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Calabi-Yau manifolds with large Picard number

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Introduction

Much has been written recently on the role of Calabi-Yau manifolds in superstring theory (see for example [4, 5, 18, 19, 39, 48]), and many specific examples have been studied in great detail [12, 11, 13, 14, 17, 36, 37]. Recall that a *Calabi-Yau manifold* is a smooth complex projective threefold with trivial canonical bundle and no global 1-forms or 2-forms – equivalently it is a projective manifold with $SU(3)$ holonomy [2]. Recall also that the general structure theorems of [2, 3] imply that a Calabi-Yau manifold has finite fundamental group.

Coincidentally, these objects have also arisen recently in algebraic geometry, as it is essentially these threefolds which constitute the big remaining gap in the threefold classification programme of Mori, Kawamata and others, carried out in the last ten or so years. In the terminology of the survey article [47], we wish to classify projective threefolds with $\mathbb{Q}$-factorial terminal singularities and numerically trivial canonical divisor. It may be seen easily (cf. [23]) that any such threefold has a finite covering (ramified only over the singularities) which is an abelian threefold, the product of a $K3$ surface with an elliptic curve, or a simply connected projective threefold V with only $\mathbb{Q}$-factorial compound Du Val (abreviated cDV) singularities, with zero canonical class and $h^1(\mathcal{O}_V)=h^2(\mathcal{O}_V)=0$. This latter category is a very mild generalization of the category of simply connected Calabi-Yau manifolds, and the methods of this paper will be applicable essentially unchanged to this larger class.

From the points of view of both physics and algebraic geometry therefore, Calabi-Yau manifolds are of interest. For V a Calabi-Yau manifold, it is an open question whether the Euler number $e(V)=2(h^{1,1}(V)-h^{1,2}(V))$ should be in a bounded range. Recall that $h^{1,1}$ is just the rank ρ of $\mathrm{Pic}(V)$, whilst $h^{1,2}$ is the dimension of the versal deformation space of V. There are certainly examples with $e(V)$ either large negative or large positive – the latter being more difficult to construct than the former. We can restrict ourselves if we wish to the simply connected case, since the universal cover of a Calabi-Yau manifold is one whose Euler number is an integral multiple of the original Euler number. More specifically we shall in this paper study the case when $h^{1,1}=\rho$ is large ($\rho>19$).

We shall say that a projective threefold V is a *Calabi-Yau model* if it has only rational Gorenstein singularities (i.e. canonical singularities of index 1) and for which there is a resolution of singularities (necessarily crepant) $\pi: \tilde{V} \to V$ with $\tilde{V}$ a Calabi-Yau manifold (see [31, 33] for definitions of terms here). We observe that the examples in the literature of Calabi-Yau manifolds with large Picard number ρ all arise as resolutions of Calabi-Yau models with much smaller Picard number but several singularities. For instance, taking a certain quintic hypersurface in $\mathbb{P}^4$ with 126 nodes yields a small resolution which is a Calabi-Yau manifold with $e=52$ and $\rho \geqq 26$ [17]. By taking a triple cover of $\mathbb{P}^3$ branched over six planes forming a cube, we obtain a threefold with 9 singularities, whose minimal resolution is a Calabi-Yau manifold with $e=72$ and $\rho \geqq 36$ – the exceptional locus above each singularity is a cubic Del Pezzo surface [17]. Lastly, an example is given in [37] of a fibre product of two rational elliptic surfaces which has 81 nodes, and with a small resolution which is a Calabi-Yau manifold with $e=168$ (and hence $\rho \geqq 84$). Determining the Euler number $e=2(h^{1,1}-h^{1,2})$ of these and other such examples is relatively straightforward; finding the individual values of $h^{1,1}$ and $h^{1,2}$ is a very much more subtle question. Methods for calculating $h^{1,1}$ and $h^{1,2}$ have however been developed by a number of authors and calculations performed on several examples [7, 35, 36, 44, 45, 41]. In particular, Dr. Jürgen Werner has pointed out to the author that $h^{2,1}=0$ in all the examples cited above (in the case of the quintic with 126 nodes, see the Appendix to [45]), and hence that the above inequalities on ρ are in fact equalities.

The examples referred to above suggest that Calabi-Yau manifolds with large values for ρ should arise as resolutions of Calabi-Yau models with reasonably small Picard number – this has now been proved.

Main theorem. *A Calabi-Yau manifold V is a resolution of a Calabi-Yau model $\bar{V}$ with Picard number $\rho \leqq 19$.*

One might note that the simplest class of Calabi-Yau manifolds (Calabi-Yau complete intersections of hypersurfaces in products of complex projective spaces) has (coincidentally) maximum value for ρ as 19 [12], although many such families do consist generically of small resolutions of nodal threefolds in other families [14].

For information on the singularities that can occur on a Calabi-Yau model V, I refer the reader to §2 of [31], and to [28]. In (2.12) of [31], we see that there is a partial resolution $V^* \to V$, where V^* has only cDV singularities. Furthermore the exceptional locus on V^* and the corresponding singularities on V are described in (2.13) of [31]; in particular the exceptional locus is a finite union of rational or ruled surfaces. Since each of these surfaces has discrepancy zero, there are corresponding ruled or rational surfaces on any full resolution $\tilde{V}$ of V. In the second example given above, V^* is a full resolution and the exceptional surfaces are cubic Del Pezzo.

If on the other hand $V=V^*$ has cDV singularities, then by §2 of [32] we may resolve the 1-dimensional Du Val locus (if it exists) and obtain the existence of ruled surfaces on any resolution $\tilde{V}$. If however V has only isolated cDV

singularities (which is the case for the other two examples given above), the given morphism $\tilde{V} \to V$ must be a small resolution of the singularities. For information on such small resolutions, the reader is referred to § 5 of [28] (see also [30, 32]); in particular we note that the exceptional locus consists of a finite number of smooth rational curves.

From the Main Theorem and the above discussion, we observe that a Calabi-Yau manifold with $\rho > 19$ must contain rational curves. For non-simply connected Calabi-Yau manifolds with fundamental group π_1, we deduce that there are rational curves as long as $e(V) > 38/|\pi_1|$. The physical relevance of rational curves on Calabi-Yau manifolds is discussed in [49, 10]; basically they are unwelcome! By considering the relative effective cone of 1-cycles for the morphism $V \to \bar{V}$ in the Main Theorem, it is straightforward to deduce that the rational curves on a Calabi-Yau manifold V generate a subgroup in $H_2(V, \mathbb{Z})$ of corank $\leqq 19$; this should be compared with the far stronger conjectures of Reid [34].

The theory described in this paper essentially ignores the torsion part of the Picard group, supporting the view that the most crucial case is when our Calabi-Yau manifold V is simply connected. From the theory of Wall [43], such manifolds are determined up to diffeomorphism by the integral cohomology groups $H^2(V, \mathbb{Z}) \cong \mathrm{Pic}\, V$ and $H^3(V, \mathbb{Z})$, the symmetric trilinear form μ on $H^2(V, \mathbb{Z})$ given by cup product, and the linear form on $H^2(V, \mathbb{Z})$ given by the second Chern class $c_2 \in H^4(V, \mathbb{Z})$. We shall use all this information apart from the cohomology in degree 3.

