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Inventiones mathematicae

On the local theta-correspondence*

Stephen S. Kudla

Mathematics Department, University of Maryland, College Park, MD 20742, USA

Introduction

Let W, $\langle\ ,\ \rangle$ be a non-degenerate symplectic vector space over a non-archimedian local field k of characteristic 0, and let $\tilde{S}p(W)$ be the non-trivial 2-fold central extension of the symplectic group $Sp(W)$. Fix a non-trivial additive character ψ of k and let (ω, S) be the corresponding (smooth) oscillator representation of $\tilde{S}p(W)$. If (G, G') is a reductive dual pair in $Sp(W)$, [7], and if $\tilde{G}$ and $\tilde{G}'$ are the inverse images of G and G' in $\tilde{S}p(W)$, then (ω, S) may be viewed as a representation of $\tilde{G}\times\tilde{G}'$. The local theta correspondence is defined as follows: if $\pi\in\operatorname{Irr}\tilde{G}$ is an irreducible smooth representation of $\tilde{G}$, let

$$\Theta(\pi;\tilde{G}')=\{\pi'\in\operatorname{Irr}\tilde{G}'\,|\,\operatorname{Hom}_{\tilde{G}\times\tilde{G}'}(\omega,\pi\otimes\pi')\neq 0\},$$

and define $\Theta(\pi';\tilde{G})$ analogously. The representations π and π' are said to correspond if $\pi'\in\Theta(\pi;\tilde{G}')$ or, equivalently, $\pi\in\Theta(\pi';\tilde{G})$. A fundamental property of this local correspondence is embodied in the *local Howe duality conjecture* (HDC) which asserts that, for all $\pi\in\operatorname{Irr}\tilde{G}$ and $\pi'\in\operatorname{Irr}\tilde{G}'$

$$|\Theta(\pi;\tilde{G}')|\leqq 1 \quad\text{and}\quad |\Theta(\pi',\tilde{G})|\leqq 1, \tag{HDC}$$

so that the local theta correspondence is a bijection on its domain of definition. Recent progress toward this conjecture has been made by Howe [10] and Rallis [13] but it remains open in the general case.

In the present paper we will prove that, at least for a reductive dual pair of type (O, Sp), the local theta correspondence is compatible with induction. More precisely, recall that if G is the group of k-rational points of a connected reductive algebraic group over k, then Bernstein and Zelevinsky show that to each $\pi\in\operatorname{Irr}G$ there is a Levi subgroup M and a (super)cuspidal $\rho\in\operatorname{Irr}M$, such that π is a constituent of the induced representation $I(M,\rho)=\operatorname{ind}_P^G\rho$, where $P=MN$ is a parabolic subgroup with Levi factor M. Moreover the data $[\pi]$

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$=[M,\rho]$ is unique up to associativity. The result of [3] extends, with minor modification, when G or G' is not (algebraically) connected, to the groups $\tilde{G}$ and $\tilde{G}'$. The local theta correspondence should then be natural in the sense that there is a *bijection* between certain subsets of the sets of equivalence classes $\{[M,\rho]\}$ for $\tilde{G}$ and $\{[M',\rho']\}$ for $\tilde{G}'$ i.e.

$$\theta([M,\rho],\tilde{G}')=[M',\rho']$$

and

$$\theta([M',\rho'],\tilde{G})=[M,\rho]$$

such that

$$\pi'\in\Theta(\pi,G')\Rightarrow[\pi']=\theta([\pi]). \tag{I}$$

If π is cuspidal, so that $[\pi]=[G,\pi]$, property (I) is a consequence of the local Howe duality conjecture; but, in general, the two properties are somewhat complementary. Of course, one would like to have an explicit recipe for the bijection θ: $[M,\rho]\to[M',\rho']$.

For the dual pair $(O(V),Sp(W))$ in $Sp(\mathbf{W})$, with $\mathbf{W}=V\otimes W$, we prove a precise form of (I)-Theorem 2.5. The proof involves essentially two facts. First, if we consider the family of dual pairs $(O(V_m),Sp(W_n))$ where V_m runs over a fixed Witt class and $\dim W_n=2n$, then for each cuspidal $\pi\in\operatorname{Irr}\tilde{O}(V_n)$ there is a unique minimal $n=n(\pi)$ such that $\Theta(\pi,\tilde{S}p(W_n))$ is non-empty, and

$$\Theta(\pi,\tilde{S}p(W_n))=\begin{cases}\phi & \text{if } n<n(\pi)\\ \{\theta(\pi)\} \text{ with } \theta(\pi)\in\operatorname{Irr}\tilde{S}p(W_n) \text{ cuspidal} & \text{if } n=n(\pi)\\ \{\pi'_j\},\ \pi'_j \text{ induced } \forall j, & \text{if } n>n(\pi).\end{cases}$$

Similarly for each cuspidal $\pi'\in\operatorname{Irr}\tilde{S}p(W_n)$ there exists an $m(\pi)$ such that

$$\Theta(\pi',\tilde{O}(V_m))=\begin{cases}\phi & \text{if } m<m(\pi)\\ \{\theta(\pi')\} \text{ with } \theta(\pi')\in\operatorname{Irr}\tilde{O}(V_m) \text{ cuspidal} & \text{if } m=m(\pi)\\ \{\pi_j\},\ \pi_j \text{ induced } \forall j, & \text{if } m>m(\pi).\end{cases}$$

Actually, if we work literally with $\tilde{O}(V_m)$, we must take a slight technical modification - see Theorem 2.1 and Remark 2.4. There is a global analogue of this result due to Rallis [13] and Howe-Piatetski-Shapiro [9] which yields a non-trivial decomposition of the space of cusp forms and whose proof involves computation of Fourier coefficients. The proof in the local case involves for example, restrictions of certain induced representations to subgroups of the form $Sp(W_1)\times Sp(W_2)$ where $W=W_1+W_2$, and a study of the orbit structure of such a subgroup on a flag manifold (Prop. 3.4). This technique originates in [6] and is essential in [13, 14]. We also make use of the invariant distribution theorem (Theorem II.1.1) of [13].

The second fact involved in the proof of (I) arises from a computation of the Jacquet modules of (ω,S) with respect to the maximal parabolics of $\tilde{O}(V_m)$ and $\tilde{S}p(W_n)$. Here the key fact is that these Jacquet modules have an invariant filtration whose successive quotients are induced from oscillator representations for similar reductive dual pairs; this allows us to proceed by induction. A computation of this sort also occurs in [13], cf. also [1].

This paper amounts to an attempt to apply the outlook of [3] to the theta correspondence and was inspired by the wonderful lectures of Bernstein at Harvard during the spring of 1983. I would like to thank him for his encouragement and for a number of incisive comments. I would also like to thank J. Adams, R. Howe, S. Rallis and G. Zetler for a number of useful discussions. In particular the methods of this paper are substantially those of [12] and [13]. Finally I would like to thank the department of mathematics of Harvard University for providing a stimulating and congenial environment during the academic year 1982–1983 and the Sloan Foundation for making that environment available.

Notation. Throughout this paper k will be a non-archimedian local field of characteristic 0. Fix a non-trivial additive character $\psi\colon k\to\mathbf{C}^1$. Here $\mathbf{C}^1=\{z\in\mathbf{C}\,|\,|z|=1\}$.

For $x\in k$,

$$|x|=q^{-\operatorname{ord}x}$$

where q is the order of the residue class field and ord is the valuation on k.

We will follow the conventions of [2, 3]. In particular, if $H\subset G$ are ℓ-groups and (ρ,E) is a smooth representation of H on a complex vector space E, then $\operatorname{ind}_H^G\rho$ will denote the representation of G by right translations on the space of functions:

$$\{f\colon G\to E\,|\,1)\,f(hg)=\delta(h)^{\frac{1}{2}}\rho(h)f(g)$$
$$2)\,f \text{ is locally constant of compact support modulo } H\}$$

where $\delta=\Delta_G/\Delta_H$. Also $\operatorname{Ind}_H^G\rho$ (resp. $\natural\text{-}\operatorname{ind}_H^G\rho$) will denote the analogous representation in which the condition on the support of f in 2) (resp. the factor $\delta^{\frac{1}{2}}$) is omitted. If (ρ,E) is any smooth representation, $(\tilde{\rho},\tilde{E})$ will denote its contragradient. All representations will be smooth representations unless otherwise indicated. For any ℓ-space X, $S(X)$ denotes the space of locally constant functions of compact support and $C^\infty(X)$ denotes the space of locally constant functions.

If G is an ℓ-group, we let $A(G)$ be the category of all smooth representations of G and $\operatorname{Irr}(G)$ the set of equivalence classes of irreducible smooth representations.

Finally we will need the following invariants from the theory of quadratic forms: If η is a non-trivial additive character of k, consider the character of the second degree, in the terminology of [17] defined by

$$x\mapsto\eta(x^2). \tag{0.1}$$

Then $\gamma(\eta)$ is the Weil invariant of (0.1), and, in the notation of [15], for $a\in k^\times$,

$$\gamma(a,\eta)=\gamma(a\eta)\,\gamma(\eta)^{-1} \tag{0.2}$$

where $a\eta(x)=\eta(ax)$. For values and properties of $\gamma(a,\eta)$ see [15].

If V, (,) is a non-degenerate inner product space over k, then $d(V)\in k^\times/(k^\times)^2$ is the determinant of V and $h_k(V)$ is the Hasse invariant of V [16].

Also $(\ ,\ )_k$ is the Hilbert symbol for the field k.

Finally, if H is $GL_n(k)$ (resp. a central extension of $GL_n(k)$), then $|\ |$ denotes the character $h\to|\det h|$ for the normalized absolute value of k.

§ 1. Preliminaries

1.0. Let $(\ ,\ )_{\mathscr{k}}$ be the Hilbert symbol of the field $\mathscr{k}$, and define a 2-fold covering $\tilde{G}L(n)=GL(n)\times\mu_2$ of $GL(n)$ with multiplication:

$$(g,\varepsilon)(g',\varepsilon')=(gg',\varepsilon\varepsilon'(\det g,\det g')_{\mathscr{k}}).$$

For convenience we will frequently write $H_n=GL(n)$ and $H'_n=\tilde{G}L(n)$. If P is any parabolic subgroup of H_n with Levi decomposition $P=MN$, and $M\simeq H_{n_1}\times\ldots\times H_{n_k}$, then the inverse image P' of P in H'_n has a decomposition $P'=M'\cdot N$ where M' is the inverse image of M and N identified with the subgroup $\{(n,1)|n\in N\}\in H'_n$. Note that there is a surjective homomorphism

$$H'_{n_1}\times\ldots\times H'_{n_k}\to M'$$

$$((g_1,\varepsilon_1),\ldots,(g_k,\varepsilon_k))\mapsto((g_1,g_2,\ldots,g_k),\varepsilon_1\ldots\varepsilon_k\alpha)$$

with

$$\alpha=\prod_{i<j}(\det g_i,\det g_j)_{\mathscr{k}}.$$

Finally, if ψ is the fixed additive character, let χ be the character of H'_n defined by:

$$\chi(g,\varepsilon)=\varepsilon\gamma(\det g,\tfrac{1}{2}\psi)^{-1} \tag{1.0.1}$$

where $\gamma(a,\eta)$ is the Weil invariant given by (0.2).

1.1. Fix an anisotropic inner product space V_0, $(\ ,\ )_0$ over $\mathscr{k}$, and for each $m\in\mathbf{Z}_{\geqq 0}$, let V_m be the orthogonal direct sum of V_0 with m hyperbolic planes. Let $\ell=\dim V_m$. We assume that V_m comes equipped with a fixed Witt decomposition

$$V_m=V'_m+V^0_m+V''_m,$$

where $V^0_m=V_0$, and with bases $\{v_1,\ldots,v_m\}$ for V'_m and $\{v'_1,\ldots,v'_m\}$ for V''_m satisfying $(v_i,v_j)=(v'_i,v'_j)=0$ and $(v_i,v'_j)=\delta_{ij}$. The orthogonal group $O(V_m)$ then has a fixed maximal $\mathscr{k}$-split torus. Of course, if $V_0=\{0\}$, then $O(V_m)$ is split for all m and is the trivial group for $m=0$.

