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Reduction of Hamiltonian Systems, Affine Lie Algebras and Lax Equations II

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The present article is the sequel to a previous paper by the same authors [1]. Its aim is to give an explicit solution of a factorization problem for groups of loops, and to establish a connection of Hamiltonian reduction methods with algebraic methods of Novikov and Krichever [2, 3] and of Mumford and van Moerbeke [4]. We also correct some erroneous statements in [1] concerning the factorization problem (see no. 2 below). To make our presentation more self-consistent, we give an elementary proof of the reduction theorem in a slightly more general form as compared to [1]. This generality corresponds to that of [3, 4] where the same equations are treated in terms of finite-difference operators. An approach based on affine Lie algebras is also described by Adler and van Moerbeke [5]. However, the Hamiltonian reduction questions are not treated there.

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1. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Clearly, $\mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$, and the Lie bracket on $\mathfrak{g}$ gives rise to a Poisson bracket on $C^\infty(\mathfrak{g}^*)$ sometimes called the Kirillov bracket for $\mathfrak{g}$. The center of $C^\infty(\mathfrak{g}^*)$ coincides with the algebra $I(\mathfrak{g}^*)$ of Ad^*G-invariant functions. Restriction of the Kirillov bracket to Ad^*G-orbits in $\mathfrak{g}^*$ induces on them the canonical symplectic structure.

Suppose that $\mathfrak{g}$ splits as a vector space into a linear sum of two its subalgebras, $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$. Let A, B be the corresponding connected subgroups. Put $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{b}$, $G_0 = A \times B$. We identify linear spaces $\mathfrak{g}_0$ and $\mathfrak{g}$ by means of the mapping $\sigma_0 \colon \mathfrak{g}_0 \to \mathfrak{g}$, $\sigma_0(x, y) = x - y$.

Hence there are two Kirillov brackets on $\mathfrak{g}^* \simeq \mathfrak{g}_0^*$.

Reduction Theorem. (i) *$I(\mathfrak{g}^*)$ is abelian with respect to both brackets on $\mathfrak{g}^*$.*

(ii) *Let $\varphi \in I(\mathfrak{g}^*)$, $\xi \in \mathfrak{g}^*$. Put $M = d\varphi(\xi)$, $(M_+, M_-) = \sigma_0^{-1}(M) \in \mathfrak{g}_0$. The Hamiltonian equations of motion defined by φ with respect to the second Poisson bracket on $\mathfrak{g}^*$ have the form*

$$\dot{\xi} = -\mathrm{ad}_{\mathfrak{g}}^* M_+ \xi = -\mathrm{ad}_{\mathfrak{g}}^* M_- \xi \tag{1}$$

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(iii) *Let* $\exp tM = a(t)b(t)$, $a(\cdot)$, $b(\cdot)$ *being smooth curves in* A *and* B, *respectively,* $a(0)=b(0)=e$. *The solution to equations* (1) *starting at* ξ *has the form*

$$\xi(t)=\mathrm{Ad}_G^* a(t)^{-1}\cdot\xi=\mathrm{Ad}_G^* b(t)\cdot\xi$$

Proof. Define a mapping $\sigma\colon G_0\to G$ by $\sigma(a\times b)=ab^{-1}$. We trivialize the tangent bundles of groups by means of left translations. Then the differential of σ at $a\times b\in G_0$ is equal to $\mathrm{Ad}_G b\circ\sigma_0$. Indeed, it is clear from the definition that $d\sigma(a\times b)=\mathrm{Ad}_G b\circ d\sigma(e)$. Evidently, $d\sigma(e)=\sigma_0$. It follows that σ is an immersion. Hence we may define a mapping $\sigma^*\colon T^*G_0\to T^*G$ putting $\sigma^*=d\sigma(a\times b)^{*-1}$ on the fiber $T^*_{a\times b}$. Given a function φ on T^*G put $\varphi^\sigma=\varphi\circ\sigma$. The mapping σ^* is a simplectic immersion and $\{\varphi^\sigma,\psi^\sigma\}=\{\varphi,\psi\}^\sigma$.

Now the Kirillov bracket on $\mathfrak{g}^*$ coincides with the canonical Poisson bracket for left-invariant functions on T^*G. Observing that for $\varphi\in I(\mathfrak{g}^*)$ φ^σ is left G_0-invariant we get (i).

