

## Everywhere non reduced moduli spaces

F. Catanese

Dipartimento di Matematica, Università di Pisa Via Buonarroti, 2, 1-56100 PISA, Italia

*Dedicated to the memory of Aldo Andreotti*

### Introduction

The purpose of this article is to show how often the moduli spaces of surfaces of general type can be everywhere non reduced in the case when the canonical bundle  $K_S$  is not ample. On the other hand, by giving a simple criterion which implies that this must happen, we are in fact able to subsume almost all the previously known examples of obstructed deformations in dimension 2 as particular issues of a very general situation; we produce also infinite series of examples, showing in particular that all non Cartier divisors of rigid 3-dimensional weighted projective spaces give rise to this pathology.

To be more precise, let  $V$  be an algebraic variety with a finite group  $\text{Aut}(V)$  of automorphisms (e.g., cf. [Ma], if  $V$  is of general type); then, if  $V$  admits a space of moduli  $\mathfrak{M}(V)$  (cf. [Mu 2]), locally (i.e. in an analytic neighbourhood of the point of  $\mathfrak{M}(V)$  corresponding to  $V$ )  $\mathfrak{M}(V)$  is the quotient of the base  $\text{Def}(V)$  of the Kuranishi family of deformations of  $V$  by the group  $\text{Aut}(V)$  (cf. [Wa]). It is clear that in this case  $\mathfrak{M}(V)$  is (locally) everywhere non reduced, e.n.r. for short (i.e., everywhere singular) if and only if  $\text{Def}(V)$  is e.n.r..

We recall the classical terminology:  $V$  is said to have obstructed deformations if the germ  $\text{Def}(V)$  is singular. The stronger condition that  $\text{Def}(V)$  be e.n.r., i.e. everywhere singular, can thus be referred to as  $V$  having “everywhere obstructed deformations”, and has been regarded up to now as a very pathological phenomenon.

The first example of algebraic varieties  $V$  with  $\text{Def}(V)$  e.n.r. is due to Kodaira and Mumford ([Ko], [Mu 1]): here, though,  $V$  and its deformations are blow-ups of  $\mathbb{P}^3$ , hence there are no birational moduli.

After several examples of obstructed deformations were exhibited, e.g. by Kas ([Ka 1], [Ka 2]), by Burns and Wahl ([B–W]), by Horikawa ([Hor 1]), then Horikawa [Hor 2] and, later, Miranda ([Mi]) gave examples of surfaces of general type  $S$  (respectively with  $p_g = 4, 7$ ,  $K^2 = 6, 14$ ) for which  $\text{Def}(S)$  is e.n.r..

Their approach was through the classification of all the surfaces with those invariants (Miranda uses Castelnuovo’s classification [Cas] of surfaces with

$K^2 = 3p_g - 7$  and  $|K_S|$  birational). In both cases the outcome is that the canonical bundle  $K_S$  is not ample for all the surfaces corresponding to the points of a component of the moduli space. This research started when I tried to find a direct proof that  $\text{Def}(S)$  was e.n.r., and I noticed that for both examples the singular canonical models were hypersurfaces in a 3 dimensional projective space (respectively  $X_9 \subset \mathbb{P}(1, 1, 2, 3)$ ,  $X_7 \subset \mathbb{P}(1, 1, 1, 2)$ ) admitting a double cover ramified on the singular point and a (disjoint) canonical divisor.

It became clear that the essential point had nothing to do with the fact that these two surfaces lie on extremal lines of surface geography, it was instead the common feature of  $X$  being the quotient of a smooth surface  $Z$  by  $\mathbb{Z}/2$  in such a way that all deformations of  $X$  remain singular.

In fact, already Burns and Wahl ([B-W]) had introduced the philosophy that one may have obstructed deformations if  $K_S$  is not ample, i.e. the canonical model  $X$  is singular. I would like parenthetically to point out that in their examples of obstructed deformations  $X$  is a surface with many nodes so that, by Beauville's beautiful remark in [Be],  $X$  has many double covers ramified only at the nodes, and thus one falls again in our general picture, which is as follows.

**Main Theorem.** *Let  $Z$  be a smooth surface with  $\text{Def}(Z)$  smooth, and let  $G$  be a finite group acting on  $Z$  in such a way that  $X = Z/G$  has only R.D.P's (Rational Double Points) as singularities and is indeed singular. Let  $p: Z \rightarrow X$  be the quotient map, and assume that the following inclusion of sheaves*

$$\Omega_X^1 \otimes \omega_X \hookrightarrow (p_* \Omega_Z^1)^G \otimes \omega_X \tag{0.1}$$

*induces an injective map between the first cohomology groups. Assume also that  $H^1(\theta_Z)^G$  surjects onto  $H^1(\theta_X)$ : then, if  $S$  is the minimal resolution of singularities of  $X$ , then*

$$\text{Def}(S) \text{ is a product } \text{Def}(Z)^G \times R, \text{ where} \tag{0.2}$$

*$\text{Def}(Z)^G$  is a smooth scheme and  $R$  is a non reduced, connected, 0-dimensional scheme whose length  $n$  can in fact be arbitrarily large.  $\square$*

It is important to observe that the two cohomological conditions are quite reasonable, since the first is verified if  $\omega_X$  is sufficiently ample, and the second one is automatically verified if  $p$  is unramified in codimension 1.

Roughly speaking, the first condition guarantees that all the deformations of  $X$  are equisingular, the second one that all the equisingular deformations of  $X$  come from deformations of  $Z$  which preserve the action of  $G$ .

This is the simple idea, and the technique uses, beyond other tools, a useful criterion by Pinkham ([Pi]), which clarifies and extends previous results of Kas [Ka 2] and Burns-Wahl [B-W].

As we already mentioned, our criterion implies that  $K_S$  is not ample for all the surfaces corresponding to the points of a component of the moduli space, and so we regard our criterion as a sort of explanation of this pathology.

In fact it makes us still confident about the validity of the following conjecture [Cat]: if  $S$  is a surface of general type with  $q = 0$  and  $K_S$  ample, then the moduli space  $\mathfrak{M}(S)$  is smooth on an open dense set.

As to the further contents of the paper, §1 contains the proof and a more general version of the basic criterion above. §2 is devoted to examples where  $p: Z \rightarrow X$  is unramified in codimension 1, and we show in particular that if  $G$  acts on a 3-fold  $W$ , then for all the hypersurfaces  $Z$  of large degree we obtain e.n.r. moduli spaces. We also use a slight generalization of Kas' surfaces [Ka 2], to show that the ratio between tangent dimension and dimension of  $\text{Def}(S)$  can be arbitrarily large. Finally, most of our effort is spent in §3 to analyze the somehow "most simple" example, the one of hypersurfaces  $X$  in a weighted projective space  $\mathbb{P}$ : for the sake of brevity, though, we omit most of the proofs which are of computational character.

We have in fact that the examples of Horikawa and Miranda belong to the huge class of non-Cartier hypersurfaces  $X$  in a weighted projective space  $\mathbb{P} = \mathbb{P}(1, 1, p, q)$  with  $p, q$  relatively prime.

We first classify all the  $X$ 's as above with R.D.P.'s, then we apply our criterion.

The striking result is that in this case the deformation space of  $S$  is always e.n.r. The underlying philosophy is the following: if  $X \subset \mathbb{P}^3$ , then all small deformations of  $X$  are still surfaces in  $\mathbb{P}^3$  (cf. [K-S]); whereas (cf. §2,  $X_8 \subset \mathbb{P}(1, 1, 2, 2)$ ) this is no longer true in general for all weighted  $\mathbb{P}$ 's, it is true if  $\mathbb{P}$  has isolated singularities and is thus rigid, and then one gets an e.n.r. deformation space because of the infinitesimal deformations coming from the singularities of  $\mathbb{P}$ .

I would like to thank R. Miranda for arousing my interest about the problem of e.n.r. moduli spaces and for sending me his preprint [Mi].

