\mathbf{x}_0$, then the natural mapping of $G(\Gamma_0, \phi)$ into $G(\Gamma, \phi)$ is an embedding.*

(iii) *If $\mathbf{x}_1$ is a subset of $\mathbf{x}$ and no two elements of $\mathbf{x}_1$ are adjacent in Γ, then the subgroup of $G(\Gamma, \phi)$ generated by $\mathbf{x}_1$ is free with free basis $\mathbf{x}_1$.*

(iv) *Any relation between the elements x^2 $(x \in \mathbf{x})$ is a consequence of the relations*

$$s^2 t^2 = t^2 s^2,$$

where $\{s, t\}$ ranges over all edges of Γ for which $\mathbf{r}_{st}$ is the symmetrized closure of one of the following: $\{sts^{-1}t^{-1}\}$, $\{st^{-1}s^{-1}t^{-1}\}$, $\{stst^{-1}\}$, $\{(st)^2, (st^{-1})^2\}$.

§ 2. Preliminaries

For simplicity, throughout the rest of the paper we write $\mathscr{G}$ and G for $\mathscr{G}(\Gamma, \phi)$ and $G(\Gamma, \phi)$ respectively.

If W is a word on $\mathbf{x}$ then we denote the length of W by $|W|$. We call the elements of $\mathbf{x} \cup \mathbf{x}^{-1}$ *letters*. We say that a word *involves* a letter a if a or a^{-1} occurs in the word. If a and b are letters with $a \neq b^{\pm 1}$, then we denote the set of elements of $\mathbf{r}$ which involve both a and b by $\mathbf{r}_{ab}$.

Lemma. *Suppose a and b are letters with $a \neq b^{\pm 1}$.*

(i) *If no element of $\mathbf{r}_{ab}$ has first symbol a and last symbol b^{-1}, then $\mathbf{r}_{ab} = \{(ab)^m\}^*$ for some integer m.*

(ii) *If there is an element of $\mathbf{r}_{ab}$ with first symbol a and last symbol b^{-1}, and if $b^{-1}a$ is not a piece, then $\mathbf{r}_{ab} = \{(\alpha\beta^{-1})^l, (ab)^m\}^*$, where l, m are integers, $l \neq 0$, α is a non-empty word of the form $aba\ldots$, β is a non-empty word of the form $bab\ldots$ (α and β need not have the same length).*

The proof of (i) is left to the reader.

To prove (ii), suppose $R \in \mathbf{r}$ has first symbol a and last symbol b^{-1}. Write $R = U^l$, where $l \geqq 1$ and U is not a proper power. Let α be the longest initial segment of U of the form $aba\ldots$, and let β be the longest initial segment of U^{-1} of the form $bab\ldots$. Then $U = \alpha\gamma\beta^{-1}$ for some word γ. Now if γ is not empty then the first symbol of γ must be one of a^{-1}, b^{-1}. However, not all the symbols of γ belong to the set $\{a^{-1}, b^{-1}\}$, for the last symbol must be one of a, b. Thus somewhere in γ there must be a subword $b^{-1}a$ or $a^{-1}b$. But this implies that $b^{-1}a$ is a piece, a contradiction. Thus γ is empty. Now $\mathbf{r}_{ab} - \{(\alpha\beta^{-1})^l\}^*$ can have no element which starts with a and ends with b^{-1} (otherwise $b^{-1}a$ would be a piece), so $\mathbf{r}_{ab} - \{(\alpha\beta^{-1})^l\}^* = \{(ab)^m\}^*$ for some m. Hence $\mathbf{r}_{ab} = \{(\alpha\beta^{-1})^l, (ab)^m\}^*$.

In our proof of Theorem 1 we will make use of the *star complex* [11] $\mathscr{G}^{st}$ of $\mathscr{G}$. This is the 1-complex with vertices the letters, edges the elements of $\mathbf{r}$, and where for any edge R, $\iota(R)$ is the first symbol of R, $\tau(R)$ is the inverse of the last symbol of R, $\bar{R} = R^{-1}$. The presentation $\mathscr{G}$ satisfies $T(4)$ if and only if there is no closed path in $\mathscr{G}^{st}$ of length 3. For further information, see [11].

