Invent. math. 77, 489–515 (1984)

Inventiones mathematicae

The 4-class ranks of quadratic fields

Frank Gerth III

Department of Mathematics, The University of Texas, Austin, TX 78712, USA

1. Introduction

Let K be a quadratic extension field of the rational numbers $\mathbf{Q}$. Let C_K be the 2-class group of K in the narrow sense. It is a classical result that rank $C_K = t-1$, where t is the number of primes that ramify in $K/\mathbf{Q}$. Now let $C_K^i = \{a^i\colon a \in C_K\}$, and let R_K denote the 4-class rank of K in the narrow sense; i.e., $R_K = \text{rank } C_K^2 = \dim_{\mathbf{F}_2}(C_K^2/C_K^4)$. Here $\mathbf{F}_2$ is the finite field with two elements, and C_K^2/C_K^4 is an elementary abelian 2-group which we are viewing as a vector space over $\mathbf{F}_2$. Given a quadratic field K, one can compute R_K by computing the rank (over $\mathbf{F}_2$) of a certain matrix of Legendre symbols (cf. [11]).

Now assume K is imaginary quadratic. So $K = \mathbf{Q}(\sqrt{-m})$, where m is a square-free positive integer. For each positive integer t, each nonnegative integer e, and each positive real number x, we define

$$\begin{aligned} A_t &= \{K = \mathbf{Q}(\sqrt{-m})\colon \text{exactly } t \text{ primes ramify in } K/\mathbf{Q}\}, \\ A_{t;x} &= \{K \in A_t\colon m \leqq x\}, \\ A_{t,e} &= \{K \in A_t\colon R_K = e\}, \\ A_{t,e;x} &= \{K \in A_{t,e}\colon m \leqq x\}. \end{aligned}$$

Next we define the density $d_{t,e}$ of $A_{t,e}$ in A_t by

$$d_{t,e} = \lim_{x \to \infty} \frac{|A_{t,e;x}|}{|A_{t;x}|} \tag{1.1}$$

where $|S|$ denotes the cardinality of a set S.

In this paper we derive an effective algorithm for computing $d_{t,e}$ for $t \geqq 1$ and $e \geqq 0$. We also determine the limiting density

$$d_{\infty,e} = \lim_{t \to \infty} d_{t,e} \tag{1.2}$$

and obtain asymptotic formulas (as $x \to \infty$) for each $|A_{t;x}|$ and $|A_{t,e;x}|$.

If K is real quadratic, we write $K=\mathbf{Q}(\sqrt{m})$, where m is a square-free positive integer. We then define

$$\begin{aligned} B_t &= \{K=\mathbf{Q}(\sqrt{m})\colon \text{exactly } t \text{ primes ramify in } K/\mathbf{Q}\}, \\ B_{t;x} &= \{K\in B_t\colon m\leqq x\}, \\ B_{t,e} &= \{K\in B_t\colon R_K=e\}, \\ B_{t,e;x} &= \{K\in B_{t,e}\colon m\leqq x\}, \end{aligned}$$

$$d'_{t,e}=\lim_{x\to\infty}\frac{|B_{t,e;x}|}{|B_{t;x}|}, \tag{1.3}$$

$$d'_{\infty,e}=\lim_{t\to\infty} d'_{t,e}. \tag{1.4}$$

We shall derive an effective algorithm for computing $d'_{t,e}$; we shall determine $d'_{\infty,e}$; and we shall obtain asymptotic formulas (as $x\to\infty$) for $|B_{t;x}|$ and $|B_{t,e;x}|$.

Some numerical results for $d_{t,e}$, $d_{\infty,e}$, $d'_{t,e}$, and $d'_{\infty,e}$ appear in Appendix II and Appendix IV. Note that for imaginary quadratic fields, the 4-class rank $e=1$ occurs most frequently, followed by $e=0$, $e=2$, $e=3$, ... For large t, $d_{t,1}$ is approximately twice $d_{t,0}$ (in fact $d_{\infty,1}=2d_{\infty,0}$). Also for each t,

$$d_{t,0}+d_{t,1}+d_{t,2}>0.99.$$

For real quadratic fields the 4-class rank $e=0$ occurs most frequently, followed by $e=1$, $e=2$, $e=3$, ... For large t, $d'_{t,0}$ is approximately 1.5 times $d'_{t,1}$ (in fact $d'_{\infty,0}=1.5d'_{\infty,1}$). Also for each t,

$$d'_{t,0}+d'_{t,1}+d'_{t,2}>0.997.$$

Our formulas for the limiting densities are as follows: For imaginary quadratic fields

$$d_{\infty,r}=\frac{2^{-r^2}\prod_{k=1}^{\infty}(1-2^{-k})}{\prod_{k=1}^{r}(1-2^{-k})^2} \quad \text{for } r=0,1,2,\ldots; \tag{1.5}$$

for real quadratic fields

$$d'_{\infty,r}=\frac{2^{-r(r+1)}\prod_{k=1}^{\infty}(1-2^{-k})}{\prod_{k=1}^{r}(1-2^{-k})\prod_{k=1}^{r+1}(1-2^{-k})} \quad \text{for } r=0,1,2,\ldots. \tag{1.6}$$

Since the 4-class rank of a quadratic field is the same as the 2-rank of the principal genus of the quadratic field, our formulas can be viewed as being in perfect accordance with heuristic predictions of Cohen and Lenstra (made only for primes $p\geqq 3$) on the p-class ranks of quadratic fields. (See [1] and [2].)

Although we have calculated density results for the 4-class ranks in the narrow sense, the density results for the 4-class ranks in the usual sense are close to our density results. In fact, for quadratic fields one can show that the

4-class ranks in the narrow sense and in the usual sense can be different only if the quadratic field is real, no ramified prime is congruent to 3 (mod 4), and the fundamental unit has norm $+1$. If $\bar{d}'_{t,e}$ denotes the density for the 4-class rank in the usual sense, then one can show that

$$\sum_{e=0}^{\infty} |\bar{d}'_{t,e} - d'_{t,e}| \leqq 2^{-(t-1)} \quad \text{for } t \geqq 2.$$

(For $t=1$, $\bar{d}'_{t,e} = d'_{t,e}$.) So $\bar{d}'_{\infty,e} = d'_{\infty,e}$.

We close this section by mentioning a few other papers that consider densities of 4-class ranks of quadratic fields. [12] considers real quadratic fields in which each ramified prime is congruent to 1 (mod 4) and obtains certain densities for the 4-class ranks of these types of quadratic fields. In [10] imaginary quadratic fields $\mathbf{Q}(\sqrt{-m})$ with $m = p_1 \dots p_{t-1} q$ are considered, where each prime $p_i \equiv 1 \pmod 4$ and the prime $q \equiv 3 \pmod 4$. Some density results are obtained in terms of the density of the set of primes q having certain properties. In [6] the density $d_{t,e}$ has been computed for $t \leqq 2$. [3] contains various results and conjectures about the 2-class groups of quadratic fields, and it also contains an extensive list of references.

Acknowledgements. The author thanks S.W. Graham for many helpful discussions. The author also thanks J. Vaaler and the referee for helpful suggestions.

2. Preliminary results for imaginary quadratic fields

We let notations be as in Sect. 1. In this section we consider imaginary quadratic fields $K = \mathbf{Q}(\sqrt{-m})$, where m is a square-free positive integer. We let $p_1 < p_2 < \dots$ be the odd prime numbers dividing m. If $K \in A_t$, then it is easy to see that

$$m = p_1 \dots p_t \text{ with an odd number of } p_i \equiv 3 \pmod 4, \text{ or} \tag{2.1}$$

$$m = p_1 \dots p_{t-1} \text{ with an even number of } p_i \equiv 3 \pmod 4, \text{ or} \tag{2.2}$$

$$m = 2 p_1 \dots p_{t-1}. \tag{2.3}$$

If x is a positive real number, we let $N_{i;x}$ denote the number of $m \leqq x$ satisfying Eq. (2.1) ($i = 1, 2, 3$). We let N_x be the number of square-free positive integers up to x with t prime factors. It is well known (see [7], Theorem 437) that

$$N_x \sim \frac{1}{(t-1)!} \frac{x (\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty). \tag{2.4}$$

Next we note that $N_{1;x} \sim \frac{1}{2} N_x$ since the number of $p_i \equiv 3 \pmod 4$ that divide m is assumed to be odd in this case. Also $N_{2;x} = o(N_x)$ and $N_{3;x} = o(N_x)$. So

$$|A_{t;x}| \sim \frac{1}{2} \frac{1}{(t-1)!} \frac{x (\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty), \tag{2.5}$$

and we may confine our attention to those m which satisfy Eq. (2.1).

To each field $K=\mathbf{Q}(\sqrt{-m})$ with m satisfying Eq. (2.1), we associate a $t\times(t-1)$ matrix $M'_K=[a_{ij}]$, where each $a_{ij}\in\mathbf{F}_2$ is defined in the following way. Let $P_i=p_i$ if $p_i\equiv 1 \pmod 4$, and let $P_i=-p_i$ if $p_i\equiv 3 \pmod 4$. Let $\bar{P}_i=-m/P_i$, and let $(-)$ denote the Legendre symbol. Then

$$(-1)^{a_{ij}}=\begin{cases}\left(\dfrac{P_j}{p_i}\right) & \text{if } i\neq j\\[2ex] \left(\dfrac{\bar{P}_j}{p_i}\right) & \text{if } i=j\end{cases} \qquad \text{for } 1\leqq i\leqq t \text{ and } 1\leqq j\leqq t-1. \tag{2.6}$$

The matrix M'_K is the Rédei matrix (see [11]) written in additive notation. Then the 4-class rank of K satisfies $R_K=t-1-\operatorname{rank} M'_K$. (Remark: Actually Rédei expresses the 4-class rank in terms of what we now call the null space of M'_K, but of course dim (null space of M'_K)$=t-1-\operatorname{rank} M'_K$.)

There are two other matrices closely related to M'_K that we shall use. We let M_K be the $t\times t$ matrix whose entries are defined by Eq. (2.6), except with $1\leqq j\leqq t$ instead of $1\leqq j\leqq t-1$. Using properties of Legendre symbols, we see that the sum of the entries in each row of M_K is zero. So $\operatorname{rank} M_K=\operatorname{rank} M'_K$. By using quadratic reciprocity, we can also verify that the sum of the entries in each column of M_K is zero. So we could omit any row and any column of M_K without changing the rank of the matrix. If p_g is the largest of the primes $\equiv 3 \pmod 4$ that divide m, we let M''_K denote the $(t-1)\times(t-1)$ matrix obtained from M_K by discarding the g-th row and g-th column of M_K. Then

$$R_K=t-1-\operatorname{rank} M'_K=t-1-\operatorname{rank} M_K=t-1-\operatorname{rank} M''_K. \tag{2.7}$$

By using quadratic reciprocity and properties of Legendre symbols, we see that the matrices M_K, M'_K, and M''_K are determined by the set of values

$$\left\{\left(\frac{p_j}{p_i}\right) \text{ for } 1\leqq i<j\leqq t\right\},$$

provided we know which primes are congruent to 1 (mod 4) and which primes are congruent to 3 (mod 4).

Now let

$$S_{t,l}=\{K=\mathbf{Q}(\sqrt{-m})\in A_t\colon m \text{ satisfies Eq. (2.1) with exactly } l \text{ primes } p_i\equiv 3 \pmod 4\}, \qquad 1\leqq l\leqq t \text{ with } l \text{ odd}.$$

Then for each positive real number x, let

$$S_{t,l;x}=\{K\in S_{t,l}\colon m\leqq x\},$$

and for $0\leqq r\leqq t-1$, let

$$S_{t,l,r;x}=\{K\in S_{t,l;x}\colon \operatorname{rank} M_K=r\}.$$

Then $|A_{t;x}|\sim\sum_{\substack{1\leqq l\leqq t\\ l \text{ odd}}}|S_{t,l;x}|$ (as $x\to\infty$) and

$$|A_{t,e;x}|\sim\sum_{\substack{1\leqq l\leqq t\\ l \text{ odd}}}|S_{t,l,t-1-e;x}| \qquad (\text{as } x\to\infty). \tag{2.8}$$

Now we note that

$$|S_{t,l;x}| \sim \binom{t}{l} \cdot 2^{-t} \cdot \frac{1}{(t-1)!} \frac{x(\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty) \tag{2.9}$$

where $\binom{t}{l}$ is the binomial coefficient $t!/l!(t-l)!$, and the factor 2^{-t} comes from the fact that each of the t primes is congruent to 1 or 3 (mod 4). Next we note that for $0 \leqq r \leqq t-1$,

$$|S_{t,l,r;x}| = \sum_{p_1 \ldots p_t \leqq x}^{(l)} \delta_{M_{K},r}. \tag{2.10}$$

Here $\sum_{p_1 \ldots p_t \leqq x}^{(l)}$ means that only those products $p_1 \ldots p_t \leqq x$ are considered where exactly l of the $p_i \equiv 3 \pmod 4$. The symbol $\delta_{M_K,r}$ is defined as follows: $\delta_{M_K,r} = 1$ if rank $M_K = r$, and $\delta_{M_K,r} = 0$ if rank $M_K \neq r$, where $K = \mathbf{Q}(\sqrt{-p_1 \ldots p_t})$.

