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Differentiability and rigidity of Möbius groups

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A. Introduction

Let G be a group of Möbius tranformations of $\bar{R}^n = R^n \cup \{\infty\}$; the action of G extends to the $(n+1)$-dimensional hyperbolic space $H^{n+1} = \{(x_1, \ldots, x_{n+1}) \in R^{n+1}: x_{n+1} > 0\}$. A point $x \in \bar{R}^n$ is a *radial point* of G if there is a sequence of elements $g_i \in G$ such that, given $z \in H^{n+1}$ and a hyperbolic line L with endpoint x, we can find $M > 0$ such that the hyperbolic distances from L are bounded,

$$d(g_i(z), L) \leqq M, \qquad \text{and that } g_i(z) \to x \tag{A1}$$

in $\bar{H}^{n+1} = H^{n+1} \cup \bar{R}^n$ as $i \to \infty$; if (A1) is true, we say that $g_i(z)$ *approach radially* x. Since the elements of G preserve the hyperbolic metric d, it follows that if (A1) is true for some z and L, then it is true for all z and L, possibly with different M.

Thus as we approach a radial point along a hyperbolic ray, we come infinitely often to a compact set in the quotient space H^{n+1}/G. If we look at a G-invariant set of $\bar{R}^n$, similar images come infinitely often. Radial points are somehow generic points for G. The whole complexity of G must be present in every neighbourhood of a radial point. Thus it is natural to suspect that if a map f of $\bar{R}^n$ is compatible with G (see (A2)) and if it is differentiable at a radial point with a non-vanishing Jacobian, then f is, up to composition with Möbius transformations, an affine map of R^n. This is indeed so as we will see. Furthermore, if some often occurring conditions are met, then f is a Möbius transformation (Theorems A and D).

We now express this more precisely. Let $A \subset \bar{R}^n$ be a G-invariant set (i.e. $g(A) = A$ for $g \in G$) and let $f\colon A \to \bar{R}^n$ be a map such that there is a homomorphism $\varphi\colon G \to G'$ (G' another group of Möbius transformations of $\bar{R}^n$) such that

$$\varphi(g) f(x) = f g(x) \tag{A2}$$

for every $x \in A$ and $g \in G$. In this case we say that f is *G-compatible* or that f *induces* φ.

We will prove in this paper

Theorem A. *Let G be a group of Möbius transformations of $\bar{R}^n$ and let f: $\bar{R}^n \to \bar{R}^n$ be a G-compatible map which is differentiable with a non-vanishing Jacobian at a radial point of G. Then f is a Möbius transformation unless there is a point $z \in \bar{R}^n$ fixed by every $g \in G$. If there is such a point z, then there are Möbius transformations h and h' such that $h(\infty)$ is fixed by every $g \in G$ and that $h' f h | R^n$ is an affine homeomorphism of R^n.*

This is a special case of Theorem D. The idea of the proof is to look at f from the points $g_i(z)$, when $g_i(z)$ approach radially a point at which f is differentiable. As $i \to \infty$, f looks more and more like an affine map, and going back with the maps $\varphi(g_i)^{-1}$, φ as in (A2), we can show that f is indeed affine modulo composition with Möbius transformations.

We remark that for Theorem A one needs only the first part of the proof of Theorem D so that one has formula (D13) which says that there are Möbius transformations h and h' such that $h' f h | R^n$ is affine. If $h(\infty)$ is not fixed by every $g \in G$, f is continuous by G-compatibility also at $h(\infty)$ and we can use Lemma C2 to conclude that f is a Möbius transformation.

The final sections give some related theorems. Roughly, Theorem E says that we can replace in Theorems A and D the radial limit points by limit points (see (B3)) if we require that f is continuously differentiable at the limit point. In Theorem F we consider a quasiconformal and G-compatible map f of $\bar{R}^n$ and show that if the matrix dilation (cf. (F0)) of f is approximately continuous at a radial point of G, or continuous at a limit point, then f is a Möbius transformation if no $z \in \bar{R}^n$ is fixed by every $g \in G$; if this condition is not true then f is affine modulo Möbius transformations.

Mostow's rigidity theorem. We can use our theorems to give an alternative proof of Mostow's rigidity theorem, and we now comment on this. Let φ: $G_1 \to G_2$ be an isomorphism of discrete Möbius groups of $\bar{R}^n$, $n \geqq 2$, such that H^{n+1}/G_i has finite hyperbolic volume. Then Mostow's theorem says that φ is a conjugation by a Möbius transformation. The first step in the proof is to show that there is a quasiconformal map f: $\bar{R}^n \to \bar{R}^n$ inducing φ. Then an argument based on ergodicity of the action of G_1 and on absolute continuity of quasiconformal maps shows that f is in fact a Möbius transformation and rigidity follows.

Our theorems assume the existence of a G-compatible map of $\bar{R}^n$ and so the first step is as before. However, once we know that there is a quasiconformal map of $\bar{R}^n$ inducing φ, we can simply observe that quasiconformal maps are a.e. differentiable with a non-vanishing Jacobian. In the finite-volume case a point $x \in \bar{R}^n$ is a radial point of G_1 unless it is fixed by a parabolic element of G_1 (see Lemma B0). Since the set of such fixed points is countable, we can find a radial point at which f is differentiable with a nonvanishing Jacobian, and we get the rigidity by Theorem A; in the present case no $x \in \bar{R}^n$ can be fixed by every $g \in G_1$ by Theorem B2 (see also Remark B2), the limit set being now obviously $\bar{R}^n$. Hence f is indeed a Möbius transformation and not only affine modulo Möbius transformations.

Alternatively, we can observe that the matrix dilation of f is a.e. approximately continuous (Federer [7, p. 159]) and we get the rigidity by Theorem F. This proof of the rigidity theorem is especially simple, modulo the theory of quasiconformal mappings, and gives a very clear expression to our basic idea.

More generally, we obtain that if a discrete Möbius group G is of the so-called divergence type, then it is Mostow-rigid. That is, if $f: \bar{R}^n \to \bar{R}^n$, $n \geqq 2$, is quasiconformal and G-compatible, then f is a Möbius transformation. This follows by the preceding argument since the radial point set of a discrete group has positive measure if and only if it is of the divergence type (see [4, Chapter VII]). Other proofs of this theorem are due to Sullivan [14] and Agard [1].

It is interesting to observe that, in contrast to Mostow's original proof, here we make use neither of ergodicity nor of absolute continuity of quasiconformal maps. It suffices to know that quasiconformal maps are a.e. differentiable with a non-vanishing Jacobian. Actually, it would suffice that f is differentiable with a non-vanishing Jacobian at a single point $x \in \bar{R}^n$ not fixed by some parabolic $g \in G$.

If $n=1$, the argument given above for the finite-volume case fails since in this case the map $f: \bar{R}^1 \to \bar{R}^1$ inducing φ may be very irregular (although it is still a so-called quasisymmetric map). However, it follows by our theorem that if f is differentiable with a finite, non-zero derivative at a single point $x \in \bar{R}^1$ which is not fixed by some parabolic $g \in G_1$, then f must be a Möbius transformation (see also Remarks D2 and D5). This result has been already obtained by Mostow [11, (22.14)] and, if H^2/G_1 is compact, by Agard [3].

The literature on Mostow's rigidity theorem is quite extensive and we now mention only Agard's paper [1], which discusses also some earlier results, and our sister paper [17] which discusses aspects of rigidity connected with the absolute continuity of the map f and of which this paper was earlier a part.

Some definitions and notations. We use cl. ∂ and int to denote the closure, the boundary and the interior, respectively. Usually these operations are taken in $\bar{H}^{n+1}$ but if some other space is meant we indicate this by an appropriate subindex.

The hyperbolic metric of H^{n+1} is d, and the euclidean distance of two points in of R^{n+1} is $|x-y|$.

The standard basis of R^n is $e_1, \ldots, e_n$. Affine and linear maps of R^n are extended to $\bar{R}^n$ by means of the rule $\infty \mapsto \infty$. A *similarity* is a map of a subset of R^n into R^n which multiplies euclidean distance by a constant.