Let ν: $O(V_m)\to\mu_2$ be the determinant character and define a 2-fold covering $G_m=\tilde{O}(V_m)=O(V_m)\times\mu_2$ with $(g,\varepsilon)(g',\varepsilon')=(gg',\varepsilon\varepsilon'(\nu(g),\nu(g'))_{\mathscr{k}})$. Of course G_m is just the inverse image of $O(V_m)$ in $\tilde{G}L(V_m)$, and is a split extension if the residue characteristic of $\mathscr{k}$ is not 2. Note that the group G_m has an automorphism

$$G_m\xrightarrow{\sim}G_m \tag{1.1.1}$$

$$(g,\varepsilon)\mapsto(g,\varepsilon\nu(g))$$

covering the identity map of $O(V_m)$.

We denote by χ the restriction to G_m of the character of $\widetilde{GL(V_m)}$ defined by (1.0.1). It will also be convenient to consider the characters

$$\zeta_{a,b}(h,\varepsilon)=((-1)^{b(a-b)+b(b-1)/2},\det h)_{\mathscr{k}} \tag{1.1.2}$$

and

$$\kappa(a,b)=\zeta_{a,a-b}\chi^{a-b}. \tag{1.1.3}$$

It is then easy to check the following.

Lemma 1.1.1. (i) $\kappa(a,b)$ *depends only on* a *and* b mod 2.

(ii) $\kappa(a,b)=\begin{cases}1 & a\equiv b(2)\\ \chi & a\equiv 1(2), b\equiv 0(2)\\ \chi^{-1} & a\equiv 0(2), b\equiv 1(2).\end{cases}$

(iii) $\kappa(n,a)^{-1}\kappa(n,b)=\kappa(a,b)$. ☐

1.2. For j with $1\leqq j\leqq m$, let $V_j'=\operatorname{span}\{v_1,\dots,v_j\}$, $V_j''=\operatorname{span}\{v_1',\dots,v_j'\}$, and

$$V_j^0=\operatorname{span}\{v_{j+1},\dots,v_m,v_{j+1}',\dots,v_m'\}+V_m^0,$$

so that we obtain a Witt decomposition

$$V_m=V_j'+V_j^0+V_j''$$

with $V_j^0\simeq V_{m-j}$. If $P(V_j'')$ is the subgroup of $O(V_m)$ which stabilizes V_j'', then there is Levi decomposition $P(V_j'')=M(V_j'')N(V_j'')$ where

$$M(V_j'')\simeq GL(V_j')\times O(V_j^0),$$

and we call $P(V_j'')$ a maximal parabolic subgroup of $O(V_m)$ - see Remark 1.2.1 below for a discussion of this choice. Let P_j (resp. M_j) be the inverse image of $P(V_j'')$ (resp. $M(V_j'')$) in G_m and let N_j be the image of $N(V_j'')$ under the natural splitting $n\mapsto(n,1)$. Then $P_j=M_jN_j$ and

$$M_j\simeq H_j\times G_{m-j}, \tag{1.2.0}$$

where we note that the $GL(V_j)\simeq H_j$ factor of $M(V_j'')$ lies in $SO(V_m)$ and hence may be identified with a subgroup of G_m. Every maximal parabolic subgroup of $O(V_m)$ is conjugate to one of the $P(V_j'')$'s. By abuse of language (e.g. if the covering $G_m\to O(V_m)$ is non-trivial), we will call the groups P_j a standard set of maximal parabolic subgroups.

A standard set of parabolic subgroups of $O(V_m)$ is then indexed by sequences of positive integers $\mathbf{s}=(s_1,\dots,s_k)$ with $|\mathbf{s}|=\sum_j s_j\leqq m$. The parabolic subgroup $P(V_{\mathbf{s}}'')$ is the stabilizer of the isotropic flag

$$V_{s_1}''\subset V_{s_1+s_2}''\subset\dots\subset V_{|\mathbf{s}|}'', \tag{1.2.1}$$

and $P(V_{\mathbf{s}}'')$ has Levi decomposition $P(V_{\mathbf{s}}'')=M(V_{\mathbf{s}}'')N(V_{\mathbf{s}}'')$ where

$$M(V_{\mathbf{s}}'')\simeq H_{s_1}\times\dots\times H_{s_k}\times O(V_{m-|\mathbf{s}|}).$$

Let $P_{\mathbf{s}}$ (resp. $M_{\mathbf{s}}$) denote the inverse image of $P(V_{\mathbf{s}}'')$ (resp. $M(V_{\mathbf{s}}'')$) in G_m and let $N_{\mathbf{s}}$ be the image of $N(V_{\mathbf{s}}'')$ under the natural splitting. Then, as before, $P_{\mathbf{s}}=M_{\mathbf{s}}N_{\mathbf{s}}$ with $M_{\mathbf{s}}\simeq H_{s_1}\times\dots\times H_{s_k}\times G_{m-|\mathbf{s}|}$. Formally we let $P_0=G_m$, and we note that

$$P_{\mathbf{s}} = P_{s_1} \cap P_{s_1+s_2} \cap \dots \cap P_{|\mathbf{s}|}. \tag{1.2.2}$$

In particular, $P_{|\mathbf{s}|}$ (resp. P_{s_1}) is the unique maximal parabolic subgroup P_j with j maximal (resp. minimal) which contains $P_{\mathbf{s}}$. Note that $P_{\mathbf{s}}$ and $P_{\mathbf{s}'}$ are associate if and only if $|\mathbf{s}| = |\mathbf{s}'|$ and the sets $\mathbf{s}$ and $\mathbf{s}'$ differ by a permutation.

Remark 1.2.1. When $\dim V_0 = 0$, so that V_m is split, a technical remark is in order. Let

$$V_m'^{-} = \mathrm{span}\{v_1, \dots, v_{m-1}, v_m'\}$$

and

$$V_m''^{-} = \mathrm{span}\{v_1', \dots, v_{m-1}', v_m\}.$$

For each j, let

$$P^0(V_j'') = P(V_j'') \cap SO(V_m)$$

and let

$$P^0(V_m''^{-}) = P(V_m''^{-}) \cap SO(V_m).$$

Then the conjugacy classes of maximal parabolic subgroups of $SO(V_m)$ are represented by $P^0(V_j'')$, for $1 \leqq j \leqq m-2$, $P^0(V_m'')$ and $P^0(V_m''^{-})$. Moreover:

$$P^0(V_{m-1}'') = P^0(V_m'') \cap P^0(V_m''^{-}).$$

is *not* maximal. On the other hand, we have

$$P^0(V_m'') = P(V_m'')$$

and

$$P^0(V_m''^{-}) = P(V_m''^{-}),$$

and these two subgroups are *conjugate* in $O(V_m)$. On the other hand the parabolic subgroup $P(V_{m-1}'')$ is no longer contained in $P(V_m'')$ or $P(V_m''^{-})$ and is now *maximal*! Note that the "extra" parabolic $P(V_{m-1}'')$ has Levi factor

$$M(V_{m-1}'') = H_{m-1} \times O(V_1).$$

Similarly, when $|\mathbf{s}| = m$, we may define flags $V_{\mathbf{s}}''^{-}$ by replacing V_m'' with $V_m''^{-}$ in (1.2.1), and the above remarks carry over verbatim to the parabolic subgroups $P^0(V_{\mathbf{s}}'')$, $P^0(V_{\mathbf{s}}''^{-})$ etc. For example

$$P^0(V_{\mathbf{s}}'') \cap P^0(V_{\mathbf{s}}''^{-}) = P^0(V_{\mathbf{t}}'')$$

where $\mathbf{t} = (s_1, \dots, s_k - 1)$.

1.3. For each $n \in \mathbf{Z}_{\geqq 0}$, let W_n, $\langle\ ,\ \rangle$ be a non-degenerate symplectic vector space of dimension $2n$ over k. We assume that W_n comes equipped with a fixed complete polarization

$$W_n = W_n' + W_n'' \tag{1.3.1}$$

and bases $\{e_1, \dots, e_n\}$ for W_n' and $\{e_1', \dots, e_n'\}$ for W_n'' such that $\langle e_i, e_j' \rangle = \delta_{ij}$, i.e. $\{e_1, \dots, e_n'\}$ is a fixed symplectic basis for W_n, $\langle\ ,\ \rangle$. The symplectic group $Sp(W_n)$ then has a fixed maximal k-split torus, and is the trivial group if $n=0$.

Let $G'_n = \tilde{S}p(W_n)$ be the unique non-trivial 2-fold central extension of $Sp(W_n)$ (or the group μ_2 if $n=0$). Then the polarization (1.3.1) gives rise to an isomorphism

$$G'_n = \tilde{S}p(W_n) \xrightarrow{\sim} Sp(W_n) \times \mu_2 \tag{1.3.2}$$

with $(g, \varepsilon)(g', \varepsilon') = (gg', \varepsilon\varepsilon' c(g, g'))$ where the cocycle $c(g, g')$ is given explicitly by Rao [15].

1.4. For j with $1 \leqq j \leqq n$, let $W'_j = \operatorname{span}\{e_1, \ldots, e_j\}$, $W''_j = \operatorname{span}\{e'_1, \ldots, e'_j\}$, and $W^0_j = (W'_j + W''_j)^{\perp}$, so that there is a decomposition $W_n = W'_j + W^0_j + W''_j$ with $W^0_j \simeq W_{n-j}$. For any sequence $\mathbf{s} = (s_1, \ldots, s_k)$ of positive integers with $|\mathbf{s}| \leqq n$, let $P(W''_{\mathbf{s}})$ be the stabilizer of the isotropic flag

$$W''_{s_1} \subset W''_{s_1+s_2} \subset \ldots \subset W''_{|\mathbf{s}|}.$$

and let $P'_{\mathbf{s}}$ be the inverse image of $P(W''_{\mathbf{s}})$ in G'_n. Then $P(W''_{\mathbf{s}})$ has a Levi decomposition $P(W''_{\mathbf{s}}) = M(W''_{\mathbf{s}}) N(W''_{\mathbf{s}})$ with

$$M(W''_{\mathbf{s}}) \simeq H_{s_1} \times \ldots \times H_{s_k} \times Sp(W_{n-|\mathbf{s}|}).$$

Let $M'_{\mathbf{s}}$ be the inverse image of $M(W''_{\mathbf{s}})$ in G'_n and let $N'_{\mathbf{s}}$ be the image of $N(W''_{\mathbf{s}})$ under the natural section given by (1.3.2). Then $M'_{\mathbf{s}}$ normalizes $N'_{\mathbf{s}}$ and $P'_{\mathbf{s}} \simeq M'_{\mathbf{s}} \ltimes N'_{\mathbf{s}}$. Moreover there is a natural homomorphism

$$H'_{s_1} \times \ldots \times H'_{s_k} \times G'_{n-|\mathbf{s}|} \rightarrow M'_{\mathbf{s}} \tag{1.4.1}$$

given by:

$$((h_1, \varepsilon_1), \ldots, (h_k, \varepsilon_k), (g, \varepsilon)) \longmapsto ((h_1, \ldots, h_k, g), \varepsilon_1 \ldots \varepsilon_k \varepsilon \alpha)$$

where

$$\alpha = \prod_{i<j} (\det h_i, \det h_j)_k \cdot \left(\prod_{j=1}^{k} (\det h_j, x(g))_k \right)$$

where $x(g)$ is as in [15, Lemma 5.1].