One of the main techniques of this paper is to study the cubic form on Pic(V) given by μ; with slight abuse of notation we have μ: Pic(V) $\to \mathbb{Z}$ given by $\mu(D) = D^3$. The paper looks at the Diophantine Geometry of the cubic hypersurface in $\mathbb{P}^{\rho-1}$ defined by μ and the relative position of the hyperplane defined by c_2, and relates these to the geometry of the Calabi-Yau manifold V. Using (hard) facts about solutions of integral cubic forms, we obtain the number 19 in the above theorem. If the cubic hypersurface defined by μ is non-singular, we shall have a contraction so long as $\rho > 9$. The author should confess however to his feeling that by using more specific information on the cup product, one might hope for a result along the lines that any Calabi-Yau manifold is the resolution of a Calabi-Yau model with $\rho \leqq 3$. The obstruction to proving such a result is the need to show that the cubic hypersurfaces which turn up have enough rational points.

The other main techniques of the paper are those developed in the last 10 or so years for studying the birational classification of threefolds. The reader will observe the obvious influence of the ideas of Mori, especially in Sect. 3. The reader unfamiliar with these techniques might wish to consult the various survey articles, for instance [24, 26, 33, 47].

Finally the author wishes to thank Dr. Roger Heath-Brown for the benefit of some extremely useful comments, which in particular enabled him to extract the desired result on cubic forms from the papers of Davenport. With the proof below in its present form, it is essentially the power of the Hardy-Littlewood method from analytic number theory which enables us to deduce the existence of the desired contractions.

1. Structure of the proof and further notation

Suppose that V is a C-Y model, as defined in the Introduction (from now on we shall usually abreviate Calabi-Yau to C-Y). For any extension field K of $\mathbb{Q}$, we denote by $\mathrm{Pic}_K(V)$ the vector space $\mathrm{Pic}(V) \otimes K$ over K. The Picard number $\rho(V)$ is just the dimension of any such vector space.

Definition. Given a C-Y model V, a *Calabi-Yau contraction* is a birational morphism $f\colon V \to \bar{V}$ where $\bar{V}$ is a normal projective threefold with $\rho(\bar{V}) < \rho(V)$. It follows automatically (notation and terminology as in [31, 33]) that $K_{\bar{V}} = 0$ and that $\bar{V}$ has index 1 canonical (i.e. rational Gorenstein) singularities; thus $\bar{V}$ will also be a C-Y model.

If V is a C-Y model with $\pi\colon \tilde{V} \to V$ the C-Y resolution of singularities, we can define a linear form $c_2\colon \mathrm{Pic}(V) \to \mathbb{Z}$ by sending a Cartier divisor D to the integer $\pi^* D \cdot c_2(\tilde{V})$; we shall denote this simply by $D \cdot c_2(V)$ or $D \cdot c_2$. Observe that even in the slightly more general case when $\tilde{V}$ has $\mathbb{Q}$-factorial cDV singularities, we can still define the number $D \cdot c_2$ in the natural way (cf. [46], Sect. 5). It is well-known that for any C-Y manifold, the linear form c_2 is non-trivial (see Theorem 1.5 of [25]). We shall be able to restrict attention to Calabi-Yau models for which this is also true.

The main result as stated in the Introduction will follow by induction from (1.1) below.

Theorem 1.1. *If V is a Calabi-Yau model with $\rho(V) > 19$ and $c_2(V)$ non-trivial, then there exists a Calabi-Yau contraction $f\colon V \to \bar{V}$ with $c_2(\bar{V})$ non-trivial.*

Remark. We might observe in passing that, as we are dealing with varieties whose canonical class is trivial, we do not run into the problems which arise from small contractions in the Minimal Model programme – see [47], Proposition 7.1. Our varieties here do not remain $\mathbb{Q}$-factorial under the contractions, but the singularities do remain canonical of index 1.

Our first step towards finding C-Y contractions is contained in a Key technical lemma, the proof of which constitutes Sect. 2 of this paper. Recall first that a Cartier divisor $D \in \mathrm{Pic}(V)$ is called *numerically effective* or *nef* if $D \cdot C \geqq 0$ for all curves C on V (see [47] for more details).

Key lemma. *If a Calabi-Yau model V contains an ample divisor $H \in \mathrm{Pic}(V)$ and a divisor $D \in \mathrm{Pic}(V)$ which is not nef but for which $D^3 > 0$, $D^2 \cdot H > 0$ and $D \cdot H^2 > 0$, then there exists a Calabi-Yau contraction $f\colon V \to \bar{V}$.*

In Sect. 3, we use the Key lemma and other results from classification theory to prove:

Proposition 3.2. *Suppose V is a Calabi-Yau model and $D \in \mathrm{Pic}(V)$ with $D^3 = 0$, $D \cdot c_2 \neq 0$ and $D^2 \cdot H > 0$ for some ample divisor $H \in \mathrm{Pic}(V)$, then either a Calabi-Yau contraction exists, or V has the structure of an elliptic fibre space $\phi\colon V \to S$ with S a normal surface and ϕ corresponding to the linear system $|nD|$ for some integer n.*

In the case above of an elliptic fibre space $\phi\colon V \to S$, we have that $\phi^* \mathrm{Pic}_{\mathbb{Q}}(S)$ is a linear subspace of $\mathrm{Pic}_{\mathbb{Q}}(V)$ containing D and of dimension $\rho(S)$. If $\rho(S)$

$<\rho(V)-1$, an easy argument on the relative effective cone $NE(V/S)$ yields the existence of a C-Y contraction on V.

We now consider the cubic hypersurface W in $\mathbb{P}(\mathrm{Pic}_{\mathbb{C}}(V))=\mathbb{P}^{\rho-1}$ consisting of points representing divisors D with $D^3=0$. The above results tell us that under the assumptions of (3.2), either there exists a C-Y contraction on V or D is contained in a *hyperplane* component of W. Observe here the slight abuse of notation (which occurs throughout Sect. 4) whereby for $D\in\mathrm{Pic}(V)$ not numerically trivial, we shall use the same symbol to denote the corresponding points of $\mathrm{Pic}_K(V)$ and $\mathbb{P}^{\rho-1}(K)=\mathbb{P}(\mathrm{Pic}_K(V))$, K an extension field of $\mathbb{Q}$.

The strategy is now clear: if we can find a rational point on W (i.e. $D\in\mathrm{Pic}_{\mathbb{Q}}(V)$ with $D^3=0$) which is neither on any hyperplane component of W nor on the hyperplane defined by the linear form c_2, and such that $D^2\cdot H>0$ for some ample divisor H, then the above results yield a C-Y contraction. This is a problem in the Diophantine geometry of the cubic hypersurface W. The geometric properties of W which we use are those derived from the Hodge index theorem on V – this for instance tells us that W is not a cone. We shall see in Sect. 4 that a rational point of the required type exists in the following cases:

(1) W irreducible and contains a singular rational point.