Now suppose we consider a fixed $K_1 = \mathbf{Q}(\sqrt{-p'_1 \ldots p'_t}) \in S_{t,l;x}$. Let $u_{ij} \in \mathbf{F}_2$ be defined by

$$(-1)^{u_{ij}} = \left(\frac{p'_j}{p'_i}\right) \quad \text{for } 1 \leqq i < j \leqq t.$$

If $K = \mathbf{Q}(\sqrt{-p_1 \ldots p_t}) \in S_{t,l;x}$, we will call K equivalent to K_1 if $\left(\frac{p_j}{p_i}\right) = (-1)^{u_{ij}}$ for $1 \leqq i < j \leqq t$ and $p_i \equiv p'_i \pmod 4$ for $1 \leqq i \leqq t$. Thus we can decompose $S_{t,l;x}$ into equivalence classes of fields. From our earlier observations we know that $M_K = M_{K_1}$ if K is equivalent to K_1.

Now we let $\delta(p_i, p_j) = 1$ if $\left(\frac{p_j}{p_i}\right) = (-1)^{u_{ij}}$, and we let $\delta(p_i, p_j) = 0$ if $\left(\frac{p_j}{p_i}\right) \neq (-1)^{u_{ij}}$. Next we observe that the conditions $p_1 \ldots p_t \leqq x$ and $p_1 < p_2 < \ldots < p_t$ imply

$$p_1 \leqq x^{1/t}, \quad p_1 < p_2 \leqq (x/p_1)^{1/(t-1)}, \ldots, p_{t-2} < p_{t-1} \leqq (x/p_1 \ldots p_{t-2})^{1/2},$$
$$p_{t-1} < p_t \leqq x/p_1 \ldots p_{t-1}.$$

If we let $N(K_1)$ denote the number of fields K in $S_{t,l;x}$ with K equivalent to K_1, then

$$N(K_1) = \sum_{\substack{p_1 \leqq x^{1/t} \\ p_1 \equiv p'_1 \pmod 4}} \sum_{\substack{p_1 < p_2 \leqq (x/p_1)^{1/(t-1)} \\ p_2 \equiv p'_2 \pmod 4}} Y_2 \cdots \sum_{\substack{p_{t-1} < p_t \leqq x/p_1 \ldots p_{t-1} \\ p_t \equiv p'_t \pmod 4}} Y_t, \tag{2.11}$$

where $Y_j = \prod_{i=1}^{j-1} \delta(p_i, p_j)$ for $2 \leqq j \leqq t$. Now it can be proved that

$$N(K_1) \sim 2^{-(t^2+t)/2} \cdot \frac{1}{(t-1)!} \frac{x(\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty). \tag{2.12}$$

An intuitive explanation of Formula (2.12) might proceed as follows. A factor of $\frac{1}{2}$ is introduced by each condition $p_i \equiv p'_i \pmod 4$ for $1 \leqq i \leqq t$ and by each condition $\left(\frac{p_j}{p_i}\right) \equiv (-1)^{u_{ij}}$ for $1 \leqq i < j \leqq t$. Since there are $(t^2+t)/2$ such conditions, we get the factor $2^{-(t^2+t)/2}$. The other factors in Formula (2.12) come from Formula (2.4). The actual derivation of Formula (2.12) from Eq. (2.11) uses the same types of calculations used in proving Lemma 3 in [5], and we refer the reader to [5] for details. We remark that a significant part of the derivation of Formula (2.12) is to show that

$$\sum_{p_1}\sum_{p_2}\cdots\sum_{\substack{p_t\\ p_1\dots p_t\leqq x}}\chi_{p_1\dots p_{t-1}}(p_t)=o\left(\frac{x(\log\log x)^{t-1}}{\log x}\right), \tag{2.13}$$

where $\chi_{p_1\dots p_{t-1}}$ is a nonprincipal quadratic Dirichlet character with modulus $p_1\dots p_{t-1}$. The analogous calculations in [5] use results from [4], Chap. 20. Alternately one may view Eq. (2.13) as an analog for a hyperbolic region of a result of Heilbronn [8] for a rectangular region. This analogy is considered in detail in [6] for the case $t=2$. Finally we observe that Formula (2.12) is valid for any $K_1\in S_{t,l;x}$; i.e., Formula (2.12) is valid for every equivalence class of fields in $S_{t,l;x}$.

Now to evaluate the right side of Eq. (2.10), we need to know how many equivalence classes in $S_{t,l;x}$ contain a field K_1 with rank $M_{K_1}=r$. We consider any $K_1=\mathbf{Q}(\sqrt{-p'_1\dots p'_t})\in S_{t,l;x}$ with the usual ordering $p'_1<p'_2<\dots<p'_t$. We now reorder the primes as follows: $p'_{i_1}<p'_{i_2}<\dots<p'_{i_l}$ and $p'_{i_{l+1}}<p'_{i_{l+2}}<\dots<p'_{i_t}$, where $p'_{i_j}\equiv 3 \pmod 4$ for $1\leqq j\leqq l$ and $p'_{i_j}\equiv 1 \pmod 4$ for $l+1\leqq j\leqq t$. For sufficiently large x we can choose $K_2=\mathbf{Q}(\sqrt{-p''_1\dots p''_t})\in S_{t,l;x}$, where $p''_1<p''_2<\dots<p''_t$, $p''_j\equiv 3 \pmod 4$ if $1\leqq j\leqq l$, $p''_j\equiv 1 \pmod 4$ if $l+1\leqq j\leqq t$, and $\left(\frac{p''_J}{p''_I}\right)=(-1)^{u_{i_I i_J}}$ for $1\leqq I<J\leqq t$. Then M_{K_2} is obtained from M_{K_1} by certain row exchanges and corresponding column exchanges, and hence rank M_{K_2} $=$ rank M_{K_1}. We associate the equivalence class of each field K_1 in $S_{t,l;x}$ with the equivalence class of the corresponding field K_2 described above. We note that for a given K_2, there are $\binom{t}{l}$ equivalence classes associated to the equivalence class of K_2.

Now we recall that rank $M''_{K_2}=$ rank M_{K_2}, where M''_{K_2} is the matrix obtained from M_{K_2} omitting the l-th row and l-th column. Because of the congruence conditions (mod 4) for each p''_i, we see that $M''_{K_2}=[a_{ij}]$ with $a_{ij}\neq a_{ji}$ when $1\leqq i<j\leqq l-1$ and with $a_{ij}=a_{ji}$ when $l\leqq i\leqq t-1$ and $1\leqq j\leqq t-1$. We let $N(t-1, l-1, r)$ denote the number of $(t-1)\times(t-1)$ matrices $M=[a_{ij}]$, each $a_{ij}\in\mathbf{F}_2$, with $a_{ij}\neq a_{ji}$ when $1\leqq i<j\leqq l-1$, with $a_{ij}=a_{ji}$ when $l\leqq i\leqq t-1$ and $1\leqq j\leqq t-1$, and with rank $M=r$. Now from Eq. (2.10), Formula (2.12), and the above discussion, we have

$$|S_{t,l,r;x}|\sim N(t-1,l-1,r)\cdot\binom{t}{l}\cdot 2^{-(t^2+t)/2}\cdot\frac{1}{(t-1)!}\,\frac{x(\log\log x)^{t-1}}{\log x}\quad(\text{as } x\to\infty). \tag{2.14}$$

Then from Formulas (2.5), (2.8), and (2.14), and from Eq. (1.1), we obtain the following proposition.

Proposition 2.1. *Let $A_{t;x}$, $A_{t,e;x}$, and $d_{t,e}$ be defined as in Sect.* 1, *and let l be a positive odd integer. Let $N(t-1, l-1, r)$ be defined as above. Then*

$$|A_{t;x}| \sim \frac{1}{2} \cdot \frac{1}{(t-1)!} \frac{x(\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty);$$

$$|A_{t,e;x}| \sim \sum_{\substack{1 \leq l \leq t \\ l \text{ odd}}} N(t-1, l-1, t-1-e) \cdot \binom{t}{l} \cdot 2^{-(t^2+t)/2} \cdot \frac{1}{(t-1)!} \frac{x(\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty);$$

$$d_{t,e} = \sum_{\substack{1 \leq l \leq t \\ l \text{ odd}}} N(t-1, l-1, t-1-e) \cdot \binom{t}{l} \cdot 2^{1-(t^2+t)/2}. \quad //$$

In Sect. 3 we shall derive an effective algorithm for computing $N(t-1, l-1, t-1-e)$. Although the calculations in Sect. 3 involve only elementary linear algebra, Sect. 3 is somewhat lengthy, and the reader may wish to skip to Sect. 4 to see how the results from Sect. 3 will be used before the reader examines the details in Sect. 3.

3. Algorithm for computing $N(t-1, l-1, t-1-e)$

Throughout this section all matrices have entries in $\mathbf{F}_2$. To simplify notation we let $n=t-1$, $k=l-1$, and $r=t-1-e$. Then $n \geq 0$, $0 \leq k \leq n$ with k even, and $0 \leq r \leq n$. We shall develop in this section an algorithm for computing $N(n, k, r)$. We recall that $N(n, k, r)$ is the number of $n \times n$ matrices $M=[a_{ij}]$, each $a_{ij} \in \mathbf{F}_2$, with the following properties: (1) $a_{ij} \neq a_{ji}$ for $1 \leq i < j \leq k$; (2) $a_{ij} = a_{ji}$ when $k+1 \leq i \leq n$ and $1 \leq j \leq n$; (3) rank $M = r$. One method for determining $N(n, k, r)$ is to compute the rank of each matrix M having properties (1) and (2), and then to count those M with property (3). However there are $2^{(n^2+n)/2}$ matrices M having properties (1) and (2), and hence this method is not feasible except for very small n. What we want is an algorithm for computing $N(n, k, r)$ such that the number of computations required grows like a polynomial in n rather than exponentially with n.

We first consider the case when $n=k$. At this point we shall consider odd k as well as even k. So $M=[a_{ij}]$ is an $n \times n$ matrix with $a_{ij} \neq a_{ji}$ for all $i \neq j$. We shall call such a matrix antisymmetric. We note that the transpose of M satisfies $M^T = M + I + J$, where I is the $n \times n$ identity matrix, and J is the $n \times n$ matrix each of whose entries equals 1. Let $H \in \mathbf{F}_2^n$ be the vector with each component equal to 1. Our first lemma is an easy exercise.

Lemma 3.1. *If n is even, then* rank$(I+J)=n$. *If n is odd, then* rank$(I+J)=n-1$. //

For any matrix A, we let $c(A)$ denote the column space of A. Then we have the following result.

Lemma 3.2. *Let* $r = \operatorname{rank} M$, *where* M *is an* $n \times n$ *antisymmetric matrix. If* n *is even, then*

$$\dim [c(M) + c(M^T)] = n \quad \text{and} \quad \dim [c(M) \cap c(M^T)] = 2r - n.$$

If n *is odd, then*

$$\dim [c(M) + c(M^T)] \geqq n-1 \quad \text{and} \quad \dim [c(M) \cap c(M^T)] \leqq 2r - n + 1.$$

Proof. First we assume n is even. If $V \in \mathbf{F}_2^n$, then by Lemma 3.1, there exists $W \in \mathbf{F}_2^n$ such that $(I+J)W = V$. So

$$V = MW + (M+I+J)W = MW + M^T W \in [c(M) + c(M^T)].$$

Hence $[c(M) + c(M^T)] = \mathbf{F}_2^n$, and $\dim [c(M) + c(M^T)] = n$. Then

$$\dim [c(M) \cap c(M^T)] = \dim [c(M)] + \dim [c(M^T)] - \dim [c(M) + c(M^T)] = 2r - n.$$

When n is odd, the above arguments show $V \in [c(M) + c(M^T)]$ if $V \in c(I+J)$. Since $\operatorname{rank}(I+J) = n-1$ when n is odd, then

$$\dim [c(M) + c(M^T)] \geqq n-1 \quad \text{and} \quad \dim [c(M) \cap c(M^T)] \leqq 2r - n + 1. \quad //$$

Remark. Since $2r - n \geqq 0$ (resp., $2r - n + 1 \geqq 0$) when n is even (resp., odd) in Lemma 3.2, we get the following corollary.

Corollary 3.3. *Suppose* M *is an* $n \times n$ *antisymmetric matrix. If* n *is even, then* $\operatorname{rank} M \geqq n/2$. *If* n *is odd, then* $\operatorname{rank} M \geqq (n-1)/2$. //

Now recall that $H \in \mathbf{F}_2^n$ is the vector with each component equal to 1. When n is even, Lemma 3.2 implies that $H \in [c(M) + c(M^T)]$. Furthermore since $\dim [c(M) \cap c(M^T)] = 2r - n$, there exist 2^{2r-n} pairs of vectors $\{V, W\}$ with $V \in c(M)$, $W \in c(M^T)$, and $H = V + W$. Then $V + H = W \in c(M^T)$.