The reader is assumed to be familiar with the use of **r**-*diagrams*. See [9, Chap. V] for details. Usually when working with **r**-diagrams one removes interior vertices of degree 2 (see [9, p. 242]), but it is convenient to assume here that we do not do this. However, quite often when *drawing* **r**-diagrams we *will* omit some (or all) of the vertices of degree 2. Thus lines will represent *segments*. (A segment is a path $e_1, e_2, \ldots, e_k$ $(k \geqq 1)$ such that the *intermediate vertices*, that is, the initial and terminal vertices of the edges $e_2, \ldots, e_{k-1}$, are all of degree 2.)

Suppose R is an element of **r**, say $R = a_1 a_2 \ldots a_n$ where the a's are letters. The *separation number* of R is the total number of words $a_1 a_2$, $a_2 a_3$, ..., $a_{n-1} a_n$, $a_n a_1$ which are *not* pieces. If some region Δ of an **r**-diagram is labelled by R (where $\partial\Delta$ is a simple closed curve):

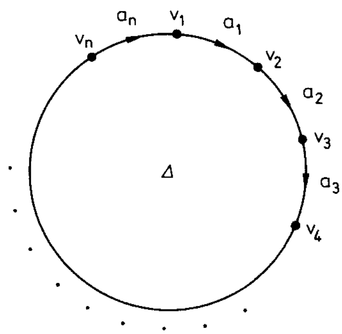


Fig. 1

then the vertices v_i for which $a_i a_{i+1}$ is not a piece are called *separating vertices* of Δ.

Example

Let $R = ((xyx)(yxyxy)^{-1})^l$, $S = (y(zyz)^{-1})^m$, and let $\mathscr{G} = \langle x, y, z; \mathbf{r} \rangle$ where $\mathbf{r} = \{R, S\}^*$. Then the separation number of R is $2l$, and the separation number of S is $4m$. (The case $l = 2$, $m = 1$ is illustrated below: only separating vertices are drawn.)

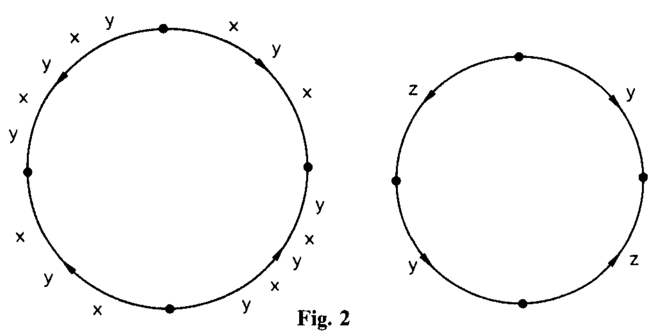


Fig. 2

An important point to note is the following: *no separating vertex of a region of an* **r**-*diagram can be an intermediate vertex of a segment labelled by a piece.* We will make considerable use of this remark in §4, without further comment.

§ 3. Proof of Theorem 1

To prove (i), suppose first that every triangle in Γ is coherently degenerate. Now using the fact that each relator of $\mathscr{G}$ is cyclically square-free one easily sees that: for any $s \in \mathbf{x}$, the set of vertices adjacent to s^{ε} $(\varepsilon = \pm 1)$ in $\mathscr{G}^{st}$ is contained in the set $\{t, t^{-1}: \{s, t\}$ an edge of $\Gamma\}$. It follows from this that if we had a closed path θ in $\mathscr{G}^{st}$ of length 3:

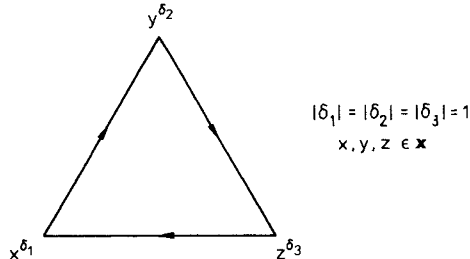


Fig. 3

then $\{x, y\}$, $\{y, z\}$, $\{z, x\}$ would be edges of Γ. Thus $\mathbf{r}_{xy}, \mathbf{r}_{yz}, \mathbf{r}_{zx}$ would be the symmetrized closures of words $(x y^{\varepsilon})^k$, $(y^{\varepsilon} z^{\delta})^l$, $(z^{\delta} x)^m$ $(k, l, m \geqq 2, |\varepsilon| = |\delta| = 1)$, and so the subcomplex of $\mathscr{G}^{st}$ consisting of the vertices x, x^{-1}, y, y^{-1}, z, z^{-1} together with all the edges of $\mathscr{G}^{st}$ joining these vertices would look as follows (only one of each edge pair $R, \bar{R}$ is drawn):