(2) W reducible and $\rho>5$.

(3) W irreducible and contains a rational linear space of dimension >2.

(4) W smooth and $\rho>9$.

(5) W arbitrary and $\rho>19$.

Of these results, (1) is essentially obvious, (2) and (3) depend crucially on the well-known result ([38], p. 43) that an indefinite quadratic form in 5 or more variables has a non-trivial rational solution, (4) follows from the results of [16], and (5) follows from (3) and results of Davenport [8, 9]. Using [8] it is also possible to prove the existence of a point of the required type in the case when W has a real singularity and $\rho>15$, but we shall not need this and so do not prove it.

An easy geometric argument shows that we may choose our divisor D to satisfy $D\cdot c_2>0$, and this is sufficient to ensure that c_2 is non-trivial on the Picard group of the contracted variety.

In all the cases under consideration we shall show that the rational points are dense in the real locus $W(\mathbb{R})$ of our cubic hypersurface W in $\mathbb{P}^{\rho-1}$. The conditions that we require from the rational point of W sought are all open conditions, and so in fact there are infinitely many rational points of the required type. Whether one can exploit this feature to produce infinitely many rational curves on V seems an interesting question (cf. the case of the quintic hypersurface [6]).

2. Proof of the key lemma

In this section, we prove the Key Lemma as stated in Sect. 1. The first observation to make is that for Cartier divisors D on V, we can use Riemann-Roch and Vanishing theorems on cohomology in the same formal way as we could if V was smooth. If $\pi: \tilde{V} \to V$ denotes the C-Y resolution of singularities, we have $R^i \pi_* \mathcal{O}_{\tilde{V}} = 0$ for all $i > 0$ since the singularities of V are rational. Thus for any Cartier divisor D on V, we have $R^i \pi_* \mathcal{O}_{\tilde{V}}(\pi^* D) = 0$ for all $i > 0$ (using the projection formula as on p. 253 of [15]), and so $h^i(\mathcal{O}_V(D)) = h^i(\mathcal{O}_{\tilde{V}}(\pi^* D))$ for all i (using the degenerate Leray spectral sequence). Adopting the definition of $c_2 \cdot D$ introduced in Sect. 1, we see that Riemann-Roch remains true on V.

In the Lemma, we may clearly assume H to be a very ample divisor, and in particular from [31, 33] that it is a surface with at worst Du Val singularities (i.e. rational double points). We may also use Riemann-Roch in a formal way for Cartier divisors on H.

Since $D^2 \cdot H > 0$ and $D \cdot H^2 > 0$, Riemann-Roch shows that $h^0(\mathcal{O}_H(mD)) > 0$ for some $m > 0$. Write $mD|_H = \Delta + E$, where E is the fixed part of the complete linear system containing $mD|_H$, and $|\Delta|$ the mobile part. Note the above standard abuse of notation whereby we use equality to denote linear equivalence (alternatively we are failing in notation to distinguish between a divisor and its linear equivalence class).

We now consider the two possibilities:

(a) $D|_H$ is not nef.

(b) $D|_H$ is nef.

We concentrate first on the former case. There are clearly only finitely many curves C on H for which $D \cdot C < 0$ (only the components of E are candidates). We can therefore choose a rational number $\lambda > 0$ for which the $\mathbb{Q}$-divisor $D' = D + \lambda H$ is nef on H, but has degree zero intersection with some non-empty (finite) set of curves on H. Observe also that $(D'|_H)^2 = D^2 \cdot H + 2\lambda D \cdot H^2 + \lambda^2 H^3 > 0$, and so $D'|_H$ is nef and big (recall [47] that a nef divisor is called *big* if its top power is strictly positive).

In case (a) we shall set L to be a suitable integral multiple of D' so as to be a Cartier divisor on V; in case (b) we merely set $L = D$.

Claim 1. *Given that $L|_H$ is nef and big, we have $h^1(\mathcal{O}_V(-L)) = 0$.*

Proof. Consider the short exact sequences of sheaves

$$0 \to \mathcal{O}_V(-L-H) \to \mathcal{O}_V(-L) \to \mathcal{O}_H(-L) \to 0.$$

Since $L|_H$ is nef and big, an appropriate form [20, 44] of Kodaira Vanishing shows that $h^1(\mathcal{O}_H(-L)) = 0$. Taking cohomology of the above sequence, we see that

$$h^1(\mathcal{O}_V(-L-H)) \geqq h^1(\mathcal{O}_V(-L)).$$

Continuing by induction on m, we see that

$$h^1(\mathcal{O}_V(-L-mH)) \geqq h^1(\mathcal{O}_V(-L)) \quad \text{for all } m > 0.$$

But $h^1(\mathcal{O}_V(-L-mH)) = 0$ for m sufficiently large ([15], p. 244), and so Claim 1 follows.

With L as defined above, we observe that $L^3 > 0$, $L^2 \cdot H > 0$ and $L \cdot H^2 > 0$, since the inequalities were assumed true for D. Applying Riemann-Roch on V, we have

$$\chi(\mathcal{O}_V(nL)) = \tfrac{1}{6} n^3 L^3 + \tfrac{1}{12} n L \cdot c_2,$$

and Claim 1 applied to nL shows that $h^2(\mathcal{O}_V(nL))=h^1(\mathcal{O}_V(-nL))=0$ for $n>0$. Therefore

$$h^0(nL)\geqq\tfrac{1}{6}n^3 L^3+\tfrac{1}{12}nL\cdot c_2 \qquad \text{for } n>0,$$

and so for n sufficiently large, the linear system $|nL|$ is non-empty; we now fix such an n.

Claim 2. *In the case when L is not nef, we can find a rational $\delta>0$ such that the $\mathbb{Q}$-divisor $L+\delta H$ is nef on V, but has zero intersection with some (possibly infinitely many) curves.*

Remark. If L is nef, then we were in case (a) above and L itself is zero on some curves (including necessarily some curves on H).

Proof of Claim 2. Since nL is effective, we have that L is a $\mathbb{Q}$-Cartier effective $\mathbb{Q}$-divisor. The singularities of V are canonical, and so for sufficiently small rational $\varepsilon>0$, the pair $(V, \varepsilon L)$ has only log-terminal singularities (for a discussion of the condition log-terminal, see § 0–2 of [24]).

Choose a small rational number $\eta>0$ so that the $\mathbb{Q}$-divisor $L+\eta H$ remains not nef on V. Applying the Theorem of the Cone for log-terminal varieties (see Theorem 4-2-1 of [24]), we observe that the part of the cone of effective divisors $\overline{NE}(V)$ on which $(K_V+\varepsilon L)+\varepsilon\eta H$ is non-positive is finite polyhedral.

