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Deformation of Kähler matrics to Kähler-Einstein metrics on compact Kähler manifolds

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0. Introduction

In the recent work [2] Hamilton proved that for any compact 3-manifold with positive Ricci curvature one can deform the initial metric along the heat flow defined by the equation:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \frac{2}{3} r g_{ij} \tag{0.1}$$

to an Einstein metric of positive scalar curvature. Here R_{ij} is the Ricci curvature of the metric g_{ij} and r is the avarage of the scalar curvature. In this paper we shall show that the heat flow method also works for the well-known Calabi conjecture in Kähler geometry.

Let M be a compact Kähler manifold of dimension n with the Kähler metric $ds^2 = g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. (We shall use the summation convention throughout the paper.) Then the Ricci curvature $R_{i\bar{j}}$ of this metric is given by the formula

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det (g_{i\bar{j}}) \tag{0.2}$$

so the $(1,1)$ form $\frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is closed and its cohomology class, as is well-known, is equal to the first Chern class $C_1(M)$ of M. Calabi conjectured that the converse is also ture. Namely, given any closed $(1,1)$ form $\frac{\sqrt{-1}}{2\pi} T_{i\bar{j}} dz^i \wedge d\bar{z}^j$ which represents the first Chern class $C_1(M)$, one can find another Kähler metric $\bar{g}_{i\bar{j}}$ on M so that $T_{i\bar{j}}$ is the Ricci tensor of $\bar{g}_{i\bar{j}}$. This conjecture was open for more than twenty years until 1978 Yau [6] gave an affirmative answer. The above Calabi conjecture can be reduced to solving a complex Monge-Ampeṙe equation and Yau proved such an equation can be solved by using the continuity method.

We consider the complex version of Hamilton's equation of the following type:

$$\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}} + T_{i\bar{j}}, \quad \tilde{g}_{i\bar{j}} = g_{i\bar{j}} \quad \text{at} \quad t=0 \tag{0.3}$$

where $\tilde{R}_{i\bar{j}}$ denotes the Ricci tensor of the metric $\tilde{g}_{i\bar{j}}$.

If we can prove that the solution for Eq. (0.3) exists for all time and $\tilde{g}_{i\bar{j}}(t)$ converges to the limit metric $\tilde{g}_{i\bar{j}}(\infty)$ as t goes to the infinity and that $\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t}$ converges uniformly in t to a constant, then $\tilde{g}_{i\bar{j}}(\infty)$ is a metric we want.

The Eq. (0.3) can be reduced to a scalar equation as follows: by the assumption on $T_{i\bar{j}}$, $\frac{\sqrt{-1}}{2\pi} T_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $\frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ belong to the same cohomology class $C_1(M)$, so we know that there exists a smooth function f on M so that

$$T_{i\bar{j}} - R_{i\bar{j}} = \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}. \tag{0.4}$$

Therefore, if we let

$$\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \tag{0.5}$$

where u is a smooth function on $M \times [0, T)$, $0 < T \leqq \infty$, with $u(0) = 0$, then Eq. (0.3) becomes

$$\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\frac{\partial u}{dt}\right) = -\tilde{R}_{i\bar{j}} + R_{i\bar{j}} + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \tag{0.6}$$

and consequently, according to (0.2),

$$\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\log\det\left(g_{i\bar{j}} + \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\right) - \log\det(g_{i\bar{j}})\right) + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \tag{0.7}$$

or equivalently,

$$\partial\bar{\partial}\left(\frac{\partial u}{\partial t}\right) = \partial\bar{\partial}\log\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) - \partial\bar{\partial}\log\det(g_{i\bar{j}}) + \partial\bar{\partial} f. \tag{0.8}$$

By the maximum principle for compact manifolds, it follows that the function u satisfies the equation

$$\frac{\partial u}{\partial t} = \log\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) - \log\det(g_{i\bar{j}}) + f + \varphi(t) \tag{0.9}$$

where $\varphi(t)$ is a smooth function in t satisfying the compatibility condition

$$\int_M \exp\left(\frac{\partial u}{\partial t} - f\right) dV = \exp(\varphi(t)) \operatorname{Vol}(M) \tag{0.10}$$

where dV is the volume element of the metric $g_{i\bar{j}}$.

Since Eq. (0.9) is a nonlinear parabolic equation we know from standard theory that the solution exists for a short time. To show that the solution actually exists for all time it sufficies to prove the a priori estimates for the solution upto third order. This is done in Sect. 1, based on Yau's work in [6]. In Sect. 2 we apply Hanack's inequality to prove that $u(x,t)$ converges to a function $u_\infty(X)$ on M under normalization and that $\frac{\partial u}{\partial t}$ converges uniformly in t to a constant as t tends to the infinity. In Sect. 3 we shall briefly discuss the negative case of the Calabi conjecture.