The orthogonal group of R^n is $O(n)$.

The identity map of a set X is id.

Remark. After the first version of this paper was completed (which was then contained in [17]), we received Agard's papers [1–3] which contain related material. In [3] he proves, as mentioned above, our theorem in a special case, and [1, 2] contain auxiliary results on Möbius groups which to some extent overlap with our Sections B and C.

B. Möbius groups

We denote the group of Möbius transformations of $\bar{R}^n$ by Möb(n); it includes also orientation reversing elements. Every $g \in$ Möb(n) can be extended in a unique manner to a Möbius transformation of the closed hyperbolic $(n+1)$-space $\bar{H}^{n+1}$ (and this can be further extended to $\bar{R}^{n+1}$); we do not distinguish between g and its extension. An element $g \in$ Möb(n)$\setminus\{\mathrm{id}\}$ can be classified as *loxodromic, parabolic*, or *elliptic* (see, for instance [1, 2.2]). The map g is elliptic if g is a conjugation (in Möb($n+1$)) of an orthogonal linear map of R^{n+1}; it is parabolic if it is a conjugation (in Möb(n)) of an affine map h of R^n of the form

$$h(x) = \beta(x) + a \tag{B1}$$

where $a \in R^n$, $a \neq 0$, and $\beta \in O(n)$ fixes a [(1)]; it is loxodromic if it is a conjugation (in Möb(n)) of a map of the form

$$h(x) = \lambda\beta(x) \tag{B2}$$

where $\lambda > 0$ and $\beta \in O(n)$. A map g is *hyperbolic* if (B2) is true with $\beta = \mathrm{id} \in O(n)$.

Equation (B2) implies that a loxodromic g has exactly two fixed points in $\bar{R}^n$. One of these is the *attractive* fixed point and is denoted $P(g)$, and the other is the *repulsive* fixed point and is denoted $N(g)$; these names should be self-explanatory. A parabolic g has exactly one fixed point which is denoted by $P(g) = N(g)$.

A *Möbius group* of $\bar{R}^n$ is a subgroup of Möb(n). Such a group is a topological group, the topology being given by the compact-open topology. It is not necessary for us to assume that the groups under consideration are discrete. Therefore we do not make this assumption although we do not know whether the non-discrete case leads to interesting situations.

The *limit set* of a Möbius group G is

$$L(G) = \bar{R}^n \cap \mathrm{cl}\, Gz \tag{B3}$$

where $z \in H^{n+1}$; since elements of G preserve the hyperbolic metric of H^{n+1}, this definition is independent of the choice of z. If G is discrete, this is the usual limit set. Obviously it is G-invariant and closed. By [9, 13.15], $L(G)$ is a perfect set if it contains more than two points.

Note that the radial points of G defined in the Introduction are also limit points but that the converse need not be true. However, there are some

(1) We have not found a reference to the representation (B1) and therefore indicate it here. If g fixes only ∞, then it is clear that it has the representation (B1) where one assumes only that $a \notin V = (\beta - \mathrm{id})(R^n)$. Conjugating by a translation one obtains that a is orthogonal to V and hence to $\beta(a) - a$. But then $\beta(a) = a$

important special cases in which we can characterize when a limit point is a radial point.

A discrete Möbius group of $\bar{R}^n$ is *geometrically finite* if its action in H^{n+1} has a finite sided hyperbolic fundamental polyhedron; for a more exact definition see [16, 1B].

Lemma B0. *Let G be a geometrically finite Möbius group of $\bar{R}^n$. Then $x \in L(G)$ is a radial point of G unless it is fixed by some parabolic $g \in G$.*

This holds also if H^{n+1}/G has finite hyperbolic volume.

Proof. The first paragraph was proved for $n=2$ by Beardon and Maskit [6, Theorem 2]. The same proof applies also for $n=1$ and the proof was generalized by Apanasov [5, Theorem 5.2] for $n>2$. They use actually a slightly less general definition of a geometrically finite group, assuming that G has a finite-sided hyperbolically convex fundamental polyhedron. But the proof boils down to the existence of so-called parabolic cusps at parabolic fixed points, whose existence follows also from the present definition [16, Theorem 2.4].

If H^{n+1}/G has finite volume, then G is geometrically finite [8, 13, 19] and so the lemma holds also in this case.

We will now present some general theorems involving Möbius groups and their limit sets; especially we need information on loxodromic elements in the group. We use Gottschalk-Hedlund [9, pp. 121–123] as our reference. In this book $n=1$ but the proofs are valid for $n>1$ as well. In fact, the only change needed seems to be to interpret intervals and arcs of [9] as closed or open n-balls of $\bar{R}^n$. Another reference is the set of notes [2, Section 5] by Agard.

Theorem B1. *Let G be a Möbius group of $\bar{R}^n$ and suppose that no $x \in \bar{R}^n$ is fixed by every $g \in G$ and that $L(G)$ consists of more than two points. Then $L(G)$ is an infinite perfect set such that if $x_1, x_2 \in L(G)$ and U_i is a neighbourhood of x_i in $\bar{R}^n$, then there is a loxodromic $g \in G$ with one fixed point in U_1 and the other in U_2.*

In particular, no $x \in \bar{R}^n$ is fixed by every loxodromic $g \in G$.

Proof. This follows from [9, 13.15 and 13.24].

A discrete Möbius group G whose limit set consists of more than two points, is usually called a non-elementary group. In this case the assumption that no $x \in \bar{R}^n$ is fixed by every $g \in G$ is automatically satisfied. In view of the importance of the discrete case we give this as a separate theorem.

Theorem B2. *Let G be a discrete Möbius group of $\bar{R}^n$ and suppose that $L(G)$ consists of more than two points. Then no $x \in \bar{R}^n$ is fixed by every loxodromic $g \in G$, and hence the conclusions of Theorem B1 hold.*

Proof. Since $L(G)$ contains more than two points, there are in G two loxodromic elements g and h with at most one common fixed point by [9, 13.21]. If $x \in \bar{R}^n$ is fixed by every element of G, then x is fixed also by g and h. Consequently g and h have exactly one common fixed point which is impossible by

Lemma B1. *Two loxodromic elements in a discrete Möbius group of $\bar{R}^n$ have either two common fixed points or no common fixed points.*

Proof. We must show that if g and h are loxodromic and have one, and only one, common fixed point x, then the group G generated by g and h is not discrete. We can assume that x is the repulsive fixed point of g. Let $g'=hgh^{-1}$ and let $g_i=g^i g' g^{-i}$. Let L be the hyperbolic line with endpoints $P(g)$ and $N(g)$ and let L_i be the line with endpoints $P(g_i)$ and $N(g_i)=N(g)$. Then $d(z,g(z)) = d(z',g_i(z'))$ for $z\in L$ and $z'\in L_i$. Since $L_i\to L$ in an obvious sense, we have for $z\in L$

$$\lim_{i\to\infty} g_i^{-1} g(z)=z.$$

This is a contradiction since $\{g_i^{-1}g\colon i>0\}$ is infinite and since discreteness of G is equivalent to the fact that G acts discontinuously in H^{n+1}, see Ahlfors [4, p. 79].

Finally, we prove a couple of lemmas concerning loxodromic elements and G-invariant sets.

Lemma B2. *Let G be a Möbius group of $\bar{R}^n$ and suppose that G has radial points. Then it is true that*

(a) *there are loxodromic elements in G,*

(b) *$L(G)$ contains at least two points,*

(c) *if $X\subset\bar{R}^n$ is a set with two elements which is fixed setwise by every $g\in G$, then $L(G)=X=$ the set of radial points of G,*

(d) *if $A\subset\bar{R}^n$ is a G-invariant set containing at least three points, then A is infinite.*

Proof. If $L(G)$ contains more than two points, we get (a) by [9, 13.21]. Hence, since radial points are also limit points, we can assume that $L(G)$ consists of one or two points. We can assume that $\infty\in L(G)$. Let $H=\{g\in G\colon g(\infty)=\infty\}$ which is a subgroup of index one or two. If $g\in H$ is non-loxodromic, then g is a euclidean isometry of R^n and R^{n+1}. Hence H preserves sets of the form $R^n\times\{t\}$, $t>0$, if every $g\in G$ is non-loxodromic and we would have that if $z\in H^{n+1}$, then $Gz\subset R^n\times\{t,t'\}$ for some $t,t'>0$. It would follows that G does not have radial points and hence (a) is true.