The paper grew out of a talk I gave at the problems seminar of Columbia University, during the special year on Algebraic Geometry; there H. Pinkham pointed out that the infinitesimal computation of the obstruction map that I needed was already available in his earlier paper ([Pi]), and I thank him heartily for this precious help.

Thanks for the warm hospitality go to Columbia University and to the Max-Planck-Institut, where I could finish the fastidious computations of §3 and write the final version. Also, I am grateful to a referee for pointing out some minor errors.

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**Notation**

$S$  a complete smooth minimal surface of general type over  $\mathbb{C}$

For any variety  $X$ ,

$\Omega_X^1$  is the sheaf of Kähler differentials,

$\theta_X = \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$  is the sheaf of vector fields, and, if  $B$  is any subscheme of  $X$  with ideal sheaf  $I_B$ ,  $\theta_X(\log B)$  is the subsheaf of vector fields carrying  $d(I_B)$  into  $I_B$ .

If  $B$  is reduced,  $N'_{B|X}$ , the equisingular normal sheaf of  $B$  in  $X$ , is the quotient  $\theta_X/\theta_X(-\log B)$  (cf. [Cat], 9.16, 9.17).

$\omega_X$  is the dualizing sheaf (here an invertible sheaf, since  $X$  shall be Gorenstein)

$\text{Def}(X)$ , if  $X$  is complete, is the basis of the semi-universal family of deformations of  $X$  (cf. [Gr]).

For a vector space  $V$ ,  $V^*$  denotes its dual space.

If  $G$  is a finite group acting on a vector space  $V$ ,  $V^G$  denotes the subspace of invariant vectors.

If  $G$  is a finite group acting on a variety  $Z$ ,  $G_z$  denotes the stabilizer of a point  $z$ .

If  $p: Z \rightarrow X = Z/G$  is the quotient map, and  $F$  is a  $G$ -linearized sheaf,  $p_*(F)^G$  denotes the  $G$ -invariant part of the direct image.

A 1-point nilpotent scheme  $R$  shall be the spectrum of a local Artin  $\mathbb{C}$ -algebra of length  $> 1$ .

**§1. The basic criterion**

Let a finite group  $G$  act on a smooth (complete) variety  $Z$ .

If  $z \in Z$  has non trivial stabilizer  $G_z$ , then

$$z \in W = \bigcup_{\substack{g \in G \\ g \neq \text{id}}} \text{Fix}(g), \text{ where } \text{Fix}(g) = \{z' | g(z') = z'\} .$$

Moreover, let  $W' \subset W$  be the locus in  $W$ -Sing  $W$  which has codimension 1: then for  $z \in W'$   $G_z$  is cyclic and generated by a pseudo-reflection  $g$ , where we recall (cf. [Ch]) that

**Definition 1.1.**  $g$  is a pseudo-reflection if  $g$  is an automorphism of  $Z$  of finite order  $n$  s.t. for each point  $z$  of  $Z$  either a)  $g^i(z) \neq z$  for  $i < n$ , or b) there exist local coordinates  $(t_1, \dots, t_m)$  around  $z$  such that

$$g(t_1, \dots, t_m) = (t_1, \dots, t_{m-1}, \varepsilon t_m) \tag{1.2}$$

(where  $\varepsilon$  is a primitive  $n^{\text{th}}$  root of unity)

**Lemma 1.3.** If  $g$  is a pseudo-reflection, let  $X' = Z/g$ , let  $p': Z \rightarrow X'$  be the quotient map and let  $B'$  be the (smooth) branch locus of  $p'$ .

Then there are natural isomorphisms

$$\Omega_{X'}^1 \xrightarrow{j^v} p'_*(\Omega_Z^1)^g \quad \text{and} \quad (p_*\theta_Z)^g \xrightarrow{j} \theta_{X'}(-\log B')$$

*Proof.* Both  $j^v, j$  are naturally defined and are clearly isomorphisms on the open sets where  $p'$  is unramified.

If  $p'(z) = x, g(z) = z$ , we take local coordinates as in (1.2) and then the proof follows from a direct computation.  $\square$

We set now  $Z'' = Z - (W - W') = \{z | G_z \text{ is generated by a pseudo-reflection}\}$ , so that  $\text{codim}(Z - Z'') \geq 2, X = Z/G$ , let  $p: Z \rightarrow X$  be the quotient map, and finally set  $X'' = p(Z'')$ .

**Lemma 1.4.** Let  $B$  be the (reduced) branch locus of  $p: Z \rightarrow X = Z/G, X^0 = X - \text{Sing } X, i: X^0 \hookrightarrow X$  the inclusion map. Then the natural homomorphism  $(p_*\theta_Z)^G \xrightarrow{j} \theta_X$

$(-\log B)$  is an isomorphism, whereas  $\Omega_X^1 \xrightarrow{j^v} p_*(\Omega_Z^1)^G$  is injective, an isomorphism on  $X^0$ , and  $p_*(\Omega_Z^1)^G$  equals  $i_*(\Omega_{X^0}^1)$ .

*Proof.* By (1.3) the assertion is true on the open set  $X'' \subset X^0$ . Since  $\text{codim}(Z - Z'') \geq 2$  and  $\theta_Z, \Omega_Z^1$  are locally free, if  $i'': X'' \rightarrow X$  is the inclusion,  $p_*(\theta_Z)^G = (i'')_*(\theta_{X''}(-\log B))$  and  $p_*(\Omega_Z^1)^G = (i'')_*(\Omega_{X''}^1) = i'_*(\Omega_{X^0}^1)$ .

We have now to show that  $\theta_X(-\log B) \rightarrow (i'')_*(\theta_{X''}(-\log B))$  is an isomorphism. This follows since  $\theta_X(-\log B)$  is torsion free (being a subsheaf of  $\theta_X$ ), hence we have an injective homomorphism of which  $j$  gives an inverse.

Finally, the inclusion  $\mathcal{O}_X \subset p_*\mathcal{O}_Z$  shows that  $j^v$  is injective Q.E.D.

We let now  $\pi: S \rightarrow X$  be a minimal resolution of singularities of  $X$ , and  $Y$  be the normalization of the fibre product  $Z_{X^0}S$ .

We then have a diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & S \\ \downarrow \varepsilon & & \downarrow \pi \\ Z & \xrightarrow{p} & X \end{array} \tag{1.5}$$

and we set  $Y^0 = Y - \text{Sing}(Y)$ ,  $S^0 = S - \varphi(\text{Sing } Y)$ ,  $Z^0 = Z - \varepsilon(\text{Sing}(Y))$ .

**Proposition 1.6.** *For any invertible sheaf  $\mathcal{L}$  on  $X$ , we have*

$$H^0(\Omega_S^1 \otimes \pi^*\mathcal{L}) \cong H^0(\Omega_Z^1 \otimes p^*\mathcal{L})^G.$$

*Proof.* By abuse of notation we shall identify  $\mathcal{L}$  with its pull-backs, which we shall consider endowed of their natural linearization (notice that  $G$  acts on  $Y$ , with quotient  $S$ ).

Since  $S - S^0$  has  $\text{codim} \geq 2$ ,  $H^0(\Omega_S^1 \otimes \mathcal{L}) = H^0(\Omega_{S^0}^1 \otimes \mathcal{L})$ ; by (1.4)  $(\varphi_*\Omega_{Y^0}^1)^G = \Omega_{S^0}^1$ , hence  $H^0(\Omega_{S^0}^1 \otimes \mathcal{L}) = H^0(\Omega_{Y^0}^1 \otimes \mathcal{L})^G$  and this last clearly equals  $H^0(\Omega_Z^1 \otimes \mathcal{L})^G$  since  $\varepsilon|_{\varepsilon^{-1}(Z^0)}$  is a modification and  $Z$  is smooth. Q.E.D

**Corollary 1.7.** *Let  $Z, G, \dots$  be as above and assume  $X$  Gorenstein,  $\dim Z = 2$ . Let  $T^*$  be the cokernel of the sequence*

$$0 \rightarrow \Omega_X^1 \otimes \omega_X \rightarrow (p_*\Omega_Z^1)^G \otimes \omega_X \xrightarrow{r} T^* \rightarrow 0.$$

*Then  $H^0(T^*) \cong H^0(\mathcal{E}_{X^0} \otimes^L (\Omega_X^1, \mathcal{O}_X))^*$  and the image of  $H^0(r)$  is isomorphic to the cokernel of*

$$H^0(\Omega_X^1 \otimes \omega_X) \rightarrow H^0(\Omega_S^1 \otimes \Omega_S^2).$$

*Proof.* First of all  $\Omega_S^2 \cong \omega_S \cong \pi^*(\omega_X)$  since the singularities of  $X$  are Gorenstein quotient singularities, hence R.D.P.'s; so that the second statement follows directly from (1.6) letting  $\mathcal{L}$  be  $\omega_X$ .

