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Elementary construction of perverse sheaves

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Let X be a stratified topological space with only even (real) dimensional strata (see section 0 for precise definitions). Beilinson, Bernstein, and Deligne have defined a category $\mathbf{P}(X)$ called the category of perverse sheaves on X constructible with respect to the given stratification [BBD]. It was defined to be the full subcategory of $\mathbf{D}^b(X)$ whose objects are complexes of sheaves satisfying three conditions (see section 0 below), where $\mathbf{D}^b(X)$ denotes the (bounded) derived category of the category of sheaves on X. The category $\mathbf{P}(X)$ is important for several reasons: If X is a complex analytic manifold with an analytic stratification, then $\mathbf{P}(X)$ is equivalent to the category of holonomic r.s. $\mathscr{D}$-modules whose characteristic variety is contained in the union of conormal bundles to the strata ([BK], [Me1], [Me2], [Br]). If X is a special fiber of an algebraic variety Z over a curve, then a perverse sheaf on $Z-X$ specializes to a perverse sheaf on X, with appropriate stratifications. For any X, the category of perverse sheaves on X forms an abelian category whose simple objects are the intersection homology sheaves of the strata of X.

The definition of $\mathbf{P}(X)$ as a subcategory of $\mathbf{D}^b(X)$ has several drawbacks. Two objects may be very different as complexes of sheaves and still be equivalent in $\mathbf{D}^b(X)$; similarly different chain maps may represent the same morphism. The kernel and cokernel of a morphism are not the naive ones. In this paper, we give a construction of $\mathbf{P}(X)$ which is "elementary" in two senses: it is not subject to these drawbacks of the definition through $\mathbf{D}^b(X)$, and the structure of $\mathbf{P}(X)$ may be read off from the topology of X and its stratification.

Our construction is inductive on the strata of X, starting with the largest ones. The inductive step constructs $\mathbf{P}(X)$ from $\mathbf{P}(X-S)$ under the assumption that S is a closed stratum. (This suffices because we can always order the strata so that each one is closed in the union of it and the preceding ones.) Then $\mathbf{P}(X)$ is constructed as a category whose objects consist of an object A of $\mathbf{P}(X-S)$ together with a

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commutative triangle of ordinary sheaves

$$\begin{array}{ccc} F(A) & \xrightarrow{T_A} & G(A) \\ & \searrow \quad \nearrow & \\ & B & \end{array}$$

where F and G are certain functors on $\mathbf{P}(X-S)$ and T is a natural transformation. We give two versions of this inductive step: one for a topological stratified space using combinatorial topology (Sect. 3 and 4) and one for a complex analytic manifold using conormal geometry (Sect. 5). To complete the induction, we must describe the F, G, and T for adding the next stratum in terms of our construction of $\mathbf{P}(X)$. This is carried out in principle at the end of Sect. 4, where it is shown how to reconstruct a combinatorial complex of sheaves from our data. It is done explicitly in an example in [MV2]. In Sect. 6 we give some examples and some general theorems about $\mathbf{P}(X)$ which can be proved using our results.

Both of the versions of the inductive step require some extra structure normal to the stratum S. In the first version, a subspace of the link of S called a perverse link must be chosen. The second version uses conormal vectors to S. It appears to be impossible to give a complete elementary construction of $\mathbf{P}(X)$ without such an extra structure, but Sect. 2 gives a partial result in this direction. The objects of $\mathbf{P}(X)$ are determined and, for any two of them, the group of homomorphisms from one to the other is determined up to an extension. In Sect. 7, this is extended to study the structure of $\mathbf{D}^b(X)$ in terms of $\mathbf{D}^b(X-S)$.

We benefitted from conversations with Beilinson and Deligne, and from a letter of Deligne [D1] containing an elementary description of $\mathbf{P}(X)$ for two dimensional X. Galligo, Granger, and Maisonobe [GGM] have given a non-inductive elementary construction of $\mathbf{P}(X)$ for X stratified by normal crossings, and Verdier [V1], [V2] has studied extensions of perverse sheaves across a principal divisor and has applied this to prove some of our results of Sect. 5. The results of Verdier were independently obtained by Beilinson [Be]. Further work along these lines has been done by several people [GrM], [GK], [N] and [MV2].

The main theorems of this paper were announced in [Mac] and [MV1].

0. Notations

We will consider a Thom-Mather stratified space X with a fixed stratification $\mathcal{S}$ (see [T], [Ma] for the definitions). For example, all analytic spaces and all Whitney stratified spaces can be endowed with a Thom-Mather structure. As part of the structure, each stratum S of $\mathcal{S}$ has "control data" consisting of a tubular neighborhood T_S with a projection π_S to S and a function ρ_S measuring the distance to S. Since we will only consider the middle perversity, we will assume that all of our spaces have only even codimensional strata. In Sect. 5 and 6, where we consider complex analytic spaces, we will impose additional assumptions on the stratification $\mathcal{S}$.

We fix a field k. In this paper all sheaves will be sheaves of k-vector spaces and all vector spaces will be k-vector spaces unless otherwise stated.