Equations of motion on T^*G corresponding to a Hamiltonian $\varphi\in I(\mathfrak{g}^*)$ are $\dot\xi=0$, $\dot g=d\varphi(\xi)$. Hence the trajectories are given by

$$(g(t),\xi(t))=(g(0)\exp t\,d\varphi(\xi),\xi(0))$$

Trajectories of the Hamiltonian φ^σ on $T^*G_0\simeq G_0\times\mathfrak{g}_0^*$ are obtained from these by the change of variables σ^*. We are only interested in their projections to $\mathfrak{g}_0^*$. If $\exp t\,d\varphi(\xi)=a(t)b(t)$, then $d\sigma^*(\exp t\,d\varphi(\xi))=\sigma_0^*\circ\mathrm{Ad}_G^* b(t)$, whence we get (iii). Differentiating the trajectory with respect to t we get (ii). ■

If there is an invariant scalar product on $\mathfrak{g}$ we may identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Then $\mathrm{ad}_{\mathfrak{g}}^*=\mathrm{ad}_{\mathfrak{g}}$ and the equations of motion have the Lax form.

The "reduced flows" of the Hamiltonians φ^σ may be restricted to Ad^*G_0-invariant submanifolds in $\mathfrak{g}^*$. In particular, we get as a corollary Theorem 10 and Proposition 12 of [1].

Corollary. *Let* f *be a character of* $\mathfrak{a}$.

(i) *Functions on* $\mathfrak{b}^*$ *of the form* $\varphi_f(\xi)=\varphi(\xi+f)$, $\varphi\in I(\mathfrak{g}^*)$ *commute with respect to the Kirillov bracket on* $\mathfrak{b}^*$.

(ii) *Equations of motion corresponding to the Hamiltonian* φ_f *have the form*

$$\dot\xi=-\mathrm{ad}_{\mathfrak{g}}^* M_\pm\cdot(\xi+f),\qquad M=d\varphi(\xi+f) \tag{2}$$

(iii) *Let* $\exp tM=a(t)b(t)$, $a(\cdot)$, $b(\cdot)$ *being smooth curves in* A *and* B, *respectively. Solution to equations* (2) *starting at* ξ *is*

$$\xi(t)=\mathrm{Ad}_G^* b(t)(\xi+f)-f$$

or, equivalently,

$$\xi(t)=\mathrm{Ad}_B^* b(t)\cdot\xi.$$

2. In the rest of the paper we shall be concerned with the following example of the above construction. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Put $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbf{C}[z,z^{-1}]$, $\mathfrak{a}=\mathfrak{g}\otimes z\mathbf{C}[z]$, $\mathfrak{b}=\mathfrak{g}\otimes\mathbf{C}[z^{-1}]$. We equip $\tilde{\mathfrak{g}}$ with a nondegenerate invariant bilinear form $(X,Y)=\mathrm{Res}_{z=0}\,z^{-1}B(X,Y)$ and identify $\mathfrak{a}^*$ with

$\mathfrak{g}\otimes z^{-1}\mathbf{C}[z^{-1}]$ and $\mathfrak{b}^*$ with $\mathfrak{g}\otimes\mathbf{C}[z]$. Invariant functionals on $\tilde{\mathfrak{g}}^*$ are given by

$$\varphi_n(X)=\operatorname{Res}_{z=0} z^{-n}\varphi(X(z)), \qquad \varphi\in I(\mathfrak{g}^*),\ n\in\mathbf{Z}.$$

The reduction theorem as applied to $\tilde{\mathfrak{g}}=\mathfrak{a}+\mathfrak{b}$ amounts to the following statement.

Let $L\in\tilde{\mathfrak{g}}^*$, $\varphi\in I(\tilde{\mathfrak{g}}^*)$, put $M=d\varphi(L)$, and let $\exp tM=g_+(t)^{-1}g_-(t)$, the factors g_+,g_- being analytic inside (outside) the unit circle. The trajectory of the Hamiltonian φ passing through L is given by

$$L(t)=\operatorname{Ad} g_\pm(t)\,L. \tag{3}$$

In the present context the relevant groups are $\tilde{G}=C^\infty(S^1,G)$ and two of its subgroups $G_\pm$ consisting of functions analitic inside (respectively, outside) the unit circle. For a careful formulation of the reduction theorem we must equip $\tilde{G}$ with a topology of a Banach Lie group such that $\sigma\colon G_+\times G_-\to\tilde{G}$, $\sigma(g_+\times g_-)=g_+g^{-1}$ maps some neighborhood of the unit element in $G_+\times G_-$ onto an open set in $\tilde{G}$. A particular choice of topology is to a large extent arbitrary. One of the simplest possibilities is described below.