Again the $P'_{\mathbf{s}}$ give a standard set of parabolic subgroups of G'_n and satisfy the analogue of (1.2.2). Again we formally let $P'_0 = G'_n$.

1.5. If G is one of the groups above and if $P = MN$ is any parabolic subgroup of G, we have induction and localization functors as usual [2, 3] R: $A(G) \rightarrow A(M)$ and I: $A(M) \rightarrow A(G)$ which we assume to be normalized so that $I(M, \tilde{\rho}) = \widetilde{I(M, \rho)}$ and $\operatorname{Hom}_G(\pi, I(M, \rho)) = \operatorname{Hom}_M(R(M, \pi), \rho)$. We also have the functor $\bar{R}$: $A(G) \rightarrow A(M)$ given by

$$\bar{R}(M, \pi) = \widetilde{R(M, \tilde{\pi})}$$

and which satisfies

$$\operatorname{Hom}_G(I(M, \rho), \pi) = \operatorname{Hom}_M(\rho, \bar{R}(M, \pi)).$$

A representation $\pi \in A(G)$ will be called cuspidal if the representations $R(M, \pi)$ are zero for all proper Levi subgroups; and $\operatorname{Irr}_c G$ will denote the set of equivalence classes of cuspidal irreducible representations. If P is indexed by a set $\mathbf{s}$ as in 1.2 and 1.4, we will write $I_{\mathbf{s}}$ and $R_{\mathbf{s}}$ for $I(M, \cdot)$ and $R(M, \cdot)$ etc.

It is easily checked that Theorem 2.9 of [3] extends immediately to the group G'_n. In particular, if $\pi' \in \operatorname{Irr} G'_n$, then there exists a set $\mathbf{s}=(s_1,\dots,s_k)$ with $s_i \in \mathbf{Z}_{>0}$ and $|\mathbf{s}| \leqq n$ and representations $\sigma'_i \in \operatorname{Irr}_c H'_i$ and $\rho' \in \operatorname{Irr}_c G'_{n-|\mathbf{s}|}$ such that

$$\pi' \in JH(I_{\mathbf{s}}(\sigma'_1 \otimes \dots \otimes \sigma'_k \otimes \rho')).$$

The data $(\mathbf{s}, \boldsymbol{\sigma}, \rho)$ is unique up to associativity in G'_n and we write

$$[\pi'] = [\sigma'_1, \dots, \sigma'_r, \rho],$$

the index $\mathbf{s}$ being understood.

The situation for G_m is a little more delicate due to the (algebraic) disconnectedness of $O(V_m)$, (except in the trivial case). Nonetheless it is easy to prove the following result by restriction to $SO(V_m)$ and its inverse image in G_m:

Proposition 1.5.1. *Suppose M and N are standard Levi subgroups of G_m and suppose that $\rho \in \operatorname{Irr}_c M$ and $\eta \in \operatorname{Irr}_c N$. Let ν be the lift to G_m of the determinant character of $O(V_m)$.*

(i) *If*

$$JH(I(M,\rho)) \cap JH(I(N,\eta)) \neq \emptyset$$

then

$$(M,\rho) \sim (N,\eta) \quad \text{or} \quad (N, \nu\eta)$$

(ii) *If $(M,\rho) \sim (N,\eta)$, then*

$$JH^0(I(M,\rho)) = JH^0(I(N,\eta)) \qquad \text{if } \rho \simeq \nu\rho$$

and

$$JH^0(I(M,\rho) \oplus I(M,\nu\rho)) = JH^0(I(N,\eta) \oplus I(N,\nu\eta)) \qquad \text{if } \rho \not\simeq \nu\rho.$$

Here $(M,\rho) \sim (N,\eta)$ means associativity in G_m and JH^0 etc. are as in [3]. □

Therefore, if $\pi \in \operatorname{Irr} G_m$, there exist $\mathbf{s}=(s_1,\dots,s_r)$ with $s_i \in \mathbf{Z}_{>0}$, $|\mathbf{s}| \leqq m$, $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_r)$, with $\sigma_i \in \operatorname{Irr}_c H_i$, and $\rho \in \operatorname{Irr}_c G_{m-|\mathbf{s}|}$ such that

$$\pi \in JH(I_{\mathbf{s}}(\sigma_1 \otimes \dots \otimes \sigma_r \otimes \rho)) \tag{1.5.1}$$

and this data is unique up to associativity in G_m and possible multiplication by ν.

Note that if $\dim V_m$ is odd, then $JH(I(M,\rho))$ and $JH(I(M,\nu\rho))$ are disjoint and so no multiplication by ν is involved.

Remark 1.5.1. In the case $\dim V_0 = 0$ the group G_1 has *no* cuspidal representations and, as a result, the Levi subgroups $M_{\mathbf{s}}$ with $|\mathbf{s}| = m-1$ have no cuspidal representations. Thus such $M_{\mathbf{s}}$ will never appear as the first member of a pair (M,ρ) with ρ cuspidal, i.e. the 'extra' parabolics for G_m - see Remark 1.2.1 - are not involved in the classification. In any case we write

$$[\pi] = \begin{cases} [\sigma_1,\dots,\sigma_r,\rho] \\ [\sigma_1,\dots,\sigma_r,\rho] + [\sigma_1,\dots,\sigma_r,\nu\rho] \end{cases} \tag{1.5.2}$$

if (1.5.1) holds (resp. if (1.5.1) holds for both ρ and $\nu\rho$ with $\rho \not\simeq \nu\rho$.)

Remark 1.5.2. We will call a representation $\pi \in \operatorname{Irr} G_m$ *ambiguous* if $[\pi]$ is given by the second case in (1.5.2).

For any $\pi \in \operatorname{Irr} G_m$ there exists $\mathbf{s}$, $\boldsymbol{\sigma}$ and ρ such that

$$\pi \hookrightarrow I_{\mathbf{s}}(\sigma_1 \otimes \ldots \otimes \sigma_r \otimes \rho).$$

If π is ambiguous, it may also happen that, for some permutation $\mathbf{s}'$ of $\mathbf{s}$ and corresponding permutation $\boldsymbol{\sigma}'$ of $\boldsymbol{\sigma}$, we have

$$\pi \hookrightarrow I_{\mathbf{s}'}(\sigma'_1 \otimes \ldots \otimes \sigma'_r \otimes \nu \rho)$$

as well. We will call such π *strongly ambiguous* and will write

$$[\pi]_+ = [\sigma_1, \ldots, \sigma_r, \rho] + [\sigma_1, \ldots, \sigma_r, \nu \rho].$$

Similarly, if π is ambiguous but not strongly ambiguous, or if π is not ambiguous, we write

$$[\pi]_+ = [\sigma_1, \ldots, \sigma_r, \rho].$$

1.6. The center of the group G_m contains the group $\{(1, \varepsilon) | \varepsilon \in \mu_2\} = \langle \varepsilon \rangle$, and for any $\pi \in \operatorname{Irr} G_m$, we define $a = a(\pi) \in \mathbf{Z}/2\mathbf{Z}$ by

$$\pi(1, \varepsilon) = \varepsilon^a$$

so that $\operatorname{Irr} G_m = \operatorname{Irr}^0 G_m \coprod \operatorname{Irr}^1 G_m$ with $\operatorname{Irr}^0 G_m \simeq \operatorname{Irr} O(V_m)$. Moreover there is a bijection between $\operatorname{Irr}^0 G_m$ and $\operatorname{Irr}^1 G_m$ given by multiplication by the character χ defined by (1.0.1):

$$\operatorname{Irr}^0 G_m \to \operatorname{Irr}^1 G_m$$
$$\pi \mapsto \chi \cdot \pi.$$

For later convenience we let

$$\pi[0] = \begin{cases} \pi & \text{if } a(\pi) = 0 \\ \chi^{-1} \pi & \text{if } a(\pi) = 1 \end{cases}$$

and let

$$\pi[1] = \chi \pi[0].$$

Of course the representation theory of G_m is thus reduced to that of $O(V_m)$, but it is the group G_m which arises most naturally from the dual pair construction, and we want to avoid making ad hoc adjustments in that formalism!

§2. The theta correspondence

For $m, n \in \mathbf{Z}_{\geqq 0}$ let

$$\mathbf{W}_{m,n} = V_m \otimes W_n,$$

and let $\langle\!\langle\ ,\ \rangle\!\rangle = (\ ,\ ) \otimes \langle\ ,\ \rangle$, so that there is, as usual a natural homomorphism α: $O(V_m) \times Sp(W_n) \to Sp(\mathbf{W}_{m,n})$. The space $\mathbf{W} = \mathbf{W}_{m,n}$ inherits the polarization

$$\mathbf{W} = \mathbf{W}' + \mathbf{W}''$$

where $\mathbf{W}' = V_m \otimes W_n'$ and $\mathbf{W}'' = V_m \otimes W_n''$ and hence there is an isomorphism

$$\tilde{S}p(\mathbf{W}) \xrightarrow{\sim} Sp(\mathbf{W}) \times \mu_2.$$

There is a lift $\tilde{\alpha}$ of α to

$$\tilde{\alpha}\colon\ G_m \times G_n' \to \tilde{S}p(\mathbf{W})$$

which is not quite unique due to the automorphism (1.1.1) of G_m, but we fix a choice given by $\tilde{\alpha}((h,\varepsilon)) = (h \otimes 1_{W_n}, \varepsilon^n)$ for $(h,\varepsilon) \in G_m = O(V_m) \times \mu_2$. Note that the diagram

$$\begin{tikzcd}
G_n' \arrow[r, "\tilde{\alpha}"] & \tilde{S}p(\mathbf{W}) \\
\tilde{G}L(W_n') \arrow[u] \arrow[r, "\tilde{\alpha}'"] & \tilde{G}L(\mathbf{W}') \arrow[u]
\end{tikzcd}$$

commutes where

$$\tilde{\alpha}'(g,\varepsilon) = (1_V \otimes g, \varepsilon^{\ell}((-1)^{\frac{1}{2}\ell(\ell-1)} d(V_m), x(g))_k). \tag{2.1}$$

Let (ω, S) be the smooth oscillator representation of $\tilde{S}p(\mathbf{W})$ determined by the character ψ and define

$$\omega_{m,n} = \tilde{\alpha}^*(\omega),$$

so that $(\omega_{m,n}, S)$ is a smooth representation of $G_m \times G_n'$. Note that if V_m or W_n is the zero space then so is $\mathbf{W}$, and (ω, S) reduces to the non-trivial character of $\tilde{S}p(\mathbf{W}) \simeq \mu_2$. As a result,

$$\omega_{m,0} \simeq \mathbf{1}_{G_m} \otimes \omega^{\ell} \tag{2.2}$$

where $\mathbf{1}_{G_m}$ is the trivial representation of G_m and ω is the non-trivial character of G_0', while, if $V_0 = 0$,

$$\omega_{0,n} \simeq \mu^n \otimes \mathbf{1}_{G_n'} \tag{2.3}$$

where $\mathbf{1}_{G_n'}$ is the trivial representation of G_n' and μ is the non-trivial character of $G_0 \simeq \mu_2$.

As in the introduction, if $\pi \in \operatorname{Irr} G_m$, let

$$\Theta_n(\pi) = \{\pi' \in \operatorname{Irr} G_n' \mid \operatorname{Hom}_{G_m \times G_n'}(\omega_{m,n}, \pi \otimes \pi') \neq 0\}$$

and if $\pi' \in \operatorname{Irr} G_n'$, let

$$\Theta_m(\pi') = \{\pi \in \operatorname{Irr} G_m \mid \operatorname{Hom}_{G_m \times G_n'}(\omega_{m,n}, \pi \otimes \pi') \neq 0\}.$$

Note that $\omega_{m,n}|_{\langle\varepsilon\rangle} = \varepsilon^n$ where $\langle\varepsilon\rangle$ is as in §1.6, and so $\Theta_n(\pi) = \emptyset$ for n and $a(\pi)$ of opposite parity. Thus it turns out to be more natural to study the sets $\Theta_n(\pi[n])$ where $\pi[n]$ is defined in §1.6.