1. The long time existance

Throughout this section we assume that u is the solution of the initial value problem

$$\frac{\partial u}{\partial t}=\log\det\left(g_{i\bar{j}}+\frac{\partial^2 u}{\partial z^i\,\partial \bar{z}^j}\right)-\log\det(g_{i\bar{j}})+f$$

$$u(x,t)=0 \quad \text{at} \quad t=0 \tag{1.1}$$

on the maximal time interval $[0,T)$ such that $\tilde{g}_{i\bar{j}}=g_{i\bar{j}}+\frac{\partial^2 u}{\partial z^i\,\partial \bar{z}^j}$ is positive definite and hence defines a Kähler metric on M for any time $t\in[0,T)$.

Differentiating Eq. (1.1), we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)=\tilde{g}^{i\bar{j}}\frac{\partial^2}{\partial z^i\,\partial \bar{z}^j}\left(\frac{\partial u}{\partial t}\right)=\tilde{\Delta}\left(\frac{\partial u}{\partial t}\right) \tag{1.2}$$

where $\tilde{g}^{i\bar{j}}$ is the inverse of $\tilde{g}_{i\bar{j}}$ and $\tilde{\Delta}$ is the normalized Laplace operator of $\tilde{g}_{i\bar{j}}$. It follows from the maximum principle for the parabolic equation that

$$\max_M\left|\frac{\partial u}{\partial t}\right|\leqq\max_M|f|. \tag{1.3}$$

Let $\Delta=g^{i\bar{j}}\frac{\partial^2}{\partial z^i\,\partial \bar{z}^j}$ be the normalized Laplacian of $g_{i\bar{j}}$ and let $\square$ denote the operator $\tilde{\Delta}-\frac{\partial}{\partial t}$.

Lemma 1. *There exist positive constants C_0 and C_1 such that*

$$0<n+\Delta u\leqq C_1\exp\big(C_0(u-\inf_{M\times[0,T)} u)\big), \quad \textit{for all } t\in[0,T). \tag{1.4}$$

Proof. The first inequality in (1.4) follows from the fact that $\tilde{g}_{i\bar{j}}$ is positive definite and $n+\Delta u$ is the trace of $\tilde{g}_{i\bar{j}}$ with respect to $g_{i\bar{j}}$.

For the second inequality we have, based on the calculation in (6), the following inequality

$$\square(\exp(-Cu)(n+\Delta u)) \geqq -\exp(-C_0 u)(\Delta f + n^2 \inf_{i \neq 1}(R_{i\bar{i}1\bar{1}}))$$
$$- C_0 \exp(-C_0 u)\left(n - \frac{\partial u}{\partial t}\right)(n+\Delta u)$$
$$+ (C_0 + \inf_{i \neq 1}(R_{i\bar{i}1\bar{1}}) \exp(-C_0 u)$$
$$\cdot \exp\left(-\frac{\partial u}{\partial t} + f\right)(n+\Delta u)^{\frac{n}{n-1}} \tag{1.5}$$

where $R_{i\bar{i}1\bar{1}}$ is the bisectional curvature of the metric $g_{i\bar{j}}$ and C_0 is a positive constant such that $C_0 + \inf_{i=1} R_{i\bar{i}1\bar{1}} > 0$.

For any given $t \in (0, T)$, we assume function $\exp(-C_0 u)(n+\Delta u)$ achieves its maximum at point (p, t_0), with $t_0 > 0$, on $M \times [0, t]$. Then at this point the left hand side of (1.5) is nonpositive and hence

$$0 \geqq -(\Delta f + n^2 \inf_{i \neq 1}(R_{i\bar{i}1\bar{1}})) - C_0 \left(n - \frac{\partial u}{\partial t}\right)(n+\Delta u)$$
$$+ (C_0 + \inf_{i \neq 1}(R_{i\bar{i}1\bar{1}})) \exp\left(-\frac{\partial u}{\partial t} + \frac{f}{n-1}\right)(n+\Delta u)^{\frac{n}{n-1}} \tag{1.6}$$

therefore, by (1.3) we have

$$(n+\Delta u)^{\frac{n}{n-1}} \leqq C'(1 + (n+\Delta u)) \tag{1.7}$$

where C' is a positive constant independent of t. From (1.7) we conclude that

$$n + \Delta u(p, t_0) \leqq C_1.$$

Hence on $M \times [0, t)$

$$\exp(-C_0 u)(n+\Delta u) \leqq C_1 \exp(-C_0 u(p, t_0))$$

and it follows that

$$n + \Delta u < C_1 \exp(C_0(u - \inf_{M \times [0,T)} u)). \tag{1.8}$$

Since the constants C_1 and C_0 in (1.8) are independent of t we finish the proof of Lemma 1.