Now suppose $M_1 = [b_{ij}]$, each $b_{ij} \in \mathbf{F}_2$, is an $(n+1) \times (n+1)$ antisymmetric matrix with $b_{ij} = a_{ij}$ for $1 \leqq i \leqq n$ and $1 \leqq j \leqq n$. We may write

$$M_1 = \begin{bmatrix} M & V \\ (V+H)^T & v \end{bmatrix}, \quad \text{where } V \in \mathbf{F}_2^n \text{ and } v \in \mathbf{F}_2. \tag{3.1}$$

If we assume $\operatorname{rank} M = r$, then $\operatorname{rank} M_1 = r$, $r+1$, or $r+2$. Given M, we would like to know how many of these matrices M_1 have $\operatorname{rank} M_1 = r$, $\operatorname{rank} M_1 = r+1$, and $\operatorname{rank} M_1 = r+2$. Suppose n is even. If $\operatorname{rank} M_1 = r$, then $V \in c(M)$ and $(V+H)^T \in$ (row space of M). But then $\{V, V+H\}$ must be one of the 2^{2r-n} pairs with $V \in c(M)$ and $V + H \in c(M^T)$. Write $V = MX$ and $V + H = M^T Y$ with $X, Y \in \mathbf{F}_2^n$. For $v = (V+H)^T X$, we have $\operatorname{rank} M_1 = r$. If we wrote $V = M(X+Z)$ with $Z \in$ (null space of M), then

$$\begin{aligned}(V+H)^T(X+Z) &= (V+H)^T X + (V+H)^T Z = (V+H)^T X + (M^T Y)^T Z \\ &= (V+H)^T X + Y^T MZ = (V+H)^T X.\end{aligned}$$

Hence given V, there is only one choice of v such that $\operatorname{rank} M_1 = r$. We have proved the first part of the following proposition.

Proposition 3.4. *Let M be an $n \times n$ antisymmetric matrix with* rank $M = r$, *and let M_1 be any $(n+1) \times (n+1)$ antisymmetric matrix satisfying Eq.* (3.1). *Assume n is even. Of all possible M_1,*

(i) 2^{2r-n} *have* rank $M_1 = r$;
(ii) $2^{r+2} - 3 \cdot 2^{2r-n}$ *have* rank $M_1 = r+1$;
(iii) $2^{n+1} + 2^{2r-n+1} - 2^{r+2}$ *have* rank $M_1 = r+2$.

Proof. We have already proved (i). To prove (ii) we describe the ways in which rank $M_1 = r+1$. First $\{V, V+H\}$ could be one of the pairs with $V \in c(M)$ and $V+H \in c(M^T)$, but we could choose $v \neq (V+H)^T X$ when $V = MX$. This situation gives 2^{2r-n} choices for M_1. Next we could have $V \in c(M)$, $V+H \notin c(M^T)$, and v arbitrary. This situation gives $(2^r - 2^{2r-n}) \cdot 2$ choices for M_1. Finally we could have $V \notin c(M)$, $V+H \in c(M^T)$, and v arbitrary. This situation gives another $(2^r - 2^{2r-n}) \cdot 2$ choices for M_1. Since

$$2^{2r-n} + (2^r - 2^{2r-n}) \cdot 2 + (2^r - 2^{2r-n}) \cdot 2 = 2^{r+2} - 3 \cdot 2^{2r-n},$$

we have proved (ii). Finally (iii) follows from calculating $2^{n+1} - (2^{2r-n} + 2^{r+2} - 3 \cdot 2^{2r-n})$.

We now suppose n is odd. When $n = 1$, it is trivial to calculate $N(1, 1, r)$. So we suppose $n \geqq 3$. If $\dim[c(M) + c(M^T)] = n$, then Proposition 3.4 will also be valid for M. However because n is odd, rank$(I+J) = n-1$, and it is possible that $\dim[c(M) + c(M^T)] = n-1$. So we need to find the M for which $\dim[c(M) + c(M^T)] = n-1$. Now recall that $M^T = M + I + J$, and hence $c(I+J) \subset [c(M) + c(M^T)]$. If $\dim[c(M) + c(M^T)] = n-1$, then since rank$(I+J) = n-1$, we must have $c(M) \subset c(I+J)$ and $c(M^T) \subset c(I+J)$. Since n is odd, we note that the sum of the entries in each column of $I+J$ is zero (in $\mathbf{F}_2$). Then $c(M) \subset c(I+J)$ implies that the sum of the entries in each column of M is zero, and $c(M^T) \subset c(I+J)$ implies that the sum of the entries in each row of M is zero. So

$$M = \begin{bmatrix} M_0 & V_0 \\ (V_0 + H_0)^T & v_0 \end{bmatrix} \tag{3.2}$$

where M_0 is an $(n-1) \times (n-1)$ antisymmetric matrix with rank $M_0 = r$, $H_0 \in \mathbf{F}_2^{n-1}$ is the vector with each component equal to 1, $V_0 = M_0 H_0$, and $v_0 = (V_0 + H_0)^T H_0$.

Suppose we have a matrix M satisfying Eq. (3.2), and suppose M_1 satisfies Eq. (3.1). From our previous discussion the sum of the entries in each row of M is zero, and the sum of the entries in each column of M is zero. Note however that the sum of the entries in H is 1 (in $\mathbf{F}_2$) since n is odd. The it is impossible for the sum of the entries in both V and $V+H$ to be zero. So we cannot have both $V \in c(M)$ and $V+H \in c(M^T)$, and hence rank $M_1 \neq r$. If we now replace 2^{2r-n} by 0 in the proof of Proposition 3.4, we obtain the following result.

Proposition 3.5. *Let M be an $n \times n$ antisymmetric matrix with* rank $M = r$, *and let M_1 be any $(n+1) \times (n+1)$ antisymmetric matrix satisfying Eq.* (3.1). *Assume n is odd and $n \geqq 3$. If M does not satisfy Eq.* (3.2), *then of all possible M_1,*

(i) 2^{2r-n} *have* rank $M_1 = r$;
(ii) $2^{r+2} - 3 \cdot 2^{2r-n}$ *have* rank $M_1 = r+1$;
(iii) $2^{n+1} + 2^{2r-n+1} - 2^{r+2}$ *have* rank $M_1 = r+2$.

If M does satisfy Eq. (3.2), *then of all possible* M_1,

(iv) 2^{r+2} *have* rank $M_1 = r+1$;
(v) $2^{n+1} - 2^{r+2}$ *have* rank $M_1 = r+2$. //

Remark. For $n=1$ and 2 and for $0 \leqq r \leqq n$, we can compute $N(n, n, r)$ directly by examining all antisymmetric 1×1 and 2×2 matrixes. In fact, $N(1,1,0)=1$, $N(1,1,1)=1$, $N(2,2,0)=0$, $N(2,2,1)=6$, $N(2,2,2)=2$. Then we can use Proposition 3.4 and Proposition 3.5 to compute $N(n, n, r)$ for $n=3,4,5,\ldots$, and $0 \leqq r \leqq n$. Let n_0 be a positive integer. To compute $N(n, n, r)$ for all $1 \leqq n \leqq n_0$ and $0 \leqq r \leqq n$ requires $O(n_0^2)$ calculations.

We may combine Propositions 3.4 and 3.5 as follows.

Proposition 3.6. *Suppose* $M = [a_{ij}]$ *is an* $n \times n$ *antisymmetric matrix with n even and* rank $M = r$. *Suppose* $M_2 = [b_{ij}]$ *is an* $(n+2) \times (n+2)$ *antisymmetric matrix with* $b_{ij} = a_{ij}$ *when* $1 \leqq i \leqq n$ *and* $1 \leqq j \leqq n$. *Then of all possible* M_2,

(i) $2^{4r-2n-1} - 2^{2r-n-1}$ *have* rank $M_2 = r$;
(ii) $3 \cdot 2^{3r-n+2} - 15 \cdot 2^{4r-2n-1} + 3 \cdot 2^{2r-n-1}$ *have* rank $M_2 = r+1$;
(iii) $13 \cdot 2^{2r+2} - 21 \cdot 2^{3r-n+2} + 35 \cdot 2^{4r-2n} - 2^{2r-n}$ *have* rank $M_2 = r+2$;
(iv) $3 \cdot 2^{r+n+4} - 39 \cdot 2^{2r+2} + 21 \cdot 2^{3r-n+3} - 15 \cdot 2^{4r-2n+2}$ *have* rank $M_2 = r+3$;
(v) $2^{2n+3} - 3 \cdot 2^{r+n+4} + 13 \cdot 2^{2r+3} - 3 \cdot 2^{3r-n+5} + 2^{4r-2n+5}$ *have* rank $M_2 = r+4$. //

We are now ready to determine $N(n, k, r)$ when $n > k$. We start with the case $k=0$. Then our $n \times n$ matrix M is symmetric. For our $(n+1) \times (n+1)$ matrix M_1, we write

$$M_1 = \begin{bmatrix} M & V \\ V^T & v \end{bmatrix}, \quad \text{where } V \in \mathbf{F}_2^n \text{ and } v \in \mathbf{F}_2. \tag{3.3}$$

As before we let $c(M)$ denote the column space of M. Since M is symmetric, then $V \in c(M) \Leftrightarrow V \in c(M^T) \Leftrightarrow V^T \in$(row space of M). Then proceeding as we did in proving Proposition 3.4, we have the following result.

Proposition 3.7. *Suppose M is an* $n \times n$ *symmetric matrix with* rank $M = r$. *Let* M_1 *be any* $(n+1) \times (n+1)$ *symmetric matrix satisfying Eq.* (3.3). *Then among all possible* M_1, (i) 2^r *have* rank $M_1 = r$; (ii) 2^r *have* rank $M_1 = r+1$; (iii) $2^{n+1} - 2^{r+1}$ *have* rank $M_1 = r+2$. //

Remark. For other approaches to computing the number of symmetric $n \times n$ matrices with a given rank, see [10] and [12].

We now want to compute $N(n, k, r)$ with k even and $k \geqq 2$. As described in the remark following Proposition 3.5, we can compute $N(k, k, r)$. Now for fixed k we shall describe an algorithm for computing $N(n, k, r)$ for $n > k$. So we consider an $n \times n$ matrix of the form

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{3.4}$$

where A is a $k\times k$ antisymmetric matrix, B is a $k\times(n-k)$ matrix, and C is an $(n-k)\times(n-k)$ symmetric matrix. Let M_1 be an $(n+1)\times(n+1)$ matrix of the form

$$M_1=\begin{bmatrix} M & V \\ V^T & v \end{bmatrix}, \quad \text{where } V\in\mathbf{F}_2^n \text{ and } v\in\mathbf{F}_2. \tag{3.5}$$

Alternately we may write

$$M_1=\begin{bmatrix} A & B & V_1 \\ B^T & C & V_2 \\ V_1^T & V_2^T & v \end{bmatrix}, \quad \text{where } V_1\in\mathbf{F}_2^k,\ V_2\in\mathbf{F}_2^{n-k}, \text{ and } v\in\mathbf{F}_2. \tag{3.6}$$

We let

$$D=\begin{bmatrix} B \\ C \end{bmatrix} \tag{3.7}$$

and

$$D_1=\begin{bmatrix} B & V_1 \\ C & V_2 \\ V_2^T & v \end{bmatrix}. \tag{3.8}$$

The following lemma is analogous to Lemma 3.2, and we omit the proof.

Lemma 3.8. *Let* $r=\operatorname{rank} M$ *and* $s=\operatorname{rank} D$ *in Eqs.* (3.4) *and* (3.7). *Assume* k *is even. Then*

$$\dim[c(M)+c(M^T)]=k+s \quad \text{and} \quad \dim[c(M)\cap c(M^T)]=2r-k-s. \quad //$$

We note that $\dim[c(M)\cap c(M^T)]\geqq\operatorname{rank} D=s$. So $2r-k-s\geqq s$, and hence $0\leqq s\leqq r-(k/2)$. Also $r\leqq k+s$, and hence $r-k\leqq s$. Also $s\leqq n-k$. Thus we have the following bounds on s.

Lemma 3.9. $\operatorname{Max}(0,r-k)\leqq s\leqq\min(r-(k/2),n-k)$. //

Now given M satisfying Eq. (3.4) with $\operatorname{rank} M=r$, we want to know how many M_1 satisfying Eq. (3.5) have $\operatorname{rank} M_1=r$, $\operatorname{rank} M_1=r+1$, and $\operatorname{rank} M_1=r+2$. Our next proposition is analogous to Proposition 3.4, and we omit the proof.