Therefore for $\delta>\eta$ a rational number, the $\mathbb{Q}$-Cartier divisor $L+\delta H$ is nef if and only if it is non-negative on some finite set of (numerical equivalence classes of) curves. Thus for some rational number $\delta>\eta$, the $\mathbb{Q}$-Cartier divisor $L+\delta H$ is nef on V, but $(L+\delta H)\cdot C=0$ for some curves C on V; thus Claim 2 is proved.

We are now in a position to prove the Key Lemma. If L defined above is not nef, we observe that the nef $\mathbb{Q}$-divisor $L+\delta H$ produced in Claim 2 is also big. We then set M to be a Cartier divisor obtained by taking a suitable integral multiple of this $\mathbb{Q}$-divisor. If on the other hand L is nef, we merely set $M=L$. In both cases, M is nef and big, but nevertheless zero on some curves. The results of Kawamata [21] (see also Theorem 5.1 of [47]) imply that for m sufficiently large, the linear system $|mM|$ is free, and that the corresponding morphism $\phi=\phi_{mM}\colon V\to\bar{V}$ is a birational morphism to a normal projective threefold $\bar{V}$, which contracts down precisely those curves C with $M\cdot C=0$.

It is now an easy check to see that $\phi^*\,\mathrm{Pic}_{\mathbb{Q}}(\bar{V})$ is a linear subspace of $\mathrm{Pic}_{\mathbb{Q}}(V)$ of dimension $\rho(\bar{V})<\rho(V)$. Thus $\phi\colon V\to\bar{V}$ is the required C-Y contraction.

3. Consequences of key lemma

For V a C-Y model, we can use the Key Lemma to deduce important results on the structure of V, which follow once we know that certain divisors on V exist.

Corollary 3.1. *If $H\in\mathrm{Pic}(V)$ ample and $D\in\mathrm{Pic}(V)$ with $D^3=0$, $D^2\cdot H>0$, then either a C-Y contraction exists, or else one of $\pm D$ is nef.*

Proof. From the Hodge index theorem on H ([15] p. 364), we have $(D^2 \cdot H)(H^3) \leq (D \cdot H^2)^2$. Our hypotheses then ensure that $D \cdot H^2 \neq 0$. By working with $\pm D$ as appropriate, we may assume that $D \cdot H^2 > 0$.

Suppose now that D is not nef; then for small rational $\varepsilon > 0$, the $\mathbb{Q}$-divisor $D + \varepsilon H$ is not nef. Since $D^3 = 0$, $D^2 \cdot H > 0$ and $D \cdot H^2 > 0$, we have that $(D + \varepsilon H)^3 > 0$, $(D + \varepsilon H)^2 \cdot H > 0$ and $(D + \varepsilon H) \cdot H^2 > 0$. The appropriate multiple of $D + \varepsilon H$ therefore satisfies the conditions of the Key Lemma, and so a C-Y contraction does exist. □

This Section will deal with the case above when one of D or $-D$ is nef. The main result will be Proposition 3.2 as stated in Sect. 1. Given (3.1), we are reduced to proving the following:

(3.2)′ Suppose there exists a nef divisor $D \in \mathrm{Pic}(V)$ with $D^3 = 0$, $D \cdot c_2 \neq 0$ and $D^2 \cdot H > 0$ for some ample $H \in \mathrm{Pic}(V)$, then for some $n > 0$, $|nD|$ is free and the corresponding morphism $\phi = \phi_{nD}: V \to S$ defines an elliptic fibre space structure on V (where S is a normal surface).

Proof. As in the proof of (3.1), we have $D \cdot H^2 \neq 0$; since D is assumed nef, it follows that $D \cdot H^2 > 0$. Since D is nef, it also follows from Theorem 1.1 of [27] that $D \cdot c_2 \geq 0$; under the assumptions of (3.2)′ therefore, we have $D \cdot c_2 > 0$.

As before, we may clearly assume that H is very ample, and in particular that it is a surface with at worst Du Val singularities [31, 33]. Since D is nef and $D^2 \cdot H > 0$, it follows from an appropriate form of Kodaira Vanishing [20, 42] that $h^1(\mathcal{O}_H(-nD)) = 0$ for all $n > 0$. Since $H + nD$ is ample for all $n \geq 0$ (cf. [47], Proposition 2.3), we see that $h^1(\mathcal{O}_V(-H - nD)) = 0$. By taking cohomology of the short exact sequences of sheaves

$$0 \to \mathcal{O}_V(-H - nD) \to \mathcal{O}_V(-nD) \to \mathcal{O}_H(-nD) \to 0,$$

we have that $h^2(\mathcal{O}_V(nD)) = h^1(\mathcal{O}_V(-nD)) = 0$ for all $n > 0$. Applying Riemann-Roch to nD, we deduce that $h^0(\mathcal{O}_V(nD)) \geq \frac{1}{12} nD \cdot c_2$.

Claim. *In the terminology of* [21, 22], *the nef divisor D is good; i.e. if ϕ_{nD} denotes the rational map determined by the complete linear system $|nD|$, then the image $\phi_{nD}(V)$ is a surface for some n sufficiently large.*

Remark. The proof of this very natural Claim is a little technical, and so the reader may wish to omit it on first reading. The proof is similar in style to that of (7.3) of [22], but also involves the theory of divisors of *canonical type* on a surface as developed in [29].

Proof of Claim. With $\pi: \tilde{V} \to V$ denoting the C-Y resolution of V, we can consider $\pi^* D$ on $\tilde{V}$; proving the Claim for $\pi^* D$ will imply it for D. We may assume therefore that V is smooth.

Suppose that the Claim is incorrect – then $\phi_{nD}(V)$ will be a curve for all n sufficiently large, since $h^0(\mathcal{O}_V(nD))$ is at least linear in n. Since this curve is normal and the irregularity of V is zero, we have $\phi_{nD}(V) = \mathbb{P}^1$ for large n. By taking n sufficiently large, we may assume that $\phi_{nD}^*: \mathbb{C}(\mathbb{P}^1) \hookrightarrow \mathbb{C}(V)$, with $\phi_{nD}^* \mathbb{C}(\mathbb{P}^1)$ algebraically closed in $\mathbb{C}(V)$ ([40], § 5).

We now fix such an n and resolve the base locus of the mobile part of $|nD|$, as in §7 of [40]. We have a birational modification μ: $V^* \to V$ with V^* smooth and with $\mu^*(nD) = mF + \sum r_i E_i$ say, where F is a smooth fibre of the morphism determined by $|\mu^* nD|$, and $\sum r_i E_i$ the fixed part of the linear system. Moreover, if we set $\mathscr{E} = \sum r_i E_i$, the theory of [40] (stated in Theorem 5.10 and proved in §7) shows that $\kappa(F, \mathscr{E}|_F) = 0$ (note slight difference of notation compared with [40]), i.e. no multiple of $\mathscr{E}|_F$ moves on F.