Now we proceed to derive the zero order estimate of u under the normalization. We put

$$v = u - \frac{1}{\mathrm{Vol}(M)} \int_M u \, dV. \tag{1.9}$$

Then, as shown in [6], we have the following

Lemma 2. $\sup_{M \times [0,T)} v \leqq C_2, \quad \sup_{M \times [0,T)} \int_M |v| \, dV \leqq C_3.$

Based on Eq. (1.1) and Lemma 2 we can use the Nash-Morse iteration arguement to get the lower bound for function v.

Lemma 3. *There exists a constant* $C_4 > 0$ *so that*

$$\sup_{M \times [0,T)} |v| < C_4$$

Proof. Let

$$\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \qquad \tilde{\omega} = \frac{\sqrt{-1}}{2} \tilde{g}_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

be the Kähler forms of $g_{i\bar{j}}$, and $\tilde{g}_{i\bar{j}}$, respectively. Then the volume forms dV and $d\tilde{V}$ are given by

$$dV = \det(g_{i\bar{j}}) \bigwedge_{i=1}^{n} \left(\frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^j\right) = \frac{\omega^n}{n!}$$

$$d\tilde{V} = \det(\tilde{g}_{i\bar{j}}) \bigwedge_{i=1}^{n} \left(\frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^j\right) = \frac{\tilde{\omega}^n}{n!}.$$

According to Eq. (1.1) we know then

$$d\tilde{V} = \exp\left(\frac{\partial u}{\partial t} - f\right) dV. \tag{1.11}$$

So it follows from (1.10) and (1.11) that, for $p > 1$,

$$\begin{aligned} -\frac{1}{n!} \int_M \frac{(-v)^{p-1}}{p-1} (\omega^n - \tilde{\omega}^n) &= -\int_M \frac{(-v)^{p-1}}{p-1} (dV - d\tilde{V}) \\ &= \int_M \frac{(-v)^{p-1}}{p-1} \left(\exp\left(\frac{\partial u}{\partial t} - f\right) - 1\right) dV \end{aligned} \tag{1.12}$$

where we have renormalized v so that $v < -1$ which is certainly possible by Lemma 2.

On the other hand

$$\begin{aligned} -\int_M \frac{(-v)^{p-1}}{p-1} (\omega^n - \tilde{\omega}^n) &= -\int_M \frac{(-v)^{p-1}}{p-1} \left(\omega^n - \left(\omega + \frac{-1}{2} \partial\bar{\partial} v\right)^n\right) \\ &= \int_M \frac{(-v)^{p-1}}{p-1} \left(\frac{\sqrt{-1}}{2} \partial\bar{\partial} v\right) \wedge \sum_{j=1}^{n-1} \omega^j \wedge \tilde{\omega}^{n-j-1} \\ &= \int_M (-v)^{p-2} \left(\frac{\sqrt{-1}}{2} \partial v \wedge \bar{\partial} v\right) \wedge \sum_{j=1}^{n-1} \omega^j \wedge \tilde{\omega}^{n-j-1} \\ &\geqq \int_M (-v)^{p-2} \left(\frac{\sqrt{-1}}{2} \partial v \wedge \bar{\partial} v\right) \wedge \omega^{n-1} \end{aligned} \tag{1.13}$$

where the last inequality follows from the fact that the terms $\frac{\sqrt{-1}}{2} (\partial v \wedge \bar{\partial} v)\, \omega^j \wedge \tilde{\omega}^{n-j-1}$ are all nonnegative.

Combining (1.12) and (1.13) we obtain

$$\int_M (-v)^{p-2} |\nabla v|^2 \, dV \leqq n! \int_M \frac{(-v)^{p-1}}{p-1} \left(\exp\left(\frac{\partial u}{\partial t} - f\right) - 1\right) dV \tag{1.14}$$

where $|\nabla v|^2 = g^{i\bar{j}} \dfrac{\partial v}{\partial z^i} \dfrac{\partial v}{\partial \bar{z}^j}$.

Since

$$(-v)^{p-2} |\nabla v|^2 = 4p^{-2} |\nabla(-v)^{p/2}|^2$$

it follows from (1.14) that

$$\int_M |\nabla(-v)^{p/2}|^2 \, dV \leqq C \frac{p^2}{p-1} \int_M (-v)^{p-1} \, dV$$

and hence

$$\begin{aligned} \|(-v)^{p/2}\|_{H^1}^2 &= \int_M |\nabla(-v)^{p/2}|^2 \, dV + \int_M (-v)^p \, dV \\ &\leqq C \frac{p^2}{p-1} \int_M (-v)^p \, dV \\ &\leqq Cp \int_M (-v)^p \, dV, \qquad \text{for } p \gg 1. \end{aligned} \tag{1.15}$$

We remark that inequality (1.15) also holds for $p=1$, simply replacing $\dfrac{(-v)^{p-1}}{p-1}$ by $\log(-v)$ in the arguement.