Since fixed points of a loxodromic $g\in G$ are in $L(G)$, (b) follows. To get (c), observe that if a loxodromic $g\in G$ does not fix $x\in X$, then X is infinite. Hence X is the fixed point set of any loxodromic $g\in G$ and so, by (a), $X\subset L(G)$. If $X\neq L(G)$, there would be by [9, 13.21] a loxodromic $g\in G$ such that g does not fix some $x\in X$. Hence $L(G)=X$ which is clearly also the radial point set of G. Thus (c) is proved.

Finally, (d) follows by [9, 13.14].

Lemma B3. *Let $g_i\in$* Möb(n) *fix some $y\in\bar{R}^n$ and suppose that, for some $z\in H^{n+1}$, $g_i(z)$ approach radially $x\in\bar{R}^n$* (i.e. (A1) *is true*). *Then there are loxodromic elements in the sequence.*

Proof. Assuming that $y=\infty$ one sees as in the proof of the preceding lemma that if all g_i:s are non-loxodromic, then always $g_i(z)\in R^n\times\{t\}$ for some $t>0$ which is impossible.

Remarks. B1. If G is non-discrete, then $L(G)$ may contain more than two points, and still there may be a point $x \in \bar{R}^n$ fixed by every $g \in G$. An example is the group of similarities of R^n for which the limit set is $\bar{R}^n$ but ∞ is fixed by every $g \in G$.

Note that if $L(G)$ contains at most two points, then G fixes either a point $x \in \bar{H}^{n+1}$ or a point-pair $\{x, y\} \subset \bar{R}^n$; this is clear if $L(G) \neq \emptyset$. If $L(G) = \emptyset$, then Gz is bounded in the hyperbolic metric for $z \in H^{n+1}$. Hence the center of the hyperbolic disk D containing Gz with minimal radius is well-defined (see [15, p. 75] or [18, Lemma E] for the easy argument). Then the center of D is fixed by every $g \in G$.

B2. It is interesting to note that the theorems on Möbius groups presented in this section are not needed in the first part of Theorem D, to prove that a G-compatible map differentiable at a radial point can be composed with Möbius transformations in such a way that it becomes affine in R^n. The only exception is Lemma B2 (d) which is needed to show that the set A is actually infinite (it would suffice to know that it contains at least four points). The results of this section are needed in the second part, to show that this affine map under certain circumstances is in fact a Möbius transformation.

In the most important application of Theorem D, in Mostow's rigidity theorem (corresponding to case (a) of Theorem D), one needs only that a group G with a finite-volume hyperbolic quotient space contains an element not fixing a given point of $\bar{R}^n$. In this case obviously $L(G) = \bar{R}^n$ and the group is discrete and hence the existence of such an element (which is even loxodromic) follows by Theorem B2.

If G is torsionless and H^{n+1}/G is compact, the existence of $g \in G$ not fixing a given $x \in \bar{R}^n$ follows even more simply. In this case every $g \in G \setminus \{\mathrm{id}\}$ is loxodromic and one easily sees that there are in G two loxodromic elements g and h with at most one common fixed point. Now the simple Lemma B1 implies that they can have no common fixed points. Hence, if $x \in \bar{R}^n$, either $g(x) \neq x$ or $h(x) \neq x$.

Thus in these cases there is actually a loxodromic $g \in G$ not fixing a given x. Then, for instance, the geometric argument of Sect. C shows (see Remark C1) that the mapping occurring in Mostow's rigidity theorem is indeed a Möbius transformation.

C. Affine conjugations of Möbius transformations

We now consider conjugations of a Möbius transformation g by an affine homeomorphism α of R^n. If $\alpha g \alpha^{-1}$ is again a Möbius transformation, then α must be a similarity provided that g does not fix ∞ (Agard [1, Lemma 2.2]). We need the following variant of this theorem.

Lemma C1. *Let $g, g' \in \mathrm{Möb}(n)$ where g is loxodromic, let α be an affine homeomorphism of R^n and let $a \in \bar{R}^n$. Suppose that*

$$\alpha g^k(a) = g'^k \alpha(a) \qquad (C1)$$

for all $k \in Z$. Then $\alpha|V$ is a similarity when V is the affine subspace of minimal dimension such that $V \cup \{\infty\}$ contains a and the points fixed by g.

Proof. If g fixes ∞, then V is at most one-dimensional and the lemma is clear. Hence we can assume, by composing with similarities, that g fixes $\pm e_1$ ($e_1 = (1, 0, \dots, 0)$), that e_1 is the attractive fixed point of g and that α fixes pointwise the x_1-axis. Furthermore, we can assume that $a = a_1 e_1 + a_2 e_2$ and that $\alpha(a) = a'_1 e_1 + a'_2 e_2$.

The case that a is on the x_1-axis is again clear. Hence we can assume that $V = \alpha(V)$ is the $x_1 x_2$-plane. Thus g' is loxodromic with $\pm e_1$ as the fixed points, e_1 being the attractive fixed point. Furthermore, α has in V the expression

$$\alpha(x_1, x_2) = (x_1, \lambda x_1 + \mu x_2) \tag{C2}$$

where λ, $\mu \in R$.

If $u \in \bar{R}^n \setminus \{\pm e_1\}$, let $S(u)$ be the circle passing through the points u and $\pm e_1$. If both g and g' are hyperbolic, then g preserves $S(a)$ and g' preserves $S(\alpha(a))$, and it follows that $g^k(a) \in S(a)$ and $g'^k \alpha(a) = \alpha g^k(a) \in S(\alpha(a))$ for all integers k. Hence

$$\alpha(S(a)) = S(\alpha(a)). \tag{C3}$$

Thus α maps a circle of V onto another circle and the lemma follows in this case.

The proof for general loxodromic g and g' is similar but more complicated. First we augment the definition of $S(u)$ by setting for $\varepsilon > 0$

$S_\varepsilon(u) = \{S$: S is a circle passing through $\pm e_1$ such that the angle between S and $S(u)$ at the points $\pm e_1$ is less than $\varepsilon\}$.

Then we recall that there are h, $h' \in \text{Möb}(n)$ such that

$$hgh^{-1}(z) = \lambda\beta(z), \qquad h'g'h'^{-1}(z) = \lambda'\beta'(z)$$

for $z \in R^n$, where $0 < \lambda$, $\lambda' < 1$ and β, $\beta' \in O(n)$. The crucial fact is that there arbitrarily large p such that $\beta^{\pm p}$ and $\beta'^{\pm p}$ are in a prescribed neighbourhood of the neutral element of the compact group $O(n)$.

It follows that, given $\varepsilon > 0$, there are arbitrarily large p such that $S(g^{\pm p}(a)) \in S_\varepsilon(a)$ and $S(g'^{\pm p}\alpha(a)) \in S_\varepsilon(\alpha(a))$. But $g'^{\pm p}\alpha(a) = \alpha g^{\pm p}(a)$ and it is thus a point of $S_\varepsilon(\alpha(a)) \cap \alpha(S_\varepsilon(a))$. Letting $\varepsilon \to 0$, we get that the circle $S(\alpha(a))$ and the ellipse $\alpha(S(a))$ are tangent to each other at the points $\pm e_1$.

In particular, the tangents of $\alpha(S(a))$ at $\pm e_1$ are symmetric with respect to the x_2-axis. In view of (C2), this is possible only if $\lambda = 0$ in (C2). Hence the equation of $\alpha(S(a))$ is of the form

$$x_1^2 + ((x_2 - u)/b)^2 = 1 + u^2/b^2 \qquad (u \in R,\ b > 0)$$

since $\pm e^1 \in S(a))$. On the other hand, the equation of the circle $S(\alpha(a))$ is of the form

$$x_1^2 + (x_2 - v)^2 = 1 + v^2 \qquad (v \in R).$$

Since they are tangent to each other at the points $\pm e_1$, we get the condition

$$v = u/b^2.$$

If $b \neq 1$, then it follows that $\alpha(S(a)) \cap S(\alpha(a)) = \{\pm e_1\}$. This contradicts the fact that $\alpha(a) \notin \{\pm e_1\}$ is also a point of this intersection. Hence $b=1$ and we get again (C3) and the lemma follows.