We denote by $\mathbf{D}^b(X)$ the derived category of complexes of k-sheaves on X whose cohomology is bounded. As in [BBD] we define the category of perverse sheaves $\mathbf{P}(X)$ as a full subcategory of $\mathbf{D}^b(X)$ consisting of complexes of sheaves $\mathbf{A}^\bullet$ satisfying the following three conditions:

i) $\mathbf{H}^k(i^*\mathbf{A}^\bullet)$ is a local system of finite rank on S
ii) $\mathbf{H}^k(i^*\mathbf{A}^\bullet)=0$ for $k>-\dim S/2$
iii) $\mathbf{H}^k(i^!\mathbf{A}^\bullet)=0$ for $k<-\dim S/2$

for all strata $S\in\mathscr{S}$ where $i\colon S\hookrightarrow X$ denotes the inclusion. The category of perverse sheaves $\mathbf{P}(X)$ defined above is that associated to the middle perversity $\bar{m}$ ([GM 2]). We treat only the case of the middle perversity in this paper although the results with the exception of Sect. 5 are valid for an arbitrary perversity. We denote by $\mathbf{D}^b_{\mathscr{S}}(X)$ the full subcategory of $\mathbf{D}^b(X)$ whose objects satisfy the condition i) above.

If $\mathbf{A}^\bullet$ is a complex of sheaves on a space Z and $i\colon Y\hookrightarrow Z$ is a subspace we write $\mathbf{A}^\bullet\,|\,Y$ for $i^*\mathbf{A}$. If Y is closed in Z and $j\colon Z-Y\hookrightarrow Z$ denotes the inclusion of the complement we write $\mathbb{H}^*(Z,Y;\mathbf{A}^\bullet)$ for $\mathbb{H}^*(Z,j_!\mathbf{A}^\bullet\,|\,(Z-Y))$. We denote by $\delta\colon i_*\mathbf{A}^\bullet\,|\,Y\to j_!\mathbf{A}^\bullet\,|\,(Z-Y)\,[1]$ the degree raising map in the triangle

$$j_!\mathbf{A}^\bullet\,|\,(Z-Y)\to\mathbf{A}^\bullet\to i_*\mathbf{A}^\bullet\,|\,Y\xrightarrow{[1]}j_!\mathbf{A}^\bullet\,|\,(Z-Y).$$

which gives rise to the long exact sequence of the pair (Z,Y).

1. A construction of abelian categories

In this section we will present a purely category theoretic construction which is the key to our work on perverse sheaves.

We will consider the following data: two categories $\mathscr{A}$ and $\mathscr{B}$, two functors F and G from $\mathscr{A}$ to $\mathscr{B}$, and a natural transformation T from F to G. Symbolically $F,G\colon\mathscr{A}\to\mathscr{B}$ and $F\xrightarrow{T}G$. We define the category $\mathscr{C}(F,G;T)$ to be the category whose objects are pairs $(A,B)\in Obj\,\mathscr{A}\times Obj\,\mathscr{B}$ together with a commutative triangle

$$\begin{array}{ccc} FA & \xrightarrow{T_A} & GA \\ & {\scriptstyle m}\searrow \quad \nearrow{\scriptstyle n} & \\ & B & \end{array}$$

and whose morphisms are pairs $(a,b)\in\operatorname{Mor}\mathscr{A}\times\operatorname{Mor}\mathscr{B}$ such that

$$\begin{array}{ccccc} FA & \searrow & \xrightarrow{TA} & \nearrow & GA \\ {\scriptstyle Fa}\downarrow & & B & & \downarrow{\scriptstyle Ga} \\ FA' & \text{——} & {\scriptstyle b}\downarrow & \xrightarrow{TA'} & GA' \\ & \searrow & B' & \nearrow & \end{array}$$

commutes.

Proposition 1.1. *If $\mathscr{A}$ and $\mathscr{B}$ are abelian categories and if F is right exact and G is left exact then the category $\mathscr{C}(F,G;T)$ is abelian and the functors taking $(A,B)\mapsto A$ and $(A,B)\mapsto B$ from $\mathscr{C}(F,G;T)$ to $\mathscr{A}$ and $\mathscr{B}$ are exact.*

Proof. It is clear that $\mathscr{C}(F, G; T)$ is an additive category. We must show the existence of kernels, the existence of cokernels and that the canonical map from the coimage to the image is an isomorphism ([P], p. 27). First we construct kernels in $\mathscr{C}(F, G; T)$. Let (a, b): $(A, B) \to (A', B')$ be a morphism in $\mathscr{C}(F, G; T)$. Consider the diagram

$$\begin{array}{ccccc} F(\operatorname{Ker} a) & & \xrightarrow{T(\operatorname{Ker} a)} & & G(\operatorname{Ker} a) \\ \downarrow & & & & \downarrow \cong \\ \operatorname{Ker}(F(a)) & \longrightarrow & \operatorname{Ker} b & \longrightarrow & \operatorname{Ker}(G(a)) \end{array}$$

where the vertical maps are the canonical maps arising from the definition of the kernel. The map on the right is an isomorphism because G is left exact. The diagram of compositions

$$\begin{array}{ccc} F(\operatorname{Ker} a) & \longrightarrow & G(\operatorname{Ker} a) \\ & \searrow \quad \nearrow & \\ & \ker b & \end{array}$$

is the kernel of (a, b) in $\mathscr{C}(F, G; T)$.