Let $G\subset GL(n,\mathbf{C})$ be a connected matrix group, $\mathfrak{g}\subset\mathfrak{gl}(n\,\mathbf{C})$ be its Lie algebra. Denote by $\mathscr{W}$ the algebra of absolutely convergent Fourier series with coefficients in $\mathfrak{gl}(n,\mathbf{C})$. Put

$$\mathfrak{g}_{\mathscr{W}}=\{u\in\mathscr{W}\colon u(z)\in\mathfrak{g} \text{ for } |z|=1\}$$
$$G_{\mathscr{W}}=\{g\in W\colon g(z)\in G \text{ for } |z|=1\}$$

Denote by $\mathscr{W}_+(\mathscr{W}_-)$ subalgebras in $\mathscr{W}$ consisting of functions analytic inside (outside) the unit circle. Put

$$G_\pm=\{g\in G_{\mathscr{W}}\cap W_\pm\colon g(z)\in G \text{ for } |z^{\pm 1}|\leqq 1\}$$

Put $A=\{g\in G_+\colon g(0)=1\}$, $B=G_-$, $\tilde{G}_0=A\times B$. The set ${}^0G_{\mathscr{W}}=AB$ is open in $G_{\mathscr{W}}$ and contains a neighborhood of the unit element. The scheme of no. 1 can be directly applied to the present case.

Remark. We point out an error in [1]: in general, ${}^0G_W$ is not a group (Lemma 19 of [1] is false). As a consequence the reduced Hamiltonian flows may be incomplete. The general theorems [6, Lemma 1.5.1] imply that $\exp tM\in{}^0G_{\mathscr{W}}$ only for t sufficiently small. We shall see below that in fact $\exp tM\in{}^0G_{\mathscr{W}}$ for all $t\in\mathbf{C}$ except possibly for a discrete set (which depends on M).

3. We now proceed to the study of the factorization problem. From [1, proposition 23] it follows that with no loss of generality we can restrict ourselves to the case $G=GL(n,\mathbf{C})$.

We begin with a brief exposition of the algebro-geometric pattern of solution of "Lax equations with a spectral parameter" based on papers [2], [3], [4]. For the sake of completeness we include the proofs of some basic results. The main novelty consists in a simple proof of correspondence between Lax equations and linear flows on the Jacobian of the spectral curve.

For $L\in\widetilde{\mathfrak{gl}(n,\mathbf{C})}$ let X^a be an affine curve defined by equation $\det(L(z)-\lambda)=0$. Assume that $L(z)$ has a simple spectrum for general z and also that X^a is nonsingular and irreducible. Let X be the smooth compactification of X^a. Coordinates z, λ are meromorphic functions on X. Let $(z)=P_+-P_-$ be the divisor of z, $P_\pm$ being effective divisors of degree n. Put $U_\pm=X\setminus P_\mp$, $X_0=U_+\cap U_-$. Let R_0 be the algebra of regular functions in X_0. Clearly, $R_0=\mathbf{C}[z,z^{-1},\lambda]$. Put $R=\mathbf{C}[z,z^{-1}]$.

For any $x\in X$ except for a finite set of branch points there exists a unique one-dimensional eigenspace of L i.e. a subspace $E_L(x)\subset\mathbf{C}^n$ such that $L(z(x))\psi=\lambda(x)\psi$ for $\psi\in E_L(x)$. The mapping $x\mapsto E_L(x)$ is clearly a meromorphic mapping of X into $\mathbf{C}P_{n-1}$. Since any such mapping is actually holomorphic we get a holomorphic line bundle $E_L\to X$.

Put $\mathfrak{g}_{p,q}=\left\{\sum_{-p}^{q} x_i z^i, x_i\in\mathfrak{g}\right\}$. An element $L\in\mathfrak{g}_{p,q}$ will be called generic if its lowest and highest coefficients have simple spectrum.

Proposition 1. *Suppose* $L=\sum l_i z^i\in\mathfrak{g}_{p,q}$ *is generic. Then*

(i) *The genus of the spectral curve is equal to*

$$g=\tfrac{1}{2}n(n-1)(p+q)-n+1$$

(ii) *The degree of the dual bundle is equal to* $\deg E_L^*=g+n-1$ *and* $H^0(X, E_L^*(-P_+))=0$.

Proof. (i) The only singular points of the closure of X^a in $\mathbf{C}P_1\times\mathbf{C}P_1\supset\mathbf{C}^2=\{z,\lambda\}$ are $P=(\infty,\infty)$, $Q=(0,\infty)$. The genus of the nonsingular curve X is given by the Hurwitz formula

$$g=d_1d_2-d_1-d_2+1-\nu(P)-\nu(Q)$$

d_1,d_2 being the degrees of the defining equation, $\nu(P)$, $\nu(Q)$ the indices of the singular points. In our case $d_1=n(p+q)$, $d_2=n$; the indices are easily computed using the principal part of the defining equation at P,Q: $\nu(P)=\frac{1}{2}n(n-1)p$, $\nu(Q)=\frac{1}{2}n(n-1)q$.