If (C1) is true for all $a \in \bar{R}^n$, and if ∞ is not fixed by g, then $\alpha|V$ is a similarity for all 2-planes V containing the fixed points of g. Hence α is a similarity. Actually, it is not now necessary to assume that g is loxodromic, g can be any Möbius transformation not fixing ∞ and we have

Lemma C2. *Let $A = V \cup \{\infty\}$ for some affine subspace V of R^n, let α be an affine homeomorphism of R^n and suppose that $g \in$ Möb(n) does not fix ∞ and that $g(A) = A$. If*

$$\alpha g|V = g'\alpha|V$$

for some $g' \in$ Möb(n), then $\alpha|V$ is a similarity.

Proof. This is clear if $n=1$ and follows from Agard [1, Lemma 2.2] if $n \geqq 2$. Agard considers only orientation preserving Möbius transformations but the proof is valid even without this assumption. If $n=2$ (and everything is orientation preserving), then Agard's lemma is a fairly direct consequence of the composition rule for the complex dilatation.

Remarks. C1. We use Lemma C2 for case (a) of Theorem D. If $L(G)$ contains more than two points, we could then by Theorem B1 assume that g in Lemma C2 is loxodromic. So in this case (which covers most important applications) we could be self-contained and get Lemma C2 from Lemma C1 as indicated above.

C2. Lemma C1 is valid also for parabolic g, provided that we interpret the set V as follows. One sees from (B1) that there is a 1-circle S_g through $P(g)$ such that $g(S_g) = S_g$; there may be more than one such circle but they all are tangent to each other at $P(g)$. Let now V be the affine subspace of minimal dimension such that $V \cup \{\infty\}$ contains $\{P(g), a\}$ and such that V and S_g are tangent to each other at $P(g)$. With this modification, Lemma C1 is valid with a similar proof.

If we modify Lemma C1 in this manner for parabolic g, it is then possible to get Lemma C2 for parabolic g not fixing ∞ in quite the same manner as we indicated above for loxodromic g.

However, for elliptic g our method seems difficult since $\{g^k(z): k \text{ integer}\}$ may be finite even if z is not fixed by g.

D. Differentiability and rigidity

Now we come to our main theorem. This theorem is quite general and we need to know what one means by the differentiability of a map defined at an arbitrary subset of $\bar{R}^n$. Assume that $f\colon A \to \bar{R}^n$, $A \subset \bar{R}^n$, is a map and that $x \in R^n$. Then we say that f is *differentiable* at x if there is an affine map α of R^n such that

$$\frac{|f(y)-\alpha(y)|}{|y-x|} \to 0 \tag{D1}$$

as $y \to x$ in A; we do not require that $x \in A$ but if $x \in A$ we also require that f is continuous at x. If $x \notin \operatorname{cl} A$, then (D1) is satisfied vacuously, and hence we regard f differentiable at x.

If β is a linear map such that $\alpha = \beta + \text{constant}$, then β is called the *derivative* of f at x. If the map α can be chosen to be an affine homeomorphism of R^n, then f is differentiable with a *non-vanishing Jacobian* at x; if α can be chosen to be a similarity, we say that f has a *conformal derivative* at x.

We extend these definitions to the case that $x = \infty$ or $f(y) \to \infty$ as $y \to x$ by means of auxiliary Möbius transformations. Note that these definitions do not depend on the dimension of the space $\bar{R}^n$ into which A is embedded: if n varies, the differentiability properties of f do not change.

A *sphere* (or a k-sphere) of $\bar{R}^n$ is a set of the form $h(\bar{R}^k)$ for some $h \in \text{Möb}(n)$ and some integer $k \in [0, n]$. If $A \subset \bar{R}^n$ contains at least two points, then the *minimal sphere* of $\bar{R}^n$ containing A is well-defined. If a Möbius group G has radial points, then $L(G)$ contains at least two points by Lemma B2, and hence the minimal spheres occurring later are well-defined.

In the following we also call restrictions of maps $f \in \text{Möb}(n)$ Möbius transformations. Recall that affine maps are extended to $\bar{R}^n$ by means of the rule $\infty \mapsto \infty$.

We will now see that if a G-compatible map f is differentiable at a radial point of G with a non-vanishing Jacobian, then f is affine modulo Möbius transformations, apart from a special set of circumstances, and even then with at most one exceptional point; in many natural situations f is even a Möbius transformation.

Theorem D. *Let G be a Möbius group of $\bar{R}^n$ and let $A \subset \bar{R}^n$ be a G-invariant set containing at least three points. Let $f\colon A \to \bar{R}^n$ be a map inducing a homomorphism φ of G onto another Möbius group. Suppose that f is differentiable with a non-vanishing Jacobian at a radial point x of G. Then there are Möbius transformations h and h' of $\bar{R}^n$ and an affine homeomorphism β of R^n such that*

$$h' f h(y) = \beta(y) \tag{D2}$$

for all $y \in h^{-1}(A) \cap R^n$. If f has a conformal derivative at x, then β can be chosen to be a similarity.

Furthermore, if $a = h(\infty) \in A$, then (D2) is true also for $y = \infty$ except in the following case: a and the radial point x (we have $x \neq a$) are fixed by every $g \in G$ and f is non-continuous at a; setting

$$f' = h'^{-1} \beta h^{-1},$$

then $f'|A$ also induces φ and $f(a) = f'(x)$ $(= f(x)$ if $x \in A)$.

Regarding the question when f is a Möbius transformation, we have

(a) *if A is a k-sphere for some $k \leqq n$ and if the point $a = h(\infty)$ is not fixed by every $g \in G$, then f is a Möbius transformation,*

(b) *if A is a subset of a 1-sphere, then $f'|A$ is a Möbius transformation.*

Quite generally, if $a = h(\infty)$ is not fixed by every $g \in G$, then

(c) *$f|S_c \cap A$ is a Möbius transformation for every $c \in \bar{R}^n$ when S_c is the minimal sphere containing $L(G) \cup \{c\}$.*

Remark. The condition that A must be a k-ssphere in (a) is far too strong. The most general, if somewhat cumbersome, condition that we get is the following. Let S_A be the minimal sphere containing $A \cup L(G)$ and let $S' = \bigcup_{a \in A} S_a$, S_a as in (c). Suppose now that for all affine homeomorphisms β of R^n and $g_i, g_i' \in \text{Möb}(n)$, $i = 1, 2$, such that $\infty \in \operatorname{cl} g_i(A)$, the condition

(d) $g_1' \beta g_1 | S' = g_2' \beta g_2 | S'$ implies $g_1' \beta g_1 | S_A = g_2' \beta g_2 | S_A$.

Then f is a Möbius transformation provided that $a = h(\infty)$ is not fixed by every $g \in G$.

Since everything here is real-analytic (β-outside ∞), (d) is true if, for instance,

(e) $\operatorname{int}_{S_A}(\operatorname{cl} S') \neq \emptyset$.

Proof. Before starting the proof proper, we introduce notation and list some facts to which we will refer later.