Proposition 3.10. *Suppose* $k\geqq 2$ *is an even integer. Suppose* M, M_1, D, *and* D_1 *are given by Eqs.* (3.4) *through* (3.8). *Let* $r=\operatorname{rank} M$ *and* $s=\operatorname{rank} D$. *Of all possible* M_1 *and* D_1,

(i) 2^s *have rank* $M_1=r$ *and rank* $D_1=s$;
(ii) $2^{2r-k-s}-2^s$ *have rank* $M_1=r$ *and rank* $D_1=s+1$;
(iii) $2^{r+2}-3\cdot 2^{2r-k-s}$ *have rank* $M_1=r+1$ *and rank* $D_1=s+1$;
(iv) $2^{n+1}-2^{k+s+1}$ *have rank* $M_1=r+2$ *and rank* $D_1=s+2$;
(v) $2^{k+s+1}+2^{2r-k-s+1}-2^{r+2}$ *have rank* $M_1=r+2$ *and rank* $D_1=s+1$. //

Remark. Let k be a fixed even positive integer. Let $N(n,k,r,s)$ be the number of matrices M of the form specified by Eq. (3.4) such that $\operatorname{rank} M=r$ and $\operatorname{rank} D=s$, where D is given by Eq. (3.7). We can use Propositions 3.4 and 3.5 to compute $N(k,k,r,0)$ and then use Proposition 3.10 to compute $N(n,k,r,s)$ for $n=k+1,k+2,\dots$. Using Lemma 3.9, we then get

$$N(n,k,r)=\sum_{s=\max(0,r-k)}^{\min(r-(k/2),n-k)} N(n,k,r,s).$$

Let n_0 be a positive integer. The computation of $N(n,k,r,s)$ for all $1\leqq n\leqq n_0$, $0\leqq k\leqq n$ with k even, $0\leqq r\leqq n$, and $\max(0,r-k)\leqq s\leqq \min(r-(k/2),n-k)$ requires at most $O(n_0^4)$ calculations.

Remark. Although we have assumed k is even in Proposition 3.10, one can obtain results for k odd by some modifications of our arguments. In particular suppose k is even and we replace V_1^T by $(V_1+H_1)^T$ in Eq. (3.6), where $H_1\in\mathbf{F}_2^k$ is the vector with each component equal to 1. Then by interchanging rows $k+1$ and $n+1$ and by interchanging columns $k+1$ and $n+1$, we see that the matrix M_1 will have a $(k+1)\times(k+1)$ antisymmetric submatrix. By going through calculations similar to those we have performed in proving Proposition 3.10, we eventually see that $N(n+1,k+1,r)=N(n+1,k,r)$ for $0\leqq r\leqq n+1$ when k is even. Since we know how to compute $N(n+1,k,r)$ with k even, we can compute $N(n+1,k+1,r)$ with $k+1$ odd.

4. Calculation of $d_{t,e}$ and $d_{\infty,e}$

Proposition 2.1 and our algorithms in Sect. 3 for computing $N(t-1,l-1,r)$ provide us with algorithms for computing $d_{t,e}$ for $t=1,2,\dots$, and $e=0,1,2,\dots$. To investigate the behavior of $d_{t,e}$, we first rewrite $d_{t,e}$ from Proposition 2.1 as follows.

$$d_{t,e}=\sum_{\substack{1\leqq l\leqq t\\ l\text{ odd}}} c_{t,l}f_{t,l,e}, \tag{4.1}$$

where

$$c_{t,l}=\binom{t}{l}\cdot 2^{-(t-1)} \tag{4.2}$$

and

$$f_{t,l,e}=N(t-1,l-1,t-1-e)\cdot 2^{-t(t-1)/2}. \tag{4.3}$$

We observe that

$$\sum_{\substack{1\leqq l\leqq t\\ l\text{ odd}}} c_{t,l}=1. \tag{4.4}$$

Now we let $N(t-1,l-1)$ denote the number of $(t-1)\times(t-1)$ matrices $M=[a_{ij}]$, each $a_{ij}\in\mathbf{F}_2$, with $a_{ij}\neq a_{ji}$ when $1\leqq i<j\leqq l-1$, and with $a_{ij}=a_{ji}$ when $l\leqq i\leqq t-1$ and $1\leqq j\leqq t-1$. Then it is easy to see that $N(t-1,l-1)=2^{t(t-1)/2}$. Thus $f_{t,l,e}$ in Eq. (4.3) represents the probability that a randomly chosen matrix M of the specified form has rank $M=t-1-e$.

We now use our results from Sect. 3 to generate denumerable Markov processes for computing the quantities $f_{t,l,e}$ for $t=1,2,3,\dots$; $l=1,3,5,\dots$; $e=0,1,2,\dots$. We refer the reader to [9] for properties of denumerable Markov processes. To be consistent with the notation of [9], we shall write our matrices on the right in our Markov processes. First we let

$$u_{t,i}=f_{t,t,i} \quad \text{for } t=1,3,5,\ldots, \quad \text{and} \quad i=0,1,2,\ldots.$$

We shall think of the values of i as the states of a Markov process and the values of t as discrete time points. (Of course i actually represents values of the 4-class ranks of imaginary quadratic fields, and t represents the number of ramified primes in the imaginary quadratic fields.) If we divide each term in (i) through (v) of Proposition 3.6 by 2^{2n+3} (Note: 2^{2n+3} is the sum of the terms in (i) through (v) of Proposition 3.6), and if we let $n=t-1$ and $r=t-1-i$, we get Markov Process C specified in Appendix I. In Markov Process C, j corresponds to $(t+2)-1-\operatorname{rank} M_2$. Markov Process C is closely related to another Markov process, which we have called Markov Process D in Appendix I. One can check that

$$Q_C^{(t)}=Q_D Q_D+2^{-t}Q_C' \tag{4.5}$$

where $Q_C'=[q_{ij}']$ with $i=0,1,2,\ldots;\ j=0,1,2,\ldots;$

$$q_{ij}'=\begin{cases}-2^{-2-2i} & \text{if } j=i\\ 3\cdot 2^{-3-2i} & \text{if } j=i+1\\ -2^{-3-2i} & \text{if } j=i+2\\ 0 & \text{otherwise.}\end{cases}$$

Before we specify our next Markov process, we recall from Sect. 3 that $N(n,k,r,s)$ is the number of matrices M of the form specified in Eq. (3.4) with rank $M=r$ and rank $D=s$, where D is given by Eq. (3.7). We define

$$g_{t,l,e,w}=N(t-1,l-1,t-1-e,t-l-w)\cdot 2^{-t(t-1)/2}. \tag{4.6}$$

Using Lemma 3.9 with $n=t-1$, $k=l-1$, $r=t-1-e$, and $s=t-l-w$, we see that

$$\max(0,e-(l-1)/2)\leqq w\leqq \min(e,t-l).$$

Then

$$f_{t,l,e}=\sum_w g_{t,l,e,w}. \tag{4.7}$$

We let l be a fixed odd positive integer. For $t=l, l+1, l+2,\ldots;\ i=0,1,2,\ldots;$ and $w_i=0,1,\ldots,i$; we define

$${}_l v_{t,(i,w_i)}=\begin{cases}g_{t,l,i,w_i} & \text{if } \max(0,i-(l-1)/2)\leqq w_i\leqq\min(i,t-l)\\ 0 & \text{otherwise.}\end{cases} \tag{4.8}$$

If we divide each term in (i) through (v) of Proposition 3.10 by 2^{n+1} (Note: The sum of the terms in (i) through (v) of Proposition 3.10 is 2^{n+1}), and if we let $n=t-1$, $k=l-1$, $r=t-1-i$, and $s=t-l-w_i$, then we get Markov Process E_l in Appendix I. In Markov Process E_l, j corresponds to $(t+1)-1-\operatorname{rank} M_1$ and w_j corresponds to $(t+1)-l-\operatorname{rank} D_1$. Note that Markov Process E_l is initialized by using the vector U_l from Markov Process C. This initialization forces ${}_l v_{t,(i,w_i)}=0$ if w_i does not satisfy $\max(0,i-(l-1)/2)\leqq w_i\leqq\min(i,t-l)$. Markov Process

E_l is closely related to another Markov process, which we have called Markov Process F_l. In fact

$$Q_{E_l} = Q_{F_l} + 2^{-l} Q''_{E_l} \tag{4.9}$$

where $Q''_{E_l} = [{}_l q''_{(i,w_i),(j,w_j)}]$ with $i=0,1,2,\ldots;\ j=0,1,2,\ldots;$
$0 \leqq w_i \leqq i;\ 0 \leqq w_j \leqq j;$

$$ {}_l q''_{(i,w_i),(j,w_j)} = \begin{cases} -2^{-w_i} & \text{if } j=i+1, \quad w_j = w_i \\ 2^{-w_i} & \text{if } j=i+1, \quad w_j = w_i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Although the Markov Processes C and E_l are the ones that arise naturally in our problem, the Markov Processes D and F_l are easier to analyze, and we shall approximate C and E_l by D and F_l. Before analyzing these processes, we introduce one more notation. If a vector $X=(x_0, x_1, x_2, \ldots)$ with each x_i a real number, we define $\|X\| = |x_0| + |x_1| + |x_2| + \ldots$. Now in the terminology of [9], Markov Process D is a recurrent Markov chain which is noncyclic and ergodic. So there is an invariant probability vector $Y=(y_0, y_1, y_2, \ldots)$; i.e., $YQ_D = Y$; and $\lim_{t\to\infty} \|X(Q_D)^t - Y\| = 0$ for every probability vector $X=(x_0, x_1, x_2, \ldots)$. In fact one can verify by induction that the components of the invariant probability vector Y satisfy

$$y_i = 2^{1-2i}(1-2^{-i})^{-2} y_{i-1} \qquad \text{for } i=1,2,3,\ldots.$$

So

$$Y = \alpha^{-1}\left(1, 2, 4/9, \ldots, 2^{-i^2} \prod_{m=1}^{i} (1-2^{-m})^{-2}, \ldots\right), \tag{4.10}$$

$$\text{where } \alpha^{-1} = \left[1 + 2 + 4/9 + \ldots + 2^{-i^2} \prod_{m=1}^{i} (1-2^{-m})^{-2} + \ldots\right]^{-1} = \prod_{m=1}^{\infty} (1-2^{-m}),$$

cf. [7], Theorem 351. Numerically $\alpha^{-1} \approx 0.288788095$.

Lemma 4.1. *Let* U_t (for $t=1,3,5,\ldots$) *be specified by Markov Process C, and let Y be defined by Eq.* (4.10). *Then* $\lim_{t\to\infty} \|U_t - Y\| = 0$.

Proof. Let $\varepsilon > 0$ be given. We choose a positive odd integer t_0 so that $2^{-t_0} < \varepsilon/4$. From our previous discussion we know that $\lim_{t\to\infty} \|U_{t_0}(Q_D)^t - Y\| = 0$. Now from the definition of Markov Process C and from Eq. (4.5), $U_{t_0+2} = U_{t_0}(Q_D)^2 + X_2$, where $X_2 = 2^{-t_0} U_{t_0} Q'_C$, and hence $\|X_2\| \leqq 2^{-t_0}$. Next

$$U_{t_0+4} = U_{t_0+2}(Q_D)^2 + 2^{-t_0-2} U_{t_0+2} Q'_C = U_{t_0}(Q_D)^4 + X_4,$$

where $X_4 = X_2(Q_D)^2 + 2^{-t_0-2} U_{t_0+2} Q'_C$. Note that $\|X_4\| \leqq 2^{-t_0} + 2^{-t_0-2}$. In general

$$U_{t_0+2i} = U_{t_0}(Q_D)^{2i} + X_{2i}$$

with

$$\|X_{2i}\| \leqq 2^{-t_0} + 2^{-t_0-2} + \ldots + 2^{-t_0-2i+2} < 2^{-t_0+1} \qquad \text{for } i=1,2,3,\ldots.$$

Hence $\|U_{t_0+2i}-U_{t_0}(Q_D)^{2i}\|<2^{-t_0+1}<\varepsilon/2$ for $i=1,2,3,\ldots$. Now choose I so that $\|U_{t_0}(Q_D)^{2i}-Y\|<\varepsilon/2$ for all $i\geqq I$. Then for $i\geqq I$, $\|U_{t_0+2i}-Y\|<\varepsilon$, and hence the lemma is proved. //

Lemma 4.2. *Let ${}_lV_t$ (for $t=l, l+1, l+2, \ldots$) be specified by Markov Process E_l, and let ${}_lV'_{l+m}$ (for $m=0,1,2,\ldots$) be the vector with components*

$$ {}_lv'_{l+m,i}=\sum_{w_\cdot=0}^{i}({}_lv_{l+m,(i,w_\cdot)}) \qquad \textit{for } i=0,1,2,\ldots. $$