Now the Sobolov inequality implies that, since $\int_M v \, dV = 0$,

$$\|(-v)^{p/2}\|_{L^{\frac{2n}{n-1}}}^2 \leqq C \|(-v)^{p/2}\|_{H^1}^2. \tag{1.16}$$

Putting (1.15), (1.16) together we see that

$$\|v\|_{L^{\frac{np}{n-1}}}^p \leqq Cp \|v\|_{L^p}^p, \qquad \text{for } p=1, \quad \text{or} \quad p \gg 1. \tag{1.18}$$

Let $p = \gamma^j$ in (1.18), where $\gamma = \dfrac{n}{n-1}$ and $j = 0, 1, 2, \ldots$,

Then by induction on j we obtain

$$\|v\|_{L^{\gamma^{j+1}}} < \tilde{C}^{\sum_{k=0}^{j} \frac{1}{\gamma^k}} \cdot \gamma^{\sum_{k=0}^{j} \frac{k}{\gamma^k}} \cdot C_3. \tag{1.19}$$

Letting $j \to \infty$, we have

$$\|v\|_{L^\infty} \leqq C_4. \tag{1.20}$$

Notice that the constant C_4 is independent of time t so we conclude that

$$\sup_{M \times [0,T)} |v| \leqq C_4.$$

This proves Lemma 3.

Combining Lemma 1 and Lemma 3, we have

$$
\begin{aligned}
0<n+\Delta v=n+\Delta u & \\
&\leqq C_1 \exp(C_0(u-\inf_{M\times[0,T)} u)) \\
&= C_1 \exp(C_0(v-\inf_{M\times[0,T)} v)) \\
&\leqq C_5. \qquad (1.21)
\end{aligned}
$$

The first order estimate then follows from the Schauder estimate, Lemma 3 and (1.21): (see Gilbart and Trudinger [3])

$$
\begin{aligned}
\sup_{M\times[0,T)} |\nabla v| &\leqq C_6(\sup_{M\times[0,T)} |\Delta v| + \sup_{M\times[0,T)} |v|). \\
&\leqq C_7 \qquad (1.22)
\end{aligned}
$$

Now we also can estimate all the second order derivatives. (1.21) implies that $1+u_{i\bar{i}}$ is bounded from above for all i. On the other hand Eq. (1.11) tells us that $\prod_{i=1}^{n}(1+u_{ii})$ are also bounded from above. These together imply that there exist positive constants A and B so that

$$A\leqq 1+u_{ii}\leqq B, \qquad \text{for all } i. \qquad (1.23)$$

In particular we know that the metrics $\tilde{g}_{i\bar{j}}(t)$ are uniformly equivalent to the initial metric $g_{i\bar{j}}$.

Let us go on to prove the third order estimate for the function v. Following Calabi and Yau we consider the quantity

$$S=g^{i\bar{r}}g^{\bar{j}s}g^{k\bar{t}}v_{i\bar{j}k}v_{\bar{r}s\bar{t}}. \qquad (1.24)$$

Modifying the calculation carried out in [6] we find

$$\Box\,(S+C_8\,\Delta v)\geqq C_9 S - C_{10} \qquad (1.25)$$

where C_8 and C_9, C_{10} are positive constants that can be estimated.

At the maximum point $p(t)$ of $S+C_8\,\Delta v$ at time t, (1.25) then shows that

$$0\geqq C_9 S - C_{10}$$

hence

$$C_9(S+C_8\,\Delta v)\leqq C_{10}+C_8\,C_9\,\Delta v, \qquad \text{at } p(t). \qquad (1.26)$$

Since we have already estimated Δv in (1.21) it follows that $\sup_{M\times[0,T)} S+C_8\,\Delta v$ is bounded and therefore $\sup_{M\times[0,T)} S$ is bounded. This gives us the estimate for all the third order derivatives of v.

We are now in the position to prove the long time existence.

Proposition 1.1. *Let u be the solution of* (1.1) *on the maximum time interval $0\leqq t<T$ and let v be the normalization of u defined as in* (1.9). *Then the C^∞-*

norm of v are uniformly bounded for all $t \in (0, T)$ and consequentely $T = \infty$. Moreover there exists a time sequence $t_n \to \infty$ such that $v(x, t_n)$ converges in C^∞ topology to a smooth function $v_\infty(x)$ on M as $n \to \infty$.

Proof. So far we have already estimated the derivatives of v up to the third order.