Theorem 2.1. (i) *For* $\pi \in \operatorname{Irr}_c G_m$, *let*

$$n(\pi) = \min\{n \mid \Theta_n(\pi[n]) \neq \emptyset\}.$$

Then if $n(\pi) < \infty$,

$$\Theta_n(\pi[n]) = \begin{cases} \emptyset & \text{if } n < n(\pi) \\ \{\theta(\pi)\} & \text{if } n = n(\pi) \\ \{\pi'_j\} & \text{if } n > n(\pi) \end{cases}$$

where $\theta(\pi) \in \operatorname{Irr}_c G'_{n(\pi)}$, *and the* $\pi'_j \in \operatorname{Irr} G'_n$ *are not cuspidal. In particular,* $\theta(\pi)$ *is the unique cuspidal representation in the set* $\coprod_{n=0}^{\infty} \Theta_n(\pi[n])$.

(ii) *For* $\pi' \in \operatorname{Irr}_c G'_n$, *let*

$$m(\pi') = \min\{m \mid \Theta_m(\pi') \neq \emptyset\}.$$

Then, if $m(\pi') < \infty$,

$$\Theta_m(\pi') = \begin{cases} \emptyset & \text{if } m < m(\pi) \\ \{\theta(\pi')\} & \text{if } m = m(\pi) \\ \{\pi_j\} & \text{if } m > m(\pi) \end{cases}$$

where $\theta(\pi') \in \operatorname{Irr}_c G_{m(\pi')}$ *and the* $\pi_j \in \operatorname{Irr} G_m$ *are not cuspidal. In particular,* $\theta(\pi')$ *is the unique cuspidal representation in the set* $\coprod_{m=0}^{\infty} \Theta_m(\pi')$.

Corollary 2.2. *If* $\pi \in \operatorname{Irr}_c G_m$ *and* $\pi' \in \operatorname{Irr}_c G_n$, *then*

$$\theta(\theta(\pi)) = \pi[n(\pi)]$$

and

$$\theta(\theta(\pi')) = \pi'.$$

Remark 2.3. We could, as is traditional, have renormalized the representation $\omega_{m,n}$ by setting

$$\omega^{\natural}_{m,n} = \begin{cases} \omega_{m,n} & \text{for } n \text{ even} \\ \chi^{-1}\,\omega_{m,n} & \text{for } n \text{ odd.} \end{cases}$$

where χ is the character of $G_m \subset \widetilde{GL(V_m)}$ given in Sect. 1.0. Since $\tilde{\alpha}$ already factors through $O(V_m)$ for n even, $\omega^{\natural}_{m,n}$ is a smooth representation of $O(V_m) \times G'_n$. In fact, in the Schrödinger model associated to the polarization $(\mathbf{W}', \mathbf{W}'')$ of $\mathbf{W}$, $S \simeq S(\mathbf{W}') \simeq S(V^n)$ and, for $h \in O(V_m)$ and $\varphi \in S$

$$\omega^{\natural}_{m,n}(h)\,\varphi(x) = \varphi(h^{-1}x).$$

For $\pi \in \operatorname{Irr}^0 G_m \cong \operatorname{Irr} O(V_m)$,

$$\begin{aligned} \Theta_n(\pi[n]) &= \{\pi' \in \operatorname{Irr} G'_n \mid \operatorname{Hom}_{G_m \times G'_n}(\omega_{m,n}, \pi[n] \otimes \pi') \neq \emptyset\} \\ &= \{\pi' \in \operatorname{Irr} G'_n \mid \operatorname{Hom}_{O(V_m) \times G'_n}(\omega^{\natural}_{m,n}, \pi \otimes \pi') \neq 0\}. \end{aligned}$$

Thus this renormalization removes the 'twists' from Theorem 2.1, and elsewhere, but, as remarked in Sect. 1.6, we wanted to avoid this convenient but ad hoc adjustment.

In the Schrödinger model just described the action of $\widetilde{GL(W'_n)} \subset G'_n$ is given by

$$\omega_{m,n}(g,\varepsilon)\,\phi(x) = \chi_V(g,\varepsilon)\,|\det g|^{\frac{1}{2}\ell}\,\phi(xg) \tag{2.4}$$

where $\ell = \dim V_m$ and

$$\chi_V(g, \varepsilon) = \varepsilon^{\ell}\,\gamma(\det g, \tfrac{1}{2}\psi)^{-\ell}(d(V_m), \det g)_{\ell}. \tag{2.5}$$

Here $\gamma(t, \frac{1}{2}\psi)$ is the Weil invariant given by (0.2). We will frequently write simply χ_V for χ_{V_m} since this character is, in fact, independent of m.

If $\sigma \in \mathrm{Irr}_c H_j$, define $\theta(\sigma) \in \mathrm{Irr}_c H_j$ by

$$\theta(\sigma)(g) = \tilde{\sigma}({}^t g^{-1}). \tag{2.6}$$

We may now state our main result:

Theorem 2.5. (i) *If $\pi \in \mathrm{Irr}\, G_m$ is strongly ambiguous, as defined in Remark* 1.5.2, *then $\Theta_n(\pi[n]) = \emptyset$ for all n.*

(ii) *If $\pi \in \mathrm{Irr}\, G_m$ is not strongly ambiguous, let*

$$[\pi]_+ = [\sigma_1, \ldots, \sigma_r, \rho]$$

with $\sigma_i \in \mathrm{Irr}_c H_{s_i}$, $\rho \in \mathrm{Irr}_c G_{m-|\mathbf{s}|}$ and suppose that $\pi' \in \Theta_n(\pi[n])$. Then, if $n \geqq n(\rho) + |\mathbf{s}|$,

$$[\pi'] = [\chi_V \theta(\sigma_1), \ldots, \chi_V \theta(\sigma_r), \chi_V|\ |^{\frac{1}{2}\ell - n}, \chi_V|\ |^{\frac{1}{2}\ell - n + 1}, \ldots, \chi_V|\ |^{\frac{1}{2}\ell - |\mathbf{s}| - n(\rho) - 1}, \theta(\rho)].$$

If $n < n(\rho) + |\mathbf{s}|$, then there exists a sequence $i_1, \ldots, i_t$ with $t = n(\rho) + |\mathbf{s}| - n$, such that

$$\sigma_{i_1} = |\ |^{\frac{1}{2}\ell - n - 1}, \ldots, \sigma_{i_t} = |\ |^{\frac{1}{2}\ell - n - t}$$

and then

$$[\pi'] = [\chi_V \theta(\sigma_1), \ldots, \widehat{\chi_V \theta(\sigma_{i_1})}, \ldots, \widehat{\chi_V \theta(\sigma_{i_t})}, \ldots, \chi_V \theta(\sigma_r), \theta(\rho)].$$

This expression for $[\pi']$ is indepencent of the choice of sequence.

Applying the symmetry of the correspondence and noting that

$$\pi \in \Theta_n(\pi') \Rightarrow \pi = \pi[n]$$

we obtain:

Corollary 2.6. *For $\pi' \in \mathrm{Irr}\, G'_n$, let*

$$[\pi'] = [\sigma'_1, \ldots, \sigma'_r, \rho']$$

with $\sigma'_i \in \mathrm{Irr}_c H'_{s_i}$ and $\rho' \in \mathrm{Irr}_c G'_{n-|\mathbf{s}|}$, and suppose that $\pi \in \Theta_m(\pi')$. Then if $m \geqq m(\rho') + |\mathbf{s}|$,

$$[\pi]_+ = [\theta((\chi_V)^{-1} \sigma'_1), \ldots, \theta((\chi_V)^{-1} \sigma'_r), |\ |^{\frac{1}{2}\ell - n - 1}, \ldots, |\ |^{\frac{1}{2}\ell - n - m + m(\rho') + |\mathbf{s}|}, \theta(\rho')[n]].$$

If $m < m(\rho') + |\mathbf{s}|$, there exists a sequence $i_1, \ldots, i_t$ with $t = m(\rho') + |\mathbf{s}| - m$ such that

$$\sigma'_{i_1} = \chi_V|\ |^{\frac{1}{2}\ell - n}, \ldots, \sigma'_{i_t} = \chi_V|\ |^{\frac{1}{2}\ell - n + t - 1}$$

and then

$$[\pi]_+ = [\theta((\chi_V)^{-1} \sigma'_1), \ldots, \widehat{\theta((\chi_V)^{-1} \sigma'_{i_1})}, \ldots, \widehat{\theta((\chi_V)^{-1} \sigma'_{i_t})}, \ldots, \theta((\chi_V)^{-1} \sigma'_r), \theta(\rho')[n]].$$

Again this expression for $[\pi]_+$ is independent of the choice of sequence.

Actually the proof of Theorem 2.5 yields a little more information.

Corollary 2.7. (i) *If* $\pi \in \operatorname{Irr} G_m$ *with*

$$[\pi]_+ = [\sigma_1, \ldots, \sigma_r, \rho]$$

and if $n < n(\rho) + |\mathbf{s}|$, *then* $\Theta_n(\pi) = \emptyset$ *unless* π *occurs as a subrepresentation of* $I_{\mathbf{s}}(\sigma_1 \otimes \ldots \otimes \sigma_r \otimes \rho)$ *for some ordering of the* σ_i's *such that there is a sequence* $i_1 < \ldots < i_t$ *with*

$$\sigma_{i_1} = |\ \ |^{\frac{1}{2}\ell - n - 1}, \sigma_{i_2} = |\ \ |^{\frac{1}{2}\ell - n - 2}, \ldots, \sigma_{i_t} = |\ \ |^{\frac{1}{2}\ell - n - t}$$

and $t = n(\rho) + |\mathbf{s}| - n$.

(ii) *If* $\pi' \in \operatorname{Irr} G'_n$ *with*

$$[\pi'] = [\sigma'_1, \ldots, \sigma'_r, \rho']$$

and if $m < m(\rho') + |\mathbf{s}|$, *then* $\Theta_m(\pi') = \emptyset$ *unless* π' *occurs as a subrepresentation of* $I'_{\mathbf{s}}(\sigma'_1 \otimes \ldots \otimes \sigma'_r \otimes \rho')$ *for some ordering of the* σ'_i's *such that there is a sequence* $i_1 < \ldots < i_t$ *with*

$$\sigma'_{i_1} = \chi_V |\ \ |^{\frac{1}{2}\ell - n}, \ldots, \sigma'_{i_t} = \chi_V |\ \ |^{\frac{1}{2}\ell - n + t - 1},$$

with $t = m(\rho') + |\mathbf{s}| - m$.

The proofs of these theorems will be given in the following sections. In particular, the proof of Theorem 2.5 is based on a computation of the Jacquet modules (Theorem 2.8) of $\omega_{m,n}$ with respect to the maximal parabolic subgroups of G_m and G'_n. The result illustrates the inductive properties of the oscillator representation.

Let $(\boldsymbol{\sigma}_k, S(H_k))$ denote the representation of $H_k \times H_k$ on $S(H_k)$ given by:

$$\boldsymbol{\sigma}_k(h_1, h_2)\, \varphi(x) = \varphi({}^t h_1 x h_2).$$

Note that if $\sigma \in \operatorname{Irr}_c H_k$, then

$$\operatorname{Hom}_{H_k \times H_k}(\boldsymbol{\sigma}_k, \sigma \otimes \sigma') \neq 0$$

if and only if $\sigma' = \theta(\sigma)$ where, as above, $\theta(\sigma)(h) = \tilde{\sigma}({}^t h^{-1})$.