The construction of cokernels using the right exactness of F is dual. The canonical map $\operatorname{coim}(a, b) \to \operatorname{Im}(a, b)$ is an isomorphism because the maps $\operatorname{coim} a \to \operatorname{Im} a$ and $\operatorname{coim} b \to \operatorname{Im} b$ are.

We want to compare two categories $\mathscr{C}(F, G; T)$ and $\mathscr{C}(F', G'; T')$ constructed from different data. Assume now that $\mathscr{B}$ is an abelian category. Suppose that we have functors F, G, F' and G' from $\mathscr{A}$ to $\mathscr{B}$ and natural transformations T, T', f and g according to the following commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{T} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F' & \xrightarrow[T']{} & G' \end{array}$$

An (f, g)-map from $FA \to B \to GA$ in $\mathscr{C}(F, G; T)$ to $F'A' \to B' \to G'A'$ in $\mathscr{C}(F', G'; T')$ is a pair of maps (a, b), a: $A \to A'$ and b: $B \to B'$ such that the diagram

$$\begin{array}{ccccc} F(A) & \longrightarrow & B & \longrightarrow & G(A) \\ {\scriptstyle f \circ F(a)}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle g \circ G(a)} \\ F'(A') & \longrightarrow & B' & \longrightarrow & G'(A') \end{array}$$

commutes.

Consider the following commutative diagram

$$\begin{array}{ccccc} F(A) & \xrightarrow{m} & B & \xrightarrow{n} & G(A) \\ \downarrow & & \downarrow & & \downarrow \\ F'(A) & \longrightarrow & B \amalg_{FA} F'A & \xrightarrow{n'} & G'(A) \end{array}$$

where the map n' is given by the universal property for the fiber sum $\amalg$. We define a functor θ_1: $\mathscr{C}(F, G, T) \to \mathscr{C}(F', G'; T')$ by $\theta_1(FA \to B \to GA) = F'A \to B \amalg_{FA} F'A \to$

$G'A$. The above diagram gives us an (f, g)-map from $X \in \mathscr{C}(F, G; T)$ to $\theta_1(X) \in \mathscr{C}(F', G'; T')$ which by the universal property of $\amalg$ induces an isomorphism

$$\mathrm{Hom}_{(f,g)}(X, Y) \cong \mathrm{Hom}(\theta_1 X, Y)$$

for $X \in \mathscr{C}(F, G; T)$ and $Y \in \mathscr{C}(F', G'; T')$.

Dually we define $\theta_2\colon \mathscr{C}(F', G'; T') \to \mathscr{C}(F, G; T)$ by sending $F'A \to B \to G'A$ to $FA \to B \times_{G'A} GA \to GA$ and we have

$$\mathrm{Hom}_{(f,g)}(X, Y) \cong \mathrm{Hom}(X, \theta_2 Y)$$

for $X \in \mathscr{C}(F, G; T)$ and $Y \in \mathscr{C}(F', G'; T')$.

Therefore (θ_1, θ_2) is a pair of adjoint functors.

Proposition 1.2. *If for every object $A \in \mathscr{A}$ the natural transformations f and g induce isomorphisms* $\mathrm{Ker}\, TA \to \mathrm{Ker}\, T'A$ *and* $\mathrm{Coker}\, TA \to \mathrm{Coker}\, T'A$ *then the functors θ_1 and θ_2 are equivalences of categories.*

Proof. We have to show that the adjunction maps N: $\mathrm{Id} \to \theta_2 \circ \theta_1$ and N^*: $\theta_1 \circ \theta_2 \to \mathrm{Id}$ are natural isomorphisms. The map N is (Id_A, N_B) where N_B: $B \to (B \amalg_{FA} F'A) \times_{G'A} GA$ is given by $N_B(x) = (x, 0, n(x))$. It remains to be checked that N_B is an isomorphism. Let $N_B(x) = 0$. By definition this means that $n(x) = 0$ and there exists $y \in FA$ such that $f(A)(y) = 0$ and $m(y) = x$. But now $T(A)(y) = n(m(y)) = n(x) = 0$. Because $f(A) \mid \mathrm{Ker}\, T(A)$ is an injection $y = 0$ and therefore $x = 0$. Similarly we can show that N_B is a surjection.

The proof that N^* is a natural isomorphism is dual.

Example 1.3. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and let E^* be a complex of exact functors from $\mathscr{A}$ to $\mathscr{B}$. Then $F = \mathrm{Coker}(E^{-2} \to E^{-1})$, $G = \mathrm{Ker}(E^0 \to E^1)$ and $T = d^{-1}$: $F \to G$ satisfies the hypotheses of proposition 1.1. If e: $E^* \to E'^*$ is a quasi-isomorphism of complexes of exact functors then proposition 1.2 applies to give an equivalence of categories e: $\mathscr{C}(F, G; T) \to \mathscr{C}(F', G'; T')$.