Differentiating the Eq. in (1.1) with respect to z^k we get

$$\square \left(\frac{\partial u}{\partial z^k}\right) = g^{i\bar{j}} \frac{\partial}{\partial z^k}(g_{i\bar{j}}) - \tilde{g}^{i\bar{j}} \frac{\partial}{\partial z^k}(g_{i\bar{j}}). \tag{1.27}$$

Then we know that the coefficients of operator $\square$ are bounded in $C^{0,\alpha}$ norm and also the right hand side of (1.27) also has estimate in $C^{0,\alpha}$ norm for all $0 < \alpha < 1$. From the Schauder regularity theory (see Ladyzenskaja et al. [4]) we know then $\frac{\partial u}{\partial z^k}$ has uniform $C^{2,\alpha}$ estimate and similarly for $\frac{\partial u}{\partial \bar{z}^k}$. So the coefficients of $\square$ and the right hand side of (1.27) has uniform $C^{1,\alpha}$ estimate. Apply the Schauder theory again we see that $\frac{\partial u}{\partial z^k}$ and $\frac{\partial u}{\partial \bar{z}^k}$ have uniform $C^{2,\alpha}$ estimates. By iteration we conclude that the C^∞-norm of $v(x, t)$ are uniformly bounded for all $t \in (0, T)$ and consequencely we can select a time sequence $t_n \to \infty$ so that $v(x, t_n)$ converge to a smooth function $v_\infty(x)$ as $t \to \infty$. On the other hand, since $\frac{\partial u}{\partial t}$ is uniformly bounded in t, the function u can not below up in finite time so the solution u exists for all time. This finishes the proof of Proposition 1.1.

2. The uniform convergence

We again assume that function u is the solution for the initial value problem

$$\frac{\partial u}{\partial t} = \log \det \left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) - \log \det (g_{i\bar{j}}) + f$$
$$u(x, t) = 0, \quad \text{at } t = 0 \tag{2.1}$$

on $M \times [0, \infty)$ and let $v = u - \frac{1}{\mathrm{Vol}(M)} \int_M u\, dV$. In this section we shall prove the uniform convergence of $v(x, t)$ and that $\frac{\partial u}{\partial t}$ converge to a constant as t tends to the infinity.

In [5] Li-Yau derived the Hanack inequality for the positive solution of the heat equation on compact manifolds. For our application we shall present here the generalized version in the following form

Theorem 2.1. *Let M be a compact manifold of dimension n and let $g_{ij}(t)$, $0 \leqq t < \infty$, be a family of Riemannian metrics on M with the following properties:*

$$\begin{aligned}&\text{(a)}\ \ C_1\, g_{i\bar{j}}(0) \leqq g_{i\bar{j}}(t) \leqq C_2\, g_{i\bar{j}}(0)\\ &\text{(b)}\ \ \left|\frac{\partial g_{ij}}{\partial t}\right|(t) \leqq C_3\, g_{ij}(0)\\ &\text{(c)}\ \ R_{ij}(t) \geqq -K\, g_{ij}(0)\end{aligned} \tag{2.2}$$

where C_1, C_2, C_3, and K are positive constants independent of t. Let Δ_t denote the Laplace operator of the metric $g_{ij}(t)$. If $\varphi(x,t)$ is a positive solution for the equation

$$\left(\Delta_t - \frac{\partial}{\partial t}\right) \varphi(x,t) = 0 \tag{2.3}$$

on $M \times [0, \infty)$, then for any $\alpha > 1$, we have

$$\begin{aligned}\operatorname*{Sup}_{x\in M} \varphi(x, t_1) \leqq \inf_{x\in M} \varphi(x, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{n}{2}} \exp\Bigg(&\frac{1}{4(t_2-t_1)}\, C_2^2\, d^2\\ &+ \left(\frac{n\alpha K}{2(\alpha-1)} + C_2\, C_3 (n+A)\right)(t_2 - t_1)\Bigg)\end{aligned}$$

where d is the diameter of M measured by the metric $g_{ij}(0)$, $A = \sup \|\nabla^2 \log \varphi\|$ and $0 < t_1 < t_2 < \infty$.

The proof of Theorem 2.1 is a straightforward modificton of Li-Yau's argument. We refer the reader to [5] for the detail.

Now recall that $F = \dfrac{\partial u}{\partial t}$ satisfies the equation

$$\begin{aligned}&\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right) F = 0\\ &F(x,t) = f(x), \quad \text{at } t = 0.\end{aligned} \tag{2.4}$$

It follows from the maximum principle for the parabolic equation that for $t_2 > t_1 > 0$,

$$\begin{aligned}&\sup_{x\in M} F(x,t_2) < \sup_{x\in M} F(x,t_1) < \sup_{x\in M} f(x)\\ &\inf_{x\in M} F(x,t_2) > \inf_{x\in M} F(x,t_1) > \inf_{x\in M} f(x).\end{aligned} \tag{2.5}$$

From Proposition 1.1 it is also clear that the condition (2.2) in Theorem 2.1 is satisfied for $\tilde{g}_{i\bar{j}}$.