Theorem 2.8. *For* $1 \leqq j \leqq m$, (resp. $1 \leqq j \leqq n$), *let* $\tau_j = R_j(\omega_{m,n})$ (resp. $\tau'_j = R'_j(\omega_{m,n})$) *where* R_j (resp. R'_j) *is the localization functor as described in Sect.* 1.5.

(i) *There is an* $M_j \times G'_n$ *invariant filtration* $\tau_j = \tau_j^{(0)} \supset \ldots \supset \tau_j^{(r)} \supset \{0\}$ *of* τ_j *with successive quotients* $\tau_{jk} \simeq \tau_j^{(k)} / \tau_j^{(k+1)}$, *with* $0 \leqq k \leqq \min(n, j) = r$ *such that*

$$\tau_{jk} \cong \operatorname{ind}_{Q_{jk} \times P'_k}^{M_j \times G'_n} \xi_{jk} \boldsymbol{\sigma}_k \otimes \omega_{m-j, n-k}$$

where the character ξ_{jk} *is given by:*

$$\xi_{jk} = \begin{cases} \chi_V |\ \ |^{\frac{1}{2}\ell - j + \frac{k-1}{2}} & \text{on } H'_k \\ |\ \ |^{\frac{1}{2}\ell - j + \frac{k-1}{2}} & \text{on } H_k \\ |\ \ |^{\frac{1}{2}\ell - \frac{1}{2}j - n + \frac{k-1}{2}} & \text{on } H_{j-k} \\ \kappa(n, n-k) & \text{on } G_{m-j} \end{cases}$$

with $\kappa(\cdot, \cdot)$ *given by* (1.1.3).

(ii) *There is a* $G_m \times M'_j$ *invariant filtration of* τ'_j:

$$(\tau'_j)^{(0)} \supset \ldots \supset (\tau'_j)^{(r')} \supset \{0\}$$

with successive quotients $\tau'_{jk} = (\tau'_j)^{(k)} / (\tau'_j)^{(k+1)}$ *with* $0 \leqq k \leqq \min(m, j) = r'$ *such that*

$$\tau'_{jk} \simeq \operatorname{ind}_{P_k \times Q_{jk}}^{G_m \times M'_j} \xi'_{jk} \boldsymbol{\sigma}_k \otimes \omega_{m-k, n-j}$$

where

$$\xi'_{jk} = \begin{cases} |\ \ |^{\frac{1}{2}\ell - \frac{k+1}{2}} & \text{on } H_k \\ \kappa(n, n-j) & \text{on } G_{m-k} \\ \chi_V |\ \ |^{\frac{1}{2}\ell - \frac{k+1}{2}} & \text{on } H'_k \\ \chi_V |\ \ |^{\frac{1}{2}\ell - n + \frac{1}{2}(j-k-1)} & \text{on } H'_{j-k}. \end{cases}$$

Here the identification of Sects. 1.2 and 1.3 are used on Levi factors. Also $Q_{jk} \subset M_j$ is the parabolic subgroup:

$$Q_{jk} \simeq \left\{ \begin{array}{c} \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \\ {\scriptstyle k \quad j-k} \end{array} \right\} \times G_{m-j}$$

and $Q'_{jk} \subset M'_j$ is the parabolic subgroup

$$Q'_{jk} \simeq \left\{ \left(\begin{array}{c} \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \\ {\scriptstyle j-k \quad k} \end{array}, \varepsilon \right) \in H'_j \right\} \cdot G'_{n-j}.$$

Thus the Levi factor of Q_{jk} (resp. Q'_{jk}) is isomorphic to $H_k \times H_{j-k} \times G_{m-j}$ (resp. $H'_{j-k} \cdot H'_k \cdot G'_{n-j}$).

§3. Proof of Theorem 2.1

An L^2 variant of the type of argument we give here appears in [13], which motivated our proof. A finite field version appeared much earlier in [6]. Related global calculations appear in [4] and [5].

We begin by showing that for $\pi \in \operatorname{Irr}_c G_m$ (resp. $\pi' \in \operatorname{Irr}_c G'_n$) the set $\coprod_n \Theta_n(\pi[n])$ (resp. $\coprod_m \Theta_m(\pi')$) contains at most one cuspidal element. First note that if G is any ℓ-group and if (π, E) is an irreducible square integrable representation, then there exists a **C**-anti-linear isomorphism

$$\xi \colon E \to \tilde{E} \tag{3.1}$$

which intertwines π and $\tilde{\pi}$. Explicitly, if for $v \in E$ and $\tilde{v} \in \tilde{E}$ we let

$$\phi_{v, \tilde{v}}(g) = \langle \pi(g^{-1}) v, \tilde{v} \rangle,$$

then ξ is defined by

$$\langle v', \xi(v)\rangle = \int_G \phi_{v',\tilde{v}}(g)\, \overline{\phi_{v,\tilde{v}}(g)}\, dg$$

for some choice of $\tilde{v} \in \tilde{E}$, $\tilde{v} \neq 0$.

For the oscillator representation (ω, S) of $\tilde{S}p(W)$, realized in the Schrödinger model associated to a polarization W', W'' of W, define, for $\varphi \in S$ and $x \in W'$,

$$\bar{\omega}(g)\,\varphi(x) = \overline{(\omega(g)\,\bar{\varphi})(x)}. \tag{3.2}$$

Note that $(\bar{\omega}, S)$ is just the oscillator representation associated to the character $\bar{\psi} = \psi^{-1}$.

Returning to a dual pair $\tilde{\alpha}_{V,W}$: $G_m \times G'_n \to \tilde{S}p(\mathbf{W})$ suppose that $\pi \in \mathrm{Irr}_c G_m$ and $\pi' \in \mathrm{Irr}_c G'_n$ and that $\lambda \in \mathrm{Hom}_{G_m \times G'_n}(\omega_{m,n}, \pi \otimes \pi')$. Let

$$\bar{\lambda}(\varphi) = (\xi \otimes \xi')(\lambda(\bar{\varphi})) \tag{3.3}$$

where ξ and ξ' are defined as above, for some fixed choices of $\tilde{v}$, $\tilde{v}'$. Then $\bar{\lambda} \in \mathrm{Hom}_{G_m \times G'_n}(\bar{\omega}_{m,n}, \tilde{\pi} \otimes \tilde{\pi}')$ where

$$\bar{\omega}_{m,n} = \tilde{\alpha}^*_{V,W}(\bar{\omega}). \tag{3.4}$$

If we let $\bar{V}$ (resp. $\bar{W}$) denote the space V, $-(\ ,\ )$ (resp. W, $-\langle\ ,\ \rangle$), then it is easily checked that

$$\tilde{\alpha}_{\bar{V},W} = \tilde{\alpha}_{V,\bar{W}} \tag{3.5}$$

and that

$$\tilde{\alpha}^*_{\bar{V},W}(\omega) = \tilde{\alpha}^*_{V,\bar{W}}(\omega) = \tilde{\alpha}^*_{V,W}(\bar{\omega}). \tag{3.6}$$

Now suppose that $\pi \in \mathrm{Irr}_c G_m$, and $\pi'_i \in \mathrm{Irr}_c G'_{n_i}$, $i = 1, 2$, with

$$\pi[n_i] \in \Theta_m(\pi'_i), \qquad i = 1, 2$$

and fix a non-zero $\lambda_i \in \mathrm{Hom}_{G_m \times G'_{n_i}}(\omega_{m,n_i}, \pi[n_i] \otimes \pi'_i)$. For convenience we write $V = V_m$, $G = G_m$, $W = W_n$, $G' = G'_n$, $W_i = W_{n_i}$, and $G'_i = G'_{n_i}$, $i = 1, 2$, where $W_n = W_{n_1} + \bar{W}_{n_2}$. The isomorphism

$$\begin{aligned} \mathbf{W} &= V \otimes W \\ &\simeq V \otimes W_1 + V \otimes \bar{W}_2 \end{aligned} \tag{3.7}$$

induces, in the terminology of [11], a see-saw dual pair in $Sp(\mathbf{W})$:

$$\begin{tikzcd} O(V) \times O(V) \arrow[drr, dash] & & Sp(W) \arrow[dll, dash] \\ O(V) \arrow[u, "\Delta"] & & Sp(W_1) \times Sp(W_2). \arrow[u, "i"'] \end{tikzcd} \tag{3.8}$$

Here we identify $Sp(\bar{W}_2) = Sp(W_2)$, and the horizontal sloping lines connect members of a dual pair. Setting $W_i = V \otimes W_i$ we obtain a commutative diagram:

$$\begin{tikzcd} G \times G'_1 \times G \times G'_2 \arrow[r, "\tilde{\beta}"] \arrow[d, "\tilde{\alpha}_{V,W_1} \times \tilde{\alpha}_{\bar{V},W_2}"'] & \tilde{S}p(\mathbf{W}) \arrow[d, equal] \\ \tilde{S}p(\mathbf{W}_1) \times \tilde{S}p(\mathbf{W}_2) \arrow[r, "\tilde{j}"] & \tilde{S}p(\mathbf{W}) \end{tikzcd} \tag{3.9}$$

which defines $\tilde{\beta}$, and since

$$\tilde{j}^*(\omega)=\omega_1\otimes\omega_2 \tag{3.10}$$

we have

$$\tilde{\beta}^*(\omega)=\omega_{m,n_1}\otimes\bar{\omega}_{m,n_2}. \tag{3.11}$$

Let $\tilde{\Delta}$: $G\to G\times G$, be the diagonal map. A routine calculation then shows:

Lemma 3.1. *As homomorphisms from* $G\times G'_1\times G'_2$ *to* $\tilde{S}p(\mathbf{W})$:

$$\tilde{\beta}\circ(\tilde{\Delta}\times 1)=(\delta\otimes 1)(\tilde{\alpha}_{V,W}\circ(1\times\tilde{i}))$$

where, for $(h,\varepsilon)\in G$

$$\delta(h,\varepsilon)=((-1)^{n_2(n_1+1)},\det h)_k. \quad \square$$

Now consider, for λ_i, $i=1,2$, as above:

$$\lambda_1\otimes\bar{\lambda}_2\in\mathrm{Hom}_{G\times G'_1\times G\times G'_2}(\omega_{m,n_1}\otimes\bar{\omega}_{m,n_2},\pi[n_1]\otimes\pi'_1\otimes\pi[n_2]\otimes\tilde{\pi}'_2).$$

Viewing this as a homomorphism for $\Delta(G)\times G'_1\times G'_2\simeq G\times G'_1\times G'_2$ and applying (3.11) and the lemma, we obtain a non-zero homomorphism:

$$\lambda_1\otimes\bar{\lambda}_2\in\mathrm{Hom}_{G\times G'_1\times G'_2}((1\times\tilde{i})^*\,\omega_{m,n},(\delta^{-1}\,\pi[n_1]\otimes\pi[n_2])\otimes\pi'_1\otimes\tilde{\pi}'_2).$$

Recall that $\pi[n]=\kappa(n,a)\cdot\pi$, for $\kappa(n,a)$ defined by (1.1.3) and $a=a(\pi)$ defined in §1.6. Also, applying Lemma 1.1.1, we have:

$$\delta^{-1}\cdot\kappa(n_1,a)\,\kappa(n_2,a)^{-1}=\kappa(n,0) \tag{3.12}$$

and so

$$\lambda_1\otimes\bar{\lambda}_2\in\mathrm{Hom}_{G\times G'_1\times G'_2}((1\times\tilde{i})^*\,\omega^{\natural}_{m,n},(\pi\otimes\tilde{\pi})\otimes\pi'_1\otimes\tilde{\pi}'_2)$$

where $\omega^{\natural}_{m,n}$ is as in Remark 2.4.