(ii) We divide the proof into several steps.

(a) Let V denote the subspace of $H^0(X,E_L^*)$ generated by linear coordinates in $\mathbf{C}^n$.

Let $\psi\in V$. If $\psi z\in H^0(X,E_L^*)$ or $\psi z^{-1}\in H^0(X,E_L^*)$ then $\psi=0$.

Indeed, $\psi z\in H^0(X,E_L^*)$ means that ψ vanishes on the eigenspaces $E_L(P_-^i)$[1], which are just the eigenspaces of l_{-p}. By our assumption $E_L(P_-^i)$ span $\mathbf{C}^n$, so that ψ vanishes on $\mathbf{C}^n$. The treatment of the second case is similar.

(b) Now, following [4] we prove that the natural mapping r: $V\otimes R\to H^0(X_0,E_L^*)$ is surjective. To this end observe that $r(V\otimes R)$ is an R_0-module: if $\psi=(\psi^1,\dots,\psi^n)$ is the standard basis in V then $\lambda\psi=L\psi$ so that $\lambda\psi^i\in r(V\otimes R)$. Suppose that $r(V\otimes R)$ is a proper R_0-submodule in $H^0(X_0,E_L^*)$. Then by a theorem from Commutative Algebra there exists a point $x\in X_0$, such

1 Since L is generic, the divisor $P_\pm$ contains n distinct points $P_\pm^i$.

that $\varphi(x)=0$ for every $\varphi\in r(V\otimes R)$. On the other hand, there exists a $\psi\in V$ which does not vanish on the eigenspace $E_L(x)$, a contradiction.

(c) Next we prove that $V=H^0(X,E_L^*)$. Let $\varphi\in H^0(X,E_L^*)$. Since $\varphi\in r(V\otimes R)$, we have $\varphi=\sum_k^m \psi_i z^i$, $\psi_i\in V$. We may assume that $\psi_k\neq 0$, $\psi_m\neq 0$; we shall prove that $k=m=0$. Suppose that $m>0$. Then we may write $\psi_m z=\psi z^{1-m} - \sum_{i\leqq 0}\psi_{i+m-1} z^i$ whence $\psi_m z\in H^0(X,E_L^*)$. Now, (a) implies that $\psi_m=0$, a contradiction. The assumption $k<0$ leads to a similar contradiction.

(d) The statement of (a) combined with $V=H^0(X,E_L^*)$ implies $H^0(X, E_L^*(-P_+))=0$. Since $\deg P_+=n$, it follows from the Riemann-Roch theorem that $\deg E_L^*\leqq g+n-1$. To prove the opposite inequality, consider the bundle $E_k=E_L^*(kP_+)$. For k sufficiently large,

$$\dim H^0(X,E_k)=\deg E_L^*+kn-g+1$$

Now, $r\left(V\otimes\left\{\sum_0^k c_i z^i\right\}\right)\subset H^0(X,E_k)$. We shall see below that r is injective (cf. Proposition 2), so $\dim H^0(X,E_k)\geqq(k+1)n$, or $\deg E_L^*\geqq g+n-1$. ■

Definition. The line bundle $E\to X$ of degree $g+n-1$ is called regular (or, more precisely, z-regular) if $H^0(X,E(-P_+))=0$.

Clearly, regular bundles form a Zarisky open set in the space $\mathscr{L}_{g+n-1}$ of all line bundles of given degree. We shall readily see that the regularity condition completely characterizes line bundles of eigenspaces of matrices $L\in\tilde{\mathfrak{g}}$ with spectrum X.

Proposition 2. *Let E be a line bundle over X of degree $g+n-1$. The following properties of E are equivalent:*

(i) *E is regular.*

(ii) *The natural mapping $r\colon H^0(X,E)\otimes R\to H^0(X_0,E)$ is an isomorphism of R-modules.*

Proof. (i)$\Rightarrow$(ii) Injectivity. Let $\sum_{i\geqq k}\varphi_i z^i=0$, $\varphi_i\in H^0(X,E)$. Then $\varphi_k z^{-1}= -\sum_{i\geqq 0}\varphi_{i+k-1} z^i$ and the right hand side is regular at P_+ so that $\varphi_k\in H^0(X, E(-P_+))$. Hence $\varphi_k=0$ and by induction $\varphi_i=0$ for every i. Surjectivity. We denote $E_k=E(k(P_++P_-))$ and prove that for all k $H^0(X,E_k)\subset r(H^0(X,E)\otimes R)$ which is clearly sufficient. Since E is regular, for any $\varphi\in H^0(X,E_k)$ there exist $\varphi_1,\varphi_2\in H^0(X,E)$ such that $\varphi-\varphi_1 z^k-\varphi_2 z^{-k}\in H^0(X,E_{k-1})$. Our claim now follows by induction.