Set $L = \mu^* nD$; then L is nef with $L^3 = 0$. Observe that $K_{V^*} = \sum a_i E_i$, where as in the proof of the Key Lemma we may assume that $r_i \geqq a_i$ (i.e. we take an 'economical' resolution and do not blow up unnecessarily; to make things easier, we may assume n has been chosen so that $h^0(\mathcal{O}_V(nD)) > 2$, and so the multiplicity of the fibre $m > 1$). Set $\mathscr{E}' = \sum a_i E_i$, $G_i = E_i|_F$, $G = \mathscr{E}|_F$ and $G' = \mathscr{E}'|_F$. Since no multiple of G moves and $K_F = G' \leqq G$, we have that $\kappa(F) = 0$. We should also observe that $G \neq 0$; to see this consider the smooth surface $H^* = \mu^* H$ on V^* and the corresponding fibre space map $H^* \to \mathbb{P}^1$. If $G = 0$, then $\mathscr{E}|_{H^*}$ is contained in fibres, and so the same is true for $\mu^*(nD)|_{H^*}$. A standard lemma from the theory of surfaces ([1], page 90) implies that $(\mu^*(nD)|_{H^*})^2 \leqq 0$, contradicting our assumption that $D^2 \cdot H > 0$.

Let $\bar{F}$ denote the smooth minimal model of F, with α: $F \to \bar{F}$ say. Note that $K_{\bar{F}} = 0$, i.e. all the components of G' are contracted under α. Let $\bar{G}$ denote the image of G in $\bar{F}$; then $\bar{G}$ is nef and $G = \alpha^* \bar{G} + \Delta_1 - \Delta_2$ say, where Δ_1, Δ_2 are effective divisors supported on the exceptional locus of α.

Since $\mu^* nD$ is nef with $(\mu^* nD)^3 = 0$, it follows that $\mathscr{E}^2 \cdot F = 0$. Thus

$$\begin{aligned} 0 = G^2 &= (\alpha^* \bar{G} + \Delta_1 - \Delta_2)^2 \\ &= \bar{G}^2 + (\Delta_1 - \Delta_2)^2 \end{aligned}$$

where $(\Delta_1 - \Delta_2)^2 = \Delta_1^2 - 2\Delta_1 \cdot \Delta_2 + \Delta_2^2 \leqq 0$, with equality only if $G = \alpha^* \bar{G}$ (an easy deduction from the Hodge index theorem on F, [15], p. 364). Therefore $\bar{G}^2 \geqq 0$, with equality only if $G = \alpha^* \bar{G}$.

Since $G^2 = 0$ and G is nef and effective, we have that $G \cdot G_i = 0$ for all components G_i of G. Since any exceptional curve under α is a component of $G' \leqq G$, it is clear that any component of Δ_1 or Δ_2 is a G_i. Thus $G \cdot \alpha^* \bar{G} = G^2 = 0$. Suppose now that $\bar{G}^2 > 0$; the Hodge index theorem would then imply that $G^2 < 0$, a contradiction. We must therefore have $\bar{G}^2 = 0$ and $G = \alpha^* \bar{G}$.

On $\bar{F}$, the divisor $\bar{G}$ is nef with $\bar{G}^2 = 0$. Since $K_{\bar{F}} = 0$, we have in the terminology of [29] that $\bar{G}$ is a divisor of *canonical type*, and so using Step III on p. 334 of [29], we deduce that some multiple of $\bar{G}$ moves on $\bar{F}$. From this it follows that some multiple of G moves on F, contrary to assumptions. The Claim has now been proved.

(3.2)′ now follows easily from the above Claim and Theorem 6.1 of [22]. We have shown that the nef divisor D is good, and the other conditions of the theorem are clear in our case. Hence for some large n, the linear system $|nD|$ is free, and the corresponding morphism ϕ_{nD}: $V \to S$ will exhibit V as a fibre space over a normal surface S, and the general fibre will be an elliptic curve since $K_V = 0$. □

Remark. Given the properties of V, it follows immediately that S will be a rational surface.

Proposition 3.3. *In the situation of* (3.2)′ *where we have an elliptic space structure* $\phi: V \to S$, $\phi^* \mathrm{Pic}_{\mathbb{Q}}(S)$ *is precisely the linear subspace of* $\mathrm{Pic}_{\mathbb{Q}}(V)$ *consisting of* $\mathbb{Q}$-*divisors* X *such that* $X \cdot C = 0$ *whenever* $D \cdot C = 0$.

Proof. Clearly any element of $\phi^* \mathrm{Pic}_{\mathbb{Q}}(S)$ satisfies the given condition; we need to prove the converse. Suppose therefore that $L \in \mathrm{Pic}(V)$ has the property that $L \cdot C = 0$ whenever $D \cdot C = 0$.

Choose a very ample divisor P on S. Since L is relatively nef for ϕ, the divisor $M = L + s\phi^* P$ is nef for all s sufficiently large. Clearly such an M is not big, since for any fibre ℓ of ϕ, we have $M \cdot \ell = 0$. Thus $M^3 = 0$. By taking s sufficiently large, we can however assume that the numerical M-dimension $\nu(V, M) = 2$ (see Sect. 1 of [22]); i.e. $M^2 \cdot H > 0$ for some ample divisor H.

Since for our original divisor D (giving rise to the fibre space structure) we had $D \cdot c_2 > 0$, we may assume (taking s large enough) that $M \cdot c_2 > 0$. We can now apply (3.2)′ to the divisor M to deduce that for some $r > 0$, the linear system $|rM|$ is free a defines a fibre space morphism ϕ': $V \to S'$; by taking s large enough, we may assume that our original morphism ϕ factors via ϕ'.

Our assumption on L ensures that exactly the same curves are contracted by ϕ and ϕ'; since S and S' are normal, Zariski's Main Theorem ([15] p. 280) ensures that $S = S'$ and $M \in \phi^* \mathrm{Pic}(S)$. Hence $L \in \phi^* \mathrm{Pic}(S)$, and the Proposition is proved. □

In the case therefore of (3.2) giving an elliptic fibre space ϕ: $V \to S$, we have seen that $\phi^* \mathrm{Pic}_{\mathbb{Q}}(S)$ is a linear subspace of $\mathrm{Pic}_{\mathbb{Q}}(V)$ containing D and of dimension $\rho(S) < \rho(V)$. We shall consider the Sect. 4 the case when this linear space is a hyperplane (see Lemma 4.3), i.e. $\rho(S) = \rho(V) - 1$. We conclude this Section by showing that when $\rho(S) < \rho(V) - 1$, we do have a C-Y contraction on V (indeed one which respects the elliptic fibre space structure can be found).

Proposition 3.4. *With* ϕ: $V \to S$ *as in* (3.3) *and* $\rho(S) < \rho(V) - 1$, *there exists a C-Y contraction on V.*

Proof. Consider the relative effective cone of 1-cycles $NE(V/S)$ as defined in [21]; this contains the numerical class of the general fibre ℓ (an elliptic curve), but will also contain the class of some other curve C, numerically independent from that of ℓ (by (3.3), the elements of $\phi^* \mathrm{Pic}_{\mathbb{Q}}(S)$ are characterized by the conditions $X \cdot Z = 0$ for every 1-cycle $Z \in NE(X/S)$, and so our assumptions imply that $NE(X/S)$ contains at least two independent classes). We may assume that the curve C is irreducible, and choose a divisor M on V such that $M \cdot C < 0$ but $M \cdot \ell > 0$. Choosing an ample divisor P on S, we can consider the divisor $D = M + s\phi^* P$ for large s.