Finally, composing with the quotient $\pi\otimes\tilde{\pi}\to\mathbf{1}_G\to 0$, we obtain a *non-zero* element:

$$\lambda\in\mathrm{Hom}_{G\times G'_1\times G'_2}((1\times\tilde{i})^*\,\omega^{\natural}_{m,n},\mathbf{1}_G\otimes\pi'_1\otimes\tilde{\pi}'_2). \tag{3.13}$$

We may now apply the following invariant distribution theorem of Rallis [13, Thm. II, 1.1]:

Theorem 3.2 (Rallis). (i) *Realize* $(\omega_{m,n},S)$ *in the Schrödinger model associated to the polarization*

$$\mathbf{W}_{m,n}=V_m\otimes W'_n+V_m\otimes W''_n,$$

i.e. $S=S(V_m\otimes W''_n)$ *and let* $\omega^{\natural}_{m,n}$ *be as in Remark* 2.4. *Then*

$$(S^*)^{O(V_m)}=\mathrm{span}\{\omega_{m,n}(g')\,\delta_0\,|\,g'\in G'_n\}$$

where S^* *is the space of linear functionals on* S *and* δ_0 *is the delta distribution at the origin.*

(ii) *Assume that* $V_0=\{0\}$ *so that* V_m *and* G_m *are split and realize* $(\omega_{m,n},S)$ *in the Schrödinger model associated to the polarization*

$$\mathbf{W}_{m,n}=V'_m\otimes W_n+V''_m\otimes W_n,$$

i.e. $S=S(V'_m\otimes W_n)$. *Then, as a representation of* G'_n, $\omega_{m,n}$ *factors through* $Sp(W_n)$ *and*

$$(S^*)^{Sp(W_n)}=\mathrm{span}\{\omega_{m,n}(g)\delta_0\,|\,g\in G_m\}$$

where δ_0 *is, again, the delta distribution at the origin.* □

Corollary 3.3 (Rallis). *With the notation of the preceeding theorem:*

(i) *The space of* $O(V_m)$*-coinvariants* $S_{O(V_m)}$ *is isomorphic to a submodule of* $I'_n(\chi_V\cdot|\;|^{\frac{1}{2}\ell})$ i.e.

$$S_{O(V_m)}\hookrightarrow I'_n(\chi_V|\;|^{\frac{1}{2}\ell}).$$

(ii) *The space of* $Sp(W_n)$*-coinvariants,* $S_{Sp(W_n)}$ *is isomorphic to a submodule of* $I_m(|\;|^{\frac{1}{2}(\ell-m-1)-n})$, i.e.

$$S_{Sp(W_n)}\hookrightarrow I_m(|\;|^{\frac{1}{2}(\ell-m-1)-n}).\quad\square$$

Now

$$\begin{array}{c}\mathrm{Hom}_{G\times G'_1\times G'_2}((1\times\tilde{i})^*\,\omega_{m,n},\mathbf{1}_G\otimes\pi'_1\otimes\pi'_2)\\ \wr\|\\ \mathrm{Hom}_{G\times G'_1\times G'_2}(S_{O(V_m)},\pi'_1\otimes\tilde{\pi}'_2)\end{array}$$

so that $\lambda\neq 0$ implies that $\pi'_1\otimes\tilde{\pi}'_2$ is a $G'_1\times G'_2$ constituent of

$$I'_n(\chi_V|\;|^{\frac{1}{2}\ell})|_{G'_1\times G'_2}. \tag{3.14}$$

Since π'_1 and π'_2 are cuspidal, we need only consider the cuspidal part of (3.14), and this is easily done via the following result on the orbit structure of $P'_n\backslash G'_n/(G'_{n_1}\times G'_{n_2})$.

Proposition 3.4. (i) *Suppose that* $W_n=W_{n_1}+W_{n_2}$ *with* $n_1\geqq n_2$. *Let* X' *be the space of maximal isotropic subspaces of* W_n. *Then the* $Sp(W_{n_1})\times Sp(W_{n_2})$ *orbits in* X' *are parameterized by* t *with,* $0\leqq t\leqq n_2=\min(n_1,n_2)$ *where*

$$\mathcal{O}'_t=\{U\in X'|\dim(U\cap W_{n_1})=t+n_1-n_2,\dim(U\cap\bar{W}_{n_2})=t\}.$$

(ii) *Suppose that* $V=V_{m_1}+\bar{V}_{m_2}$ *with* $m_1\geqq m_2$. *Let* X *be the space of maximal isotropic subspaces of* V. *Then the* $O(V_{m_1})\times O(V_{m_2})$ *orbits in* X *are parameterized by* t *with* $0\leqq t\leqq m_2$ *where*

$$\mathcal{O}_t=\{U\in X|\dim U\cap V_{m_1}=m_1-m_2+t,\dim U\cap V_{m_2}=t\}.$$

(iii) *Moreover, the stabilizer in* $Sp(W_{n_1})\times Sp(W_{n_2})$ (resp. $O(V_{m_1})\times O(V_{m_2})$) *of a point* x *of* X' (resp. X) *is contained in a proper parabolic subgroup unless* $n_1=n_2$ (resp. $m_1=m_2$) *and* $x\in\mathcal{O}_0$.

(iv) *If* $n_1=n_2$ *then, in* X',

$$\mathcal{O}'_0\simeq Sp(W_{n_1})$$

and the action of $Sp(W_{n_1})\times Sp(W_{n_1})$ *is given by*

$$(g_1,g_2)\colon\; x\mapsto g_1\,x\,g_2^{-1}.$$

If $m_1 = m_2$, *then, in* X,

$$\mathcal{O}_0 \simeq O(V_{m_1})$$

and the action of $O(V_{m_1}) \times O(\bar{V}_{m_2}) \simeq O(V_{m_1}) \times O(V_{m_1})$ *is given by*

$$(h_1, h_2)\colon x \mapsto h_1 x h_2^{-1}.$$

Proof. To lighten the notation we write $W = W_n$, $W_i = W_{n_i}$, $i = 1, 2$, $G' = Sp(W)$ and $G'_i = Sp(W_i)$. If $U \in X'$, write

$$U_i = U \cap W_i \tag{3.15}$$

and let

$$d_i = \dim U_i. \tag{3.16}$$

Then, letting pr_i denote the projection of W to W_i,

$$W_i \supset U_i^{\perp} \supset \mathrm{pr}_i(U) \supset U_i \tag{3.17}$$

and so, counting dimensions, we see that

$$\mathrm{pr}_i(U) = U_i^{\perp} \tag{3.18}$$

$$\dim U_1^{\perp}/U_1 = \dim U_2^{\perp}/U_2,$$

and

$$d_1 - d_2 = n_1 - n_2. \tag{3.19}$$

Define an *anti-isometry* (for W_1 and $\bar{W}_2$ induced forms)

$$\phi_U\colon\ U_1^{\perp}/U_1 \xrightarrow{\sim} U_2^{\perp}/U_2 \tag{3.20}$$

by the condition, $\forall u \in U$

$$\mathrm{pr}_2(u) + U_2 = \phi_U(\mathrm{pr}_1(u) + U_1).$$

Note that

$$\begin{aligned} 0 &= \langle u, u' \rangle \\ &= \langle u_1, u'_1 \rangle + \langle u_2, u'_2 \rangle \end{aligned}$$

if $u_i = \mathrm{pr}_i(u)$, $u'_i = \mathrm{pr}_i(u')$. Clearly the integer $t = d_2$, $0 \leqq t \leqq n_2$ is an invariant of the $G'_1 \times G'_2$ orbit of U, and, letting

$$\mathcal{O}'_t = \{U \in X' \mid d_2(U) = t\} \tag{3.21}$$

we have:

$$\mathcal{O}'_t \simeq \{(U_1, U_2, \phi) \mid U_1 \in X'_{d_1}(W_1), U_2 \in X'_{d_2}(W_2) \text{ and } \phi \in \mathrm{Isom}(U_1^{\perp}/U_1, U_2^{\perp}/U_2)\}. \tag{3.22}$$

Here $X'_k(W)$ is the space of isotropic k-planes in W, etc. Note that here we use W_2 not $\bar{W}_2$. The action of $G'_1 \times G'_2 \ni (g_1, g_2)$ on the right side of (3.22) is given by

$$(g_1, g_2)\colon\ (U_1, U_2, \phi) \mapsto (g_1 U_1, g_2 U_2, g_2 \circ \phi \circ g_1^{-1}). \tag{3.23}$$

Statements (i), (ii), (iii) and (iv) are now clear in the symplectic case. The proof in the orthogonal case is analogous and hence is omitted. □

Thus the cuspidal representation $\pi'_1 \otimes \tilde{\pi}'_2$ can only occur in (3.14) if $n_1 = n_2$, and in that case it must occur as a constituent of the representation $(\sigma'_{n_1}, S(G'_{n_1}))$ where

$$\sigma'_{n_1}(g_1, g_2)\,\varphi(x) = \varphi(g_1^{-1}\, x\, g_2)$$

for $x \in G'_{n_1}$ and $\varphi \in S(G'_{n_1})$. This implies, as usual, that $\pi'_2 \simeq \pi'_1$ and so, the uniqueness of $\theta(\pi)$ is proved. The proof of uniqueness of $\theta(\pi')$ is completely analogous so we omit it.

To complete the proof of Theorem 2.1 we must show that every element of the set $\Theta_{n(\pi)}(\pi[n(\pi)])$ is cuspidal. For convenience let $n = n(\pi)$, and suppose that $\pi' \in \Theta_n(\pi[n])$ is such that

$$\pi' \hookrightarrow I'_j(\sigma' \otimes \rho')$$

for some $j > 0$. Then, using Theorem 2.8, we have:

$$\begin{aligned}
\dim \operatorname{Hom}_{G_m \times G'_n}(\omega_{m,n}, \pi[n] \otimes \pi') &\leqq \dim \operatorname{Hom}_{G_m \times G'_n}(\omega_{m,n}, \pi[n] \otimes I'_j(\sigma' \otimes \rho')) \\
&= \dim \operatorname{Hom}_{G_m \times M'_j}(\tau'_j, \pi[n] \otimes (\sigma' \otimes \rho')) \\
&\leqq \sum_{k=0}^{\min(m,j)} \dim \operatorname{Hom}_{G_m \times M'_j}(\tau'_{jk}, \pi[n] \otimes (\sigma' \otimes \rho')).
\end{aligned}$$

Since $\pi[n]$ is cuspidal only the $k = 0$ term can contribute, and that term gives:

$$\dim \operatorname{Hom}_{G_m \times H'_j \times G'_{n-j}}(\omega_{m,n-j}, (\xi'_{j0})^{-1}\, \pi[n] \otimes (\sigma' \otimes \rho')).$$

This vanishes unless $\sigma' = \chi_V |\ |^{\frac{1}{2}\ell - n + \frac{1}{2}(j-1)}$ and $\rho' \in \Theta_{n-j}(\pi[n-j])$. Here we use the fact that (Lemma 1.1.1) $\kappa(n, n-j)^{-1} \cdot \kappa(n, a) = \kappa(n-j, a)$ so that

$$\kappa(n, n-j)^{-1} \cdot \pi[n] = \pi[n-j]. \tag{3.24}$$

But, by hypothesis, $\Theta_{n-j}(\pi[n-j]) = \emptyset$, and so no such π' can occur. This finishes the proof of Theorem 2.1 (i); the proof of (ii) is analogous and so will be omitted. □

§4. Proof of Theorem 2.5

First suppose that $\pi \in \operatorname{Irr}_c G_m$. Then, via Theorem 2.1, we may assume that $n > n(\pi)$ and that $\pi' \in \Theta_n(\pi[n])$ satisfies

$$\pi' \hookrightarrow I'_j(\sigma' \otimes \rho')$$

with $j > 0$ maximal. Here $\sigma' \in \operatorname{Irr} H'_j$ and, since j is maximal, $\rho' \in \operatorname{Irr}_c G'_{n-j}$. Now the argument of the last part of the proof of Theorem 2.1 implies that $\sigma' = \chi_V |\ |^{\frac{1}{2}\ell - n + \frac{1}{2}(j-1)}$ and $\rho' \in \Theta_{n-j}(\pi[n-j])$. But now, since ρ' is cuspidal, Theorem 2.1 implies that $n - j = n(\pi)$ and $\rho' = \theta(\pi)$. Thus

$$\pi' \hookrightarrow I'_{\mathbf{s}}(\sigma'_1 \otimes \dots \otimes \sigma'_j \otimes \theta(\pi))$$

where $\mathbf{s} = (1, \dots, 1)$, $|\mathbf{s}| = j$ and, for $1 \leqq k \leqq j$,

$$\sigma'_k = \chi_V |\ |^{\frac{1}{2}\ell - n + k - 1}.$$

This proves part (ii) of Theorem 2.5 in the case when $r=0$, since a cuspidal π is never ambiguous.