(ii)$\Rightarrow$(i) Let $\varphi\in H^0(X,E(-P_+))$, i.e. let φ be a section of E which vanishes at P_+. Then $\psi=\varphi z^{-1}$ belongs to $H^0(X,E)$ and $r(\psi-\varphi z^{-1})=0$. It follows that $\varphi=0$.

Proposition 3. *A regular line bundle E corresponds to a $GL(n)$-orbit in $\mathfrak{g}_{p,q}$ consisting of matrices L with spectrum X such that $E_L^*\simeq E$.*

Proof. Multiplication by λ gives an R-linear operator in $H^0(X_0, E)$. Identifying $H^0(X_0, E)$ and $H^0(X, E)\otimes R$ we get an R-linear operator in $H^0(X, E)\otimes R$ i.e. a Laurent polynomial with coefficients in End $H^0(X, E)$. Choosing a basis in $H^0(X, E)$ we get an $L\in\tilde{\mathfrak{g}}$. It is easy to check that $L\in\mathfrak{g}_{p,q}$. ∎

Fix a Hamiltonian $\varphi\in I(\tilde{\mathfrak{g}})$. Put $M=d\varphi(L)$ and let $L(t)$ be the solution to the Hamiltonian Lax equation $\frac{d}{dt}L=[L, M_+]$, $L(0)=L$. The spectral curve X does not vary with t. The time evolution of the corresponding line bundle $E_{L(t)}$ is easy to describe. Let $\psi(x)\in E_L(x)$ be an eigenvector of $L(z(x))$ corresponding to $x\in X$. Since $[L, M]=0$, we get $M(z(x))\psi(x)=\mu(x)\psi(x)$, $\mu\in R_0$. Let F_t be the line bundle over X defined by transition function $\exp t\mu$ with respect to the covering $X=U_+\cup U_-$.

Proposition 4. $E_{L(t)}=E_L\otimes F_t$.

Proof. Let $\exp tM=g_+(t)^{-1}g_-(t)$ be the solution to the factorization problem defined a priori for sufficiently small t. The time evolution of L is given by

$$L(t)=g_+(t)Lg_+(t)^{-1}=g_-(t)Lg_-(t)^{-1}. \tag{3}$$

Now, $E_{L(t)}$ is a subbundle of $X\times\mathbf{C}^n$. Functions $g_+(t)$ give isomorphisms of E_L and $E_{L(t)}$ over $U_\pm: E_{L(t)}(x)=g_\pm(z(x), t)E_L(x)$. The transition function in $U_+\cap U_-$ which distinguishes between these two isomorphisms is

$$g_+(t)^{-1}g_-(t)|_{E_L}=\exp tM|_{E_L}=\exp t\mu. \quad ∎$$

The group of linear bundles over X of degree zero is isomorphic to the Jacobian of X and F_t is its one-parameter subgroup. So the reduction theorem readily leads to the main result of the "direct spectral problem": Lax equations generate linear flows on the Jacobian of the spectral curve. The converse is also true.

Proposition 5. *Every linear bundle over X of degree zero may be defined by a transition function* $\exp\mu$ *with respect to the covering* $\{U_+, U_-\}$.

Proof. The domains $U_\pm$ being affine curves, our bundle is trivial over $U_\pm$ and so is defined by a transition function φ. The degree of the bundle being zero, φ may be so chosen that a univalent $\mu=\log\varphi$ exists. ∎

For any $\mu\in R_0$ there exists a Hamiltonian $\varphi\in I(\tilde{\mathfrak{g}})$ such that $d\varphi(L)=\mu(z, z^{-1}, L)$. We obtain

Corollary 1. *Every linear flow on $\mathscr{L}_{g+n-1}$ is generated by a Lax equation.*

This corollary enables us to prove that Lax equations are completely integrable. Let $L=\sum_{-p}^{q} l_i z^i$ be generic and let H be the centralizer of l_{-p} in $GL(n,\mathbf{C})$ if $p>0$, or $H=GL(n,\mathbf{C})$ if $p=0$. Let $\tilde{G}_0=A\times B$ be as in n°2. Let $\mathcal{O}_L$ be the Ad* $\tilde{G}_0$-orbit of L.