An easy calculation verifies that for H any ample divisor on V and s sufficiently large, $D^3 > 0$, $D^2 \cdot H > 0$ and $D \cdot H^2 > 0$. By construction however, $D \cdot C < 0$, and so the Key Lemma can be used to provide the required contraction. □

4. Diophantine geometry of the intersection form

For V a C-Y model, we consider the cubic hypersurface W in $\mathbb{P}(\mathrm{Pic}_{\mathbb{C}}(V)) = \mathbb{P}^{\rho-1}$ consisting of points representing divisors D with $D^3 = 0$ (we shall often not distinguish notationally between a divisor D, its numerical equivalence class and the corresponding point of $\mathbb{P}^{\rho-1}$). In $\mathbb{P}^{\rho-1}$ we also have the hyperplane defined by the linear form c_2 (see Sect. 1). Summarising and rephrasing the results of Sect. 3, we obtain:

Corollary 4.1. *If W contains a rational point (represented by a divisor $D \in \operatorname{Pic}(V)$) not contained in any hyperplane component of W and not contained in the hyperplane defined by c_2, and such that $D^2 \cdot H > 0$ for some ample divisor $H \in \operatorname{Pic}(V)$, then a C-Y contraction exists on V.*

The question of what geometric properties the cubic hypersurface W and its real locus $W(\mathbb{R})$ satisfy seems rather a subtle one – essentially the only properties we use are those derived from the Hodge index theorem.

Lemma 4.2. *Given divisors L, $H \in \operatorname{Pic}(V)$, L not numerically trivial but with $L^3 = 0$, and where H is very ample, then the real valued function $g(t) = (L + tH)^3$ cannot have a triple root at $t = 0$. If moreover $L^2 \cdot H \leqq 0$, we can find a real number $\lambda \neq 0$ with $(L + \lambda H)^3 = 0$ and $(L + \lambda H)^2 \cdot H > 0$.*

Proof. Observe that $g'(0) = 3L^2 \cdot H$ and $g''(0) = 6L \cdot H^2$. If $t = 0$ were a triple root, then we would have $L^2 \cdot H = L \cdot H^2 = 0$. Using the Hodge index theorem on H (assumed general in its linear system), we deduce that $L|_H$ is numerically trivial. A slight generalization of the Lefshetz Hyperplane theorem implies that L is numerically trivial, contrary to assumption. For completeness, we give a brief proof of this last step: since $h^1(\mathcal{O}_H) = 0$, the Picard group $\operatorname{Pic}(H)$ is discrete and so $\mathcal{O}_H(dL) = \mathcal{O}_H$ for some $d > 0$. An appropriate form of Kodaira Vanishing gives $h^1(\mathcal{O}_H(-nH)) = 0$ for all $n > 0$, and an argument by induction similar to one used before then shows that $h^1(\mathcal{O}_V(dL - H)) = 0$. Taking sections of the exact sequence

$$0 \to \mathcal{O}_V(dL - H) \to \mathcal{O}_V(dL) \to \mathcal{O}_H \to 0,$$

we obtain $h^0(\mathcal{O}_V(dL)) > 0$, and thus dL is trivial in $\operatorname{Pic}(V)$.

If $L^2 \cdot H = g'(0) \leqq 0$, we observe that the cubic g must have a real root $\lambda \neq 0$ with $g'(\lambda) > 0$; hence $(L + \lambda H)^3 = 0$ and $(L + \lambda H)^2 \cdot H > 0$ as claimed. □

We deduce from (4.2) that W is not a cone. Hence if $\rho > 3$, W does not consist of 3 hyperplanes; moreover if W contains a hyperplane, it must be rational.

Lemma 4.3. *If $\rho > 5$ and W contains a hyperplane, then W contains a rational point (represented by a divisor $D \in \operatorname{Pic}(V)$) not on the hyperplane and also not on the hyperplane defined by c_2, and such that $D^2 \cdot H > 0$ for some ample divisor $H \in \operatorname{Pic}(V)$. Moreover, the rational points are dense in the real locus.*

Proof. Set $W = M \cup Q$ with M the rational hyperplane and Q the residual quadric. Let $H \in \operatorname{Pic}(V)$ be any ample divisor on V. Choose a rational point on M representing a divisor E on V with $E \cdot H^2 = 0$. The Hodge index theorem on H implies that $E^2 \cdot H \leqq 0$. Applying (4.2) with $L = E$, we obtain a real point on W but not on M for which the corresponding element $F \in \operatorname{Pic}_{\mathbb{R}}(V)$ satisfies $F^2 \cdot H > 0$. By varying E, we can clearly obtain infinitely many such real points of Q not on M.

We consider first the case when Q is singular. Since W is not a cone, Q must be a quadric cone with point vertex not on M. As there are other real points of Q apart from the vertex, we see that $Q \cap M$ has real points. Since $\dim M \geqq 4$, we deduce the existence of a rational point on $Q \cap M$, using the well-known result that an indefinite rational quadratic form in 5 or more variables must have a non-trivial rational zero ([38] p. 43). But $Q \cap M$ is non-singular, and so the rational points of $Q \cap M$ are dense in the real locus with respect to the classical topology. This in turn implies that the rational points of Q are dense in the real locus $Q(\mathbb{R})$. This result is however also true when Q is non-singular, since the above quoted standard result on quadratic forms yields a rational point on Q, and this ensures that the rational points are dense in $Q(\mathbb{R})$.

With $F \in Q(\mathbb{R})$ as obtained above, we can find (since the rational points are dense in the classical topology) a rational point on Q near to $F \in Q(\mathbb{R})$ representing a divisor $D \in \mathrm{Pic}(V)$ with $D \cdot c_2 \neq 0$ and $D^2 \cdot H > 0$, as claimed by the Lemma. □

Corollary 4.4. *If $\rho > 5$ and W is reducible, then there exists a C-Y contraction on V.*

Proof. Use (4.1) and (4.3). □

We now restrict ourselves to the case when W is irreducible. The tie-up between the Diophantine Geometry of W and the geometry on V is given by the following result.

Proposition 4.5. *If W irreducible, $\rho > 2$, and the rational points of W are dense in the classical topology on $W(\mathbb{R})$, then there exists a C-Y contraction on V.*

Proof. Since W irreducible, we can choose a rational point on W whose corresponding divisor L has $L \cdot c_2 \neq 0$. Let $H \in \mathrm{Pic}(V)$ be an ample divisor; if $L^2 \cdot H > 0$, we merely set $D = L$ and apply (4.1).