We define

$$\begin{aligned}&\varphi_n(x,t) = \sup_{x\in M} F(x, n-1) - F(x, (n-1)+t)\\ &\psi_n(x,t) = F(x, n-1+t) - \inf_{x\in M} F(x, n-1)\\ &\omega(t) = \sup_{x\in M} F(x,t) - \inf_{x\in M} F(x,t).\end{aligned} \tag{2.6}$$

Then obviously φ_n, ψ_n satisfy the Eq. (2.4) and according to (2.5) they are all positive functions. Apply Theorem 2.1 to these functions with $t_1 = \frac{1}{2}$ and $t_2 = 1$, we get

$$\sup_{x\in M} F(x,n-1)-\inf_{x\in M} F(x,n-\tfrac{1}{2})\leqq \gamma(\sup_{x\in M} F(x,n-1)-\sup_{x\in M} F(x,n)) \tag{2.7}$$

$$\sup_{x\in M} F(x,n-\tfrac{1}{2})-\inf_{x\in M} F(x,n-1)\leqq \gamma(\inf_{x\in M} F(x,n)-\inf_{x\in M} F(x,n-1)) \tag{2.8}$$

where $\gamma>1$ is a constant independent of n.

Add (2.8) to (2.7), we have

$$\omega(n-1)+\omega(n-\tfrac{1}{2})\leqq \gamma(\omega(n-1)-\omega(n))$$

and hence

$$\omega(n)\leqq \delta\omega(n-1), \quad \delta=\frac{\gamma-1}{\gamma}<1.$$

By induction we obtain

$$\omega(n)\leqq \delta^n \omega_0, \quad (\omega_0=\sup_{x\in M} f-\inf_{x\in M} f). \tag{2.9}$$

We know from (2.5) that the oscilation function $\omega(t)$ is decreasing in t, therefore we conclude from (2.9) that

$$\omega(t)\leqq C_4 e^{-at} \quad (e^{-a}=\delta) \tag{2.10}$$

Let us now define

$$\varphi(x,t)=\frac{\partial u}{\partial t}-\frac{1}{\mathrm{Vol}(M)}\int_M \frac{\partial u}{\partial t}\, d\tilde{V}. \tag{2.11}$$

Notice that

$$d\tilde{V}=\det\left(g_{i\bar{j}}+\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right)\bigwedge_{i=1}^{n}\left(\frac{-1}{2} dz^i\wedge d\bar{z}^j\right)$$

so

$$\begin{aligned}\frac{\partial}{\partial t}(d\tilde{V})&=\frac{\partial}{\partial t}\left(\det\left(g_{i\bar{j}}+\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right)\right)\bigwedge_{i=1}^{n}\left(\frac{-1}{2} dz^i\wedge d\bar{z}^j\right)\\ &=\frac{\partial}{\partial t}\log\det\left(g_{i\bar{j}}+\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) d\tilde{V}=\tilde{\Delta}\left(\frac{\partial u}{\partial t}\right) d\tilde{V}\end{aligned} \tag{2.12}$$

and then we have

$$\begin{aligned}\frac{\partial\varphi}{\partial t}(x,t)&=\frac{\partial^2 u}{\partial t^2}(x,t)-\frac{1}{\mathrm{Vol}(M)}\int_M \frac{\partial^2 u}{\partial t^2}\, d\tilde{V}-\frac{1}{\mathrm{Vol}(M)}\int_M \frac{\partial u}{\partial t}\tilde{\Delta}\left(\frac{\partial u}{\partial t}\right) d\tilde{V}\\ &=\tilde{\Delta}\left(\frac{\partial u}{\partial t}\right)-\frac{1}{\mathrm{Vol}(M)}\int_M \frac{\partial u}{\partial t}\tilde{\Delta}\left(\frac{\partial u}{\partial t}\right) d\tilde{V}.\end{aligned} \tag{2.13}$$

Consider

$$E=\frac{1}{2}\int_M \varphi^2\, d\tilde{V}$$

Then it follows from (2.12) and (2.13) that

$$\begin{aligned}\frac{dE}{dt} &= \int_M \varphi \frac{\partial \varphi}{\partial t} d\tilde{V} + \frac{1}{2} \int_M \varphi^2 \tilde{\Delta} \left(\frac{\partial u}{\partial t}\right) d\tilde{V} \\ &= \int_M \left(\frac{\partial u}{\partial t} - \frac{1}{\mathrm{Vol}(M)} \int_M \frac{\partial u}{\partial t} d\tilde{V}\right) \left(\tilde{\Delta}\left(\frac{\partial u}{\partial t}\right) - \frac{1}{\mathrm{Vol}(M)} \int_M \frac{\partial u}{\partial t} \tilde{\Delta}\left(\frac{\partial u}{\partial t}\right) d\tilde{V}\right. \\ &\quad + \frac{1}{2} \int_M \varphi^2 \tilde{\Delta} \left(\frac{\partial u}{\partial t}\right) d\tilde{V} \\ &= \int_M \frac{\partial u}{\partial t} \tilde{\Delta} \left(\frac{\partial u}{\partial t}\right) d\tilde{V} + \frac{1}{2} \int_M \varphi^2 \tilde{\Delta} \left(\frac{\partial u}{\partial t}\right) d\tilde{V} \\ &= \int_M (-1+\varphi) \left|\tilde{V} \frac{\partial u}{\partial t}\right|^2 d\tilde{V} \\ &\leqq -\frac{1}{2} \int_M \left|\tilde{V} \frac{\partial u}{\partial t}\right|^2 d\tilde{V} = -\frac{1}{2} \int_M |\tilde{V}\varphi|^2 d\tilde{V} \end{aligned} \tag{2.14}$$