If $u \in \bar{H}^{n+1}$ and $v \in \bar{R}^n$, $u \neq v$, let

$$L(u, v) = \text{the hyperbolic line or ray with endpoints } u \text{ and } v;$$

we take $L(u, v)$ be closed in H^{n+1}, i.e. $v \notin L(u, v)$ and $u \in L(u, v)$ if and only if $u \in H^{n+1}$. If $u \in H^{n+1}$ and v, $w \in \bar{R}^n$, let

$$\operatorname{ang}(u, v, w) = \text{the angle between the rays } L(u, v) \text{ and } L(u, w)\ (\in [0, \pi]).$$

This is obviously Möbius-invariant: if $g \in \text{Möb}(n)$, then

$$\operatorname{ang}(g(u), g(v), g(w)) = \operatorname{ang}(u, v, w).$$

Let

$$T^n = \{(u, v, w) \in (\bar{R}^n)^3 \colon\ u, v, w \text{ distinct}\};$$

if $t \colon \bar{R}^n \to \bar{R}^n$ is injective, it operates on T^n by

$$t(u, v, w) = (t(u), t(v), t(w)).$$

We can also define a projection $P \colon T^n \to H^{n+1}$ by

$$P(u, v, w) = \text{the orthogonal projection of } w \text{ onto } L(u, v);$$

that is, $p = P(u, v, w) \in L(u, v)$ and $L(u, v)$ and $L(p, w)$ are orthogonal. Finally, we define for $m \geqq 0$

$$C_m = \{u \in H^{n+1} \colon\ d(u, L(0, \infty)) \leqq m\}.$$

We will need the following properties of ang and P:

1°. P is Möbius compatible in the sense that if $u\in T^n$ and $g\in \text{Möb}(n)$, then

$$Pg(u)=gP(u).$$

2°. If $\alpha\colon R^n\to R^n$ is linear and non-singular, then

$$d(P(u), P\alpha(u))\leqq m'$$

for all $u\in T^n$ such that $P(u)\in C_m$ and where $m'=m'(m,\alpha)$.

3°. If $u\in H^{n+1}$ and $v\in \bar{R}^n$ are fixed and $w\in \bar{R}^n$ varies in such a way that $\text{ang}(u,v,w)\to 0$, then $w\to v$.

4°. If $m\geqq 0$ and $\delta>0$ and if $u\in C_m$, then $\text{ang}(u,v,\infty)\geqq\delta$ implies that $|v|/|u|\leqq m''$ for some $m''=m''(m,\delta)$.

5°. Given $\varepsilon>0$ and $m\geqq 0$, there is $\delta'=\delta'(\varepsilon,m)>0$ such that if $u\in C_m$, $v,w\in R^n$ and $|v-w|/|u|\leqq\delta'$, then $\text{ang}(u,v,w)\leqq\varepsilon$.

6°. Given $\varepsilon>0$, there is $\delta_\varepsilon>0$ such that if $u=(u_1,u_2,u_3)\in T^n$ and $v=(v_1,v_2,v_3)\in(\bar{R}^n)^3$ satisfy $\text{ang}(P(u),u_j,v_j)\leqq\delta_\varepsilon$ for $j=1, 2, 3$, then $v\in T^n$ and

$$d(P(u),P(v))\leqq\varepsilon.$$

7°. Given $\varepsilon>0$ and $m\geqq 0$, there is $\delta''=\delta''(\varepsilon,m)>0$ such that if w, $w'\in H^{n+1}$ and v, $v'\in\bar{R}^n$, then $d(w,w')\leqq m$ and $\text{ang}(w,v,v')\leqq\delta''$ imply that $\text{ang}(w',v,v')\leqq\varepsilon$.

Some of these claims are obvious and for others we give some explanation. For 2° we may note that

$$S_m=\{p\in C_m\colon |p|=1\}$$

is compact and hence so is $P^{-1}(S_m)$ since each $P^{-1}(p)$, $p\in H^{n+1}$, is homeomorphic to the compact space of 2-frames of R^{n+1}. Hence 2° is true if $P(u)\in S_m$ by compactness and continuity. Since $P(\lambda u)=\lambda P(u)$ for $\lambda>0$ by 1°, we have 2° for all $u\in T^n$ such that $P(u)\in C_m$.

In 4° and 5° one can replace u,v and w by λu, λv and λw where $\lambda>0$. Hence one can assume that u is in the compact set S_m defined above. Then 4° is immediate and for 5° we need only to note that for compact $K\subset H^{n+1}$ there is $c=c(K)>0$ such that $\text{ang}(u,v,w)\leqq c|u-w|$ for $u\in K$ and $v,w\in R^n$.

In 6° we can replace u by $g(u)$ and v by $g(v)$ where $g\in\text{Möb}(n)$. By the transitivity of the action of $\text{Möb}(n)$ on T^n, we can then assume that $u=(0,\infty,e_1)$ and 6° follows now by the continuity of P. Similarly in 7°, we can assume that $w=e_{n+1}$ and use the compactness of $\{p\in H^{n+1}\colon d(p,w)\leqq m\}$.

After these preliminary steps, we can start to prove Theorem D. By composing with Möbius transformations, we can assume that $x=0$ is a radial point of G and that (D1) is true for some linear homeomorphism α of R^n which is a similarity if f has a conformal derivative at x. We can also assume that

$$\infty\in A. \qquad \text{(D3)}$$

Let then $g_i\in G$ be a sequence such that if $z\in H^{n+1}$, then (A1) is true for $x=0$. By passing to a subsequence, we can assume that

$$\lim_{i\to\infty} g_i^{-1}(\infty) = a \in \bar{R}^n. \tag{D4}$$

Setting

$$g_i' = \varphi(g_i)^{-1},$$

we claim that if $y \in A \setminus \{a\}$, then

$$f(y) = \lim_{i\to\infty} g_i' \alpha g_i(y) \quad \text{and} \tag{D5}$$

$f|A\setminus\{a\}$ is an injection.

To prove (D5), we pick points $y_1 = y$, y_2, $y_3 \in A\setminus\{a\}$ such that y_j are distinct. Since A is actually infinite by Lemma B2, we can find such points. Let

$$z = P(y_1, y_2, y_3),$$

and

$$\delta_{ij} = \operatorname{ang}(g_i(z), \infty, g_i(y_j)) = \operatorname{ang}(z, g_i^{-1}(\infty), y_j).$$

In view of (D4), we can find $\delta > 0$ such that

$$\delta_{ij} \geqq \delta > 0 \tag{D6}$$

for big i and all j.

Let

$$z_i = g_i(z) = P(g_i(y_1), g_i(y_2), g_i(y_3))$$

and

$$z_i'' = P(\alpha g_i(y_1), \alpha g_i(y_2), \alpha g_i(y_3)).$$

Since $z_i = g_i(z)$ approach radially 0, there is $m \geqq 0$ such that all z_i are in the cone C_m. Hence 2° implies that for some $M'' \geqq 0$ and all i,

$$d(z_i, z_i'') \leqq M''. \tag{D7}$$

By 4° and (D6), there is $M_0 > 0$ such that

$$|g_i(y_j)|/|z_i| \leqq M_0$$

for big i and all j. Since f has the derivative α at the origin and $z_i \to 0$, it follows that

$$|f g_i(y_j) - \alpha g_i(y_j)|/|z_i| \to 0$$

as $i \to \infty$. By 5°, then

$$\operatorname{ang}(z_i, f g_i(y_j), \alpha g_i(y_j)) \to 0$$

as $i \to \infty$ for all j. In view of (D7) and 7°, we get now that

$$\operatorname{ang}(z_i'', f g_i(y_j), \alpha g_i(y_j)) \to 0 \tag{D8}$$

as $i \to \infty$ for all j. Hence 6° implies that

$$z_i' = P(f g_i(y_1), f g_i(y_2), f g_i(y_3))$$

is defined for big i and that

$$d(z_i', z_i'') \to 0$$

as $i \to \infty$. Finally, by 7°,

$$\operatorname{ang}(z_i', fg(y_j), \alpha g_i(y_j)) \to 0 \tag{D9}$$

as $i \to \infty$ for all j.

Now

$$\begin{aligned} z' = g_i'(z_i') &= P(g_i' f g_i(y_1), g_i' f g_i(y_2), g_i' f g_i(y_3)) \\ &= P(f(y_1), f(y_2), f(y_3)) \end{aligned}$$

does not depend on i. By (D9),

$$\begin{aligned} \operatorname{ang}(z', f(y), g_i' \alpha g_i(y)) &= \operatorname{ang}(g_i'(z_i'), g_i' f g_i(y), g_i' \alpha g_i(y)) \\ &= \operatorname{ang}(z_i', f g_i(y_1), \alpha g_i(y_1)) \to 0 \end{aligned}$$

as $i \to \infty$. By 3°, this proves the limit relation of (D5).