By (1.4) the above sequence is obtained by tensoring with  $\omega_X$  the exact sequence of local cohomology

$$0 \rightarrow \Omega_X^1 \rightarrow i_*(\Omega_{X^0}^1) \rightarrow H_{\text{Sing } X}^1(\Omega_X^1) \rightarrow 0$$

and, as in Pinkham's article ([Pi] page 174, (4) of theorem 1), we notice that by local duality the last term is isomorphic to

$$\mathcal{E}xt^1(\Omega_X^1, \mathcal{O}_X)^*.$$

*Q.E.D.*

*Remark.* Notice that sometimes  $Y$  is necessarily singular, no matter of which resolution  $S$  of  $X$  one takes.

I.e., there is no blow up of  $Z$  on which  $G$  acts as a group generated by pseudo-reflections. For instance, take the action of  $\mathbb{Z}/4$  such that  $(u, v) \rightarrow (iu, -iv)$ : on the first blow-up there is a point where the action has eigenvalues  $(i, -1)$ , hence this point has to be blown up, and over it lies a point where the eigenvalues are  $(-i, -1)$ , . . . hence the procedure can never end.

From now on we shall assume  $\dim Z = 2$ ,  $X$  Gorenstein (i.e., with R.D.P.'s only), and let  $S$  be a minimal resolution of singularities of  $X$ .

Also we shall use freely the results of [B-W] and [Pi], in particular we shall use the theorem of Burns and Wahl, asserting that there is a fibre product

$$\begin{array}{ccc} \text{Def}(S) & \longrightarrow & L_S \\ \downarrow & & \downarrow \beta \\ \text{Def}(X) & \xrightarrow{\psi} & L_X \end{array} \tag{1.8}$$

where  $L_X, L_S$  are the bases of the versal family of (local) deformations of the respective germs  $(X, \text{Sing } X), (S, \pi^{-1}(\text{Sing } X))$ .

We have that  $L_X \cong H^0(\mathcal{E}xt^1(\Omega_X^1, \mathcal{O}_X)) \stackrel{\text{def}}{=} T_X, L_S$  is also smooth and  $\beta$  is a finite cover with zero differential at the origin ( $\beta$  is a direct product, over the singular points of  $X$ , of the quotient maps

$$H_{E_x}^1(\theta_S) \rightarrow \mathcal{E}xt^1(\Omega_{X,x}^1, \mathcal{O}_{X,x}) \cong H_{E_x}^1(\theta_S)/W_x, \tag{1.9}$$

where  $E_x = \pi^{-1}(x)$ , and  $W_x$  is the Weyl group of the singularity acting on the vector space  $H_{E_x}^1(\theta_S)$ .

Let us now look at the maps of tangent spaces at the base points induced by diagram (1.8): it fits into a larger diagram (where  $E = \bigcup_{x \text{ singular}} E_x$ )

$$\begin{array}{ccccccc} & & H^1(\theta_S) & \longleftrightarrow & H_E^1(\theta_S) & \longleftrightarrow & 0 \\ & \nearrow & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H^1(\theta_X) & \rightarrow & \text{Ext}^1(\Omega_X^1, \mathcal{O}_X) & \rightarrow & H^0(\mathcal{E}xt^1(\Omega_X^1, \mathcal{O}_X)) \xrightarrow{\text{ob}} H^2(\theta_X) \\ & & & & \parallel & & \parallel \\ & & & & T_X & & \end{array} \tag{1.10}$$

Here the inclusion of  $H_E^1(\theta_S)$  is induced by the sequence of local cohomology ([B-W] 1.3), on the other hand Burns and Wahl prove that  $\pi_*\theta_S = \theta_X$ , so that the



- a)  $\text{Def}(Z)$  is smooth
- b)  $H^1(\theta_Z)^G$  surjects onto  $H^1(\theta_X)$
- c)  $H^0(r)$  is surjective.

*Proof.* i) is a restatement of 1.14, in view of remark 1.15. For the same reason ii) is proved if we show that  $ES\text{Def}(X)$  is smooth provided a), b) hold.

Let  $\text{Def}(Z)^G$  be the subspace of  $\text{Def}(Z)$  consisting of the deformations preserving the action of  $G$ : it is well known that, under the natural inclusion  $\text{Def}(Z) \subset H^1(\theta_Z)$ ,  $\text{Def}(Z)^G = \text{Def}(Z) \cap H^1(\theta_Z)^G$  (cf. e.g. [Cat], Lecture 3). Hence, if  $\text{Def}(Z)$  is smooth, then  $\text{Def}(Z)^G$  is smooth with tangent space equal to  $H^1(\theta_Z)^G$ . We have a natural morphism  $\text{Def}(Z)^G \rightarrow ES\text{Def}(X)$ : by b) it induces a surjective map on tangent spaces, hence  $ES\text{Def}(X)$  is also smooth, by Dini's theorem on implicit functions. Q.E.D.

Recall that, by lemma 1.4, we have an exact sequence

$$0 \rightarrow p_*(\theta_Z)^G \rightarrow \theta_X \rightarrow N'_{B|X} \rightarrow 0. \tag{1.18}$$

The map  $H^1(\theta_Z)^G \rightarrow H^1(\theta_X)$  fits thus into the cohomology exact sequence associated to (1.18), hence hypothesis b) is equivalent to the injectivity of  $H^1(N'_{B|X}) \rightarrow H^2(\theta_Z)^G$ .

We record for later use the following

**Lemma 1.19.** *Hypothesis b), that  $H^1(\theta_Z)^G$  maps onto  $H^1(\theta_X)$ , is implied by the vanishing of  $H^1(N'_{B|X})$  and is equivalent to the following numerical relation*

$$h^1(N'_{B|X}) = h^0(\Omega_Z^1 \otimes \Omega_Z^2)^G - h^0(\Omega_Z^1 \otimes p^*\omega_X)^G.$$

*Proof.* The first assertion is trivial, the second follows from Serre duality. In fact  $h^2(\theta_Z)^G = h^0(\Omega_Z^1 \otimes \Omega_Z^2)^G$ , whereas  $h^2(\theta_X) = h^2(\theta_S)$  (cf. 1.10), which equals  $h^0(\Omega_S^1 \otimes \Omega_S^2) = h^0(\Omega_Z^1 \otimes \omega_X)^G$  by 1.6. Q.E.D.

**Corollary 1.20.** *Let  $Z$  be a smooth algebraic surface with  $\text{Def}(Z)$  smooth, let  $G$  be a finite group acting on  $Z$  in such a way that the quotient map  $p: Z \rightarrow X = Z/G$  is unramified in codimension 1, and  $X$  has only R.D.P.'s as singularities (and is indeed singular!).*

*Then, letting as usual  $S$  be a minimal resolution of singularities of  $X$ , observe that  $p^*\omega_X = \Omega_S^2$ ; if  $H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G$  surjects onto  $T_X^*$ , we have  $\text{Def}(S) \cong \text{Def}(Z)^G \times R$ , where  $R$  is a 1-point nilpotent scheme of length =  $\dim T_X^*$ .*

*Proof.*  $N'_{B|X} = 0$ . Q.E.D.