where $|\tilde{V}()|^2 = \tilde{g}^{ij}()_i()_j$ and the last inequality follows from the fact that

$$\sup_{x \in M} \varphi(x,t) < \omega(t) < \tfrac{1}{2}, \qquad \text{for } t \text{ large enough}$$

By the definition (2.11) we have

$$\int_M \varphi \, d\tilde{V} = 0.$$

Then the Poincare inequality tells us that

$$\int_M |\tilde{V}\varphi|^2 d\tilde{V} \geqq \lambda_1(t) \int_M \varphi^2 d\tilde{V} \tag{2.15}$$

where $\lambda_1(t)$ is the first eigenvalue of the operator $\tilde{\Delta}$ at time t. Since the metrics $\tilde{g}_{i\bar{j}}(t)$ are uniformly equivalent to $g_{i\bar{j}}$ we know that there exists a constant $C_5 > 0$, such that $\lambda_1(t) > C_5$ for all t and therefore we have

$$\frac{d}{dt} E \leqq -C_5 E. \tag{2.16}$$

This implies that there exists a positive constant C_6 depending only on function f such that

$$E \leqq C_6 e^{-C_5 t} \tag{2.17}$$

since the volume forms $d\tilde{V}$ are uniformly equivalent to dV we also have

$$\int_M \varphi^2 dV \leqq C_6' e^{-C_5 t}. \tag{2.18}$$

Now we are in the position to prove the following

Proposition 2.2. *As $t \to \infty$ $v(x,t)$ converges to the function $v_\infty(x)$ in Proposition* 1.1 *in C^∞ topology and that $\frac{\partial u}{\partial t}$ converge to a constant in C^∞ topology as $t \to \infty$.*

Proof. For any $0<s<s'$ we have, according to (2.18) and (2.10)

$$\int_M |v(x,s')-v(x,s)|\,dV \leqq \int_M \int_s^{s'} \left|\frac{\partial v}{\partial t}(x,t)\right| dt\,dV$$
$$=\int_s^{s'} \int_M \left|\frac{\partial u}{\partial t}-\frac{1}{\mathrm{Vol}(M)}\int_M \frac{\partial u}{\partial t}\right| dV\,dt$$
$$\leqq \int_s^{s'} \int_M |\varphi|\,dV\,dt+\int_s^{s'} \int_M \frac{1}{\mathrm{Vol}(M)} \left|\int_M \frac{\partial u}{\partial t}\,d\tilde{V}-\int_M \frac{\partial u}{\partial t}\,dV\right| dV$$
$$\leqq \mathrm{Vol}(M)^{\frac{1}{2}} \int_s^{\infty} \left(\int_M \varphi^2\,dV\right)^{\frac{1}{2}} dt+\frac{1}{\mathrm{Vol}(M)}\int_s^{\infty} \omega(t)\,dt$$
$$\leqq C_7 \int_s^{\infty} e^{-C_5 2t}\,dt+C_8\int_s^{\infty} e^{-at}\,dt.$$

This shows that as $t\to\infty\, v(x,t)$ are Cauchy in L^1 norm so $v(x,t)$ converg in L^1 norm to some function $v'_\infty(x)$ as $t\to\infty$. On the other hand we know from Proposition 1.1 that for some time sequence $t_n\to\infty\, v(x,t)$ converg to the smooth function $v(x)$ in C^∞ topology as $n\to\infty$ these together imply that $v'_\infty(x)$ is identically equal to $v_\infty(x)$ and hence $v(x,t)$ converge in L^1 norm to $v_\infty(x)$ as $t\to\infty$. We claim that $v(x,t)$ actually converge to $V_\infty(x)$ in the C^∞ topology. Suppose this is not ture, then for some integer r and $\varepsilon>0$ there is a sequence t_n with