Remark 1.4. Let $\tilde{\mathscr{B}} \subset \mathscr{B}$ be a full abelian subcategory with $\tilde{\mathscr{B}} \to \mathscr{B}$ exact and $\tilde{\mathscr{B}}$ stable by extensions. Assume that $\mathrm{Ker}(FA \to GA)$ and $\mathrm{Coker}(FA \to GA)$ are in $\tilde{\mathscr{B}}$. Let $\tilde{\mathscr{C}}(F, G; T)$ be the subcategory of $\mathscr{C}(F, G; T)$ consisting of pairs (A, B) such that the kernel and cokernel of $FA \to B$ are in $\tilde{\mathscr{B}}$. It follows from the long exact sequence

$$0 \to \mathrm{Ker}\,(FA \to B) \to \mathrm{Ker}\,(FA \to GA) \to \mathrm{Ker}\,(B \to GA) \to \mathrm{Coker}\,(FA \to B)$$
$$\to \mathrm{Coker}\,(FA \to GA) \to \mathrm{Coker}\,(B \to GA) \to 0$$

that it is equivalent to define $\tilde{\mathscr{C}}(F, G; T)$ as a subcategory of $\mathscr{C}(F, G; T)$ such that $\mathrm{Ker}(B \to GA)$ and $\mathrm{Coker}(B \to GA)$ belong to $\tilde{\mathscr{B}}$. The category $\tilde{\mathscr{C}}(F, G; T)$ is a full subcategory of $\mathscr{C}(F, G; T)$ such that the inclusion functor is exact and $\tilde{\mathscr{C}}(F, G; T)$ is stable by extensions.

Appendix to Section 1.

In the remainder of this section we will give an interpretation of the constructions in this section which was pointed out to us by Deligne. These ideas will not be explicitly used in this paper.

The constructions involving θ_1 and θ_2 have nothing to do with $\mathscr{A}$. Let $\mathscr{B}$ be an abelian category. Let $Fl(\mathscr{B})$ be the category whose objects consist of morphisms in $\mathscr{B}$ and whose morphisms are the appropriate commutative squares. Let $Fl_2(\mathscr{B})$ be the category whose objects consist of pairs of morphisms (m, n) such that $n \circ m$ exists. There is a canonical functor $Fl_2(\mathscr{B}) \to Fl(\mathscr{B})$ given by conposing the morphisms. This functor gives $Fl_2(\mathscr{B})$ the structure of a fibered and cofibered category over $Fl(\mathscr{B})$ [SGA1]. The fiber above the point $F \xrightarrow{T} G$ in $Fl(\mathscr{B})$ is $\mathscr{C}(F, G; T)$. (Here F and G are objects in B and T is a morphism.) Given any map $(f, g)\colon (F \xrightarrow{T} G) \to (F' \xrightarrow{T'} G')$ we get by definition two adjoint functors θ_1: $\mathscr{C}(F, G; T) \to \mathscr{C}(F', G'; T')$ and θ_2: $\mathscr{C}(F', G'; T') \to \mathscr{C}(F, G; T)$. We have

i) If (f, g) is quasi-isomorphism then θ_1 and θ_2 are equivalences of categories (Proposition 1.2).

ii) If $H\colon G \to F'$ is a homotopy between (f, g) and (f', g') then the formula

$$\begin{array}{ccccc} F & \longrightarrow & B & \xrightarrow{n} & G \\ {\scriptstyle f}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\ F' & \xrightarrow[m']{} & B' & \longrightarrow & G' \end{array} \quad\longrightarrow\quad \begin{array}{ccccc} F & \longrightarrow & B & \longrightarrow & G \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle g'} \\ F' & \longrightarrow & B' & \longrightarrow & G' \end{array}$$

$u' = u + m'Hn$, gives an isomorphism $\mathrm{Hom}_{(f,g)}(X, Y) \cong \mathrm{Hom}_{(f',g')}(X, Y)$. This isomorphism is compatible with composition and therefore gives an isomorphism of functors $\theta_i(f, g) \cong \theta_i(f', g')$.

If $\mathscr{B}$ is the category of abelian sheaves on a topological space X then the above considerations show that $\mathscr{C}(F, G; T)$ only depends on the Picard stack $\mathscr{P}$ defined by $F \xrightarrow{T} G$ [SGA4 XVIII]. As a matter of fact we can construct it as follows. The objects of $\mathscr{C}(F, G; T)$ are morphisms $S\colon \mathscr{P} \to \mathscr{P}'$ such that S maps $H^0(\mathscr{P})$ onto $H^0(\mathscr{P}')$, where $H^0(\mathscr{P})$ stands for the sheaf generated by the presheaf of isomorphism classes of objects of $\mathscr{P}$. The morphisms are maps $\mathscr{P}' \to \mathscr{P}''$ such that

$$\begin{array}{ccc} & & \mathscr{P}' \\ & \nearrow & \downarrow \\ \mathscr{P} & & \\ & \searrow & \downarrow \\ & & \mathscr{P}'' \end{array}$$

commutes.