Next suppose that $\pi \in \operatorname{Irr} G_m$ with

$$\pi \hookrightarrow I_{\mathbf{s}}(\sigma_1 \otimes \ldots \otimes \sigma_r \otimes \rho) \tag{4.1}$$

where $\sigma_i \in \operatorname{Irr}_c H_{s_i}$ and $\rho \in \operatorname{Irr}_c G_{m-|\mathbf{s}|}$. Then $\pi \hookrightarrow I_j(\sigma_1 \otimes \rho_1)$ where $j=s_1$ and

$$\rho_1 \hookrightarrow I_{(s_2, \ldots, s_r)}(\sigma_2 \otimes \ldots \otimes \sigma_r \otimes \rho).$$

Suppose that $\pi' \in \Theta_n(\pi[n])$ and consider:

$$\begin{aligned} &\dim \operatorname{Hom}_{G_m \times G'_n}(\omega_{m,n}, \pi[n] \otimes \pi') \\ &\quad \leqq \dim \operatorname{Hom}_{G_m \times G'_n}(\omega_{m,n}, I_j(\sigma_1 \otimes \rho_1[n]) \otimes \pi') \\ &\quad = \dim \operatorname{Hom}_{M_j \times G'_n}(\tau_j, \sigma_1 \otimes \rho_1[n] \otimes \pi') \\ &\quad \leqq \sum_{k=0}^{\min(n,j)} \dim \operatorname{Hom}_{Mj \times G'_n}(\tau_{jk}, \sigma_1 \otimes \rho_1[n] \otimes \pi'). \end{aligned}$$

Since σ_1 is cuspidal, only the terms with $k=j$, or $j=1$ and $k=0$ can contribute. The first of these yields

$$\begin{aligned} &\dim \operatorname{Hom}_{M_j \times G'_n}(\operatorname{ind}_{M_j \times P'_j}^{M_j \times G'_n} \xi_{jj} \boldsymbol{\sigma}_j \otimes \omega_{m-j,n-j}, \sigma_1 \otimes \rho_1[n] \otimes \pi') \\ &\quad = \dim \operatorname{Hom}_{M_j \times M'_j}(\xi_{jj} \boldsymbol{\sigma}_j \otimes \omega_{m-j,n-j}, \sigma_1 \otimes \rho_1[n] \otimes \bar{R}'_j(\pi')). \end{aligned}$$

This can be non-zero only if there exists a constituent

$$\sigma'_1 \otimes \rho'_1 \in JH(\bar{R}'_j(\pi'))$$

such that

$$\operatorname{Hom}_{M_j \times M'_j}(\xi_{jj} \boldsymbol{\sigma}_j \otimes \omega_{m-j,n-j}, \sigma_1 \otimes \rho_1[n] \otimes \sigma'_1 \otimes \rho'_1) \neq 0.$$

Since σ_1 is cuspidal we conclude that, with the notation of (2.7),

$$\xi_{jj}^{-1} \sigma'_1 = \theta(\xi_{jj}^{-1} \sigma_1);$$

and so, using the value of ξ_{jj} given in Theorem 2.8,

$$\sigma'_1 = \chi_V \theta(\sigma_1). \tag{4.2}$$

Since $\xi_{jj}|_{G_{m-j}} = \kappa(n, n-j)$, and recalling (3.24), we must also have

$$\rho'_1 \in \Theta_{n-j}(\rho_1[n-j]). \tag{4.3}$$

Next consider the case $k=0$, $j=1$. This term contributes:

$$\dim \operatorname{Hom}_{M_1 \times G'_n}(\xi_{10}\, \omega_{m-1,n}, \sigma_1 \otimes \rho_1[n] \otimes \pi')$$

and so can be non-zero only if

$$\sigma_1 = |\ \ |^{\frac{1}{2}\ell - n - 1} \tag{4.4}$$

and

$$\pi' \in \Theta_n(\rho_1[n]). \tag{4.5}$$

An easy inductive argument on m now shows that $[\pi']$ is as in (ii) of Theorem 2.5. Here note that if $m=0$ every $\pi \in \operatorname{Irr} G_0$ is cuspidal. For example, if $n \geqq |\mathbf{s}|+n(\rho)$ the same inequality is true for $n-s_1$ and $|\mathbf{s}|-s_1+n(\rho)$, or for n and $|\mathbf{s}|-1+n(\rho)$. Therefore, the first case in the above argument and the inductive hypothesis applied to ρ'_1 and $\rho_1[n-j]$ would yield

$$[\pi'] = [\chi_V \theta(\sigma_1), \ldots, \chi_V \theta(\sigma_r), \chi_V|\ |^{\frac{1}{2}\ell-n}, \ldots, \chi_V|\ |^{\frac{1}{2}\ell-|\mathbf{s}|-n(\rho)-1}, \theta(\rho)]$$

while the second case and the inductive hypothesis applied to π' and $\rho_1[n]$ would yield

$$[\pi'] = [\chi_V \theta(\sigma_2), \ldots, \chi_V \theta(\sigma_r), \chi_V|\ |^{\frac{1}{2}\ell-1-n}, \chi_V|\ |^{\frac{1}{2}\ell-n}, \chi_V|\ |^{\frac{1}{2}\ell-|\mathbf{s}|-n(\rho)-1}, \theta(\rho)],$$

provided $\sigma_1 = |\ |^{\frac{1}{2}\ell-n-1}$. But then

$$\chi_V \theta(\sigma_1) = \chi_V|\ |^{\frac{1}{2}\ell-n-1},$$

so that the two expressions agree (up to permutation, i.e. associativity). The general argument proceeds in the same way, taking account of the various cases, so we omit it.

Note that we have used precisely the inclusion (4.1) and the fact that π' was an arbitrary element of $\Theta_n(\pi[n])$. Thus if π were strongly ambiguous, as defined in Remark 1.5.2, we would obtain two non-associate expressions for $[\pi']$ since by Theorem 2.1 (ii), $\theta(\rho)$ and $\theta(\nu\rho)$ are unequal for $\rho \neq \nu\rho$. No such π' can exist and so statement (i) of Theorem 2.5 is proved.

Finally, observe that if $n < n(\rho)+|\mathbf{s}|$, and if we iterate the inductive step above, then the 'second case' must occur at least $t=n(\rho)+|\mathbf{s}|-n$ times. This can only happen if $\pi \hookrightarrow I_{\mathbf{s}}(\sigma_1 \otimes \ldots \otimes \sigma_r \otimes \rho)$ where some subsequence $\sigma_{i_1}, \ldots, \sigma_{i_t}$ of the σ_j's is as in Corollary 2.7 (i), and so that part of Corollary 2.7 is proved.

Finally we note that Theorem 2.5 could have been proved in a symmetric way beginning with π', and this symmetric proof yields Corollary 2.7 (ii). □

§5. Proof of Theorem 2.8

In this section we will compute the Jacquet modules $R_j(\omega_{m,n})$ and $R'_j(\omega_{m,n})$ with respect to maximal parabolics of G_m and G'_n. Such a computation is also contained in the proofs of [13], but since the precise statement (Theorem 2.8) which we need is not so easy to find in [13], and for the sake of completeness, we will give a detailed exposition. Of course, since the computations in the two cases are essentially the same, we will only compute $R'_j(\omega_{m,n})$.

For convenience, since m and n are fixed, we will write $V=V_m$, $W=W_n$, and $\omega=\omega_{m,n}$.

Fix j with $1 \leqq j \leqq n$ and let $P'_j = M'_j N'_j$ and

$$W = W'_j + W^0_j + W''_j$$

be as in Sect. 1.3. Then, associated to the decomposition

$$\mathbf{W} = \mathbf{W}' + \mathbf{W}^0 + \mathbf{W}''$$

with $\mathbf{W}'=V\otimes W_j'$, $\mathbf{W}^0=V\otimes W_j^0$ and $\mathbf{W}''=V\otimes W_j''$, we have a mixed model [8] of (ω, S) with

$$S\simeq S(V^j)\otimes S^0.$$

Here (ω_0, S^0) is a model of the representation $\omega_{m,n-j}$. Let $n'(c,d)\in N_j'$, where

$$n'(c,d)=\begin{pmatrix}1 & {}^t c & d-\frac{1}{2}\langle c,c\rangle\\ & 1 & c\\ & & 1\end{pmatrix}$$

with $d={}^t d\in M_r(\mathscr{k})$, $c=(c_1,\dots,c_r)$, $c_j\in W^0$. We view c as an element of $\mathrm{Hom}(W^0, W'')$ via

$$c\colon\ w^0\to\langle w^0,c\rangle=(\langle w^0,c_1\rangle,\dots,\langle w^0,c_r\rangle)\in\mathscr{k}^r.$$

Then we have

$$\omega(n'(c,d))\,\varphi(x)=\psi(\tfrac{1}{2}\mathrm{tr}(d(x,x)))\,\rho_0(x^t c)\,\varphi(x) \tag{5.0}$$

where $x\in V^j$, $\varphi(x)\in S^0$ and ρ_0 is the action of the Heisenberg group $H(\mathbf{W}^0)$ in S^0.

Let $N_{j0}'=\{n'(0,d)\,|\,d={}^t d\in M_j(\mathscr{k})\}$. Then a standard argument yields:

Lemma 5.1. *There is an isomorphism of* $G_m\times P_j'$ *modules*

$$S_{N_{j0}'}\xrightarrow{\sim}S(X_0)\otimes S^0$$

where

$$X_0=\{x\in V^j\,|\,(x,x)=0\}$$

and such that the natural homomorphism $S\to S_{N_{j0}'}$ *is given by restriction to* $X_0\subset V^j$. *Let* $\tilde{T}=S(X_0)\otimes S^0$. *The space* X_0 *has a decomposition*

$$X_0=\coprod_{k=0}^{\min(j,m)}X_{0k} \tag{5.1}$$

where $X_{0k}=\{x\in X_0\,|\,\dim\operatorname{span}x=k\}$. *Here and elsewhere*

$$\operatorname{span}x=\operatorname{span}\{x_1,\dots,x_j\}$$

if $x=(x_1,\dots,x_j)\in V^j$. *This decomposition induces a* $G_m\times P_j'$ *invariant filtration*

$$\tilde{T}=\tilde{T}^{(0)}\supset\tilde{T}^{(1)}\supset\dots\supset\tilde{T}^{(r)}\supset\{0\} \tag{5.2}$$

of $\tilde{T}$, *where* $r=\min(j,m)$, *whose successive quotients* $\tilde{T}_k=\tilde{T}^{(k)}/\tilde{T}^{(k+1)}$ *have the form*

$$\tilde{T}_k\simeq S(X_{0k})\otimes S^0. \tag{5.3}$$

The representation of $G_m\times P_j'$ on $\tilde{T}_k$ may be identified with an induced representation as follows:

For simplicity let $G=G_m\times H_j'\times G_{n-j}'$ and $N=N_j'$. Also let $X=X_{0k}$ and fix $x_0\in X$ given by

$$x_0=(0,\dots,0,v_1',\dots,v_k') \tag{5.4}$$

where $v'_1, \ldots, v'_m$ are as in Sect. 1.1. Let H be the stabilizer of x_0 in G where we view G as acting on X via its projection to $G_m \times H'_j$. Define a representation τ of $H \ltimes N$ in S^0 by:

$$\tau(h) = \xi(h)\,\xi'(h)\,\omega_0(h) \tag{5.5}$$

where ξ and ξ' are characters given by:

$$\xi(h, \varepsilon) = \kappa(n, n-j)(h, \varepsilon) \tag{5.6}$$

for $(h, \varepsilon) \in G_m$ and,

$$\xi'(g, \varepsilon) = \chi(g, \varepsilon)^{\ell}\,|\det g|^{\ell/2}(d(V_m), \det g)_k \tag{5.7}$$

for $(g, \varepsilon) \in H'_j$, and let

$$\tau(n) = \rho_0(x_0^t c) \tag{5.8}$$

if $n = n'(c, d) \in N$.