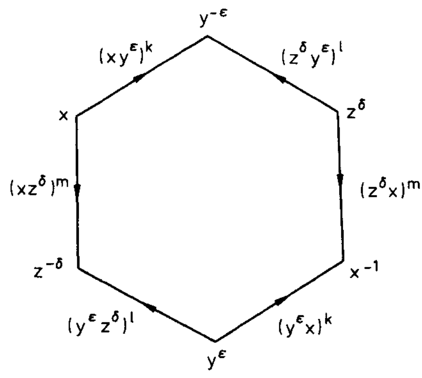
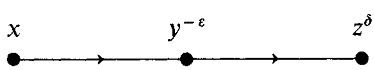


Fig. 4

But θ must lie in this subcomplex, a contradiction. Thus $\mathscr{G}$ satisfies $T(4)$.

Conversely, suppose that $\mathscr{G}$ satisfies $T(4)$, and let $\{x, y\}$, $\{y, z\}$, $\{z, x\}$ be a triangle in Γ. Now for certain $\varepsilon, \delta = \pm 1$ there are edges in $\mathscr{G}^{st}$ as follows:



By the $T(4)$ condition there can be no edge with initial vertex z^{δ} and terminal vertex x. By the Lemma of § 2, $\{z, x\}$ is degenerate, and, by symmetry, so are $\{x, y\}$, $\{y, z\}$. Thus we must have $\mathbf{r}_{xy} = \{(x y^{\varepsilon})^k\}^*$, $\mathbf{r}_{yz} = \{(y^{-\varepsilon} z^{-\delta})^l\}^*$, $\mathbf{r}_{zx} = \{(z^{\delta} x)^m\}^*$ for certain non-zero integers k, l, m, and so our triangle is coherently degenerate.

To prove (ii), first note that if $\mathscr{G}$ satisfies $C(4)$ then certainly each of the subpresentations $\langle x, y; \mathbf{r}_{xy}\rangle$ ($\{x, y\}$ a non-degenerate edge of Γ) must satisfy $C(4)$. Conversely, suppose each of these subpresentations satisfies $C(4)$. Let $R \in \mathbf{r}$, and assume that $R = P_1 P_2 \dots P_n$ where each P_i is a non-empty piece. Now $R \in \mathbf{r}_{xy}$ for some edge $\{x, y\}$, and it is easy to see that for $i = 1, 2, \dots, n$, P_i is a piece *relative to* $\langle x, y; \mathbf{r}_{xy}\rangle$ if $|P_i| > 1$ or if $|P_i| = 1$ and $\{x, y\}$ is not degenerate. It follows immediately that if $\{x, y\}$ is not degenerate then each P_i is a piece relative to $\langle x, y; \mathbf{r}_{xy}\rangle$, and so $n \geqq 4$. On the other hand, if $\{x, y\}$ is degenerate then, since there are *no* pieces relative to $\langle x, y; \mathbf{r}_{xy}\rangle$, $|P_i| = 1$ for each i, and so $n = |R| \geqq 4$.

§ 4. Proof of Theorem 2

(i) follows from general results concerning the word and conjugacy problems for $C(4)$, $T(4)$ presentations [9, pp. 259–267].

To prove (ii), let W be a word on $\mathbf{x}_0$ which defines the identity of G; we want to show that $W = 1$ in $G(\Gamma_0, \phi)$. Now W is freely equal to a product

$$\prod_{i=1}^{n} T_i^{-1} R_i T_i$$

($n \geqq 0$, $R_i \in \mathbf{r}$, T_i a word on $\mathbf{x}$ for $i = 1, \dots, n$). We denote by $\deg(W)$ the least value of n over all expressions of the above form which are freely equal to W. If $\deg(W) = 0$ then W is freely equal to 1. Suppose $\deg(W) > 0$. Let $\hat{W}$ be the freely reduced form of W. Then by small cancellation theory there is an $\mathbf{r}$-diagram with $\deg(W)$ regions and with boundary label $\hat{W}$, which has a simple boundary region Δ:

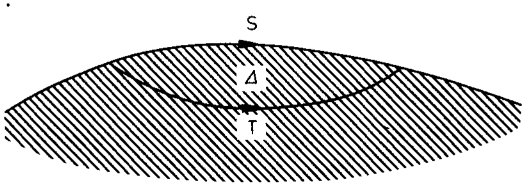


Fig. 5

such that S is a subword of $\hat{W}$ and T is a product of two pieces (one or both of which may be empty). Suppose $ST^{-1} \in \mathbf{r}_{xy}$ ($x, y \in \mathbf{x}$). Then x and y both occur in S (otherwise the $C(4)$ condition would be violated), and so the edge $\{x, y\}$ of Γ belongs to Γ_0. Let W_1 be the word obtained from $\hat{W}$ by replacing the subword S by T. Then $W_1 = W$ in $G(\Gamma_0, \phi)$, and $\deg(W_1) = \deg(W) - 1$. Now use induction.

(iii) follows from (ii), since the induced subgraph on $\mathbf{x}_1$ has no edges.

To prove (iv) requires considerably more work.

Let W be a freely reduced word in the elements x^2 ($x \in \mathbf{x}$), and suppose that W defines the identity of G. We show, by induction on the length of W, that W is a consequence of the relations

$$(1) \qquad c^2 d^2 = d^2 c^2,$$

where c and d are letters, and $\mathbf{r}_{cd}$ is the symmetrized closure of one of $\{cdc^{-1}d^{-1}\}$, $\{cd^{-1}c^{-1}d^{-1}\}$, $\{cdcd^{-1}\}$, $\{(cd)^2, (cd^{-1})^2\}$.

If W is empty the result is trivial. Suppose W is non-empty. Then there is a connected, simply connected reduced $\mathbf{r}$-diagram $\mathcal{M}$ with boundary label W. By [2, §3], $\mathcal{M}$ has a subdiagram (a "strip") $\mathcal{M}_0$ as follows:

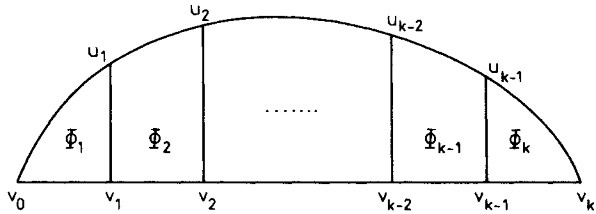


Fig. 6

Here the edges of the segments $\overrightarrow{v_i v_{i+1}}$ $(0 \leqq i < k)$ are interior edges of $\mathcal{M}$, the edges of the segments $\overrightarrow{v_0 u_1}, \overrightarrow{u_1 u_2}, \ldots, \overrightarrow{u_{k-1} v_k}$ are boundary edges of $\mathcal{M}$, the label on $\partial\mathcal{M}_0 \cap \partial\mathcal{M}$ is a subword of W. The strip must have at least two regions. (In the terminology of [2], the strip is a "compound strip". We remark that although the discussion in §3 of [2] allows for the possibility of a "singleton strip", the fact that the boundary label of $\mathcal{M}$ is a product of squares whereas the label on each region of $\mathcal{M}$ is cyclically square-free, means that this cannot occur here.)

Now let x be a generator occurring in the label on Φ_1. Since W is a product of squares, whereas the label on Φ_1 is cyclically square-free, at most one occurrence of x can arise as a label of an edge in $\partial\Phi_1 \cap \partial\mathcal{M}$. Thus one of the edges of $\partial\Phi_1 - \partial\Phi_1 \cap \mathcal{M}$ must be labelled by x, so x is a piece. It therefore follows from the $C(4)$ condition that the segment $\overrightarrow{v_0 u_1}$ cannot consist of a single edge. Hence it must consist of two edges. Similarly $\overrightarrow{u_{k-1} v_k}$ consists of two edges.