2. The zig-zag functor

In this section we give a version of the fundamental inductive step of constructing $\mathbf{P}(X)$ from $\mathbf{P}(X-S)$ for the case that S is contractible which almost works. It allows us to compute in an elementary way the set of isomorphism classes of elements of $\mathbf{P}(X)$ and, for any two elements $\mathbf{Q}^{\bullet}$ and $\bar{\mathbf{Q}}^{\bullet}$ of $\mathbf{P}(X)$, the group $\mathrm{Hom}(\mathbf{Q}^{\bullet}, \bar{\mathbf{Q}}^{\bullet})$ up to an extension. The methods of this section are used to study extensions of an arbitrary element in $\mathbf{D}^b_{\mathscr{S}}(X-S)$ in Sect. 7. The results of this section hold for any stratified topological space as in [GM2].

Suppose that S is any closed stratum of $\mathscr{S}$ of dimension $2d$. Denote by $j\colon X-S \hookrightarrow X$ and $i\colon S \hookrightarrow X$ the inclusions. We define the zig-zag category $\mathbf{Z}(X,S)$ as follows. An object in $\mathbf{Z}(X,S)$ is an object $\mathbf{P}^\bullet$ of $\mathbf{P}(X-S)$ together with an exact sequence of local systems on S:

$$\mathbf{H}^{-d-1}(i^* Rj_* \mathbf{P}^\bullet) \to K \to C \to \mathbf{H}^{-d}(i^* Rj_* \mathbf{P}^\bullet)$$

A morphism in $\mathbf{Z}(X,S)$ is a morphism $p\colon \mathbf{P}^\bullet \to \overline{\mathbf{P}}^\bullet$ together with a diagram

$$\begin{array}{ccccccc}
\mathbf{H}^{-d-1}(i^* Rj_* \mathbf{P}^\bullet) & \to & K & \to & C & \to & \mathbf{H}^{-d}(i^* Rj_* \mathbf{P}^\bullet) \\
\downarrow{\scriptstyle p_*} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p_*} \\
\mathbf{H}^{-d-1}(i^* Rj_* \overline{\mathbf{P}}^\bullet) & \to & \overline{K} & \to & \overline{C} & \to & \mathbf{H}^{-d}(i^* Rj_* \overline{\mathbf{P}}^\bullet)
\end{array}$$

(The reason for the name "zig-zag" is seen in formula 2.2.) We define the zig-zag functor $\mathscr{Z}\colon \mathbf{P}(X) \to \mathbf{Z}(X,S)$ by sending an object $\mathbf{Q}^\bullet$ of $\mathbf{P}(X)$ to $j^*\mathbf{Q}^\bullet$ together with the exact sequence

$$\mathbf{H}^{-d-1}(i^* Rj_* j^* \mathbf{Q}^\bullet) \to \mathbf{H}^{-d}(i^! \mathbf{Q}^\bullet) \to \mathbf{H}^{-d}(i^* \mathbf{Q}^\bullet) \to \mathbf{H}^{-d}(i^* Rj_* j^* \mathbf{Q}^\bullet).$$

The above exact sequence is a piece of the long exact sequence associated to the triangle

$$\begin{array}{ccc}
 & & Rj_* j^* \mathbf{Q}^\bullet \\
 & \nearrow & \\
\mathbf{Q}^\bullet & & \downarrow{\scriptstyle (1)} \\
 & \nwarrow & \\
 & & i_* i^! \mathbf{Q}^\bullet
\end{array}$$

restricted to S.

Note that if S is contractible (or simply connected) local systems on S are just vector spaces.

Theorem 2.1. *Suppose that S is contractible. Then*

i) *$\mathscr{Z}$ gives rise to a bijection from the isomorphism classes of objects of $\mathbf{P}(X)$ to the isomorphism classes of objects of $\mathbf{Z}(X,S)$.*

ii) *For any two objects $\mathbf{Q}^\bullet$ and $\overline{\mathbf{Q}}^\bullet$ of $\mathbf{P}(X)$, $\mathscr{Z}$ gives rise to a surjection $\mathscr{Z}_*$: $\mathrm{Hom}(\mathbf{Q}^\bullet, \overline{\mathbf{Q}}^\bullet) \to \mathrm{Hom}(\mathscr{Z}\mathbf{Q}^\bullet, \mathscr{Z}\overline{\mathbf{Q}}^\bullet)$. Furthermore there is a canonical injection $\mathrm{Hom}(\mathbf{H}^{-d}(i^*\mathbf{Q}^\bullet), \mathbf{H}^{-d}(i^!\overline{\mathbf{Q}}^\bullet)) \hookrightarrow \mathrm{Hom}(\mathbf{Q}^\bullet, \overline{\mathbf{Q}}^\bullet)$ whose image contains the kernel of $\mathscr{Z}$ such that the composed map*

$$\mathrm{Hom}(\mathbf{H}^{-d}(i^*\mathbf{Q}^\bullet), \mathbf{H}^{-d}(i^!\overline{\mathbf{Q}}^\bullet)) \hookrightarrow \mathrm{Hom}(\mathbf{Q}^\bullet, \overline{\mathbf{Q}}^\bullet) \twoheadrightarrow \mathrm{Hom}(\mathscr{Z}\mathbf{Q}^\bullet, \mathscr{Z}\overline{\mathbf{Q}}^\bullet)$$