Then we have:

Lemma 5.2. *The representation ω of $G \ltimes N$ on $\tilde{T}_k$ is given by*

$$\omega \cong \natural - \mathrm{ind}_{H \ltimes N}^{G \ltimes N}\,\tau.$$

Proof. Consider first the restriction of ω to G. We have

$$\omega|_G \simeq \mu \otimes \xi\,\xi'\,\omega_0$$

where $(\mu, S(X))$ is the natural action of G on $S(X)$ induced by its action on X. Thus there is a natural isomorphism:

$$\omega|_G \xrightarrow{\sim} \natural - \mathrm{ind}_H^G(\tau|_H) \tag{5.9}$$

$$\varphi \to f$$

given by

$$f(g) = (\xi\,\xi'\,\omega_0)(g)\,\varphi(g^{-1} x_0).$$

On the other hand, there is a natural isomorphism

$$\natural - \mathrm{ind}_H^G(\tau|_H) \xrightarrow{\sim} (\natural - \mathrm{ind}_{H \ltimes N}^{G \ltimes N}\tau)|_G \tag{5.10}$$

$$f \longrightarrow F$$

given by

$$F(g, n) = \tau(1, g(n))\,f(g).$$

Combining (5.9) and (5.10) we see that it is sufficient to check that the map

$$\varphi \to F$$

given by

$$F(g, n) = \tau(1, g(n))(\xi\,\xi'\,\omega_0)(g)\,\varphi(g^{-1} x_0) \tag{5.11}$$

intertwines the N-actions, and this follows by a direct calculation. □

To compute ω_N we apply the following:

Lemma 5.3. *There is a natural isomorphism*

$$(\natural-\mathrm{ind}_{H\ltimes N}^{G\ltimes N}\tau)_N \xrightarrow{\sim} \natural-\mathrm{ind}_H^G(\tau_N)$$

induced by restriction to G, i.e. by the inverse of (5.11). *In particular the natural homomorphism*

$$\natural-\mathrm{ind}_{H\ltimes N}^{G\ltimes N}\tau \to (\natural-\mathrm{ind}_{H\ltimes N}^{G\ltimes N}\tau)_N$$

is given by

$$F \mapsto \iota(f)$$

where

$$f(g)=F(g,1),$$

and where

$$\iota:\ \natural-\mathrm{ind}_H^G(\tau|_H)\to \natural-\mathrm{ind}_H^G\tau_N$$

is the homomorphism corresponding to

$$\iota:\ \tau|_H \to \tau_N.$$

Proof. It is clear that

$$(\natural-\mathrm{ind}_{H\ltimes N}^{G\ltimes N}\tau)[N]\subset \natural-\mathrm{ind}_H^G(\tau[N])$$

i.e. consider $S^0[N]$-valued functions. It then remains to check that these two spaces coincide. We omit the details. □

To compute τ_N we consider the decomposition

$$V_m = V_k' + V_k^0 + V_k''$$

as in Sect. 1.2, and we realize the representation (ω_0, S^0) in the mixed model associated to the corresponding decomposition

$$\mathbf{W}^0 = \mathbf{W}^{0\prime} + \mathbf{W}^{00} + \mathbf{W}^{0\prime\prime}$$

where $\mathbf{W}^{0\prime} = V_k' \otimes W_j^0$, $\mathbf{W}^{00} = V_k^0 \otimes W_j^0$, and $\mathbf{W}^{0\prime\prime} = V_k'' \otimes W_j^0$. Then

$$S^0 \simeq S((W_j^0)^k) \otimes S^{00} \tag{5.12}$$

where (ω_{00}, S^{00}) is a model for $\omega_{m-k,n-j}$. Since for any $n=n'(c,d)\in N$,

$$x_0^t c \in V_k'' \otimes W_j^0 = \mathbf{W}^{0\prime\prime}$$

we have:

$$\rho_0(x_0^t c)\,\varphi(z) = \psi(\mathrm{tr}\langle z, x_0^t c\rangle)\,\varphi(z) \tag{5.13}$$

for $z\in(W_j^0)^k$ and $\varphi\in S^0$. It is then easy to check:

Lemma 5.4. *There is a natural isomorphism*

$$S_N^0 \xrightarrow{\sim} S^{00}$$

such that the quotient

$$S^0 \to S_N^0$$

is given by

$$\varphi \mapsto \varphi(0)$$

where $0\in(W_j^0)^k$. □

Now observe that

$$H \subset P_k \times (H'_j \times G'_{n-j}) \ltimes N$$

where P_k is the maximal parabolic as in Sect. 1.2, and that the representation τ_N of H on $S^0_N \simeq S^{00}$ extends to a representation $\tilde{\tau}$ of this larger group as follows:

$$\tilde{\tau}|_{H_k} = \xi_0 \tag{5.14}$$

where

$$\xi_0 = |\ |^{j-n}; \tag{5.15}$$

$$\tilde{\tau}|_{G_{m-k}} = \xi\omega_{00} \tag{5.16}$$

where ξ is given by (5.6);

$$\tilde{\tau}|_{H'_j} = \xi' \tag{5.17}$$

where ξ' is given by (5.7);

$$\tilde{\tau}|_{G'_{n-j}} = \omega_{00} \tag{5.18}$$

and $\tilde{\tau}$ is trivial on $N_k \times N'_j$. In particular we have proved that

$$(\tilde{T}_k)_{N'_j} \xrightarrow{\sim} \natural - \operatorname{ind}_H^{G_m \times H'_j \times G'_{n-j}}(\tilde{\tau}|_H). \tag{5.19}$$

Now we want to identify $(\tilde{T}_k)_{N'_j}$ with a representation induced from a parabolic subgroup. Let

$$R'_{jk} = \left\{ \left(\left(\begin{array}{c|c} * & * \\ \hline 0 & * \\ \scriptstyle j-k & \scriptstyle k \end{array} \right), \varepsilon \right) \right\} \subset H'_j$$

and define homomorphisms pr: $R'_{jk} \to H_k$ by

$$\operatorname{pr}\left(\left(\begin{array}{c|c} * & * \\ \hline 0 & a \end{array} \right), \varepsilon \right) = a$$

and pr: $P_k \to H_k$ via (1.2.0). Also define an inclusion i: $H_k \to P_k$ via (1.2.0). Then, recalling that (1.2.0) was defined via $H_k \simeq GL(V'_k) \hookrightarrow M_k$, whereas the components $v'_1, \dots, v'_k$ of x_{0k} lie in V''_k, we have:

$$H = \{(h, g, g') \in P_k \times R'_{jk} \times G'_{n-j} | \operatorname{pr}(h) = {}^t\operatorname{pr}(g)^{-1}\}. \tag{5.20}$$

Then, since the set

$$\{(i({}^t x), 1, 1) \in P_k \times R'_{jk} \times G'_{n-j} | x \in H_k\} \tag{5.21}$$

gives a set of representatives for $H \backslash P_k \times R'_{jk} \times G'_{n-j}$, the representation

$$\mu_{jk} = \natural - \operatorname{ind}_H^{P_k \times R'_{jk} \times G'_{n-j}} \xi_0 \xi \omega_{00} \otimes \xi' \otimes \omega_{00}$$

may be realized, by restriction to the set (5.21), on the space

$$S(H_k) \otimes S^{00}.$$

Moreover, in that realization, the action of $P_k \times R'_{jk} \times G'_{n-j}$ is given by:

Lemma 5.4. *Let* $x \in H_k$ *and* $\varphi \in S(H_k) \otimes S^{00}$.

(i) *If* $(h, g) \in H_k \times H'_k \subset P_k \times R'_{jk}$, *then*

$$\mu_{jk}(h, g)\,\varphi(x) = \chi_V(g)\,|\det g|^{\frac{1}{2}\ell + n - j}\,\varphi({}^t h x g).$$

where

$$\chi_V(g) = \chi(g)^\ell (d(V), \det g)_\ell.$$

(ii) *If* $g \in H'_{j-k} \subset R'_{jk}$, *then*

$$\mu_{jk}(g)\,\varphi(x) = \chi_V(g)\,|\det g|^{\ell/2}\,\varphi(x).$$

(iii) *If* $(h, g) \in G_{m-k} \times G'_{n-j}$, *then*

$$\mu_{jk}(h, g)\,\varphi(x) = \kappa(n, n-j)(h)\,\omega_{00}(h)\,\omega_{00}(g)\,\varphi(x).$$

(iv) *The unipotent radicals of* P_k *and* R'_{jk} *act trivially.*

Proof. Since the factor $G_{m-k} \times H'_{j-k} \times G'_{n-j}$ of $P_k \times R'_{jk} \times G'_{n-j}$ lies in H, and centralizes the set (5.21), and since ξ_0 is trivial on this factor, statements (ii) and (iii) follow from (5.7), (5.17), (5.6) and (5.16). Statement (iv) is proved analogously. Finally, to prove (i) suppress the inclusion i: $H_k \to P_k$ and consider

$$\begin{aligned}\mu_{jk}(h, g)\,\varphi(x) &= f(({}^t x, 1, 1)(h, g, 1))\\ &= f(({}^t x h, g, 1))\\ &= f(({}^t \mathrm{pr}(g)^{-1}, g, 1)({}^t \mathrm{pr}(g)\,{}^t x h, 1, 1))\\ &= |\det g|^{n-j+\ell/2}\,\chi_V(g)\,\varphi({}^t h x g). \quad \square\end{aligned}$$

Finally we want to replace '♮−ind' with 'ind' so that we obtain Theorem 2.8(ii) with

$$\xi'_{jk} = \begin{cases} \delta_k^{-\frac{1}{2}} & \text{on } H_k \\ \kappa(n, n-j) & \text{on } G_{m-k} \\ \chi_V|\ |^{\frac{1}{2}\ell+n-j}(\delta'_{jk} \cdot \delta'_j)^{-\frac{1}{2}} & \text{on } H'_k \\ \chi_V|\ |^{\frac{1}{2}\ell}(\delta'_{jk}\delta'_j)^{-\frac{1}{2}} & \text{on } H'_{j-k}. \end{cases}$$

where δ'_j (resp. δ'_{jk}) is the module of P'_j (resp. R'_{jk}) and δ_k is that of P_k. Since

$$\delta'_{jk} = \begin{cases} |\ |^{k-j} & \text{on } H'_k \\ |\ |^k & \text{on } H'_{j-k}, \end{cases}$$

$$\delta'_j = |\ |^{2n-j+1} \quad \text{on } H'_j,$$

and

$$\delta_k = |\ |^{-(\ell-k-1)} \quad \text{on } H_k,$$

we obtain the claimed values for ξ'_{jk}.

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