Let Y be given by Eq. (4.10). Then for each $\varepsilon>0$, there exists a positive integer T (depending on ε) such that for $l\geqq T$,

$$ \|{}_lV'_{l+m}-Y\|<\varepsilon/2 \quad \textit{if } 0\leqq m\leqq 4l. $$

Proof. Using Lemma 4.1 and the fact that Y is invariant under Q_D, we can choose T large enough so that $\|U_l(Q_D)^m-Y\|<\varepsilon/4$ and $l\cdot 2^{-l}<\varepsilon/32$ for all $l\geqq T$ and $m\geqq 0$. For $l\geqq T$, we consider ${}_lV_t$ in Markov Process E_l. We have

$$ {}_lV_{l+1}={}_lV_lQ_{E_l}={}_lV_l(Q_{F_l}+2^{-l}Q''_{E_l}) $$

by Eq. (4.9). Let $X_1=2^{-l}{}_lV_lQ''_{E_l}$. Then

$$ {}_lV_{l+1}={}_lV_lQ_{F_l}+X_1 \quad \text{with } \|X_1\|\leqq 2^{-l+1}. $$

Next ${}_lV_{l+2}={}_lV_{l+1}(Q_{F_l}+2^{-l}Q''_{E_l})={}_lV_l(Q_F)^2+X_2$ with $X_2=X_1Q_{F_l}+2^{-l}{}_lV_{l+1}Q''_{E_l}$. Note that $\|X_2\|\leqq 2^{-l+1}+2^{-l+1}=2\cdot 2^{-l+1}$. In general

$$ {}_lV_{l+m}={}_lV_l(Q_{F_l})^m+X_m \tag{4.11} $$

with $\|X_m\|\leqq m\cdot 2^{-l+1}$ for $m\geqq 0$. Now we note that because ${}_lv_{l,(i,w_\cdot)}=0$ if $w_i\neq 0$ (see initialization of Markov Process E_l), then ${}_lV_l(Q_{F_l})^m$ has (i,w_i) component equal to zero if $w_i\neq 0$ (see definition of Q_{F_l} in Markov Process F_l). Furthermore we note that ${}_lq'_{(i,0),(j,0)}$ in Markov Process F_l equals q_{ij} in Markov Process D for all $i\geqq 0$ and $j\geqq 0$. We let ${}_lV'_{l+m}$ be defined as in the statement of Lemma 4.2; we let ${}_lV''_{l+m}$ be the vector with components

$$ {}_lv''_{l+m,i}=\sum_{w_\cdot=0}^{i}({}_lV_l(Q_{F_l})^m)_{(i,w_\cdot)} \qquad \text{for } i=0,1,2,\ldots; $$

and we let X'_m be the vector with components

$$ x'_{m,i}=\sum_{w_\cdot=0}^{i}(X_m)_{(i,w_\cdot)} \qquad \text{for } i=0,1,2,\ldots. $$

Then from Eq. (4.11), we have ${}_lV'_{l+m}={}_lV''_{l+m}+X'_m$ with $\|X'_m\|\leqq m\cdot 2^{-l+1}$ for $m\geqq 0$. Furthermore our above discussion shows that ${}_lv''_{l+m,i}=({}_lV_l(Q_{F_l})^m)_{(i,0)}=(U_l(Q_D)^m)_i$. So

$$ {}_lV'_{l+m}=U_l(Q_D)^m+X'_m. $$

Hence for $l \geqq T$ and $0 \leqq m \leqq 4l$, we have

$$\|{}_lV'_{l+m} - Y\| \leqq \|U_l(Q_D)^m - Y\| + \|X'_m\| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \quad //$$

Remark. From Eqs. (4.7) and (4.8) and from the definition of ${}_lV'_{l+m}$, we see that ${}_lv'_{t,i} = f_{t,l,i}$ for $t = l, l+1, l+2, \ldots$, and $i = 0, 1, 2, \ldots$.

Theorem 4.3. *Let $d_{t,e}$ be defined by Eq.* (1.1), *and let Y be given by Eq.* (4.10). *Let $G_t = (d_{t,0}, d_{t,1}, d_{t,2}, \ldots)$. Then*

$$\lim_{t \to \infty} \|G_t - Y\| = 0.$$

Proof. Let $\varepsilon > 0$ be given. From Eq. (4.1) and the remark following Lemma 4.2, we have

$$G_t = \sum_{\substack{1 \leqq l \leqq t \\ l \text{ odd}}} c_{t,l}({}_lV'_t). \tag{4.12}$$

Next let W_t (for $t \geqq 1$) be the random variable which takes on the value l (for $1 \leqq l \leqq t$, l odd) with probability $c_{t,l}$. Then one can check that the expected value of W_t is $t/2$ and the standard deviation of W_t is $\sqrt{t}/2$. For large t, W_t is approximately normally distributed. Since the standard deviation $\sqrt{t}/2$ is much smaller than the expected value $t/2$ for large t, then if

$$h_t(j) = \sum_{\substack{1 \leqq l \leqq j \\ l \text{ odd}}} c_{t,l} \text{ (the cumulative probability for } W_t), \tag{4.13}$$

we can choose t sufficiently large so that $h_t(t/4)$ is arbitrarily small. We choose t_0 so that $h_t(t/4) < \varepsilon/4$ for all $t \geqq t_0$. Now we write $G_t = G'_t + G''_t$, where

$$G'_t = \sum_{\substack{1 \leqq l \leqq t/4 \\ l \text{ odd}}} c_{t,l}({}_lV'_t) \quad \text{and} \quad G''_t = \sum_{\substack{t/4 < l \leqq t \\ l \text{ odd}}} c_{t,l}({}_lV'_t).$$

We let $Y' = h_t(t/4) \cdot Y$ and $Y'' = (Y - Y')$. Let $T_0 = \max(t_0, 4T)$, where T is specified in Lemma 4.2. Then for $t \geqq T_0$, we have

$$\|G_t - Y\| \leqq \|G'_t - Y'\| + \|G''_t - Y''\|$$

with

$$\|G'_t - Y'\| \leqq \sum_{\substack{1 \leqq l \leqq t/4 \\ l \text{ odd}}} c_{t,l}\|{}_lV'_t\| + h_t(t/4)\,\|Y\| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2,$$

and

$$\|G''_t - Y''\| \leqq \sum_{\substack{t/4 < l \leqq t \\ l \text{ odd}}} c_{t,l}\|{}_lV'_t - Y\| < \varepsilon/2$$

by Eqs. (4.4) and (4.13) and by Lemma 4.2. So

$$\|G_t - Y\| < \varepsilon \qquad \text{for all } t \geqq T_0,$$

and the proof is complete. //

Numerical values of $d_{t,e}$ for small t and e and also numerical values for the limiting density $d_{\infty,e}$ appear in Appendix II.

5. Calculations for the real quadratic case

We let notations be as in Sect. 1. In this section we shall present the results for the real quadratic case and discuss the similarities and differences of the real quadratic case and the imaginary quadratic case. For a real quadratic field K we write $K=\mathbf{Q}(\sqrt{m})$, where m is a square-free positive integer. We let $p_1<p_2<\dots$ be the odd prime numbers dividing m. Analogous to Eqs. (2.1), (2.2), and (2.3), we have

$$m=p_1\dots p_t \text{ with an even number of } p_i\equiv 3\ (\mathrm{mod}\ 4), \text{ or} \tag{5.1}$$

$$m=p_1\dots p_{t-1} \text{ with an odd number of } p_i\equiv 3\ (\mathrm{mod}\ 4), \text{ or} \tag{5.2}$$

$$m=2p_1\dots p_{t-1}. \tag{5.3}$$

Analogous to Formula (2.5) we have

$$|B_{t;x}|\sim\frac{1}{2}\frac{1}{(t-1)!}\frac{x(\log\log x)^{t-1}}{\log x}\quad(\text{as } x\to\infty), \tag{5.4}$$

and we may confine our attention to those m which satisfy Eq. (5.1). Next we associate to each $K=\mathbf{Q}(\sqrt{m})$ the $t\times(t-1)$ matrix $M'_K=[a_{ij}]$, each $a_{ij}\in\mathbf{F}_2$, where

$$(-1)^{a_{ij}}=\begin{cases}\left(\dfrac{P_j}{p_i}\right) & \text{if } i\neq j\\ \left(\dfrac{\bar{P}_j}{p_i}\right) & \text{if } i=j\end{cases}\quad \text{for } 1\leqq i\leqq t \text{ and } 1\leqq j\leqq t-1. \tag{5.5}$$

Here $P_j=p_j$ if $p_j\equiv 1\ (\mathrm{mod}\ 4)$; $P_j=-p_j$ if $p_j\equiv 3\ (\mathrm{mod}\ 4)$; and $\bar{P}_j=m/P_j$. We let M_K be the $t\times t$ matrix whose entries are defined by Eq. (5.5), except with $1\leqq j\leqq t$ instead of $1\leqq j\leqq t-1$. As in the imaginary quadratic case the sum of the entries in each row of M_K is zero. So rank $M_K=$ rank M'_K, and we could omit any column of M_K without changing the rank of the matrix. However in contrast to the imaginary quadratic case, the sum of the entries in each column of M_K is not always zero in the real quadratic case. In fact for real K one can check that the sum of the entries in the j-th column of M_K equals zero if $p_j\equiv 1\ (\mathrm{mod}\ 4)$ but equals one if $p_j\equiv 3\ (\mathrm{mod}\ 4)$. Now we let p_g denote the largest of the primes $\equiv 3\ (\mathrm{mod}\ 4)$ that divide m. (If no prime $\equiv 3\ (\mathrm{mod}\ 4)$ divides m, we take $p_g=p_t$.) Then we let M''_K be the $t\times(t-1)$ matrix obtained from M_K by first discarding the g-th column of M_K and then replacing the g-th row of the resulting matrix by the sum of the rows of that matrix. If R_K is the 4-class rank of K (in the narrow sense), then the analog of Eq. (2.7) is

$$R_K=t-1-\mathrm{rank}\, M'_K=t-1-\mathrm{rank}\, M_K=t-1-\mathrm{rank}\, M''_K. \tag{5.6}$$

Next we let $S_{t,l}=\{K=\mathbf{Q}(\sqrt{m})\in B_t\colon m \text{ satisfies Eq. (5.1) with exactly } l \text{ primes } p_i\equiv 3\ (\mathrm{mod}\ 4)\}$, $0\leqq l\leqq t$ with l even. Then we can proceed to obtain formulas analogous to Formulas (2.8) through (2.12), except with B instead of A and with l taking on the even values in the interval $0 \leqq l \leqq t$. So assume we have already obtained formulas analogous to Formulas (2.8) through (2.12), and assume we have real fields K_1 and K_2 analogous to the K_1 and K_2 in the imaginary quadratic case. Now we have reached a point where the difference between the real and imaginary quadratic cases becomes significant for subsequent calculations. The matrix M''_{K_2} in the real quadratic case is a $t \times (t-1)$ matrix instead of a $(t-1) \times (t-1)$ matrix and has the following properties when $l \geqq 2$: $M''_{K_2} = [a'_{ij}]$ with $a'_{ij} \neq a'_{ji}$ when $1 \leqq i < j \leqq l-1$; $a'_{lj} = 1$ for $1 \leqq j \leqq l-1$ and $a'_{lj} = 0$ for $l \leqq j \leqq t-1$; $a'_{(i+1)j} = a'_{ji}$ when $l \leqq i \leqq t-1$ and $1 \leqq j \leqq l-1$; and $a'_{(i+1)j} = a'_{(j+1)i}$ when $l \leqq i \leqq t-1$ and $l \leqq j \leqq t-1$.

Now by applying certain row exchanges to M''_{K_2} and then taking the transpose of the resulting matrix, we get a $(t-1) \times t$ matrix

$$\bar{M} = \left[\begin{array}{c|c} H_{l-1} & M \\ 0_{t-l} & \end{array}\right] \tag{5.7}$$

where $H_{l-1} \in \mathbf{F}_2^{l-1}$ is the vector with each component equal to 1; 0_{t-l} is the zero vector in $\mathbf{F}_2^{t-l}$; and

$$M = [a_{ij}], \quad \text{each } a_{ij} \in \mathbf{F}_2, \quad 1 \leqq i \leqq t-1,\ 1 \leqq j \leqq t-1, \tag{5.8}$$

with $a_{ij} \neq a_{ji}$ when $1 \leqq i < j \leqq l-1$, and with $a_{ij} = a_{ji}$ when $l \leqq i \leqq t-1$ and $1 \leqq j \leqq t-1$. Because of the way $\bar{M}$ was obtained from M''_{K_2}, we have $\operatorname{rank} \bar{M} = \operatorname{rank} M''_{K_2}$. For $l \geqq 2$, we now let $N'(t-1, l-1, r)$ denote the number of matrices $\bar{M}$ of the form given by Eq. (5.7) with M satisfying Eq. (5.8) and such that $\operatorname{rank} \bar{M} = r$. When $l = 0$, the appropriate $(t-1) \times t$ matrix $\bar{M}$ has as its first column the zero vector 0_{t-1} in $\mathbf{F}_2^{t-1}$, and the $(t-1) \times (t-1)$ matrix M is symmetric. We then let $N'(t-1, -1, r)$ denote the number of these matrices $\bar{M}$ with $\operatorname{rank} \bar{M} = r$. We can then obtain a result analogous to Proposition 2.1.