If $L^2 \cdot H \leqq 0$, we apply (4.2) to deduce the existence of a point in $W(\mathbb{R})$ representing $F \in \mathrm{Pic}_{\mathbb{R}}(V)$ with $F^2 \cdot H > 0$. Since by assumption the rational points of W are dense in $W(\mathbb{R})$ (and W does not contain a hyperplane), we can find a rational point of W (near to the real point representing F) representing a divisor $D \in \mathrm{Pic}(V)$ with $D \cdot c_2 \neq 0$ and $D^2 \cdot H > 0$. Now apply (4.1) again. □

Corollary 4.6. *If W is irreducible and contains a singular rational point, then a C-Y contraction on V exists.*

Proof. Using (4.1) we see that $\rho > 2$; we have also seen that W cannot be a cone. An obvious argument then shows that the rational points of W are dense in $W(\mathbb{R})$, and so (4.5) applies. □

Remark. If W is non-singular and $\rho > 9$, then the results of Heath-Brown [16] (see in particular the top of p. 230) ensure that the rational points are dense in the real locus, and so there is a C-Y contraction on V. As yet the author does not know any examples where W turns out to be irreducible and singular, but without singular rational points.

Proposition 4.7. *If $W \subset \mathbb{P}^{\rho-1}$ $(\rho > 5)$ is irreducible and contains a rational linear space Λ of dimension 3, then the rational points of W are dense in $W(\mathbb{R})$.*

Proof. Choose an arbitrary ample divisor $H \in \mathrm{Pic}(V)$ and consider the 4-dimensional rational linear space containing Λ and the point of $\mathbb{P}^{\rho-1}$ determined by H. Let $\bar{W}$ denote the intersection of W with this $\mathbb{P}^4$; then $\bar{W} = \Lambda \cup Q \subset \mathbb{P}^4$, where Q is the residual quadric hypersurface.

If Q is singular, then it is a cone; using (4.2) we observe that Q can only have one vertex, which is *rational* and does not lie on Λ (cf. also the proof of (4.3)).

However, unless W has singularities at rational points (in which case we have already observed in (4.6) that the result holds), we can choose our ample divisor H (and thus our 4-dimensional rational linear space containing Λ) so that no such singularity occurs on Q – here we are essentially just applying Bertini's theorem. We may assume therefore that our residual quadric Q is non-singular.

From this we deduce that for the general 4-dimensional rational linear space through Λ, the intersection of W with this space, $W' = \Lambda \cup Q'$ with non-singular residual quadric Q'.

For each such intersection, the previously used standard result on indefinite quadratic forms [38] shows that if Q' has real points, then the rational points of Q' are dense in the real locus $Q'(\mathbb{R})$. Since this holds for the general 4-dimensional rational linear space through Λ, we deduce that the rational points of W are dense in $W(\mathbb{R})$ as claimed. □

We now turn to the important work of Davenport [8], which uses the Hardy-Littlewood method to produce rational points on cubic hypersurfaces, provided that the number of variables is at least 16. We however need a slight generalization of the 17 variable case.

Theorem 4.8 (Davenport). *If a cubic hypersurface $W \subset \mathbb{P}^{n-1}$ does not contain any rational linear space of dimension r and if $n \geqq 17 + r$, then the rational points of W are dense in the real locus $W(\mathbb{R})$.*

Proof. The proof of this is essentially an easy modification of the 17 variable argument in [8]. The result however does appear more or less explicitly in [9]. If the reader consults p. 658 of that paper (in particular the statement of Theorem 1 and the two paragraphs following), together with the last paragraph of the paper on p. 671, he will recover the result stated above. I remark that Davenport is dealing with the affine case, and so his cubic cone will just be the affine cone over W; moreover, we are only interested in the homogeneous case (in his notation $\phi = C$). His deduction on p. 671 that the vectors from the origin to the solutions lie asymptotically everywhere dense on the cubic cone is just our statement that the rational points are dense in $W(\mathbb{R})$. His invariant $h(C)$ defined on p. 658 is the maximum integer h for which W contains no linear spaces of dimension $\geqq n-h$; the conditions of (4.8) ensure that $h(C) \geqq 17$ as required by his Theorem 1. □

Proof of Theorem 1.1, and hence the main theorem. We show first that a C-Y contraction exists. In the statement of (1.1), we are given that $\rho(V) > 19$. In the light of (4.4), we may assume that the cubic surface $W \subset \mathbb{P}^{\rho-1}$ is irreducible. Using (4.7) and (4.8) taken together, we see that the rational points of W are dense in $W(\mathbb{R})$. The required C-Y contraction $f: V \to \bar{V}$ is then provided by (4.5). Let $P \in \mathrm{Pic}(V)$ denote the pull-back of a hyperplane section $\bar{V}$; thus P is nef and big. Using Theorem 1.1 of [27], we observe that $P \cdot c_2 \geqq 0$; if $P \cdot c_2 > 0$, then we also have the required condition that $c_2(\bar{V})$ is non-trivial.

Suppose therefore that $P \cdot c_2 = 0$ (i.e. P is in the hyperplane defined by $c_2 = 0$). We shall produce a C-Y contraction of V which definitely does not trivialize the form c_2.

We choose a nef divisor $L \in \mathrm{Pic}_{\mathbb{R}}(V)$ on the hyperplane $c_2 = 0$, with the property that $L - \varepsilon P$ is not nef for any $\varepsilon > 0$. If $L^3 > 0$, it follows that $L^2 \cdot H > 0$ and $L \cdot H^2 > 0$. We can now choose $D \in \mathrm{Pic}_{\mathbb{Q}}(V)$ close to L with the properties that D is not nef but $D^3 > 0$, $D^2 \cdot H > 0$, $D \cdot H^2 > 0$ *and* $D \cdot c_2 > 0$. The C-Y contraction provided by the Key Lemma does not then trivialize c_2, since the divisor M which gives the morphism has the property that $M \cdot c_2 > 0$.

We may suppose therefore that $L \in W(\mathbb{R})$. Since L is nef, we have $L^2 \cdot H \geqq 0$, and since it is also not numerically trivial we have $L \cdot H^2 > 0$. Since $c_2 \cdot N \geqq 0$ for any nef divisor N, we know that $c_2 \cdot H > 0$.

We consider first the case above when $L^2 \cdot H > 0$. Since $\kappa(V, P) = 3$, we deduce that $L^2 \cdot P > 0$, and hence that $(L - \varepsilon P)^3 < 0$, $(L - \varepsilon P)^2 \cdot H > 0$ and $(L - \varepsilon P) \cdot H^2 > 0$ for all sufficiently small $\varepsilon > 0$. To such a divisor $L - \varepsilon P$, we may add some positive multiple of H to achieve a divisor $R \in W(\mathbb{R})$ with $R^2 \cdot H > 0$, $R \cdot H^2 > 0$ *and* $R \cdot c_2 > 0$. Both for W reducible and irreducible, we have however seen that the rational points of W are dense in $W(\mathbb{R})$, and so we can find a rational point of W close to R represented by a divisor $D \in \mathrm{Pic}(V)$ with $D^3 = 0$ and satisfying the inequalities $D^2 \cdot H > 0$, $D \cdot H^2 > 0$ *and* $D \cdot c_2 > 0$. The results of Sect. 3 then yield a C-Y contraction which does not trivialize c_2.