We can in fact restate theorem 1.16 with the weaker assumptions we have in fact used

**Theorem 1.21.** *Notation being as in 1.16,  $\text{Def}(S) = \text{Def}(Z)^G \times R'$ , with  $R'$  a 1-point nilpotent scheme of length  $\geq \text{length } T_x^*$ , provided:*

- a)  $\text{Def}(Z)^G$  is smooth
- b)  $H^1(\theta_Z)^G \rightarrow H^1(\theta_x)$  is surjective
- c) there is a singular point  $x$  s.t.  $H^0(r)$  surjects onto  $T_x^*$ .



**§2. Examples of e.n.r. moduli spaces**

In this paragraph we shall consider basically two types of examples for which cor. 1.20 applies.

As mentioned in the introduction, the first example concerns surfaces in 3-folds, while the second is a slight generalization of Kas' surfaces [Ka 2], and deals with quotients of products of curves.

In the first example we shall consider the following situation

(2.1)  $W$  is a smooth 3-fold, and  $G$  is a finite group acting on  $W$  with only a finite number of points  $w_1, \dots, w_s$  having a non trivial stabilizer. There is also a very ample divisor  $H$  such that  $\mathcal{O}_W(H)$  has a  $G$ -linearization (e.g. if  $H$  is a  $G$ -invariant effective divisor).

Assume further

(2.2) There exists an integer  $r > 0$  and  $Z \in |rH|$  s.t.

- i)  $Z$  is  $G$ -invariant and smooth
- ii)  $Z$  contains some  $w_i$
- iii) if  $Z \ni w_i$ , then  $G_{w_i}$  acts on the tangent plane  $T_{Z, w_i}$  with determinant = 1.

**Theorem 2.3.** *If (2.1), (2.2) hold and  $S$  is a minimal resolution of  $X = Z/G$ , then for  $r \gg 0$   $S$  has everywhere obstructed deformations.*

*Proof.* Since, by (2.1),  $p: Z \rightarrow X$  is unramified in codimension 1, cor. 1.20 applies: in fact, by (2.2)  $X$  has only R.D.P.'s and is singular. We have to verify that  $H^0(\Omega_Z^1 \otimes \omega_Z)^G$  maps onto  $T_X^* = H^0(T^*) = H^0((p_* \Omega_Z^1)^G / \Omega_X^1)$  (cf. 1.7). Notice that the restriction map  $\Omega_W^1 \otimes \omega_W(rH) \rightarrow \Omega_Z^1 \otimes \omega_W(rH)$  is a homomorphism of  $G$ -modules, while  $\omega_Z$  and  $\omega_Z(rH)$  differ just by a character of  $G$ .

Now  $T^*$  is a quotient of  $(p_* \Omega_W^1)^G / \Omega_{W/G}^1$ , which has finite length, hence there exists an integer  $k > 0$  s.t.  $T^*$  is a quotient of  $\bar{T} = \bigoplus_{i=1}^s \Omega_{W, w_i}^1 / \mathfrak{M}_{W, w_i}^k \cdot \Omega_{W, w_i}^1$ , where  $\mathfrak{M}_{W, w_i}$  is the maximal ideal of the point  $w_i$ .

It suffices now to choose  $r \gg 0$  s.t.  $H^0(\Omega_W^1 \otimes \omega_W(rH))$  maps onto  $\bar{T}$ . *Q.E.D.*

We show now at least that (2.1) (2.2) occur easily.

*Example 2.4.* We let  $W = A$ , where  $A$  is an abelian 3-fold and let  $H$  be any symmetric polarization.

$G = \{\pm 1\}$  acts in the standard way, so  $w_1, \dots, w_{64}$  are precisely the 64 torsion points.

We choose an even integer  $r \geq 4$ , and since  $H$  is symmetric, the invertible sheaf  $\mathcal{O}_A(rH)$  has a  $G$ -linearization, and we choose  $Z$  to be a generic surface inside  $|rH|^-$ .

By proposition 2, page 307 of [Mu 3], if  $s$  is a section of  $H^0(\mathcal{O}_A(rH))^-$ , a local expression for  $s$ , around each of the points  $w_i$ , is given, up to a unit, by an odd function. Hence  $Z$  passes through  $w_1, \dots, w_{64}$ . We claim that a general such  $Z$  is smooth.

This follows from the classical theorem of Lefschetz, asserting that the sections of  $H^0(\mathcal{O}_A(rH))$  give an embedding of  $A$ .

In fact, first of all the linear system  $|rH|^-$  has only the  $w_i$ 's as base points, by virtue of the following remark:  $H^0(\mathcal{O}_A(rH))^+$  gives a morphism which (see loc. cit.) factors through the Kummer 3-fold  $K = A/G$ , hence if  $x$  is a base point for  $H^0(\mathcal{O}_A(rH))^-$ , then  $x$  and  $-x$  have the same image under the above embedding, so that  $x = -x$ .

By Bertini's theorem, therefore, it suffices to verify that a general  $Z$  is smooth at  $w_i$ . But this follows again from Lefschetz's theorem, since if some sections give a local embedding, there are sections of multiplicity 1 at  $w_i$ , and thus also odd sections of multiplicity 1.

2.2. iii) holds since  $(-1)^2 = 1$ , so that  $X$  has 64 ordinary quadratic singularities.

It is important to remark that  $S$  is a regular surface. In fact if  $X^0$  is the smooth part of  $X$ , we have

$$H^0(\Omega_S^1) \rightarrow H^0(\Omega_{X^0}^1) \rightarrow H^0(\Omega_Z^1)^+ .$$

But, by standard exact sequences, since  $H^1(\mathcal{O}_A(-rH)) = 0$ , we have  $H^0(\Omega_Z^1) = H^0(\Omega_A^1)$ , so there are no invariant forms.

The next examples we consider are a slight generalization of an example due to Kas, [Ka 2], and we shall call the corresponding surfaces "generalized Kas surfaces".

Consider, for  $i = 1, 2$ ,  $f_i: C_i \rightarrow \Gamma_i$  a simple cyclic cover of degree  $n$  between complete smooth curves, i.e.

(2.5) There is an invertible sheaf  $\mathcal{L}_i$  on  $\Gamma_i$  and a divisor  $B_i$  consisting of distinct points s.t.  $\mathcal{O}_{\Gamma_i}(B_i) \cong \mathcal{L}_i^n$ ;  $C_i$  is the subvariety of  $L_i$ , the line bundle whose sheaf of sections is  $\mathcal{L}_i$ , obtained by taking the  $n^{\text{th}}$ -root of the section defining  $B_i$ .

Clearly the group  $\mu_n \cong \mathbb{Z}/n$  of  $n^{\text{th}}$  roots of unity acts on  $C_i$ , and  $f_i: C_i \rightarrow \Gamma_i$  is the quotient map.

Notice further that

$$f_{i*} \mathcal{O}_{C_i} = \mathcal{O}_{\Gamma_i} \oplus \mathcal{L}_i^{-1} \oplus \dots \oplus \mathcal{L}_i^{-(n-1)} . \tag{2.6}$$

To adhere to our standard notation, we let  $Z = C_1 \times C_2$  and we let  $\mu_n$  act on  $Z$  by the (twisted) action

$$\zeta(x, y) = (\zeta x, \zeta^{-1} y) . \tag{2.7}$$

It follows immediately that if  $X = Z/\mu_n$ , then the singularities of  $X$  are exactly R.D.P.'s of type  $A_{n-1}$ , and  $p: Z \rightarrow X$  is unramified in codimension 1.