Proposition 6. *The orbit $\mathscr{L}_L$ and Lax equations are invariant under the action of H in $\mathfrak{g}_{p,q}$. This action is Hamiltonian.*

Proof. For $p=0$ the statement is obviously true. For $p>0$ we use the arguments of [5]. Consider the Hamiltonians

$$\tilde{\varphi}(L) = \operatorname{Res}_{z=0} z^{-p-1} \varphi(Lz^p), \qquad \varphi \in I(\mathfrak{g}).$$

If $M = d\tilde{\varphi}(L)$, then $M_- = d\varphi(l_{-p})$, so that the Hamiltonian flow of $\tilde{\varphi}$ coincides with the adjoint action of the subgroup $\exp tM_- \subset H$. Since the spectrum of l_{-p} is simple, H is generated by these subgroups. ■

Let $\Phi\colon \mathcal{O}_L \to \mathfrak{h}^*$ be the momentum map, let $\bar{\mathcal{O}}_L$ be the reduced space over the point $\xi = \Phi(L)$, i.e. $\bar{\mathcal{O}}_L = \Phi^{-1}(\xi)/H_\xi$. Let $\bar{T}_L$ be the level surface of the reduced Hamiltonians $\bar{\varphi}$, $\varphi \in I(\mathfrak{g})$, which contains the image $\bar{L}$ of L.

Theorem 1. (i) *Reduced Hamiltonian systems on $\bar{\mathcal{O}}_L$ defined by the Hamiltonians $\varphi \in I(\tilde{\mathfrak{g}})$ are completely integrable.*

(ii) *There is a natural isomorphism of $\bar{T}_L$ onto a Zarisky open subset of $\mathscr{L}_{g+n-1}$.*

Proof. Let T_L denote the set of elements of $\mathfrak{g}_{p,q}$ which are isospectral with L; in other words, T_L is the level surface of the algebra $I(\tilde{\mathfrak{g}})$. Put $T_L^0 = T_L \cap \mathcal{O}_L$. Proposition 1 gives rise to a mapping $I\colon T_L \to \mathscr{L}_{g+n-1}$, whose fibres are the $GL(n)$-orbits in $\mathfrak{g}_{p,q}$. Clearly, the fibres of $I|_{T_L^0}\colon T_L^0 \to \mathscr{L}_{g+n-1}$ are the H-orbits. The Hamiltonian reduction with respect to H contracts the H-orbits (or rather their intersections with $\Phi^{-1}(\xi)$) into points. Hence the reduced mapping $\bar{I}\colon \bar{T}_L \to \mathscr{L}_{g+n-1}$ is one-to-one. On the other hand, corollary 1 says that the trajectories of reduced Lax equations with the inital point $\bar{L}$ cover an open part of $\mathscr{L}_{g+n-1}$. This implies complete integrability. ■

Corollary 2. *If $p>0$, $I(\tilde{\mathfrak{g}})$ is a maximal involutive algebra on $\mathcal{O}_L$ and T_L is a maximal torus. If $p=0$, a maximal involutive system on $\mathcal{O}_L$ is obtained by combining $I(\tilde{\mathfrak{g}})$ with a maximal* $\mathfrak{gl}(n)$*-involutive system of functions of l_0.*

4. Now we consider the factorization problem. Let $\psi(t) = (\psi^1(t), \dots, \psi^n(t))$ be the n-tuple of sections of $E^*_{L(t)}$ generated by the linear coordinates in $\mathbf{C}^n$. By Proposition 4 $\psi(t)$ is described by two n-tuples $\psi_\pm(t)$ of sections of $E^*_L|_{U_\pm}$ related by

$$\psi_-(t) = e^{t\mu} \psi_+(t)$$

If $g_\pm(t)$ solve the factorization problem, then it is clear from (3) that

$$\psi_\pm(t) = g_\pm(t)\, \psi(0)$$

Suppose $z \in \mathbf{C}$ is such that X is unramified over z and let $x_1, \dots, x_n$ be the points of X over z. Define the matrices $\hat{\psi}_\pm(z,t)$ by