sends the homomorphism Θ to the homomorphism which is 0 on $\mathbf{P}(X-S)$ together with the diagram

$$\begin{array}{ccccccc}
\mathbf{H}^{-d-1}(i^* Rj_* j^* \mathbf{Q}^\bullet) & \longrightarrow & \mathbf{H}^{-d}(i^! \mathbf{Q}^\bullet) & \xrightarrow{u} & \mathbf{H}^{-d}(i^* \mathbf{Q}^\bullet) & \longrightarrow & \mathbf{H}^{-d}(i^* Rj_* j^* \mathbf{Q}^\bullet) \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \Theta \circ u} & & \downarrow{\scriptstyle \bar{u} \circ \Theta} & & \downarrow{\scriptstyle 0} \\
\mathbf{H}^{-d-1}(i^* Rj_* j^* \overline{\mathbf{Q}}^\bullet) & \longrightarrow & \mathbf{H}^{-d}(i^! \overline{\mathbf{Q}}^\bullet) & \xrightarrow{\bar{u}} & \mathbf{H}^{-d}(i^* \overline{\mathbf{Q}}^\bullet) & \longrightarrow & \mathbf{H}^{-d}(i^* Rj_* j^* \overline{\mathbf{Q}}^\bullet).
\end{array}$$

Remark. The results of Sect. 7 show that this theorem holds whenever $\pi_1(S)=\pi_2(S)=0$.

Suppose that we have determined two isomorphism calsses $[\mathbf{Q}^\bullet]$ and $[\bar{\mathbf{Q}}^\bullet]$ of elements of $\mathbf{P}(X)$ by two elements of $\mathbf{Z}(X,S)$ using the first part of Theorem 2.1:

$$\mathscr{Z}\mathbf{Q}^\bullet = \{\mathbf{H}^{-d-1}(i^* Rj_* \mathbf{P}^\bullet) \longrightarrow K \xrightarrow{u} C \longrightarrow \mathbf{H}^{-d}(i^* Rj_* \mathbf{P}^\bullet)\}$$

$$\mathscr{Z}\bar{\mathbf{Q}}^\bullet = \{\mathbf{H}^{-d-1}(i^* Rj_* \bar{\mathbf{P}}^\bullet) \longrightarrow \bar{K} \xrightarrow{\bar{u}} \bar{C} \longrightarrow \mathbf{H}^{-d}(i^* Rj_* \bar{\mathbf{P}}^\bullet)\}.$$

Then the second part of Theorem 2.1 gives two ways to display $\operatorname{Hom}(\mathbf{Q}^\bullet, \bar{\mathbf{Q}}^\bullet)$ as an extension (using only knowledge of $\mathbf{P}(X-S)$).

Corollary 2.2. *There is an exact sequence*

$$0 \to \operatorname{Hom}(\operatorname{Cok} u, \operatorname{Ker} \bar{u}) \to \operatorname{Hom}(\mathbf{Q}^\bullet, \bar{\mathbf{Q}}^\bullet) \to \operatorname{Hom}(\mathscr{Z}\mathbf{Q}^\bullet, \mathscr{Z}\bar{\mathbf{Q}}^\bullet) \to 0.$$

Corollary 2.3. *There is an exact sequence*

$$0 \to \operatorname{Hom}(C, \bar{K}) \to \operatorname{Hom}(\mathbf{Q}^\bullet, \bar{\mathbf{Q}}^\bullet) \to G \to 0$$

where $G \subset \operatorname{Hom}(\mathbf{P}^\bullet, \bar{\mathbf{P}}^\bullet)$ *is the group of homomorphisms which extend to homomorphisms of* $\mathscr{Z}\mathbf{Q}^\bullet$ *to* $\mathscr{Z}\bar{\mathbf{Q}}^\bullet$.

The proof of these corollaries using Theorem 2.1 is immediate.

In the proof of Theorem 2.1 we need the following.

Lemma 2.4. *Let S be contractible. Then the derived category $D^b_s(S)$ of complexes of sheaves with bounded and (locally) constant cohomology is equivalent to the derived category of vector spaces $\mathbf{D}^b(k)$.*

Proof. We first show that any element $\mathbf{Q}^\bullet \in \mathbf{D}^b_s(S)$ is isomorphic to its cohomology i.e. that $\mathbf{Q}^\bullet \cong \oplus \mathbf{H}^k(\mathbf{Q}^\bullet)[-k]$. We will prove this by induction on the length N of an interval outside of which the cohomology groups $\mathbf{H}^k(\mathbf{Q}^\bullet)$ vanish. For $N=0$ the statement is clearly true because then $\mathbf{Q}^\bullet$ has non-vanishing cohomology for one value of k only. Assume now that the claim is true for N and $\mathbf{Q}^\bullet$ is a complex having non-vanishing cohomology $\mathbf{H}^k(\mathbf{Q}^\bullet)$ for $k \in [0, N+1]$ only. Consider the triangle

$$\begin{array}{ccc} & & \mathbf{H}^{N+1}(\mathbf{Q}^\bullet)[-N-1] \\ \mathbf{Q}^\bullet & \begin{array}{c}\nearrow \\ \nwarrow\end{array} & e \Big\downarrow (1) \\ & & \tau_{\leq N}\mathbf{Q}^\bullet \end{array}$$