As a preliminary computation, we notice that  $(\Omega = \Omega^1, \text{ for short})$

$$\Omega_{C_i} = f_i^*(\Omega_{\Gamma_i} \otimes \mathcal{L}_i^{n-1}) , \tag{2.8}$$

$$(f_i)_* \Omega_{C_i} = \Omega_{\Gamma_i} \oplus (\Omega_{\Gamma_i} \otimes \mathcal{L}_i) \oplus \dots \oplus (\Omega_{\Gamma_i} \otimes \mathcal{L}_i^{n-1})$$

(with  $\Omega_{\Gamma_i} \otimes \mathcal{L}_i^j$  being the eigensheaf corresponding to the character of  $\mu_n, \zeta \rightarrow \zeta^{-j}$ ).

$$(f_i)_* \Omega_{C_i}^{\otimes 2} = \Omega_{\Gamma_i}^{\otimes 2}(B_i) \otimes \mathcal{L}_i^{-1} \oplus \Omega_{\Gamma_i}^{\otimes 2}(B_i) \oplus \dots \oplus \Omega_{\Gamma_i}^{\otimes 2}(B_i) \otimes \mathcal{L}_i^{n-2} ,$$

where  $\Omega_{\Gamma_i}^{\otimes 2}(B_i) \otimes \mathcal{L}_i^j$  is the  $(\zeta \rightarrow \zeta^{-j})$  eigensheaf.

We apply now the Künneth formula to  $Z = C_1 \times C_2$ , keeping in mind that the action is twisted on the second factor, to compute the  $G$ -invariant sections of

$$\begin{aligned} \Omega_Z^1 \otimes \Omega_Z^2 \text{ (where we denote by the same symbol a sheaf on } C_i \text{ and its pull-back to } Z) \\ H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G = H^0((\Omega_{C_1}^{\otimes 2} \otimes \Omega_{C_2}) \oplus (\Omega_{C_1} \otimes \Omega_{C_2}^{\otimes 2}))^G \\ = (H^0(\Omega_{\Gamma_1}^{\otimes 2}(B_1) \otimes \mathcal{L}_1^{-1}) \otimes H^0(\Omega_{\Gamma_2} \otimes \mathcal{L}_2^{n-1})) \\ \otimes \left( \bigotimes_{j=0}^{n-2} \left( H^0 \left( \Omega_{\Gamma_1}^{\otimes 2}(B_1) \otimes \mathcal{L}_1^j \right) \otimes H^0(\Omega_{\Gamma_2} \otimes \mathcal{L}_2^j) \right) \right) \oplus \dots \end{aligned} \tag{2.9}$$

(symmetrical expression interchanging the roles of  $\Gamma_1, \Gamma_2$ ).

It is well known (e.g. since the Kodaira–Spencer map is onto for the family of deformations of  $Z$  of the form  $C'_1 \times C'_2$ ) that  $\text{Def}(Z)$  is smooth.

In order to check that  $H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G \rightarrow T_X^*$ , we set

$$\begin{aligned} B_1 &= b'_1 + \dots, b'_{r_1} \\ B_2 &= b''_1 + \dots, b''_{r_2} \end{aligned}$$

and we choose local coordinates  $(x, y)$  on  $Z$  vanishing at  $(b'_i, b''_j)$ . Then the first summand of (2.9), locally at  $(b'_i, b''_j)$  contributes expressions of type

$$\begin{aligned} (a_{-1} x^{n-1} dx^2 \cdot \alpha_{n-1} dy) + (a_0 x^{n-2} dx^2 \alpha_0 y^{n-1} dy) + \\ + \dots (a_j x^{n-2-j} dx^2 \alpha_j y^{n-1-j} dy) + \dots \end{aligned} \tag{2.10}$$

where  $a_h = a_h(x^n), \alpha_k = \alpha_k(y^n)$ , i.e. the  $a_h$ 's are pull-backs of local functions on  $\Gamma_1$ , and similarly for the  $\alpha_k$ 's.

We have the following lemma, whose proof is straight-forward.

**Lemma 2.11.** *By the quotient map  $\begin{cases} x \mapsto \zeta x \\ y \mapsto \zeta^{-1} y \end{cases}$ , with  $\zeta$  a generator of  $\mu_n$ , the quotient  $p_* \Omega_Z^1 / \Omega_X^1$  is locally generated by the expressions  $x^i y^{i+1} dx - y^i x^{i+1} dy$  (for  $i = 0, \dots, n-2$ ).*

**Theorem 2.12.** *Let  $S$  be a generalized Kas surface as above of degree  $n$ . Then  $\text{Def}(S) = \text{Def}(\Gamma_1) \times \text{Def}(\Gamma_2) \times R$  where  $R$  is a 1-point nilpotent scheme of length  $(n-1) \times (r_1 \times r_2)$  if  $r_i = \text{deg}(B_i) \leq g_i = \text{genus}(\Gamma_i)$  and the branch points are general.*

**Corollary 2.13.** *If  $v = h^1(\theta_S) / \dim \text{Def}(S)$  is the ratio between tangent dimension and dimension of  $\text{Def}(S)$  there do exist generalized Kas-surfaces with  $v$  arbitrarily large.*

*Proof.* If  $S$  has degree  $n$ ,  $\text{deg}(B_i) = g_i - 1$ , then  $v(S) = (n-1)(g_1-1)(g_2-1) / 3[(g_1-1) + (g_2-1)]$ .  $\square$

*Proof of 2.12.* By lemma 2.11 it suffices to verify that, for each  $h = 0, \dots, n-2$ , we have 2 surjective maps,

$$H^0(\Omega_{\Gamma_1}^{\otimes 2}(B_1) \otimes \mathcal{L}_1^h) \otimes (\Omega_{\Gamma_2} \otimes \mathcal{L}_2^h) \rightarrow \bigoplus_{\substack{j=1, \dots, r_1 \\ j=1, \dots, r_2}} \mathbb{C}_{ij} \tag{2.14}$$

(given by  $\bigoplus_{i,j} (\text{val}_{b_i} \otimes \text{val}_{b'_j})$ )

and its symmetrical.

Since  $H^1(\Omega_{\Gamma_1}^{\otimes 2} \otimes \mathcal{L}_1^h) = 0$ , we have a surjection  $H^0(\Omega_{\Gamma_1}^{\otimes 2}(B_1) \otimes \mathcal{L}_1^h) \rightarrow \bigoplus_{i=1, \dots, r_1} \mathbb{C}_{b_i'}$ , while  $H^0(\Omega_{\Gamma_2} \otimes \mathcal{L}_2^h) \rightarrow \bigoplus_{j=1, \dots, \Omega_2} \mathbb{C}_{b_j'}$  if and only if  $H^1(\Omega_{\Gamma_2}(-B_2) \otimes \mathcal{L}_2^h)$  injects into  $H^1(\Omega_{\Gamma_3} \otimes \mathcal{L}_2^h)$ .

By duality there must be a surjection  $H^0(\mathcal{L}_2^{-h}) \rightarrow H^0(\mathcal{L}_2^{n-h})$ , and this holds  $\Leftrightarrow H^0(\mathcal{L}_2^i) = \begin{cases} 0 & \text{for } i < n \\ 1 & \text{for } i = n. \end{cases}$

This condition and the symmetrical one hold if  $\text{deg}(B_i) \leq \text{genus}(\Gamma_i)$  and the branch points are general, by an easy argument on the Jacobian of  $\Gamma_i$ . *Q.E.D.*

The following is, instead, an example where the moduli space is only singular. Its purpose is to illustrate the following feature: though small deformations of hypersurfaces in  $\mathbb{P}^3$  are still hypersurfaces in  $\mathbb{P}^3$  (cf. [K-S], [Se]), the same is not true for hypersurfaces of a weighted projective space  $\mathbb{P}$  if  $\mathbb{P}$  has not isolated singularities.

In fact (as it happens in the examples by Horikawa and Miranda cf. §3) when all the deformations are surfaces in  $\mathbb{P}$ , and the surfaces have to pass through the (rigid) singular points of the weighted projective space, then the deformation space becomes everywhere not reduced.

*Example 2.15.* Let  $X$  be a general member of the family of hypersurfaces of degree 8 in  $\mathbb{P} = \mathbb{P}(1, 1, 2, 2)$ .  $X$  has 4 quadratic ordinary singular points, hence the tangent codimension of this family in the Deformation space is at least 4.