We see from the previous paragraph that $\mathcal{M}_0$ is an example of a reduced $\mathbf{r}$-diagram of the form:

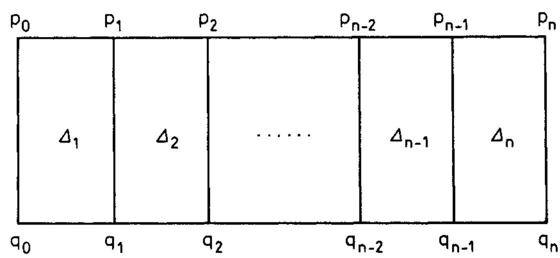


Fig. 7

where the labels on $\overrightarrow{p_0 q_0}$, $\overrightarrow{p_n q_n}$ are of length 1, and where:

(2) the label on $\overrightarrow{p_0 p_1 \cdots p_n}$ is a reduced word in the elements x^2 $(x \in \mathbf{x})$;

(3) the labels on the segments $\overrightarrow{q_{i-1} q_i}$ $(1 \leqq i \leqq n)$ are pieces.

We will determine the structure of an r-diagram $\mathscr{L}$ of the above form. We will show that:

(4) *n must be even, say $n=2n'$; moreover, if the label on $\overrightarrow{p_0q_0}$ is a, then, for $j=1,2,\ldots,n'$, the subdiagram made up of the two regions $\Delta_{2j-1}, \Delta_{2j}$ has one of the labellings depicted below.*

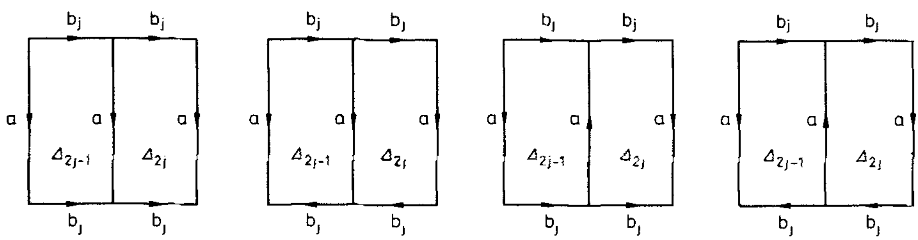


Fig. 8

(In each case b_j is a single letter.)

We note for future reference (bearing in mind that we wish to apply (4) when $\mathscr{L}=\mathscr{M}_0$), that for $j=1,2,\ldots,n'$, $a^2b_j^2=b_j^2a^2$ is a relation in (1).

To prove the above result, first observe that, by (2) and the fact that each element of $\mathbf{r}$ is cyclically square-free, we have that $n>1$, and that the label on $\overrightarrow{p_0p_1}$ is a single letter b. Moreover, the labels on $\overrightarrow{p_1p_2},\ldots,\overrightarrow{p_{n-1}p_n}$ are each of length at most two (this fact will be used often, without further comment).

Now since the label on $\overrightarrow{q_0q_1}$ is a piece, the $C(4)$ condition implies that $b^{-1}a$ cannot be a piece. Thus the label on Δ_1 is a word $R=(\alpha\beta^{-1})^l$ as in the Lemma of §2, and $\mathbf{r}_{ab}=\{(\alpha\beta^{-1})^l,(ab)^m\}^*$. We note that since Δ_1 can have at most 4 separating vertices, $l=1$ or 2.

Now $|\alpha|+|\beta|$ is even. We will examine separately the three cases $|\alpha|+|\beta|\geqq 6$, $|\alpha|+|\beta|=4$, $|\alpha|+|\beta|=2$. We will show that the first case is impossible. For the other two cases we will determine the structure of the subdiagram made up of the two regions Δ_1, Δ_2. Using the results obtained and an obvious induction, it will then follow easily that $\mathscr{L}$ satisfies (4).

Case 1. $|\alpha|+|\beta|\geqq 6$

Then R has separation number $2l$. Now since the labels on $\overrightarrow{p_0p_1}$, $\overrightarrow{p_0q_0}$ are each of length 1, whereas one of $|\alpha|,|\beta|$ is greater than 1, not both of p_1, q_0 can be separating vertices of Δ_1. Hence $l=1$. Then by the $C(4)$ condition, $|\alpha|=|\beta|$ (for if $|\alpha|<|\beta|$ then α would be a subword of β, and hence a piece, and, since $|\alpha|+|\beta|\geqq 6$, β would be the product of two pieces).

We claim that each Δ_i has label R, and that p_{i-1} is a separating vertex of Δ_i.

This is certainly true for Δ_1.

Suppose that it is true for Δ_i $(1 \leqq i < n)$ and consider Δ_{i+1}. Assume first that the label on $\overrightarrow{p_{i-1}p_i}$ has length 1. For definiteness, let the label be a.