By the induction hypothesis $\tau_{\leq N}\mathbf{Q}^\bullet \cong \bigoplus_{k=0}^{N} \mathbf{H}^k(\mathbf{Q}^\bullet)[-k]$. The map

$$\begin{aligned} e \in \operatorname{Ext}^1\left(\mathbf{H}^{N+1}(\mathbf{Q}^\bullet)[-N-1], \bigoplus_{k=0}^{N} \mathbf{H}^k(\mathbf{Q}^\bullet)[-k]\right) \\ \cong \bigoplus_{k=0}^{N} \operatorname{Ext}^{2+k}(\mathbf{H}^{N+1}(\mathbf{Q}^\bullet), \mathbf{H}^{N-k}(\mathbf{Q}^\bullet)) \\ \cong \bigoplus_{k=0}^{N} \mathbf{H}^{2+k}(\mathbf{H}^{N+1}(\mathbf{Q}^\bullet)^* \otimes \mathbf{H}^{N-k}(\mathbf{Q}^\bullet)) \\ \cong 0 \end{aligned}$$

because S is contractible. Therefore

$$\mathbf{Q}^{\bullet} \cong \bigoplus_{k=0}^{N+1} \mathbf{H}^k(\mathbf{Q}^{\bullet})[-k].$$

It remains to show that if $\mathbf{Q}^{\bullet}$ and $\overline{\mathbf{Q}}^{\bullet}$ belong to $\mathbf{D}_s^b(S)$ then $\operatorname{Hom}(\mathbf{Q}^{\bullet}, \overline{\mathbf{Q}}^{\bullet}) = \bigoplus_k \operatorname{Hom}(\mathbf{H}^k(\mathbf{Q}^{\bullet}), \mathbf{H}^k(\overline{\mathbf{Q}}^{\bullet}))$. By using the first part of the proof we see that

$$\operatorname{Hom}(\mathbf{Q}^{\bullet}, \overline{\mathbf{Q}}^{\bullet}) \cong \bigoplus_{i,j} \operatorname{Ext}^{i-j}(\mathbf{H}^i(\mathbf{Q}^{\bullet}), \mathbf{H}^j(\overline{\mathbf{Q}}^{\bullet})).$$

But as above $\operatorname{Ext}^{i-j}(\mathbf{H}^i(\mathbf{Q}^{\bullet}), \mathbf{H}^j(\overline{\mathbf{Q}}^{\bullet})) = 0$ for $i \neq j$. This proves our claim.

Proof of Theorem 2.1 i) Let $\mathbf{Q}^{\bullet} \in \mathbf{P}(X)$. Consider the triangle

$$\text{(2.1)} \qquad \begin{array}{ccc} & & Rj_*j^*\mathbf{Q}^{\bullet} \\ & \nearrow & \\ \mathbf{Q}^{\bullet} & & \Big\downarrow e \; (1) \\ & \nwarrow & \\ & & i_*i^!\mathbf{Q}^{\bullet} \end{array}$$

The object $\mathbf{Q}^{\bullet}$ is completely determined by the map $e \in \operatorname{Ext}^1(Rj_*j^*\mathbf{Q}^{\bullet}, i_*i^!\mathbf{Q}^{\bullet})$. But

$$\begin{aligned} \operatorname{Ext}^1(Rj_*j^*\mathbf{Q}^{\bullet}, i_*i^!\mathbf{Q}^{\bullet}) &\cong \operatorname{Ext}^1(i^*Rj_*j^*\mathbf{Q}^{\bullet}, i^!\mathbf{Q}^{\bullet}) \\ &\cong \bigoplus_k \operatorname{Hom}(\mathbf{H}^k(i^*Rj_*j^*\mathbf{Q}^{\bullet}), \mathbf{H}^{k+1}(i^!\mathbf{Q}^{\bullet})) \end{aligned}$$

by Lemma 2.4.

By the perversity conditions and the long exact sequence associated to the triangle (2.1) we see that for $k > -d$ the above homomorphism has to be an isomorphism and for $k < -d-1$ it is zero. Writing out this non-trivial part of the long exact sequence associated to the triangel (2.1) we get

$$\text{(2.2)} \qquad \begin{array}{ccccc} & & 0 & \longleftarrow & \mathbf{H}^{-d+1}(i^!\mathbf{Q}^{\bullet}) \\ & & & \nearrow & \\ & & \mathbf{H}^{-d}(i^*Rj_*\mathbf{P}^{\bullet}) & & \\ & \nearrow & & & \\ \mathbf{H}^{-d}(i^*\mathbf{Q}^{\bullet}) & \longleftarrow & & \longleftarrow & \mathbf{H}^{-d}(i^!\mathbf{Q}^{\bullet}) \\ & & & \nearrow & \\ & & \mathbf{H}^{-d-1}(i^*Rj_*\mathbf{P}^{\bullet}) & & \\ & \nearrow & & & \\ \mathbf{H}^{-d-1}(i^*\mathbf{Q}^{\bullet}) & \longleftarrow & & \longleftarrow & 0 \end{array}$$

where we have denoted $\mathbf{P}^{\bullet} = j^*\mathbf{Q}^{\bullet}$.