But  $\omega_X = \mathcal{O}_X(2)$ , so that the canonical map of  $X$  embeds it as the complete intersection  $G_4 \cap Q_2'$  in  $\mathbb{P}^4$  where  $G_4$  is a quartic hypersurface in  $\mathbb{P}^4$  and  $Q_2'$  is a quadric of rank 3. It is easy to see that the complete intersections  $G_4 \cap Q_2$  are all the deformations of  $X$ , and the previous family has thus codimension equal to 3.

### §3. Weighted hypersurfaces with everywhere non reduced moduli spaces

*This paragraph shows somehow that our situation is not “artificial”, in fact the examples we shall consider in this section will be hypersurfaces  $X$  of degree  $d$  in a weighted 3-dimensional projective space  $\mathbb{P}(1, 1, p, q)$ : in these examples  $S$  shall be simply connected, we shall get all the  $A_n$  singularities and  $\text{Def}(S)$  will be e.n.r. for all  $d$  satisfying certain congruences which guarantee that  $X$  is generically singular; the examples of Horikawa and Miranda will be two issues of a double infinite series of examples.*

Here  $\mathbb{P}^3 = \text{Proj}(\mathbb{C}[y_0, y_1, y_2, y_3])$ ,  $p < q$  are relatively prime integers,  $\mathbb{P} = \mathbb{P}(1, 1, p, q) = \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3])$  where  $\text{deg } x_0 = \text{deg } x_1 = 1, \text{deg } x_2 = p, \text{deg } x_3 = q$ .

We let  $G = \mathbb{Z}/pq\mathbb{Z}$  and let  $\zeta \in \mathbb{C}^*$  be a primitive  $(pq)^{\text{th}}$  root of unity;  $G$  acts on  $\mathbb{P}^3$  by

$$(y_0, y_1, y_2, y_3) \mapsto (y_0, y_1, \zeta^q y_2, \zeta^p y_3). \tag{3.1}$$

(3.2)  $\psi: \mathbb{P}^3 \rightarrow \mathbb{P}$ , given by  $\psi(y_0, y_1, y_2, y_3) = (y_0, y_1, y_2^p, y_3^q)$  is such that  $\mathbb{P}^3/G \cong \mathbb{P}$ .

(3.3) Let  $f \in \mathbb{C}[x_0, x_1, x_2, x_3]_d$  define a hypersurface  $X = \{f = 0\}$  of degree  $d$  in  $\mathbb{P}^3$ , and let  $Z = \psi^{-1}(X)$  (thus  $Z$  is the locus of zeros of  $F(y) \stackrel{\text{def}}{=} f(y_0, y_1, y_2^2, y_3^2)$ ).

**Proposition 3.4.** Assume  $d \geq (p - 1)(q - 1)$ , so that we can write  $d$  as  $d = bp + aq$  with  $a, b \geq 0$ . Then, for a general choice of  $f$ ,  $Z$  is smooth,  $X$  is singular but has only R.D.P.'s as singularities iff  $p, q, d$  are according to the following table, where  $P_2 = (0, 0, 1, 0), P_3 = (0, 0, 0, 1)$  are the singular points of  $\mathbb{P}^3$ .

**Table 3.5.**

$\mathbb{P}$	$d$	Singularity at $P_2$	Singularity at $P_3$
$\mathbb{P}(1, 1, 2, 3)$	$d = 1 + 6k$	$A_1$	$A_2$
$\mathbb{P}(1, 1, p, p + 1)$	$d = p(k(p + 1) - 1)$	/	$A_p$
$\mathbb{P}(1, 1, p, rp - 1)$	$d = (kp - 1)(rp - 1)$	$A_{p-1}$	/

**Theorem 3.6.** Let  $X \subset \mathbb{P}(1, 1, p, q) = \mathbb{P}^3$  be a general hypersurface as in table 3.5 of degree  $d$  s.t.  $X$  is of general type, i.e.  $d > 2 + p + q$ , and  $r > p - 2$ . If  $S$  is a minimal resolution of  $X$ , then every small deformation of  $S$  is the resolution of a hypersurface of degree  $d$  in  $\mathbb{P}^3$ , and  $\text{Def}(S) = \text{ES Def}(X) \times R$ , where  $\text{ES Def}(X)$  is smooth and  $R$  is a 1-point nilpotent scheme of length equal to the sum of the Milnor numbers of the singularities.

*Idea of the proof.* Of course one has to use theorem 1.16, part ii). It is well known that  $\text{Def}(Z)$  is smooth (cf. [K-S] [Se]), and one has to verify the surjections

$$H^1(\theta_Z)^G \rightarrow H^1(\theta_X)$$

$$H^0(\Omega_Z^1 \otimes p^* \omega_X)^G \rightarrow H^0((p^* \Omega_Z^1)^G / \Omega_X^1 \otimes \omega_X) = T_X^* .$$

To verify the first, we consider the exact sequence 1.18. Here  $B = B_2 \cup B_3$ , where  $B_i$  is defined by the equation  $x_i = 0$ ;  $B_2, B_3$  are smooth and intersect transversally in  $d$  points. (The only exception being the II case with  $p = 1$ , where  $B = B_3$ ). One can verify through elementary computations the following two Lemmas.

**Lemma 3.7.**  $N'_B = N'_{B_2} \oplus N'_{B_3}$ .

**Lemma 3.8.** If  $B_3$  does not contain  $P_2$ , then  $N'_{B_3} \cong \mathcal{O}_{B_3}(q)$ , otherwise  $N'_{B_3} \cong \mathcal{O}_{B_3}(q) \otimes \mathcal{O}_{B_3}(-(p - 1)P_2)$ .

Analogously we have (assuming  $p \geq 2$ )

$$N'_{B_2} \text{ equals } \begin{cases} \mathcal{O}_{B_2}(p) & \text{if } B_2 \not\ni P_3 \\ \mathcal{O}_{B_2}(p) \otimes \mathcal{O}_{B_2}(-pP_3) & \\ \text{if } B_2 \ni P_3 \text{ (here } q - 1 = p). \end{cases} \tag{3.9}$$

To finish the proof, we need to apply lemma 1.19, and to show by explicit computations that

$$h^0(\Omega_Z^1 \otimes \Omega_Z^2)^G - h^0(\Omega_Z^1 \otimes p^* \omega_X)^G = h^1(N'_{B_2}) + h^1(N'_{B_3})$$

$$(h^1(N'_{B_3}) \text{ if } p = 1). \tag{3.10}$$

In fact the following holds true:

(3.11) **Claim.** *The left hand side of (3.10) equals  $h^0(\mathcal{O}_{\mathbb{P}(1,1,p)}(d-p-q-2) + h^0(\mathcal{O}_{\mathbb{P}(1,1,q)}(d-p-q-2))$  if  $p \geq 2$ , and equals  $h^0(\mathcal{O}_{\mathbb{P}^2}(d-3-q))$  if  $p = 1$ .*

We omit the proof of the above claim: the main ingredients are just the Koszul exact sequence for 1-forms on  $\mathbb{P}^3$ , plus a careful analysis of the  $G$ -module structure.