$$\hat{\psi}^{ij}_\pm(z,t) = \psi^i_\pm(x_j, t) \tag{4}$$

Then $\hat{\psi}_\pm(t)=g_\pm(t)\hat{\psi}(0)$, so that

$$g_\pm(z,t)=\hat{\psi}_\pm(z,t)\hat{\psi}(z,0)^{-1} \tag{5}$$

This formula reconstructs $g_\pm$ in terms of $\psi_\pm$. Hence if we can write down $\psi_\pm$ explicitely, we solve the factorization problem. To this end we interprete $H^0(X,E_L^*)$ as a linear space $\mathfrak{L}(D)$ of meromorphic functions on X subordinate to a divisor D which corresponds to E_L^*. By Proposition 1 $\deg D=g+n-1$. The regularity condition allows for a simple choice of basis in $\mathfrak{L}(D)$. Since L is generic, the divisor P_+ contains n distinct points P_+^i. Let ψ^i be a (unique up to a constant factor) function on X such that $(\psi^i)\geqq -D+P_+-P_+^i$. Clearly, the functions ψ^i are linearly independent and generate the whole space $\mathfrak{L}(D)$. To motivate this choice note that if the lowest coefficient of L is diagonal and has simple spectrum, then ψ^i are just the canonical sections (i.e. those defined by standard coordinates in $\mathbf{C}^n$). It is convenient to assume that $D-P=D_0-P_0$, $D_0\geqq 0$, $\deg D_0=g$, $P_0\in X$, which is always possible.

Theorem 2 [3]. *Let $\mu\in R_0$. Then* (i) *There exist meromorphic functions $\psi^i_\pm(z,t)$ defined in $U_\pm$ such that*

$$\psi^i_-=e^{t\mu}\psi^i_+,\qquad (\psi^i_\pm)\geqq -D_0+P_0-P_+^i$$

These functions are unique up to a multiple $c_i(t)$.

(ii) *Equations*

$$L\psi_\pm=\lambda\psi_\pm$$
$$\partial_t\psi_\pm=\mp M_\pm\psi_\pm$$

define matrix-valued polynomials $L(t)\in\mathfrak{g}[z,z^{-1}]$, $M_\pm(t)\in\mathfrak{g}[z^{\pm 1}]$ whose coefficients are meromorphic functions of t.

Proof. Define a linear bundle F_t by the transition function $\exp t\mu$ for the covering $\{U_+,U_-\}$. Put $E_t=E_L\otimes F_t$. For almost all t the bundle E_t^* is regular and from the Riemann-Roch theorem it follows that for such t $\psi^i_\pm(t)$ is unique up to a factor. Hence a t-smooth function $\psi^i_\pm$ is unique up to a factor $c_i(t)$. The existence of $\psi^i_\pm$ follows from an explicit formula. To write it down we introduce some notation. Fix a basis $\{\omega^i\}$ of holomorphic differentials on X and let ω: $X\times X\to \operatorname{Jac}X$ be the Abel transform defined by $\omega(x,y)=\left\{\int_x^y\omega^i\right\}$. Choose a ϑ-function on $\mathbf{C}^g$ in such a way that there exist $(g-1)$ points $x_1,\dots,x_{g-1}$ with the property $\vartheta(\omega(x,x_i))=0$ for every $x\in X$ but $\vartheta(\omega(x,y))$ is not identically zero. There exist unique meromorphic differentials $v_\pm$ on X such that

a) $v_\pm$ are regular in $U_\pm$ and $(v_\pm-d\mu)$ are regular in $U_\mp$.

b) The $\mathbf{Z}$-linear functionals on $H_1(X,\mathbf{Z})$ defined by $\gamma\mapsto\int_\gamma v_\pm$ extend to $\mathbf{C}$-antilinear functionals on $\mathbf{C}^g(H_1(X,\mathbf{Z})$ is embedded into C^g via the period mapping $\gamma\mapsto\{\int_\gamma\omega^i\})$. Let $V_\pm\in\mathbf{C}^g$ represent these functionals with respect to the Riemann scalar product in $\mathbf{C}^g$.

Fix a point $x_0 \in X$ and choose $c \in \mathbf{C}^g$ so that $(\vartheta(\omega(x_0, .) - c)) = D_0$. Then

$$\psi^i_\pm(x) = e^{t \int_{x_0}^{x} v_\pm} \frac{\vartheta(\omega(P_0, x))\,\vartheta(\omega(x_0, x) + \omega(P^i_+, P_0) - tV_\pm - c)}{\vartheta(\omega(P^i_+, x))\,\vartheta(\omega(x_0, x) - c)}$$