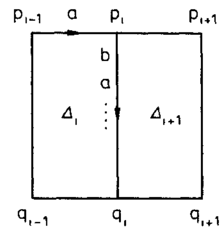


Fig. 9

Now q_i must be the other separating vertex of Δ_i, and so the label on $\overrightarrow{p_iq_i}$ must start with ba. Thus Δ_{i+1} is labelled by a word involving a and b. Now the label on $\overrightarrow{p_ip_{i+1}}$ cannot start with a^{-1}, since the label on $\overrightarrow{p_{i-1}p_ip_{i+1}}$ is freely reduced. Hence the label on $\overrightarrow{p_ip_{i+1}}$ starts with a, so Δ_{i+1} has label R, and p_i is a separating vertex, as required. Now assume that the label on $\overrightarrow{p_{i-1}p_i}$ has length 2. For definiteness, let the label be ab. Now by (2), the label on $\overrightarrow{p_ip_{i+1}}$ must start with b. Since the label on $\overrightarrow{p_iq_i}$ starts with a, again we find that the label on Δ_{i+1} is R, and p_i is a separating vertex.

Now the above gives a contradiction, for it is easy to see that p_{n-1} is not a separating vertex of Δ_n.

Case 2. $|\alpha|+|\beta|=4$

Then $\alpha\beta^{-1}$ is one of $ab(ba)^{-1}$, $a(bab)^{-1}$, $(aba)b^{-1}$.

Now consider the relator $(ab)^m$. If $m \neq 0$ then ab and ba are pieces, and so R is the product of $\leqq 3l$ pieces. Moreover R has separation number $2l$. Now since one of $|\alpha|, |\beta|$ is greater than 1, not both of p_1, q_0 are separating vertices of Δ_1, so $2l \leqq 3$. Thus $l=1$. However, by the $C(4)$ condition, $4 \leqq 3l$, a contradiction. Hence $m=0$. It follows that R has separation number $4l$. Thus $l=1$. Then, bearing in mind that the label on $\overrightarrow{p_1p_2}$ must start with b (by (2)), we see that we must have one of the following (depending on $\alpha\beta^{-1}$).

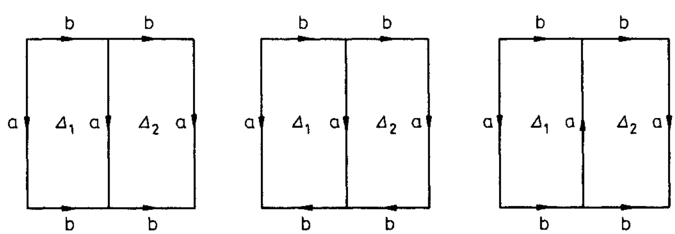


Fig. 10

Case 3. $|\alpha|+|\beta|=2$

Then $l=2$. Now the label on $\overrightarrow{p_1p_2}$ starts with b, so $m \neq 0$, and Δ_2 is labelled by $(ab)^m$. Since $(ab)^m$ has separation number $2m$, and Δ_2 has at most 5 separating vertices (the vertices on the segment $\overrightarrow{p_1 p_2}$ and the vertices q_1, q_2 – here we are using (3)), $m=2$, and we have:

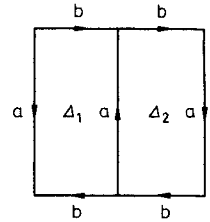


Fig. 11

Using the results just obtained and an obvious induction, it is now easily established that $\mathscr{L}$ satisfies (4).

Let us return to our word W. Using the fact that $\mathscr{M}_0$ satisfies (4), and taking note of the comment following (4), we see that W can be written in the form

$$W_1 g^{-2} h_1^2 h_2^2 \dots h_r^2 g^2 W_2,$$

where $k=2r$, $g, h_1, h_2, \dots, h_r$ are letters, and for $i=1,2,\dots,r$, $g^2 h_i^2 = h_i^2 g^2$ is a relation in (1). Thus as a consequence of relations in (1), we have $W=W'$ in G, where W' is the freely reduced form of

$$W_1 h_1^2 h_2^2 \dots h_r^2 W_2.$$

Since the length of W' is less than the length of W, it follows by induction that W' is a consequence of relations in (1).

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