$$\|v(x,t_n)-v_\infty(x)\|_{C^r}>\varepsilon.$$

But $v(x,t_n)$ are bounded in C^∞ topology so there is a subsequence which we again denote it by $v(x,t_n)$ such that $v(x,t_n)$ converge in C^∞ topology to a smooth function $\tilde{v}_\infty(x)\neq v_\infty(x)$. This is a contridiction because $v(x,t_n)$ do converge to $v_\infty(x)$ in L^1 norm. Hence as $t\to\infty\, v(x,t)$ converge to $v_\infty(x)$ in C^∞ topology. This proves the first statement and consequently it follows from Eq. (2.1) that $\frac{\partial u}{\partial t}$ converge to $\frac{\partial u}{\partial t}(x,\infty)$ which is equal to $\log\det\left(g_{i\bar{j}}+\frac{\partial v_\infty(x)}{\partial z^i\,\partial \bar{z}^j}\right)$ $-\log\det(g_{i\bar{j}})+f$ in C^∞ topology as $t\to\infty$. But we also know from (2.10) that $\frac{\partial u}{\partial t}(x,\infty)$ must be a constant function on M. This finishs the proof of Proposition 2.2.

3. The main theorem

Based on the works in previous sections we now have the following:

Main Theorem. *Let M be a compact Kähler manifold of complex dimension n with the Kähler metric $g_{i\bar{j}}\,dz^i\,d\bar{z}^j$. Then for any closed $(1,1)$ form $\frac{\sqrt{-1}}{2}\,T_{i\bar{j}}\,dz^i\wedge d\bar{z}^j$ which represents the first Chern class $C_1(M)$ of M one can deform the initial metric by the heat equation*

$$\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t}=-\tilde{R}_{i\bar{j}}+T_{i\bar{j}} \qquad (3.1)$$

to another Kähler metric $\bar{g}_{i\bar{j}}$ which is in the same Kähler class as $g_{i\bar{j}}$ so that $T_{i\bar{j}}$ is the Ricci tensor of $\bar{g}_{i\bar{j}}$.

Corollary. *If the first Chern class of M is equal to zero then one can deform the initial metric in the negative Ricci direction to a Ricci flat metric.*

Proof of the Main Theorem. Let $R_{i\bar{j}}$ be the Ricci tensor of the metric $g_{i\bar{j}}$. Then a well-known theorem of Chern [1] shows that the (1, 1) form $\frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ represents $C_1(M)$. Since we assume that $\frac{\sqrt{-1}}{2\pi} T_{i\bar{j}} dz^i \wedge d\bar{z}^j$ also represents $C_1(M)$ we know that

$$T_{i\bar{j}} - R_{i\bar{j}} = \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \tag{3.2}$$

for some real-valued smooth function f on M. According to Proposition 1.1 we can find a smooth function $u(x, t)$ on $M \times [0, \infty)$ which solves the initial problem

$$\frac{\partial u}{\partial t} = \log\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) - \log\det(g_{i\bar{j}}) + f, \qquad u(x, 0) = 0 \tag{3.3}$$

and that

$$\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}$$

define a family of Kähler metrics on M. Moreover we know from Proposition 2.2 that as $t \to \infty$ $\tilde{g}_{i\bar{j}}$ converge in C^∞ topology to the limit metric $\tilde{g}_{i\bar{j}}(\infty)$ and that $\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t}$ converge uniformely to zero.

By the well-known formula the Ricci tensor $\tilde{R}_{i\bar{j}}$ of $\tilde{g}_{i\bar{j}}$ is given by

$$\tilde{R}_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right). \tag{3.4}$$

Differentiating Eq. (3.3) we have

$$\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log\det(g_{i\bar{j}}) + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \tag{3.5}$$

i.e., the metrics $\tilde{g}_{i\bar{j}}$ satisfy the equation

$$\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}} + T_{i\bar{j}}. \tag{3.6}$$

Let $t \to \infty$ we conclude that $T_{i\bar{j}} = \tilde{R}_{i\bar{j}}(\infty)$. This proves the theorem.

Finally we remark that the heat deformation method applies equally well, and is much easier, to the existence problem of Kähler-Einstein metrics on compact Kähler manifold M with negative first Chern class, which was also proved by Yau in [6]. In this case the evolution equation for the metrics has the form

$$\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}} - \tilde{g}_{i\bar{j}} \tag{3.7}$$

where $g_{i\bar{j}} = \tilde{g}_{i\bar{j}}(0)$ is positive definite and represents the negative of the first Chern class. The corresponding scalar equation is

$$\frac{\partial u}{\partial t} = \log \det \left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \right) - \log \det (g_{i\bar{j}}) - u + f. \tag{3.8}$$

The zero order estimate for u follows immediately from (3.8) by applying the maximum principle argument.

Also, differentiating (3.8) with respect to t, we get

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = \tilde{\Delta} \left(\frac{\partial u}{\partial t} \right) - \frac{\partial u}{\partial t}. \tag{3.9}$$

Again, the maximum principle implies the exponential decay of $\frac{\partial u}{\partial t}$. So we conclude in the same way that as $t \to \infty\, \tilde{g}_{ij}$ converge to the limit metric $\tilde{g}_{ij}(\infty)$ which is a Kähler-Einstein metric.

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