We also proved that if y_1, y_2, $y_3 \in A \setminus \{a\}$ are arbitrary distinct points, then $fg_i(y_1)$, $fg_i(y_2)$, $fg_i(y_3)$ are distinct for big i. Hence $f(y_1)$, $f(y_2)$, $f(y_3)$ are distinct by G-compatibility. It follows that $f|A \setminus \{a\}$ is injective and (D5) is proved.

By passing to a subsequence we can assume that in addition to the limit (D4), also the limit

$$\lim_{i \to \infty} g_i'(\infty)$$

exists. The existence of these limits implies that we can find h, h', h_i, $h_i' \in \text{Möb}(n)$ for $i > 0$ such that both $g_i h_i$ and $h_i' g_i'$ fix ∞ and that $h_i \to h$ and $h_i' \to h'$ as $i \to \infty$ uniformly in the spherical metric of $\bar{R}^n$.

By the uniform convergence, (D4) implies that $h_i^{-1}(a) \to \infty$ and hence

$$h^{-1}(a) = \infty. \tag{D10}$$

Furthermore, fixing ∞, $g_i h_i$ and $h_i' g_i'$ are similarities of R_n and consequently

$$\alpha_i = h_i' g_i' \alpha g_i h_i \tag{D11}$$

is an affine map of R^n. Hence by (D5),

$$h' f(y) = \lim_{i \to \infty} \alpha_i h_i^{-1}(y) \tag{D12}$$

for $y \in A \setminus \{a\}$.

Let V be the affine subspace of R^n generated by $h^{-1}(A \setminus \{a\})$ (which is a subset of R^n by (D10)). Then (D12) implies that there is an affine map β of R^n such that $\lim_{i \to \infty} \alpha_i(y) = \beta(y)$ for $y \in V$ and that

$$h' f(y) = \beta h^{-1}(y) \tag{D13}$$

for $y \in A \setminus \{a\}$.

We claim that here actually $\beta | V$ is an embedding. To see this, observe that the maps $h_i' g_i'$ and $g_i h_i$ in (D11) are similarities, and hence there is $K \geqq 1$ such that

$$1/K \leqq |\alpha_i(u)-\alpha_i(u')|/|\alpha_i(v)-\alpha_i(v')| \leqq K$$

whenever $|u-u'|=|v-v'|$. Since $\beta|V$ cannot be constant by (D5), it follows that it indeed is an embedding.

Thus we can choose that β is an embedding and then, in view of (D10), (D13) is (D2) written in another manner.

If f has a conformal derivative at x, then α and α_i are similarities. Hence $\beta|V$ is also a similarity, and we can choose β to be a similarity of R^n.

This proves the first paragraph of Theorem D. We will now examine the validity of (D2) for $y=\infty$. We claim that the point $a=h(\infty)$ satisfies

$$a\in \mathrm{cl}(A\setminus\{a\}). \tag{D14}$$

By (D3) and (D4), this is clear if $\{g_i^{-1}(\infty)\colon i>0\}$ is infinite. If it is not, then by passing to a subsequence we can assume that $g_1^{-1}g_i$ fixes a. Since $g_1^{-1}g_i$ approaches radially $g_1^{-1}(x)$ by (A1), it follows by Lemma B3 that some $g_1^{-1}g_i$ is loxodromic and fixes a. Since A is G-invariant and contains at least three points, (D14) follows.

Consequently, if $a=h(\infty)\in A$ and (D2) is not true for a, f must be non-continuous at a. Suppose that f is non-continuous at a. Then every $g\in G$ must fix a since f is G-compatible and continuous at all other points of A by (D2), continuity being a G-invariant property. It follows that every $g\in\varphi(G)$ fixes $f(a)$.

Let f' be as defined in the statement of Theorem D. Then f' is a homeomorphism of $\bar{R}^n$ such that $f'|A\setminus\{a\}=f|A\setminus\{a\}$. Using (D14) and the fact $g(a)=a$ for all $g\in G$, one sees that every $g\in\varphi(G)$ fixes $f'(a)\neq f(a)$. It follows that $f'|A$ also induces φ.

Pick loxodromic $h\in G$; by Lemma B2 there is such h. Then the fixed points of h are in $\mathrm{cl}\,A$ (since A contains more than two points). It is now easy to see that $\varphi(h)$ is also loxodromic with fixed points in $\mathrm{cl}\,f'(A)$; these fixed points are $f(a)$ and $f'(a)$ which are fixed by all $g\in\varphi(G)$. Thus $f(a)$, $f'(a)\in\mathrm{cl}\,f'(A)$. It follows that every $g\in G$ fixes $x'=f'^{-1}(f(a))\neq a$. Thus $\{x',a\}$ is a doubleton fixed by every $g\in G$ and Lemma B2 implies that $x\in\{x',a\}$. Since f is continuous at x if $x\in A$, it follows that $x=x'$. Hence $f(a)=f'(x)=f(x)$ if $x\in A$.

This proves the second paragraph of the theorem. To get the rest, we replace f by $h'fh|h^{-1}(A)$ (and A by $h^{-1}(A)$, etc.). Note that then by (D10),

$$a=\infty\in\mathrm{cl}(A\setminus\{a\}). \tag{D15}$$

It is also clear that we can assume that f is continuous (by changing f at a if necessary), and we have

$$f|A=\beta|A$$

and thus, if $g\in G$,

$$\varphi(g)\,\beta|A=\beta g|A. \tag{D16}$$

Now (a) is an immediate consequence of Lemma C2 since $\infty\in\mathrm{cl}\,A$ by (D15). Similarly, (b) is clear by (D15).

Part (c) is more complicated. Note that $L(G)$ contains at least two points by Lemma B2. If $L(G)$ contains exactly two points, then S_c is a 1-circle (or a 0-circle). In this case Lemma C1 implies (c) since there are loxodromic elements in G by Lemma B2.

If $L(G)$ contains more than two points, then we get (c) as follows. Let S be the minimal sphere containing $L(G)$ and let k be its dimension. Pick now $x_0, \dots, x_p \in L(G) \cap R^n$ such that the affine subspace V of R^n generated by x_i contains $L(G) \cap R^n$. Then $p=k$ or $p=k+1$ according to whether $\infty \in S$ or $\infty \notin S$. If $(A \cap S_c) \setminus (V \cup \{\infty\}) \neq \emptyset$, pick $x_{p+1} \in (A \cap S_c) \setminus (V \cup \{\infty\})$. Let V' be the affine subspace of R^n generated by x_i ($i \leqq p$ or $i \leqq p+1$ if x_{p+1} exists). By Theorem B1, there is a loxodromic $g \in G$ such that the fixed points of g are arbitrarily near given x_i and x_j, $i, j \leqq p$. Let V_{ijq} be the affine 2-plane containing the points x_i, x_j and x_q for distinct i, j, q. Then Lemma C1 implies that every $\beta | V_{ijq}$ is a similarity. It follows that β preserves the ratios $|x_i - x_j| / |x_k - x_q|$ for distinct i, j, k, q, and hence $\beta | V'$ and $\beta | S_c \setminus \{\infty\}$ are similarities. This proves (c).

Finally, we prove claim (d) in the Remark. By (c), both sides of (D16) are Möbius transformations on S_a, coinciding on $L(G) \cup \{a\}$. Since S_a is the minimal sphere containing $L(G) \cup \{a\}$, it follows that they coincide on S_a and hence on S', too. In view of (D15), condition (d) (which is unaffected by the substitution of $h'fh$ for f) guarantees that they coincide even on S_A and hence the claim of the remark follows by case (a).

Everything is now proved.

Remarks. D1. The exceptional case of the second paragraph of Theorem D can occur. Let G_0 be the subgroup of Möb(n) consisting of elements fixing 0 and ∞, let $A = \bar{R}^n$ and define f by $f | R^n = \mathrm{id}$ and $f(\infty) = 0$. Then f induces id: $G_0 \to G_0$ and is differentiable at the radial point 0 of G_0 with a non-vanishing Jacobian. Any group G for which the exceptional case can occur is (up to conjugation) a subgroup of G_0.