The formula is derived as in [2], [4] and shows that $\psi^i_\pm$ is holomorphic in t. If E^*_t is regular the natural mapping

$$H^0(X, E^*_t) \otimes \mathbf{C}[z^{\pm 1}] \to H^0(U_\pm, E^*_t)$$

is a $\mathbf{C}[z^{\pm 1}]$-module isomorphism. The relation $\partial_t \psi_- = e^{t\mu}(\partial_t \psi_+ + \mu \psi_+)$ allows to interprete $\partial_t \psi_\pm$ as meromorphic sections of E^*_t which are regular in $U_\pm$. Hence there exist matrix polynomials $M_\pm(t) \in \mathfrak{g}[z^{\pm 1}]$ such that

$$\partial_t \psi_\pm = \mp M_\pm \psi_\pm$$

To obtain $L(t) \in \mathfrak{g}[z, z^{-1}]$ we use Proposition 3. Coefficients of $L, M_\pm$ may be computed recurrently by decomposing $\lambda \psi_\pm$, $\partial_t \psi_\pm$ into a power series in the local parameter in the vicinity of $P^i_\pm$. ■

Remark. If ψ^i_+ are normalized so that $\partial_t \psi_+(0, t) = 0$, then $M_+(t) \in \mathfrak{g} \otimes z\,\mathbf{C}[z]$.

Proposition 6. $M_+(t) + M_-(t) = \mu(z, z^{-1}, L(t))$

Proof. From $\psi_- = e^{t\mu} \psi_+$ we get

$$M_- \psi_- = \partial_t \psi_- = \mu \psi_- - M_+ \psi_-$$

hence $(M_+ + M_-)\psi_- = \mu \psi_-$. ■

Since the functions of Theorem 2 are analytic in t, the factors $g_\pm(t)$ related to them by (5) are regular for all t except for a discrete subset of $\mathbf{C}$. In what follows we give an alternative proof for the existence of a factorization which makes no appeal to general theorems of [6].

As in (4) we define the matrices $\hat{\psi}_\pm(z, t)$ by

$$\hat{\psi}^{ij}_\pm(z, t) = \psi^i_\pm(x_j, t)$$

and put

$$g_\pm(z, t) = \hat{\psi}_\pm(z, t)\,\hat{\psi}(z, 0)^{-1} \tag{5}$$

The definition of $g_\pm$ is easily seen to be correct (cf. [2]).

Theorem 3 (i) *$g_\pm$ are entire functions of $z^{\pm 1}$. If t lies outside of a discrete subset of* $\mathbf{C}$, *then* $\det g_\pm(z, t) \neq 0$.

(ii) *$g_\pm$ provide a factorization for* $\exp tM$:

$$\exp tM = g_+(t)^{-1} g_-(t)$$

Proof. (i) Let $M_\pm$ be as in Theorem 2, let Γ be the set of poles of their coefficients. If $t \in \mathbf{C} \setminus \Gamma$ then

$$\partial_t g_\pm = \mp M_\pm g_\pm$$

Thus $g_\pm$ are fundamental solutions of analytic differential equations and hence $g_\pm$ are regular in $z^{\pm 1}$ and $\det g_\pm(z,t) \neq 0$. The definition (5) of $g_\pm$ shows that they are regular for all $t \in \mathbf{C}$.

Since $\hat{\psi}_-(t) = \hat{\psi}_+(t)\,\widehat{\exp}\,t\,\mu$, we have

$$g_+(t)^{-1} g_-(t) = \hat{\psi}(0)\,\hat{\psi}_+(t)^{-1}\,\hat{\psi}_-(t)\,\hat{\psi}(0)^{-1}$$
$$= \hat{\psi}(0)\,\widehat{\exp t\,\mu}\,\hat{\psi}(0)^{-1} = \exp t M.$$

Clearly, for $t \in \mathbf{C} \setminus \Gamma$

$$L(t) = g_\pm(t)\,L\,g_\pm(t)^{-1}.$$

Remark. Theorem 2 shows that the solutions of Lax equations may have poles in time variable. Formulae (3, 5) give a regularization of trajectories.

In the algebraic approach to Lax equations $L(t)$ and $M_\pm(t)$ are determined by the power series expansion of $\lambda\psi_\pm$, $\partial_t\psi_\pm$ in $P^i_\pm$. On the other hand, the group-theoretic approach gives $L(t) = g_\pm(t)\,L\,g_\pm(t)^{-1}$. The relationship of two methods is based on a remarkable property of $g_\pm$:

$$\mathrm{Ad}^*_G\, g_\pm(t)\, L = \mathrm{Ad}^*_{G_0}\, g_+(t)\, L_+ + \mathrm{Ad}^*_{G_0}\, g_-(t)\, L_-.$$

Following Zakharov and Shabat we may say that $g_\pm$ are dressing transformations. To sum up, our approach gives a group-theoretic interpretation of the Zakharov-Shabat dressing-up and of algebraic methods of Novikov, Matveev, Dubrovin, Krichever, Mumford, Moerbeke and others. This approach also applies to partial differential equations [7].

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