To deal with the right hand side of (3.10), we use Serre duality and the fact that (cf. [Do], 3.5.2.)

$$\omega_{B_2} = \mathcal{O}_{B_2}(d-q-2), \quad \omega_{B_3} = \mathcal{O}_{B_3}(d-p-2).$$

Then, setting  $\delta_2 = 0$  if  $B_2 \not\# P_3$ ,  $\delta_2 = p = q - 1$  if  $P_3 \in B_2$ ,  $\delta_3 = 0$  if  $B_3 \not\# P_2$ ,  $\delta_3 = p - 1$  otherwise, we have

$$\begin{aligned} h^1(N'_{B_2}) &= h^0(\text{Hom}(\mathcal{O}_{B_2}(p)(-\delta_2 P_3), \mathcal{O}_{B_2}(d-q-2))) \\ &= h^0(\mathcal{O}_{B_2}(d-p-q-2)(\delta_2 P_3)), \text{ and similarly} \\ h^1(N'_{B_3}) &= h^0(\mathcal{O}_{B_3}(d-p-q-2)(\delta_3 P_2)). \end{aligned} \tag{3.12}$$

To end the proof, in view of 3.11 and 3.12, it suffices to prove that for our choices of  $d, p, q$ ,

$$h^0(\mathcal{O}_{B_2}(d-p-q-2)(\delta_2 P_3)) = h^0(\mathcal{O}_{\mathbb{P}(1,1,q)}(d-p-q-2))$$

if  $p > 1$  and

$$h^0(\mathcal{O}_{B_3}(d-p-q-2)(\delta_3 P_2)) = h^0(\mathcal{O}_{\mathbb{P}(1,1,p)}(d-p-q-2)). \tag{3.13}$$

Since  $B_3 \subset \mathbb{P}(1,1,q)$  and symmetrically  $B_2 \subset \mathbb{P}(1,1,p)$ , we are considering the following problem:

(3.14)  $B \subset \mathbb{P}(1,1,p) = \mathbb{P}$  has degree  $d$ ,  $P$  is the singular point of  $\mathbb{P}$ ,  $\delta = 0$  if  $B \not\# P$ ,  $\delta = (p-1)$  otherwise,  $i$  is an integer  $> 0$ : is there an equality

$$h^0(\mathcal{O}_{\mathbb{P}}(d-i)) = h^0(\mathcal{O}_B(d-i)(\delta P)?$$

(3.15) The answer is positive if  $\delta = 0$ , since we have the long exact cohomology sequence

$$0 \rightarrow H^0(\mathcal{O}_{\mathbb{P}}(d-i)) \rightarrow H^0(\mathcal{O}_B(d-i)) \rightarrow H^1(\mathcal{O}_{\mathbb{P}}(-i)) = 0.$$

The following lemma follows from a more general result proven in the Appendix.

**Lemma 3.16.** *Let  $\pi: \mathbb{F} = \mathbb{F}_p \rightarrow \mathbb{P} = \mathbb{P}(1,1,p)$  be the blow up of the singular point  $P$  of  $\mathbb{P}$ , and let  $E_\infty$  be the exceptional divisor. Let  $B$  be a smooth curve of  $\mathbb{P}$ , and identify  $B$  to its proper transform in the Segre–Hirzebruch surface  $\mathbb{F} = \mathbb{F}_p$ . Then if  $E_0$  is the curve defined by  $\pi^{-1}(x_2=0)$ , and  $F$  is a fiber of  $\mathbb{F}$  (proper transform of  $x_0=0$ ), we have: if  $r$  is a positive integer and  $r = r' + r''p$ , with  $r' < p$ , the  $\mathcal{O}_B(r) = \mathcal{O}_B(r'F + r''E_0)$ , and  $H^0(\mathcal{O}_{\mathbb{P}}(r)) = H^0(\mathcal{O}_{\mathbb{F}}(r'F + r''E_0))$ .*

Moreover  $\mathcal{O}_B(\delta P) = \mathcal{O}_B(\delta E_\infty)$  (recall that  $E_0 \equiv E_\infty + pF$ ).

Assume now we are in the case when  $B \ni P$ , and  $(d-i)$  (cf. 3.14) equals  $t'p$  (as can be checked), while  $d = tp + 1$  (so  $t > t'$ ).

It is then easy to see that  $B \equiv tE_0 + F$ , and, in view of 3.16 and since  $H^0(\mathcal{O}_{\mathbb{P}}(t'p)) = H^0(\mathcal{O}_{\mathbb{F}}(t'E_0)) = H^0(\mathcal{O}_{\mathbb{F}}(t'E_0 + (p-1)E_{\infty}))$ , we are looking at the surjectivity of the map (res) in the following exact sequence on  $\mathbb{F}$ :

$$\begin{aligned} H^0(\mathcal{O}_{\mathbb{F}}(t'E_0 + (p-1)E_{\infty})) &\xrightarrow{\text{res}} H^0(\mathcal{O}_B(t'E_0 + (p-1)E_{\infty})) \\ &\longrightarrow H^1(\mathcal{O}_{\mathbb{F}}(((p-1) + t' - t)E_0 - (p(p-1) + 1)F)) \\ &\xrightarrow{j} H^1(\mathcal{O}_{\mathbb{P}}((t' + p - 1)E_0 - p(p-1)F)). \end{aligned} \quad (3.17)$$

This is equivalent to requiring the injectivity of  $j$ .

Set  $b = (p-1) - (t-t')$ : if  $b \leq -1$  the domain of  $j$  is a vanishing cohomology group and we are done (cf., for the vanishing, the table at page 612 of [Kon]).

Assume instead  $b \geq 0$ , and set  $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(p)$ . Then the map  $j$  is the map  $j'$

$$H^1(\text{Sym}^b V \otimes \mathcal{O}(-p(p-1)-1)) \xrightarrow{j'} H^1(\text{Sym}^{b+t} V \otimes \mathcal{O}(-p(p-1))) \quad (3.18)$$

given by the equation of  $B$ .

$$\text{I.e., if } f = \sum_{k=0}^t c_{ijk} y_0^i y_1^j y_2^{p(t-k)} = \sum_{\substack{k=0 \\ \deg f_k = kp+1}}^t f_k(y_0, y_1) y_2^{p(t-k)}$$

is the equation of  $B$ , then, since  $\text{Sym}^b V = \bigoplus_{i=0}^b \mathcal{O}(ip)$  ( $\mathcal{O}$  stands for  $\mathcal{O}_{\mathbb{P}^1}$ ),

$\psi_i \in H^1(\mathcal{O}(ip - p(p-1) - 1))$  is mapped under  $j'$  to  $\sum_{k=0}^i f_k \psi_i$ , with  $f_k \psi_i \in H^1(\mathcal{O}((i+k)p - p(p-1)))$ .

On the other hand this last group is zero unless  $i+k = h' \leq p-2$ , and it is always better to deal with  $H^0$ 's rather than  $H^1$ 's, hence we dualize, and ask for the surjectivity of

$$\begin{aligned} \bigoplus_{h'=0}^{p-2} H^0(\mathcal{O}((p-1-h')p-2)) &\twoheadrightarrow \bigoplus_{i'=0}^b H^0(\mathcal{O}(p(p-1-i')-1)) \\ \parallel & \parallel \\ \bigoplus_{h=1}^{p-1} H^0(\mathcal{O}(hp-2)) &\twoheadrightarrow \bigoplus_{i=t-t'}^{p-1} H^0(\mathcal{O}(ip-1)) \end{aligned} \quad (3.19)$$

$$\left( \text{where } \varphi_h \mapsto \sum_{\substack{k=0 \\ t-t' \leq k+h \leq p-1}}^i f_k \varphi_h \right).$$

This map is not in general surjective, hence we have to check the three cases of table 2.5.

*Case I:* for  $p = 2$ ,  $b = -2$ ; for  $p = 3$ ,  $b = 0$  and  $H^0(\mathcal{O}(1) \oplus \mathcal{O}(4)) \twoheadrightarrow H^0(\mathcal{O}(5))$ .

*Case II:* for  $p = 1$  there's nothing to check, for  $p \geq 2$ , there remains to check the case for  $\mathbb{F}(1, 1, q)$ ,  $q = p+1$ , which follows from the following lemma 3.20, since  $b = q - 3$

Case III:  $b = p - 1 - r - 1$ , O.K. if  $r > p - 2$ .

**Lemma 3.20.** *If  $f$  has  $f_0 \neq 0, f_0, f_1$  rel. prime, then (3.19) is surjective for  $t - t' = 2$ .*

*Proof.* Since  $\sum_{h=1}^{p-1} h^0(\mathcal{O}(hp - 2)) = \sum_{i=2}^{p-1} h^0(\mathcal{O}(ip - 1)) + 1$ , it suffices to show that the kernel has dimension 1.