Finally, we consider the case when $L^3 = L^2 \cdot H = 0$. As in (4.3), we can consider the real divisors $-L + tH \in \mathrm{Pic}_{\mathbb{R}}(V)$, and the real valued function $g(t) = (-L + tH)^3$. Observe that $g(0) = 0 = g'(0)$ and $g''(0) < 0$; thus for some $t > 0$ we obtain a real divisor $R = -L + tH$ with $R^3 = 0$, $R^2 \cdot H > 0$, $R \cdot H^2 > 0$ and $R \cdot c_2 > 0$ (recall that by assumption $L \cdot c_2 = 0$). As in the previous case, we can then find a rational divisor D with $D^3 = 0$ and satisfying the previous inequalities, and then the results of Sect. 3 yield a C-Y contraction of the required type. □

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Corrigendum added in proof

Prof. N. Nakayama has pointed out to the author that the proof of (3.3) is not valid. To circumvent this, we show instead that not too many (in a sense to be made explicit below) divisors D with $D^3=0$ correspond to elliptic fibrations which cannot be dealt with easily, and then appeal to the density results of Section 4. We first replace (3.3) by:

(3.3)′ *If $\phi: V\to S$ is as in (3.2)′ and $NE(V/S)$ contains numerically independent 1-cycles, then there exists a C-Y contraction on V.*

The proof of this is as in (3.4).

We now consider the surface S; by Corollary (0.4) of Nakayama's paper [On Weierstrass Models; in Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, pp 405–431. Academic Press, 1989] we know that S has only log-terminal singularities (which for surfaces are $\mathbb{Q}$-factorial), and that $-K_S$ is an effective $\mathbb{Q}$-divisor.

We observe that if S admits a birational morphism $g: S\to\bar{S}$ contracting a curve C to a point, then V itself admits a C-Y contraction. To see this, we consider the $\mathbb{Q}$-Cartier divisor $E=\phi^* C$ on V, and show that its restriction to a general hyperplane section H gives an element of $NE(V/\bar{S})$ which is not numerically equivalent to a multiple of the general fibre of ϕ.

It is easily checked that $-K_S$ is not numerically trivial; indeed $c_2\cdot D=-12K_S\cdot M$ for an appropriate hyperplane section M of S. We then apply Mori's theory of contracting extremal curves on the log-terminal surface S (see [24]). Arguing in this way, it follows that unless $S=\mathbb{P}^1\times\mathbb{P}^1$ or S is a log Del Pezzo surface with $\rho(S)=1$, then a birational morphism g as above can be found on S, and hence that a C-Y contraction exists.

The main extra ingredient is to prove that not too many elliptic fibrations arising via (3.2)′ can have S being one of the above two types. As usual, we let H denote a general hyperplane section of V.

Proposition 3.4′. *Suppose that there is no C-Y contraction on V; then given $\varepsilon>0$,*

(i) *there exist only finitely many elliptic fibrations $\phi: V\to S$ arising via* (3.2)′ *with $\rho(S)=1$ and such that the divisor D of* (3.2)′ *satisfies $D\cdot c_2\geq\varepsilon D\cdot H^2$;*

(ii) *there are also only finitely many fibrations with $S=\mathbb{P}^1\times\mathbb{P}^1$ and such that $D\cdot c_2\geq\varepsilon D\cdot H^2$ for some D in the pullback of the nef cone from S.*

Proof. Under our assumptions, there can be only finitely many distinct elliptic fibrations $\phi_i: V\to S_i$ with non-equidimensional fibres. This follows since for each i, we have a Weil divisor E_i contracted to a point by ϕ_i and a curve $C_i=E_i|_H$ on H. If E_i and E_j were to contain a common curve for some $i\neq j$, a C-Y contraction (contracting this curve) would be easy to find. So, given our assumptions, it will follow that the C_i are disjoint, numerically independent curves on H, and hence there are no more than $\rho(H)$ of them.

In both (i) and (ii) therefore, we can restrict ourselves to the case of equidimensional fibres. Thus given an irreducible curve C on S, (with possibly finitely many exceptions) $E=\phi^{-1}(C)$ is an irreducible Weil divisor on V; moreover, since C is $\mathbb{Q}$-Cartier on S, so too is E on V.

We prove (i) and leave (ii) as an exercise. Let $\tilde{S}$ denote a minimal desingularization of S and S^* its minimal model. On S^*, we take C^* to be a general line if $S^*=\mathbb{P}^2$ and a general fibre if S^* is ruled (recall that S is rational). Let C be the curve on S corresponding to C^*; a straightforward check verifies that $0<-K_S\cdot C\leq 3$. Let D_0 be the $\mathbb{Q}$-Cartier irreducible Weil divisor $\phi^{-1}(C)$ on V. Since $\rho(S)=1$, the original D is just a rational multiple of D_0.

To calculate $D_0\cdot c_2$, we first compute $L\cdot c_2$ for L the pullback of the general hyperplane section M of S. Riemann-Roch, the adjunction formula and Corollary 12.3 from [1] together yield that $\frac{1}{12}c_2\cdot L=-K_S\cdot M$. Since $\rho(S)=1$, this holds for the pullback of any $\mathbb{Q}$-divisor on S, and in particular that $c_2\cdot D_0=-12K_S\cdot C\leq 36$.

We now let Λ denote the discrete subgroup of $\mathrm{Pic}_{\mathbb{Q}}(V)$ generated by the $\mathbb{Q}$-Cartier Weil divisors. That part of the nef cone in $\mathrm{Pic}_{\mathbb{Q}}(V)$ given by the inequality $X\cdot H^2\leq 36/\varepsilon$ is clearly compact, and hence contains only finitely many elements of Λ. Since any of the elliptic fibrations in question arise from such a Weil divisor D_0, the claimed result follows.

We prove (ii) similarly, noting that if $E_i=\phi^{-1}(C_i)$ for C_i a line in one of the two rulings of S, then $c_2\cdot E_i=24$. Moreover, the pullback under ϕ^* of the nef cone on S is generated by the divisors E_1 and E_2 on V. □

The arguments in Sect. 4 of the paper remain essentially unchanged. If we happen on a nef divisor D with $D^3=0$, $D^2\cdot H>0$ and $D\cdot c_2>0$, we can choose $\varepsilon>0$ with $D\cdot c_2>\varepsilon D\cdot H^2$. Since the rational points of W are dense in the real locus $W(\mathbb{R})$ for all the cases considered, we may assume (using (3.4)′) that D has been chosen so as not to give rise to an elliptic fibration over either $\mathbb{P}^1\times\mathbb{P}^1$ or a log Del Pezzo surface with $\rho=1$. The above arguments together with (3.2) then imply the existence of the desired C-Y contraction.