Proposition 5.1. *Let $B_{t;x}$, $B_{t,e;x}$, and $d'_{t,e}$ be defined as in Sect. 1, and let l be a nonnegative even integer. Let $N'(t-1, l-1, r)$ be defined as above. Then*

$$|B_{t;x}| \sim \frac{1}{2} \frac{1}{(t-1)!} \frac{x(\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty),$$

$$|B_{t,e;x}| \sim \sum_{\substack{0 \leqq l \leqq t \\ l \text{ even}}} N'(t-1, l-1, t-1-e) \cdot \binom{t}{l} \cdot 2^{-(t^2+t)/2}$$
$$\cdot \frac{1}{(t-1)!} \frac{x(\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty),$$

$$d'_{t,e} = \sum_{\substack{0 \leqq l \leqq t \\ l \text{ even}}} N'(t-1, l-1, t-1-e) \cdot \binom{t}{l} \cdot 2^{1-(t^2+t)/2}. \quad //$$

Our next goal is to develop an algorithm for computing $N'(t-1, l-1, t-1-e)$. To simplify notation, we let $n = t-1$, $k = l-1$, and $r = t-1-e$. First we suppose $l = 0$, and hence we want to compute $N'(n, -1, r)$. Since $\bar{M} = [0_n M]$ with M symmetric, we obtain the following result by using Proposition 3.7.

Proposition 5.2. *Let $\bar{M}=[0_n M]$ be an $n\times(n+1)$ matrix with 0_n the zero vector in $\mathbf{F}_2^n$, with $M=[a_{ij}]$ an $n\times n$ symmetric matrix, and with* rank $\bar{M}=r$. *Suppose $\bar{M}_1=[0_{n+1} M_1]$ is an $(n+1)\times(n+2)$ matrix with 0_{n+1} the zero vector in $\mathbf{F}_2^{n+1}$ and with $M_1=[b_{ij}]$ an $(n+1)\times(n+1)$ symmetric matrix such that $b_{ij}=a_{ij}$ for $1\leqq i\leqq n$ and $1\leqq j\leqq n$. Then among all such $\bar{M}_1$,* (i) 2^r *have* rank $\bar{M}_1=r$; (ii) 2^r *have* rank $\bar{M}_1=r+1$; *and* (iii) $2^{n+1}-2^{r+1}$ *have* rank $\bar{M}_1=r+2$. //

We may now suppose $l\geqq 2$, and we want to develop algorithms for computing $N'(n,k,r)$. As in the imaginary quadratic case, we next suppose $n=k$, and we shall consider both odd and even k. Now Eq. (5.7) becomes

$$\bar{M}=[H_n M] \tag{5.9}$$

where $H_n\in\mathbf{F}_2^n$ is the vector with each component equal to 1, and M is an $n\times n$ antisymmetric matrix.

We let S_0 (resp., S_1) be the subset of $\mathbf{F}_2^n$ consisting of all vectors $Y\in\mathbf{F}_2^n$ with the sum of the components of Y equal to 0 (resp., 1). Note that S_0 is a subspace of $\mathbf{F}_2^n$, and $\dim S_0=n-1$. Also note that if Y_1 is one vector in S_1, then $S_1=\{Y_1+Y\colon Y\in S_0\}$.

Lemma 5.3. *Let $c(\bar{M})$ denote the column space of $\bar{M}$, and let $c_0(M^T)=\{M^T Y\colon Y\in S_0\}$. Assume* rank $\bar{M}=r$. *Then* $\dim c_0(M^T)=r-1$. *If n is odd, then*

$$\dim[c(\bar{M})+c_0(M^T)]=n \quad \textit{and} \quad \dim[c(\bar{M})\cap c_0(M^T)]=2r-n-1.$$

If n is even, then

$$\dim[c(\bar{M})+c_0(M^T)]\geqq n-1 \quad \text{and} \quad \dim[c(\bar{M})\cap c_0(M^T)]\leqq 2r-n.$$

Proof. First we claim that $\dim c_0(M^T)=r-1$. For suppose $Y\in S_0$. Then $Y^T\bar{M}=[0\, Y^T M]=[0\,(M^T Y)^T]$. Let $Z=\{Y^T\bar{M}\colon Y\in S_0\}$. Then $\dim c_0(M^T)=\dim Z$. Since rank $\bar{M}=r$ and $\dim S_0=n-1$, then $\dim Z$ must be r or $r-1$. However each vector in Z has first component equal to 0, whereas each row of $\bar{M}$ has first component equal to 1. So in fact $\dim c_0(M^T)=\dim Z=r-1$. Next we shall show that $\dim[c(\bar{M})+c_0(M^T)]\geqq n-1$. Since M is antisymmetric, then $M^T=M+I+J$, where I is the $n\times n$ identity matrix, and J is the $n\times n$ matrix each of whose entries is 1. Let $W\in S_0$. Then $W=(I+J)W=MW+(M+I+J)W\in[c(\bar{M})+c_0(M^T)]$. So $S_0\subset[c(\bar{M})+c_0(M^T)]$, and hence $\dim[c(\bar{M})+c_0(M^T)]\geqq n-1$. Now suppose n is odd. Then $H_n\notin S_0$ but $H_n\in[c(\bar{M})+c_0(M^T)]$. So $\dim[c(\bar{M})+c_0(M^T)]=n$ if n is odd. Hence when n is odd,

$$\begin{aligned}\dim[c(\bar{M})\cap c_0(M^T)]&=\operatorname{rank}\bar{M}+\dim c_0(M^T)-\dim[c(\bar{M})+c_0(M^T)]\\&=r+(r-1)-n=2r-n-1.\end{aligned}$$

When n is even, one obtains $\dim[c(\bar{M})\cap c_0(M^T)]\leqq 2r-n$. //

Since $2r-n-1\geqq 0$ (resp., $2r-n\geqq 0$) when n is odd (resp., even) in Lemma 5.3, we get the following corollary.

Corollary 5.4. *Suppose $\bar{M}$ is given by Eq. (5.9) with M an $n\times n$ antisymmetric matrix. If n is odd, then* rank $\bar{M}\geqq(n+1)/2$. *If n is even, then* rank $\bar{M}\geqq n/2$. //

Now let Y_1 be one vector in S_1 which will be fixed throughout this discussion. Let $c_1(M^T)=\{M^T Y\colon Y\in S_1\}=\{M^T Y_1+M^T Y_0\colon Y_0\in S_0\}$. When n is odd, Lemma 5.3 implies that $H_n+M^T Y_1\in[c(\bar{M})+c_0(M^T)]$. Furthermore since $\dim[c(\bar{M})\cap c_0(M^T)]=2r-n-1$, then there exist 2^{2r-n-1} pairs of vectors $\{V, W\}$ with $V\in c(\bar{M})$, $W\in c_0(M^T)$, and $H_n+M^T Y_1=V+W$. Then $V+H_n=M^T Y_1+W\in c_1(M^T)$.

Now suppose $\bar{M}_1$ is an $(n+1)\times(n+2)$ matrix of the form

$$\bar{M}_1=\begin{bmatrix} H_n & M & V \\ 1 & (V+H_n)^T & v \end{bmatrix} \tag{5.10}$$

where H_n and M are given by Eq. (5.9), $V\in\mathbf{F}_2^n$, and $v\in\mathbf{F}_2$. If we assume rank $\bar{M}=r$, where $\bar{M}$ is given by Eq. (5.9), then rank $\bar{M}_1=r$, $r+1$, or $r+2$. Given $\bar{M}$, we want to know how many $\bar{M}_1$ have rank $\bar{M}_1=r$, rank $\bar{M}_1=r+1$, and rank $\bar{M}_1=r+2$. Suppose n is odd. If rank $\bar{M}_1=r$, then $V\in c(\bar{M})$ and $[1\,(V+H_n)^T]\in$(row space of $\bar{M}$). But then $V+H_n\in c_1(M^T)$, and hence $\{V, V+H_n\}$ must be one of the 2^{2r-n-1} pairs with $V\in c(\bar{M})$ and $V+H_n\in c_1(M^T)$. Write $V=\bar{M}X$ and $V+H_n=M^T Y$ with $X\in\mathbf{F}_2^{n+1}$ and $Y\in S_1$. For $v=[1\,(V+H_n)^T]\,X$, we have rank $\bar{M}_1=r$. If we wrote $V=\bar{M}(X+X')$ with $X'\in$(null space of $\bar{M}$), then

$$\begin{aligned}[1\,(V+H_n)^T]\,(X+X')&=[1\,(V+H_n)^T]\,X+[1\,(V+H_n)^T]\,X'\\&=[1\,(V+H_n)^T]\,X+[1\,(M^TY)^T]\,X'\\&=[1\,(V+H_n)^T]\,X+[1\;Y^TM]\,X'\\&=[1\,(V+H_n)^T]\,X+Y^T[H_n\,M]\,X'=[1\,(V+H_n)^T]\,X.\end{aligned}$$

So given V, there is only one choice of v such that rank $\bar{M}_1=r$. We have proved the first part of the following proposition.

Proposition 5.5. *Let $\bar{M}$ be given by Eq.* (5.9) *with* rank $\bar{M}=r$. *Let $\bar{M}_1$ be any matrix satisfying Eq.* (5.10). *Assume n is odd. Of all possible $\bar{M}_1$,*

(i) 2^{2r-n-1} *have* rank $\bar{M}_1=r$;
(ii) $3\cdot 2^r-3\cdot 2^{2r-n-1}$ *have* rank $\bar{M}_1=r+1$;
(iii) $2^{n+1}+2^{2r-n}-3\cdot 2^r$ *have* rank $\bar{M}_1=r+2$.

Proof. (i) has already been proved. To prove (ii), we describe the ways in which rank $\bar{M}_1=r+1$. First $\{V, V+H_n\}$ could be one of the pairs with $V\in c(\bar{M})$ and $V+H_n\in c_1(M^T)$, but we could choose $v\neq[1\,(V+H_n)^T]\,X$ when $V=\bar{M}X$. This situation gives 2^{2r-n-1} choices for $\bar{M}_1$. Next we could choose $V\in c(\bar{M})$, $V+H_n\notin c_1(M^T)$, and v arbitrary. This situation gives $(2^r-2^{2r-n-1})\cdot 2$ choices for $\bar{M}_1$. Finally we could choose $V\notin c(\bar{M})$, $V+H_n\in c_1(M^T)$, and v arbitrary. This situation gives $(2^{r-1}-2^{2r-n-1})\cdot 2$ choices for $\bar{M}_1$. Since

$$2^{2r-n-1}+(2^r-2^{2r-n-1})\cdot 2+(2^{r-1}-2^{2r-n-1})\cdot 2=3\cdot 2^r-3\cdot 2^{2r-n-1},$$

we have proved (ii). Finally (iii) follows from calculating $2^{n+1}-(2^{2r-n-1}+3\cdot 2^r-3\cdot 2^{2r-n-1})$. //

We now suppose n is even. If $\dim[c(\bar{M})+c_0(M^T)]=n$, then Proposition 5.5 will be valid for $\bar{M}$. However because n is even, it is possible that $\dim[c(\bar{M})$

$+c_0(M^T)]=n-1$. So we need to find the M for which $\dim[c(\bar{M})+c_0(M^T)]=n-1$. From the proof of Lemma 5.3, we recall that $S_0 \subset [c(\bar{M})+c_0(M^T)]$. Since $\dim S_0=n-1$, then we must have $[c(\bar{M})+c_0(M^T)]=S_0$. So the sum of the entries in each column of $\bar{M}$ (and hence of M) is 0. Then the sum of the entries in each column of $(M+I+J)$ equals 1 since n is even. Since $M^T=M+I+J$, then the sum of the entries in each row of M is 1, and hence the sum of the entries in each row of $\bar{M}$ is 0. So

$$\bar{M}=\begin{bmatrix} H_{n-1} & M_0 & V_0 \\ 1 & (V_0+H_{n-1})^T & v_0 \end{bmatrix} \tag{5.11}$$

where $H_{n-1}\in \mathbf{F}_2^{n-1}$ has each component equal to 1; M_0 is an $(n-1)\times(n-1)$ antisymmetric matrix; $V_0=\bar{M}_0 H_n$, where $\bar{M}_0=[H_{n-1}\, M_0]$; and $v_0=1+(V_0+H_{n-1})^T H_{n-1}$. We can now proceed in a straightforward manner to obtain the following analogs of Propositions 3.5 and 3.6.

Proposition 5.6. *Let $\bar{M}$ be given by Eq.* (5.9) *with* rank $\bar{M}=r$. *Let $\bar{M}_1$ be any matrix satisfying Eq.* (5.10). *Assume n is even. If $\bar{M}$ does not satisfy Eq.* (5.11), *then of all possible $\bar{M}_1$,*

(i) 2^{2r-n-1} *have* rank $\bar{M}_1=r$;
(ii) $3\cdot 2^r-3\cdot 2^{2r-n-1}$ *have* rank $\bar{M}_1=r+1$;
(iii) $2^{n+1}+2^{2r-n}-3\cdot 2^r$ *have* rank $\bar{M}_1=r+2$.

If $\bar{M}$ does satisfy Eq. (5.11) *then of all possible $\bar{M}_1$,*

(iv) $3\cdot 2^r$ *have* rank $\bar{M}_1=r+1$;
(v) $2^{n+1}-3\cdot 2^r$ *have* rank $\bar{M}_1=r+2$. //

Remark. We can directly compute $N'(1,1,1)=2$ and $N'(1,1,0)=0$. Then we can use Propositions 5.5 and 5.6 to compute $N'(n,n,r)$ for $n=2,3,4,\ldots$, and $0\leq r\leq n$. By combining Propositions 5.5 and 5.6, we get the following result.