The map is given in matrix equation by

$$\begin{bmatrix} \psi_{p-1} \\ \vdots \\ \psi_2 \end{bmatrix} = \begin{bmatrix} f_0 f_1 \cdots f_{p-2} & & & & & \\ & \ddots & & & & \\ & & f_0 & & & \\ & & & \ddots & & \\ & & & & f_0 f_1 & f_2 \\ & & & & & \ddots \\ & & & & & & f_0 & f_1 \end{bmatrix} \begin{bmatrix} \varphi_{p-1} \\ \vdots \\ \varphi_1 \end{bmatrix}$$

and it suffices to show, by induction on  $i = 1, \dots, p - 2$ , that if  $\varphi$  is in the kernel, then  $f_0^i | \varphi_1, \varphi_2, \dots, \varphi_{i+1}$  are determined by  $\varphi_1$ , and if  $\varphi_1 = f_0^i \cdot g_i$ , then  $g_i \cdot f_0^{i-(j-1)} | \varphi_j$  and  $\varphi_{i+1} \equiv \pm f_1^i g_i \pmod{f_0}$ .

In fact, if  $\varphi$  is in the kernel, then  $0 = \psi_{i+2} = f_0 \varphi_{i+2} + f_1 \varphi_{i+1} + \dots + f_{i+1} \varphi_1$ , thus  $f_0 | \varphi_{i+1}$ , hence  $f_0 | g_i, g_i = f_0 g_{i+1}$  and the other inductive assertions are clear since e.g.  $\varphi_{i+2} \equiv \pm f_1^{i+1} g_{i+1} \pmod{f_0}$  is gotten by dividing by  $f_0$  the above equation. *Q.E.D. for the Lemma*

The verification of the surjectivity of  $H^0(\Omega_Z^1 \otimes p^* \omega_X)^G \rightarrow T_X^*$  follows easily from the explicit description we gave of  $H^0(\Omega_Z^1 \otimes p^* \omega_X)^G = \{\sum u_{ij}(y) \eta_{ij} | u_{ij} \in \mathcal{A}_{d-4-p-q}, w(u_{ij}) + w(\eta_{ij}) = 0\}$  and from lemma 2.11 (applied with  $n$  equal to  $p$ , resp.  $q$ ). *Q.E.D.*

**Appendix: formulas for almost simple cyclic covers and cones  $\mathbb{P}(1, 1, p)$**

To justify the definition of almost simple cyclic covers, let's consider  $\mathbb{P} = \mathbb{P}(1, 1, n)$ , the projective cone over the rational normal curve of degree  $n$ , and  $\pi: \mathbb{F} \rightarrow \mathbb{P}$  where  $\mathbb{F}$ , the blow up of the vertex of the cone, is the Segre–Hirzebruch surface  $\mathbb{F}_n$ . We have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\psi} & \mathbb{P} \\ \uparrow \varepsilon & & \uparrow \pi \\ \mathbb{F}_1 & \xrightarrow{\varphi} & \mathbb{F}_n \end{array} \tag{A1}$$

where  $\psi(y_0, y_1, y_2) = (y_0, y_1, y_2^n)$ ,  $\varepsilon$  is the blow-up of the point  $y_0 = y_1 = 0$ , and  $\psi, \varphi$  are quotient morphisms by  $G \cong \mu_n \cong \mathbb{Z}/n$ .

Let  $E_0$  be  $\pi^{-1}(x_2 = 0)$ ,  $E_\infty = \pi^{-1}(0, 0, 1)$ ,  $F = \pi^{-1}(x_0 = 0)$ , and set  $E'_0, E'_\infty, F'$  their set theoretical inverse images under  $\varphi$ .

An easy computation gives

(A2)  $\varphi^*(E_0) = nE'_0, \varphi^*(E_\infty) = nE'_\infty, \varphi^*(F) = F'$ , and  $E_0, E_\infty$  are the branch locus of  $\varphi$ , with  $E_0 - E_\infty \equiv nF$ .



We notice that, by definition (cf. [Do])

$$(A3) \quad \mathcal{O}_{\mathbb{P}^2}(r) = \psi_* (\mathcal{O}_{\mathbb{P}^2}(r))^G, \quad \text{moreover} \quad \mathcal{O}_{\mathbb{P}^2}(r) = \varepsilon_* \mathcal{O}_{\mathbb{F}_1}(rE'_0), \quad \text{hence} \quad \mathcal{O}_{\mathbb{P}^2}(r) = \pi_* (\varphi_* (\mathcal{O}_{\mathbb{F}_1}(rE'_0))^G).$$

Writing  $r = q + r'n$ , with  $q < n$  we have  $(\varphi_* \mathcal{O}_{\mathbb{F}_1}(rE'_0))^G = \varphi_* \mathcal{O}_{\mathbb{F}_1}(qE'_0)^G \otimes \mathcal{O}_{\mathbb{F}_n}(r''E_0)$ , hence, in order to demonstrate lemma 3.31, it suffices to show that

$$\varphi_* \mathcal{O}_{\mathbb{F}_1}(qE'_0)^G = \mathcal{O}_{\mathbb{F}_n}(qF). \tag{A4}$$

This will be done in greater generality.

*Almost simple cyclic covers*

Let an algebraic variety  $Y$  and a line bundle  $L$  of rank 1 be given on  $Y$ , such that the sheaf of sections of  $L$  be isomorphic to  $\mathcal{O}_Y(F) = \mathcal{L}$  for some divisor  $F$ . Assume that there are given reduced effective divisors  $E_0, E_\infty$  on  $Y$ , which are disjoint, and are such that  $E_0 \equiv E_\infty + nF$ .

**Definition A5.** The *almost simple cyclic cover* associated to  $(Y, L, E_0, E_\infty)$  is the subvariety  $X \hookrightarrow \mathbb{P}(L \oplus \mathbb{C}_Y)$  defined by the equation  $z_1^n e_\infty = e_0 z_0^n$  where  $e_0, e_\infty$  are sections defining  $E_0$ , resp.  $E_\infty$ , and  $z_1, z_0$  are respective linear coordinates on the fibers of  $L$ , resp. the trivial bundle  $\mathbb{C}_Y$ .

The group  $G = \mu_n$  acts, if  $\zeta$  is a primitive  $n^{\text{th}}$  root of 1, by  $z_1 \rightarrow \zeta z_1, z_0 \rightarrow z_0$ .

Take, as customary,  $z = z_1/z_0$  as a coordinate on  $V_0 = (X - E_\infty)$ , and  $z' = z_0/z_1$  on  $V_\infty = X - E_0$ .

We have

**Proposition A6.**  $\varphi_* \mathcal{O}_X \cong \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}(-E_\infty)$ , where  $\mathcal{L}^{-i}(-E_\infty) \cong \mathcal{L}^{n-i}(-E_0)$  is the eigensheaf corresponding to  $\zeta \mapsto \zeta^i$ .

*Proof.* Write a function on  $V_0$  as  $f_0 + f_1 z + \dots + f_{n-1} z^{n-1}$  with  $f_i$  a section of  $\mathcal{L}^{-i}$  on  $V_0$ , and as  $g_0 + \dots + g_{n-1} (z')^{n-1}$  on  $V_\infty$ , with  $g_j$  a section of  $\mathcal{L}^j$  on  $V_\infty$ ; then notice that  $f_i z^i = f_i e_0 / e_\infty z'^{n-i}$ , hence  $f_i e_0 / e_\infty = g_{n-i}$ . Q.E.D.

Similarly one proves:

**Theorem A7.** If  $\varphi^*(E_0) = nE'_0$ , and  $q < n$ , then

$$\varphi_* \mathcal{O}_X(qE'_0) \cong \left( \bigoplus_{i=0}^q \mathcal{L}^{q-i} \right) \oplus \left( \bigoplus_{i=q+1}^{n-1} \mathcal{L}^{n+q-i}(-E_0) \right)$$

where the index  $i$  gives the  $\zeta \mapsto \zeta^i$  eigensheaf.

**Corollary A8.**  $\varphi_* \mathcal{O}_X(qE'_0)^G \cong \mathcal{L}^q$ .

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**Note added-in-proof.** While giving a talk at the Max-Planck-Institut on the above results, I learnt from Masa-Hiko Saito that he has also been studying a situation similar to the one considered in §1.