D2. Let G be a Möbius group of S^1 which has parabolic elements. Let f be a homeomorphism of S^1 inducing an isomorphism $G \to G'$, G' another Möbius group. Then f is differentiable with a non-vanishing Jacobian at parabolic fixpoints of G as a simple calculation shows. Teichmüller space theory provides examples of Fuchsian G and G-compatible homeomorphisms f such that f is not the restriction of a Möbius transformation. This example shows that it is essential in Theorem D that x is a radial point, it does not suffice that x is a limit point.

D3. In parts (a) and (c) of Theorem D it is necessary to assume that the point a is not fixed by every $g \in G$ as the following counter-example shows.

Let G be the group of similarity maps of R^n of the form $z \mapsto \lambda z + b$, $\lambda > 0$ and $b \in R^n$; now ∞ is fixed by every $g \in G$. Then $\alpha G \alpha^{-1} = G$ for any affine homeomorphism α of R^n. Hence if we take $f = \alpha$ and $A = R^n$, then every $x \in A$ is a radial point of G at which f is differentiable with a non-vanishing Jacobian but $f | A \cap L(G) = \alpha$ is not a Möbius transformation unless α is a similarity.

D4. The next example shows that, even if no $x \in \bar{R}^n$ is fixed by every $g \in G$, some conditions like (a) or (d) are necessary for the conclusion that f is a Möbius transformation.

Let G be a Fuchsian group of $\bar{R}^2$ whose limit set is $\bar{R}^1$ and extend G in the natural manner to a Möbius group of $\bar{R}^3$. Let α be an affine map of R^3 such that, when $A_1 = R^2 \times \{0\} \subset R^3$ and $A_2 = R \times \{0\} \times R \subset R^3$, then $\alpha|A_1 = \mathrm{id}$ and $\alpha|A_2$ is an isometry onto $\alpha(A_2)$ but that α is not a similarity. Let $A = A_1 \cup A_2 \cup \{\infty\}$ and let $f = \alpha|A$ which induces id: $G \to G$. Then G is non-elementary, it has radial points in $\bar{R}^1$ (for instance, we can assume that every $x \in \bar{R}^1$ is a radial point) at which f is differentiable in the sense of (D1) with a non-vanishing Jacobian but f is not a Möbius transformation.

D5. If G_1 and G_2 are geometrically finite Möbius groups of $\bar{R}^n$ and if φ: $G_1 \to G_2$ is an isomorphism which carries parabolic elements bijectively onto parabolic elements, then there is a homeomorphism f: $L(G_1) \to L(G_2)$ inducing φ ([16, Theorem 3.3]). In the geometrically finite case a point $x \in L(G_1)$ is a radial point of G_1 unless it is fixed by some parabolic $g \in G_1$ (Lemma B0). Suppose that f is not a Möbius transformation. Then the set where f is differentiable with a non-vanishing Jacobian is at most countable, being contained in the set of points fixed by some parabolic $g \in G_1$; if G_1 does not contain parabolic elements, then this set is empty.

In particular, if G_i are Fuchsian groups such that H^2/G_1 is compact, and if f: $\bar{R} \to \bar{R}$ fixes ∞, then f can have at no $x \in R$ a finite, non-zero derivative unless f is a Möbius transformation. This striking result was already obtained by Mostow [11, 22.14] and Agard [3].

D6. Actually, the requirement that f is differentiable at a radial point x is still too strong for Theorem D. What we need is that as we approach x and look at x from the points $g_i(z)$ as in (A1), then f looks more and more like an affine map but we can *disregard the scale*. This can be expressed as follows. Suppose that there is an affine homeomorphism α of R^n and numbers $\lambda_i > 0$ and a sequence $g_i(z)$ as in (A1) such that, normalizing $x = 0 = \alpha(x)$,

$$|f(y) - \lambda_i \alpha(y)| \leqq \varepsilon_i \lambda_i r_i \tag{D17}$$

if $y \in A$, $|y - x| \leqq c_i r_i$. Here $r_i = |g_i(z)|$ and $\varepsilon_i \to 0$ and $c_i \to \infty$ as $i \to \infty$.

If this is true, Theorem D remains valid with much the same proof. However, now $d(z_i, z_i')$ need not be bounded which requires some slight changes. In fact, one needs only to replace, beginning from the displayed formula immediately before (D8), z_i by $\lambda_i z_i$, z_i'' by $\lambda_i z_i''$ and α with $\lambda_i \alpha$.

In addition, one substitutes $\lambda_i \alpha$ for α in (D5) and refers to (D17) instead of the differentiability. These are the only changes needed.

If H^{n+1}/G is compact, we need not be concerned about the sequence $g_i(z)$ and can simply suppose that this is true for some sequence $r_i > 0$ such that $r_i \to 0$ as $i \to \infty$; we can forget also the requirement that x is a radial point since all $x \in \bar{R}^n$ are.

D7. I am indebted to the referee for the observation that one could assume in Theorem D that only the limit (D1) exists at the radial point x (with non-singular α) but not that f is continuous at x if $x \in A$. Otherwise the conclusions of Theorem D are valid but the exceptional case needs some revision. Now $f'(y) = f(y)$ for all $y \in A \setminus \{x, a\}$ where x is the radial point and $a = h(\infty)$ is as before. We have that $x \neq a$ and that $\{x, a\} \subset \mathrm{cl}\, A$ and that f and f' may fail to coincide at $y \in \{x, a\} \cap A$ if and only if f is non-continuous at y. We relabel $\{a, x\}$ as $\{x_0, x_1\}$ and can say that

(*) if $x_i \in A$, then $f(x_i) \neq f'(x_i)$ if and only if $f(x_i) = f'(x_j)$ where $\{i, j\} = \{0, 1\}$.

If both x_0 and x_1 are fixed by every $g \in G$, then conditions (*) for x_0 and x_1 are independent of each other. However, if there is $s \in G$ such that $s(x_0) = x_1$, then $f(x_0) \neq f(x_0)$ if and only if $f(x_1) \neq f'(x_1)$.

This can be proved by considerations similar to the proof of the exceptional case of Theorem D. One sees also that if the exceptional case can occur, then G is a subgroup (up to conjugation) of the group G_1 consisting of $g \in \text{Möb}(n)$ which map the set $\{0, \infty\}$ onto itself.

D8. Suppose that $n \geqq 2$. Then we say that f has a K-quasiconformal derivative at x if the map α in (D1) can be chosen to be a K-quasiconformal affine map. In this terminology, f has a conformal derivative at x if and only if f has a 1-quasiconformal derivative at x. Our proof shows that if f has a K-quasiconformal derivative at the radial point x of Theorem D, then the map β in (D2) can be chosen to be K-quasiconformal.

D9. Finally, a trivial remark. Obviously Theorem D is true if A is empty or contains only one point. However, if A contains exactly two points, then it need not be true. To see this, let $G = G_0$, G_0 as in Remark D1, let $A = \{0, \infty\}$ and define f by $f(\{0, \infty\}) = \{0\}$. Then the conditions of Theorem D are satisfied and although the first paragraph is true, the second is not since f is continuous.

E. Differentiability at a limit point

As we observed in Remark D2 of the preceding section, differentiability at a limit point is not sufficient for rigidity. However, if we strengthen the assumption on differentiability, we can replace radial points by limit points in Theorem D. For instance, we could require in Theorem D that f is continuously differentiable with a non-vanishing Jacobian at a limit point of G (and that $L(G)$ contains at least two points).

The proof of this fact is based on the fact that if $L(G)$ contains at least two points, then every neighbourhood of a limit point of G contains radial points. Hence the continuous differentiability implies that there is a radial point at which f is differentiable with a non-vanishing Jacobian. So the continuity is used only to find such points and we have the more general

Theorem E. *Let G and A be as in Theorem D and suppose that $L(G)$ contains at least two points. Let $f\colon A \to \bar{R}^n$ be a G-compatible map such that there is $x \in L(G)$ which has a neighbourhood U such that f is differentiable with a non-vanishing Jacobian at every $y \in U$. Then the conclusions of Theorem D hold.*

In particular, if $f\colon \bar{R}^n \to \bar{R}^n$ is a G-compatible map and if no $z \in \bar{R}^n$ is fixed by every $g \in G$, then f is a Möbius transformation as soon as it is differentiable at a neighbourhood of a single limit point x of G and the derivative is continuous and has a non-vanishing Jacobian at x.