Proposition 5.7. *Suppose $\bar{M}$ is given by Eq.* (5.9) *with n odd and* rank $\bar{M}=r$. *Write $M=[a_{ij}]$ for $1\leq i\leq n$ and $1\leq j\leq n$ in Eq.* (5.9). *Suppose $\bar{M}_2=[H_{n+2}\, M_2]$ is an $(n+2)\times(n+3)$ matrix with the following properties: $H_{n+2}\in\mathbf{F}_2^{n+2}$ is the vector with each component equal to* 1; *$M_2=[b_{ij}]$ is an $(n+2)\times(n+2)$ antisymmetric matrix; and $b_{ij}=a_{ij}$ for $1\leq i\leq n$ and $1\leq j\leq n$. Then of all possible $\bar{M}_2$,*

(i) $2^{4r-2n-3}-2^{2r-n-2}$ *have* rank $\bar{M}_2=r$;
(ii) $9\cdot 2^{3r-n-1}-15\cdot 2^{4r-2n-3}+3\cdot 2^{2r-n-2}$ *have* rank $\bar{M}_2=r+1$;
(iii) $7\cdot 2^{2r+2}-63\cdot 2^{3r-n-1}+35\cdot 2^{4r-2n-2}-2^{2r-n-1}$ *have* rank $\bar{M}_2=r+2$;
(iv) $9\cdot 2^{r+n+2}-21\cdot 2^{2r+2}+63\cdot 2^{3r-n}-15\cdot 2^{4r-2n}$ *have* rank $\bar{M}_2=r+3$;
(v) $2^{2n+3}-9\cdot 2^{r+n+2}+7\cdot 2^{2r+3}-9\cdot 2^{3r-n+2}+2^{4r-2n+3}$ *have* rank $\bar{M}_2=r+4$. //

We now want to compute $N'(n,k,r)$ with k odd. As described in the remark following Proposition 5.6, we can compute $N'(k,k,r)$. Now for fixed k we shall describe an algorithm for computing $N'(n,k,r)$ for $n>k$. So we consider an $n\times(n+1)$ matrix $\bar{M}$ with rank $\bar{M}=r$ and with

$$\bar{M}=\left[\begin{array}{c|c} H_k & \\ 0_{n-k} & M \end{array}\right]=\begin{bmatrix} H_k & A & B \\ 0_{n-k} & B^T & C \end{bmatrix} \tag{5.12}$$

where $H_k \in \mathbf{F}_2^k$ has each component equal to 1; 0_{n-k} is the zero vector in $\mathbf{F}_2^{n-k}$; A is a $k \times k$ antisymmetric matrix; B is a $k \times (n-k)$ matrix; and C is an $(n-k) \times (n-k)$ symmetric matrix. We let $\bar{M}_1$ be an $(n+1) \times (n+2)$ matrix of the form

$$\bar{M}_1 = \begin{bmatrix} H_k & A & B & V_1 \\ 0_{n-k} & B^T & C & V_2 \\ 0 & V_1^T & V_2^T & v \end{bmatrix} \tag{5.13}$$

where $V_1 \in \mathbf{F}_2^k$, $V_2 \in \mathbf{F}_2^{n-k}$, and $v \in \mathbf{F}_2$. We let

$$D = \begin{bmatrix} B \\ C \end{bmatrix} \tag{5.14}$$

and

$$D_1 = \begin{bmatrix} B & V_1 \\ C & V_2 \\ V_2^T & v \end{bmatrix}. \tag{5.15}$$

For each matrix L, we let $c(L)$ denote the column space of L. For those matrices L with h columns, where $h \geqq k$, we let $c'_0(L) = \{LY\colon Y \in \mathbf{F}_2^h$ and the sum of the first k components of Y equals $0\}$. We can then obtain the following analogs of Lemma 3.8, Lemma 3.9, and Proposition 3.10. We leave the details to the reader.

Lemma 5.8. *Suppose* $\operatorname{rank} \bar{M} = r$ *and* $\operatorname{rank} D = s$ *in Eqs.* (5.12) *and* (5.14). *Assume* k *is odd. Then*

$$\dim[c(\bar{M}) + c'_0(M^T)] = k + s \quad \text{and} \quad \dim[c(\bar{M}) \cap c'_0(M^T)] = 2r - k - s - 1. \quad //$$

Lemma 5.9. $\operatorname{Max}(0, r-k) \leqq s \leqq \min(r - (k+1)/2, n-k)$. //

Proposition 5.10. *Suppose* $k \geqq 1$ *is an odd integer. Suppose* $\bar{M}$, $\bar{M}_1$, D, *and* D_1 *are given by Eqs.* (5.12) *through* (5.15). *Let* $r = \operatorname{rank} \bar{M}$ *and* $s = \operatorname{rank} D$. *Of all possible* $\bar{M}_1$ *and* D_1,

(i) 2^s *have* $\operatorname{rank} \bar{M}_1 = r$ *and* $\operatorname{rank} D_1 = s$;
(ii) $2^{2r-k-s-1} - 2^s$ *have* $\operatorname{rank} \bar{M}_1 = r$ *and* $\operatorname{rank} D_1 = s+1$;
(iii) $3 \cdot 2^r - 3 \cdot 2^{2r-k-s-1}$ *have* $\operatorname{rank} \bar{M}_1 = r+1$ *and* $\operatorname{rank} D_1 = s+1$;
(iv) $2^{n+1} - 2^{k+s+1}$ *have* $\operatorname{rank} \bar{M}_1 = r+2$ *and* $\operatorname{rank} D_1 = s+2$;
(v) $2^{k+s+1} + 2^{2r-k-s} - 3 \cdot 2^r$ *have* $\operatorname{rank} \bar{M}_1 = r+2$ *and* $\operatorname{rank} D_1 = s+1$. //

Remark. Let $k \geqq 1$ be a fixed odd integer. Let $N'(n, k, r, s)$ denote the number of matrices $\bar{M}$ of the form specified by Eq. (5.12) such that $\operatorname{rank} \bar{M} = r$ and $\operatorname{rank} D = s$, where D is given by Eq. (5.14). We can use Propositions 5.5 and 5.6 to compute $N'(k, k, r, 0)$ and then use Proposition 5.10 to compute $N'(n, k, r, s)$ for $n = k+1, k+2, \ldots$. Then using Lemma 5.9, we get

$$N'(n, k, r) = \sum_{s = \max(0, r-k)}^{\min(r-(k+1)/2, n-k)} N'(n, k, r, s).$$

Also by slight modifications in our arguments, one can show that $N'(n+1, k+1, r) = N'(n+1, k, r)$ for $0 \leqq r \leqq n+1$ when k is odd.

We are now ready to calculate $d'_{t,e}$ and $d'_{\infty,e}$. Analogous to Eq. (4.1) through (4.4), we have

$$d'_{t,e} = \sum_{\substack{0 \leq l \leq t \\ l \text{ even}}} c_{t,l} f'_{t,l,e} \tag{5.16}$$

$$c_{t,l} = \binom{t}{l} \cdot 2^{-(t-1)}, \tag{5.17}$$

$$f'_{t,l,e} = N'(t-1, l-1, t-1-e) \cdot 2^{-t(t-1)/2}, \tag{5.18}$$

$$\sum_{\substack{0 \leq l \leq t \\ l \text{ even}}} c_{t,l} = 1. \tag{5.19}$$

Since we have an algorithm for computing $N'(t-1, l-1, t-1-e)$, we can compute $d'_{t,e}$ for $t=1,2,3,\ldots$, and $e=0,1,2,\ldots$. As in the computation of $d_{t,e}$ in Sect. 4, we actually perform the calculations by using denumerable Markov processes for computing the quantities $f'_{t,l,e}$, and from these Markov processes we then determine $d'_{\infty,e}$. Appendix III lists the Markov processes that are analogous to the Markov processes in Appendix I. Since the procedures we use now are very similar to those we used in Sect. 4, we skip directly to our final result, which is the analog of Theorem 4.3, and we let the reader fill in the details.

Theorem 5.11. *Let $d'_{t,e}$ be defined by Eq.* (1.3). *Let*

$$Y' = \beta^{-1}\left(1, 2/3, 4/63, \ldots, 2^{-i(i+1)} \prod_{m=1}^{i} (1-2^{-m})^{-1}(1-2^{-m-1})^{-1}, \ldots\right),$$

where

$$\beta^{-1} = \left[1 + 2/3 + 4/63 + \ldots + 2^{-i(i+1)} \prod_{m=1}^{i} (1-2^{-m})^{-1}(1-2^{-m-1})^{-1} + \ldots\right]^{-1}$$
$$= \prod_{m=2}^{\infty} (1-2^{-m}) = 2\alpha^{-1}.$$

Let $G'_t = (d'_{t,0}, d'_{t,1}, d'_{t,2}, \ldots)$. Then $\lim_{t\to\infty} \|G'_t - Y'\| = 0$. //

Remark. Numerically $\beta^{-1} \approx 0.577576190$.

Appendix IV contains numerical values of $d'_{t,e}$ for small t and e and also numerical values for the limiting density $d'_{\infty,e}$.

Appendix I. Markov processes for imaginary quadratic case

Markov process C: States $u_{t,i}$ with $t=1,3,5,\ldots$, and $i=0,1,2,\ldots$.

Let $U_t = (u_{t,0}, u_{t,1}, \ldots)$. Then $U_{t+2} = U_t Q_C^{(t)}$, where

$$Q_C^{(t)} = [q_{ij}^{(t)}] \quad \text{with } i=0,1,2,\ldots; \quad j=0,1,2,\ldots;$$

$$q_{ij}^{(t)} = \begin{cases} 1-3\cdot 2^{1-i}+13\cdot 2^{-2i}-3\cdot 2^{2-3i}+2^{2-4i} & \text{if } j=i-2 \\ 3\cdot 2^{1-i}-39\cdot 2^{-1-2i}+21\cdot 2^{-3i}-15\cdot 2^{-1-4i} & \text{if } j=i-1 \\ 13\cdot 2^{-1-2i}-21\cdot 2^{-1-3i}+35\cdot 2^{-3-4i}-2^{-2-2i-t} & \text{if } j=i \\ 3\cdot 2^{-1-3i}-15\cdot 2^{-4-4i}+3\cdot 2^{-3-2i-t} & \text{if } j=i+1 \\ 2^{-4-4i}-2^{-3-2i-t} & \text{if } j=i+2 \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: $U_1=(1,0,0,\ldots)$.

Markov process D: States $y_{t,i}$ with $t=1,2,3,\ldots$, and $i=0,1,2,\ldots$.
Let $Y_t=(y_{t,0}, y_{t,1}, \ldots)$. Then $Y_{t+1}=Y_t Q_D$, where

$$Q_D=[q_{ij}] \quad \text{with } i=0,1,2,\ldots; \quad j=0,1,2,\ldots;$$

$$q_{ij} = \begin{cases} 1+2^{-2i}-2^{1-i} & \text{if } j=i-1 \\ 2^{1-i}-3\cdot 2^{-1-2i} & \text{if } j=i \\ 2^{-1-2i} & \text{if } j=i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: $Y_1=(1,0,0,\ldots)$.

Markov process E_l $(l=1,3,5,\ldots)$: States ${}_lv_{t,(i,w_i)}$ with $t=l, l+1, l+2, \ldots$; $i=0,1,2,\ldots$; and $0\leqq w_i\leqq i$. Let ${}_lV_t=({}_lv_{t,(0,0)}, \ldots, {}_lv_{t,(i,w_i)}, \ldots)$, where the component ${}_lv_{t,(i,w_i)}$ precedes ${}_lv_{t,(j,w_j)}$ if $i<j$, or if $i=j$ and $w_i<w_j$. Then ${}_lV_{t+1}={}_lV_t Q_{E_l}$, where

$$Q_{E_l}=[{}_lq_{(i,w_i),(j,w_j)}] \quad \text{with } i=0,1,2,\ldots; \quad j=0,1,2,\ldots; \quad 0\leqq w_i\leqq i; \quad 0\leqq w_j\leqq j;$$

$${}_lq_{(i,w_i),(j,w_j)} = \begin{cases} 1-2^{-w_i} & \text{if } j=i-1,\ w_j=w_i-1 \\ 2^{-w_i}+2^{-2i+w_i}-2^{1-i} & \text{if } j=i-1,\ w_j=w_i \\ 2^{1-i}-3\cdot 2^{-1-2i+w_i} & \text{if } j=i,\ w_j=w_i \\ 2^{-1-2i+w_i}-2^{-w_i-l} & \text{if } j=i+1,\ w_j=w_i \\ 2^{-w_i-l} & \text{if } j=i+1,\ w_j=w_i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: ${}_lV_l$ has $\begin{cases} {}_lv_{l,(i,0)}=u_{l,i} & \text{for } 0\leqq i\leqq (l-1)/2 \\ {}_lv_{l,(i,w_i)}=0 & \text{otherwise.} \end{cases}$

Markov process F_l $(l=1,3,5,\ldots)$: States ${}_lz_{t,(i,w_i)}$ with $t=l, l+1, l+2, \ldots$; $i=0,1,2,\ldots$; and $0\leqq w_i\leqq i$. Let ${}_lZ_t=({}_lz_{t,(0,0)}, \ldots, {}_lz_{t,(i,w_i)}, \ldots)$, where the component ${}_lz_{t,(i,w_i)}$ precedes ${}_lz_{t,(j,w_j)}$ if $i<j$, or if $i=j$ and $w_i<w_j$. Then ${}_lZ_{t+1}={}_lZ_t Q_{F_l}$, where

$$Q_{F_l}=[{}_lq'_{(i,w_i),(j,w_j)}] \quad \text{with } i=0,1,2,\ldots; \quad j=0,1,2,\ldots; \quad 0\leqq w_i\leqq i; \quad 0\leqq w_j\leqq j;$$

$${}_lq'_{(i,w_i),(j,w_j)} = \begin{cases} 1-2^{-w_i} & \text{if } j=i-1,\ w_j=w_i-1 \\ 2^{-w_i}+2^{-2i+w_i}-2^{1-i} & \text{if } j=i-1,\ w_j=w_i \\ 2^{1-i}-3\cdot 2^{-1-2i+w_i} & \text{if } j=i,\ w_j=w_i \\ 2^{-1-2i+w_i} & \text{if } j=i+1,\ w_j=w_i \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: ${}_lZ_l={}_lV_l$.