Proof. As observed above, apart from a minor remark (see below) concerning the case when f has a conformal derivative at x, it suffices to show that every neighbourhood of x contains radial points of G. We prove this by showing that every neighbourhood of x contains fixed points of loxodromic $g \in G$ which are radial points.

If $L(G)$ contains at least three points, this follows from [9, 13.20 and 13.14]. If $L(G)$ contains exactly two points, then no $g \in G$ can be parabolic. If $g \in G$ is non-loxodromic, then either g fixes pointwise the hyperbolic line L joining the points of $L(G)$ or then g interchanges the points of $L(G)$. If no $g \in G$ is loxodromic, it would follow that $L(G) = \emptyset$. Hence there is loxodromic $g \in G$ whose fixed point set is $L(G)$.

The proof is complete if we remark that if f has a conformal derivative at x, then, by continuity, for every $\varepsilon > 0$, there is a radial point of G at which f is differentiable with $(1+\varepsilon)$-quasiconformal derivative and hence the map β in (D2) can be chosen to be $(1+\varepsilon)$-quasiconformal (see Remark D8). A limit process now shows that the map β in (D2) can in fact be chosen to be 1-quasiconformal, that is a similarity.

Note that in the second paragraph of the theorem we need not assume that $L(G)$ consists of at least two points since if $L(G)$ consists of one point, then this point is fixed by every $g \in G$.

F. Matrix dilatation and rigidity

We now prove a theorem similar to Theorems D and E by considering the matrix dilatation of a quasiconformal map. If $f\colon U \to V$ (U, V domains of R^n) is differentiable at $x \in U$, the *matrix dilatation* of f at x is

$$\mu_f(x) = |\det f'(x)|^{-2/n} f'(x)^T f'(x) \tag{F0}$$

when $f'(x)$ is the differential at x and $f'(x)^T$ its transpose. Thus μ_f is defined a.e. in U.

If $n=2$, one sees that μ_f gives the directions of the principal axes of the dilatation ellipsoid of $f'(x)$ as well as their ratios. Thus the matrix dilatation and the complex dilatation of a quasiconformal f determine uniquely each other. We conclude that the matrix dilatation is the generalization of the complex dilatation for $n \geqq 2$.

The composition rule for the matrix dilatation is

$$\mu_{fg}(x) = |\det g'(x)|^{-2/n} g'(x)^T \mu_f(g(x))\, g'(x) \tag{F1}$$

which is valid a.e. in U. If $f \in \text{Möb}(n)$, $\mu_f(g(x))$ is the unit matrix and so one sees that composition on left with Möbius transformations does not change the matrix dilatation.

This latter observation enables us to extend $\mu_f(x)$ to the case that $x \neq \infty$ but $f(x) = \infty$ by means of auxiliary Möbius transformations. On the other hand, if

$x=\infty$, then $\mu_f(x)$ cannot be defined by this method but the properties of μ_f like continuity and approximate continuity (cf. [7, p. 159] or [18, (A2)]) are well-defined also for $x=\infty$.

In considerations involving the complex or matrix dilatation, it is usually best to identify two such dilatations if their domain of definition and values are the same up to a null-set. Hence we regard μ_f continuous or approximately continuous at a point if it can be made to have this property after a redefinition in a null-set. Thus even if μ_f is continuous at a point, f need not be differentiable at it (Näätänen [12, 5.4]) and so case (a) of the next theorem is not a special case of Theorem D.

We will now show that if f is a G-compatible and quasiconformal map of $\bar{R}^n$ and if the matrix dilatation of f is sufficiently continuous at a limit point or at a radial point of G, then f is, if not a Möbius transformations, at least affine modulo Möbius transformations.

Theorem F. *Let G be a Möbius group of $\bar{R}^n$, $n\geqq 2$, and let f be a G-compatible quasiconformal map of $\bar{R}^n$. Suppose that the matrix dilatation μ_f of f is either*

(a) *approximately continuous at a radial point of G, or*

(b) *continuous at a limit point of G.*

Then f is a Möbius transformation provided that no $z\in\bar{R}^n$ is fixed by every $g\in G$. If there is such a point z, then there are Möbius transformations h and $\bar{h}$ such that $h(\infty)$ is fixed by every $g\in G$ and that $\bar{h}fh$ is affine in R^n.

Proof. We prove case (a). Case (b) is similar.

Suppose that 0 is a radial point of G at which μ_f is approximately continuous. Pick $z\in H^{n+1}$ and $g_i\in G$ such that $g_i(z)$ approach radially 0. Choose then $h_i\in\text{Möb}(n)$ such that h_i fixes z and g_ih_i fixes ∞. Thus g_ih_i is a similarity of R^n. Furthermore, one sees easily that the maps h_i can be chosen in such a way that

$$g_ih_i(x)=\lambda_i x+b_i \tag{F2}$$

for some $\lambda_i>0$ and $b_i\in R^n$. Since the set of Möbius transformations fixing z is compact, one can assume by passing to a subsequence that there is $h\in\text{Möb}(n)$ such that

$$h_i\to h \tag{F3}$$

as $i\to\infty$.

Suppose that f induces φ. Then

$$fh_i=\varphi(g_i)^{-1}fg_ih_i.$$

Here $\varphi(g_i)^{-1}\in\text{Möb}(n)$ and hence composing with it does not change the matrix dilatation of fg_ih_i. In view of (F2), the composition rule (F1) now gives

$$\mu_{fh_i}(x)=\mu_f(g_ih_i(x)). \tag{F4}$$

Let μ_f have the approximate limit μ_0 at 0. Since $g_i(z)=g_ih_i(z)$ approach radially 0, (F4) implies that, as $i\to\infty$,

$$\mu_{fh_i} \to \mu_0 \tag{F5}$$

in measure (with respect to the spherical measure of $\bar{R}^n$).

Since $h_i \to h$, also $fh_i \to fh$. In view of (F5), one can now show that

$$\mu_{fh} = \mu_0 = \text{a constant.} \tag{F6}$$

This is a consequence of the so called good approximation theorem for quasiconformal mappings (see [18, Corollary D] if $n \geqq 2$ or [10, IV.5.6] if $n=2$). This theorem is fairly deep and in the present case one gets (F6) by the following much simpler argument.

If A and B are non-singular $n \times n$-matrices, let $A[B] = |\det A|^{-2/n} A^T BA$. Then the composition rule (F1) gives

$$\mu_{fh_i}(x) = h_i'(x)[\mu_f(h_i(x))].$$

which is valid for a.e. x. That is, since $h_i'(x)$ is orthogonal,

$$(h_i'(h_i^{-1}(x)))^{-1}[\mu_{fh_i}(h_i^{-1}(x))] = \mu_f(x).$$

which is also valid a.e. in R^n. Since $h_i \in \text{Möb}(n)$ and $h_i \to h$, also the derivatives converge: $(h_i' \circ h_i^{-1})^{-1} \to (h' \circ h^{-1})^{-1}$ as $i \to \infty$. Hence it follows by (F5) that

$$\mu_f(x) = (h'(h^{-1}(x)))^{-1}[\mu_0]$$

a.e. in R^n. That is, $h'(x)[\mu_f(h(x))] = \mu_{fh}(x) = \mu_0$ a.e. and (F6) is proved.

We must now only to choose $\bar{h} \in \text{Möb}(n)$ such that $\bar{h}fh(\infty) = \infty$. Then $\bar{h}fh$ also has the matrix dilatation μ_0 and hence is affine. A reference to Lemma C2 completes the proof.

Remarks. F1. Sullivan [14] has proved the related theorem which says that if the action of a discrete G is conservative in $\bar{R}^n$ and if f is a G-compatible and quasiconformal map of $\bar{R}^n$, then f is a Möbius transformation.

F2. A proof of case (a) of Theorem F could be based also on Theorem D and Remark D6.

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