Appendix II. Densities for 4-class ranks of imaginary quadratic fields

In the following table, t denotes the number of ramified primes for an imaginary quadratic field; e denotes the 4-class rank of the imaginary quadratic field; and $d_{t,e}$ is the density defined by Eq. (1.1).

Values of $d_{t,e}$

t \ e	0	1	2	3	4	5	6
1	1.0						
2	0.5	0.5					
3	0.4375	0.46875	0.09375				
4	0.375	0.515625	0.101563	0.007813			
5	0.350586	0.523682	0.117188	0.008240	0.000305		
6	0.331299	0.538666	0.120659	0.009079	0.000292	5.7×10^{-6}	
7	0.319630	0.547553	0.123847	0.008670	0.000296	4.9×10^{-6}	5.2×10^{-8}
8	0.311068	0.555336	0.125113	0.008225	0.000253	4.5×10^{-6}	3.9×10^{-8}
9	0.305101	0.560985	0.126023	0.007673	0.000215	3.4×10^{-6}	3.4×10^{-8}
10	0.300759	0.565305	0.126574	0.007182	0.000178	2.6×10^{-6}	2.3×10^{-8}
⋮							
15	0.291461	0.574843	0.127857	0.005758	0.000080	5.8×10^{-7}	3.1×10^{-9}
⋮							
20	0.289408	0.576950	0.128222	0.005365	0.000055	2.0×10^{-7}	4.6×10^{-10}
⋮							
∞	0.288788	0.577576	0.128350	0.005239	0.000047	9.7×10^{-8}	4.9×10^{-11}

(Rows may not add up to 1 because of roundoff error.)

Appendix III. Markov processes for real quadratic case

Markov process C': States $u_{t,i}$ with $t=2, 4, 6, \ldots$, and $i=0, 1, 2, \ldots$.
Let $U_t=(u_{t,0}, u_{t,1}, u_{t,2}, \ldots)$. Then $U_{t+2}=U_t Q_{C'}^{(t)}$, where

$$Q_{C'}^{(t)}=[q_{ij}^{(t)}] \quad \text{with} \quad i=0,1,2,\ldots; \quad j=0,1,2,\ldots;$$

$$q_{ij}^{(t)}=\begin{cases} 1-9\cdot 2^{-1-i}+7\cdot 2^{-2i}-9\cdot 2^{-1-3i}+2^{-4i} & \text{if } j=i-2 \\ 9\cdot 2^{-1-i}-21\cdot 2^{-1-2i}+63\cdot 2^{-3-3i}-15\cdot 2^{-3-4i} & \text{if } j=i-1 \\ 7\cdot 2^{-1-2i}-63\cdot 2^{-4-3i}+35\cdot 2^{-5-4i}-2^{-3-2i-t} & \text{if } j=i \\ 9\cdot 2^{-4-3i}-15\cdot 2^{-6-4i}+3\cdot 2^{-4-2i-t} & \text{if } j=i+1 \\ 2^{-6-4i}-2^{-4-2i-t} & \text{if } j=i+2 \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: $U_2=(1, 0, 0, \ldots)$.

Markov process D': States $y_{t,i}$ with $t=2, 3, 4, \ldots$, and $i=0, 1, 2, \ldots$.
Let $Y_t=(y_{t,0}, y_{t,1}, y_{t,2}, \ldots)$. Then $Y_{t+1}=Y_t Q_{D'}$, where

$$Q_{D'}=[q_{ij}] \quad \text{with } i=0,1,2,\ldots; \quad j=0,1,2,\ldots;$$

$$q_{ij}=\begin{cases} 1+2^{-1-2i}-3\cdot 2^{-1-i} & \text{if } j=i-1 \\ 3\cdot 2^{-1-i}-3\cdot 2^{-2-2i} & \text{if } j=i \\ 2^{-2-2i} & \text{if } j=i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: $Y_2=(1,0,0,\ldots)$.

Markov process E'_0: States ${}_0v_{t,i}$ with $t=1,2,3,\ldots$, and $i=0,1,2,\ldots$.
Let ${}_0V_t=({}_0v_{t,0},{}_0v_{t,1},{}_0v_{t,2},\ldots)$. Then ${}_0V_{t+1}={}_0V_tQ_{E'_0}$, where

$$Q_{E'_0}=[{}_0q_{ij}] \quad \text{with } i=0,1,2,\ldots; \quad j=0,1,2,\ldots;$$

$${}_0q_{ij}=\begin{cases} 1-2^{-i} & \text{if } j=i-1 \\ 2^{-1-i} & \text{if } j=i \\ 2^{-1-i} & \text{if } j=i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: ${}_0V_1=(1,0,0,\ldots)$.

Markov process E'_l $(l=2,4,6,\ldots)$: States ${}_lv_{t,(i,w_i)}$ with $t=l,l+1,l+2,\ldots$; $i=0,1,2,\ldots$; and $0\leqq w_i\leqq i$. Let ${}_lV_t=({}_lv_{t,(0,0)},\ldots,{}_lv_{t,(i,w_i)},\ldots)$, where the component ${}_lv_{t,(i,w_i)}$ precedes ${}_lv_{t,(j,w_j)}$ if $i<j$, or if $i=j$ and $w_i<w_j$. Then ${}_lV_{t+1}={}_lV_tQ_{E'_l}$, where

$$Q_{E'_l}=[{}_lq_{(i,w_i),(j,w_j)}] \quad \text{with } i=0,1,2,\ldots; \quad j=0,1,2,\ldots; \quad 0\leqq w_i\leqq i; \quad 0\leqq w_j\leqq j;$$

$${}_lq_{(i,w_i),(j,w_j)}=\begin{cases} 1-2^{-w_i} & \text{if } j=i-1,\ w_j=w_i-1 \\ 2^{-1-2i+w_i}+2^{-w_i}-3\cdot 2^{-1-i} & \text{if } j=i-1,\ w_j=w_i \\ 3\cdot 2^{-1-i}-3\cdot 2^{-2-2i+w_i} & \text{if } j=i,\ w_j=w_i \\ 2^{-2-2i+w_i}-2^{-w_i-l} & \text{if } j=i+1,\ w_j=w_i \\ 2^{-w_i-l} & \text{if } j=i+1,\ w_j=w_i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: ${}_lV_l$ has $\begin{cases} {}_lv_{l,(i,0)}=u_{l,i} & \text{for } 0\leqq i\leqq (l-2)/2 \\ {}_lv_{l,(i,w_i)}=0 & \text{otherwise.} \end{cases}$

Markov process F'_l $(l=2,4,6,\ldots)$: States ${}_lz_{t,(i,w_i)}$ with $t=l,l+1,l+2,\ldots$; $i=0,1,2,$; and $0\leqq w_i\leqq i$. Let ${}_lZ_t=({}_lz_{t,(0,0)},\ldots,{}_lz_{t,(i,w_i)},\ldots)$, where the component ${}_lz_{t,(i,w_i)}$ precedes ${}_lz_{t,(j,w_j)}$ if $i<j$, or if $i=j$ and $w_i<w_j$. Then ${}_lZ_{t+1}={}_lZ_tQ_{F'_l}$, where

$$Q_{F'_l}=[{}_lq'_{(i,w_i),(j,w_j)}] \quad \text{with } i=0,1,2,\ldots; \quad j=0,1,2,\ldots; \quad 0\leqq w_i\leqq i; \quad 0\leqq w_j\leqq j;$$

$${}_lq'_{(i,w_i),(j,w_j)}=\begin{cases} 1-2^{-w_i} & \text{if } j=i-1,\ w_j=w_i-1 \\ 2^{-1-2i+w_i}+2^{-w_i}-3\cdot 2^{-1-i} & \text{if } j=i-1,\ w_j=w_i \\ 3\cdot 2^{-1-i}-3\cdot 2^{-2-2i+w_i} & \text{if } j=i,\ w_j=w_i \\ 2^{-2-2i+w_i} & \text{if } j=i+1,\ w_j=w_i \\ 0 & \text{otherwise.} \end{cases}$$

Initial vector: ${}_lZ_l={}_lV_l$.

Appendix IV. Densities for 4-class ranks of real quadratic fields

In the following table, t denotes the number of ramified primes for a real quadratic field; e denotes the 4-class rank of the real quadratic field; and $d'_{t,e}$ is the density defined by Eq. (1.3).

Values of $d'_{t,e}$

t \ e	0	1	2	3	4	5	6
1	1.0						
2	0.75	0.25					
3	0.6875	0.28125	0.03125				
4	0.648438	0.3125	0.037109	0.001953			
5	0.627930	0.328369	0.041504	0.002136	6.1×10^{-5}		
6	0.613953	0.341690	0.042076	0.002222	5.8×10^{-5}	9.5×10^{-7}	
7	0.604473	0.351600	0.041839	0.002032	5.5×10^{-5}	7.8×10^{-7}	7.5×10^{-9}
8	0.597573	0.359503	0.041062	0.001818	4.4×10^{-5}	6.7×10^{-7}	5.3×10^{-9}
9	0.592507	0.365631	0.040228	0.001599	3.5×10^{-5}	4.7×10^{-7}	4.2×10^{-9}
10	0.588735	0.370372	0.039456	0.001410	2.7×10^{-5}	3.3×10^{-7}	2.7×10^{-9}
⋮							
15	0.580189	0.381558	0.037357	0.000888	8.4×10^{-6}	5.1×10^{-8}	2.5×10^{-10}
⋮							
20	0.578191	0.384230	0.036831	0.000745	4.2×10^{-6}	1.1×10^{-8}	2.6×10^{-11}
⋮							
∞	0.577576	0.385051	0.036672	0.000699	3.0×10^{-6}	3.1×10^{-9}	7.7×10^{-13}

(Rows may not add up to 1 because of roundoff error.)

References

1. Cohen, H.: Sur la distribution asymptotique des groupes de classes. C.R. Acad. Sc. Paris **296**, 245–247 (1983)
2. Cohen, H., Lenstra, H.: Heuristics on class groups, to appear in the proceedings of the Journées arithmétiques in Leiden (Holland) 1983
3. Cohn, H., Lagarias, J.: On the existence of fields governing the 2-invariants of the classgroup of $\mathbf{Q}(\sqrt{dp})$ as p varies. Math. Comp. in press (1984)
4. Davenport, H.: Multiplicative number theory, 2nd ed. New York: Springer 1980
5. Gerth, F.: Counting certain number fields with prescribed l-class numbers. J. Reine Angew. Math. **337**, 195–207 (1982)
6. Gerth, F., Graham, S.: Application of a character sum estimate to a 2-class number density. J. Number Theory in press (1984)
7. Hardy, G., Wright, E.: An Introduction to the Theory of Numbers, 4th ed. London: Oxford Univ. Press 1965
8. Heilbronn, H.: On the averages of some arithmetical functions of two variables. Mathematika **5**, 1–7 (1958)
9. Kemeny, J., Snell, J., Knapp, A.: Denumerable Markov Chains. New York: Springer 1976
10. Morton, P.: Density results for the 2-classgroups of imaginary quadratic fields. J. Reine Angew. Math. **332**, 156–187 (1982)
11. Rédei, L.: Arithmetischer Beweis des Satzes über die Anzahl der durch vier teilbaren Invarianten der absoluten Klassengruppe im quadratischen Zahlkörper. J. Reine Angew. Math. **171**, 55–60 (1934)
12. Rédei, L.: Über einige Mittelwertfragen im quadratischen Zahlkörper. J. Reine Angew. Math. **174**, 15–55 (1936)

Oblatum 25-I-1984 & 16-IV-1984