We now try to understand all possible extensions of $\mathbf{P}^{\bullet} \in \mathbf{P}(X-S)$ to $\mathbf{P}(X)$ up to isomorphism. But we see that given an extension $\mathbf{Q}^{\bullet}$ of $\mathbf{P}^{\bullet}$ is equivalent to giving two groups $\mathbf{H}^{-d}(i^!\mathbf{Q}^{\bullet})$ and $\mathbf{H}^{-d+1}(i_*i^!\mathbf{Q}^{\bullet})$ and homomorphisms $\mathbf{H}^{-d}(i^*Rj_*\mathbf{P}^{\bullet}) \to \mathbf{H}^{-d+1}(i_*i^!\mathbf{Q}^{\bullet})$ and $\mathbf{H}^{-d-1}(i^*Rj_*\mathbf{P}^{\bullet}) \to \mathbf{H}^{-d}(i_*i^!\mathbf{Q}^{\bullet})$. On the other hand this is equivalent to giving the diagram (2.2) because all the cohomology groups are just vector spaces. Finally this is equivalent to giving an element in $\mathbf{Z}(X,S)$.

ii) Consider the triangle

$$\begin{array}{ccc} & & Rj_*j^*\overline{\mathbf{Q}}^\bullet \\ & \nearrow & \downarrow{\scriptstyle (1)} \\ \overline{\mathbf{Q}}^\bullet & \nwarrow & \\ & & i_*i^!\overline{\mathbf{Q}}^\bullet \end{array}$$

Apply the functor $R\mathrm{Hom}(\mathbf{Q}^\bullet, -)$ to this triangle to get

$$\begin{array}{ccc} & & R\mathrm{Hom}(\mathbf{Q}^\bullet, Rj_*j^*\overline{\mathbf{Q}}^\bullet) \\ & \nearrow & \downarrow{\scriptstyle (1)} \\ R\mathrm{Hom}(\mathbf{Q}^\bullet, \overline{\mathbf{Q}}^\bullet) & \nwarrow & \\ & & R\mathrm{Hom}(\mathbf{Q}^\bullet, i_*\overline{\mathbf{Q}}^\bullet) \end{array}$$

By using the fact that f^* is a left adjoint of Rf_* we get the triangle

$$\begin{array}{ccc} & & R\mathrm{Hom}(j^*\mathbf{Q}^\bullet, j^*\overline{\mathbf{Q}}^\bullet) \\ & \nearrow & \downarrow{\scriptstyle (1)} \\ R\mathrm{Hom}(\mathbf{Q}^\bullet, \overline{\mathbf{Q}}^\bullet) & \nwarrow & \\ & & R\mathrm{Hom}(i^*\mathbf{Q}^\bullet, i^!\overline{\mathbf{Q}}^\bullet). \end{array}$$

Now we observe that $\mathrm{Ext}^{-1}(j^*\mathbf{Q}^\bullet, j^*\overline{\mathbf{Q}}^\bullet) = 0$, because $\mathbf{Q}^\bullet$ and $\overline{\mathbf{Q}}^\bullet$ are perverse and by Lemma 2.4 and using the perversity conditions we see that

$$\mathrm{Hom}(i^*\overline{\mathbf{Q}}^\bullet, i^!\overline{\mathbf{Q}}^\bullet) = \mathrm{Hom}(\mathbf{H}^{-d}(i^*\mathbf{Q}^\bullet), \mathbf{H}^{-d}(i^!\overline{\mathbf{Q}}^\bullet))$$

$$\mathrm{Ext}^1(i^*\mathbf{Q}^\bullet, i^!\overline{\mathbf{Q}}^\bullet) = \mathrm{Hom}(\mathbf{H}^{-d-1}(i^*\mathbf{Q}^\bullet)), \mathbf{H}^{-d}(i^!\overline{\mathbf{Q}}^\bullet)) \oplus \mathrm{Hom}(\mathbf{H}^{-d}(i^*\mathbf{Q}^\bullet), \mathbf{H}^{-d-1}(i^!\overline{\mathbf{Q}}^\bullet)).$$

Putting this all together we get an exact sequence

$$0 \to \mathrm{Hom}(\mathbf{H}^{-d}(i^*\mathbf{Q}^\bullet), \mathbf{H}^{-d}(i^!\overline{\mathbf{Q}}^\bullet)) \to \mathrm{Hom}(\mathbf{Q}^\bullet, \overline{\mathbf{Q}}^\bullet) \to \mathrm{Hom}(j^*\mathbf{Q}^\bullet, j^*\overline{\mathbf{Q}}^\bullet)$$
$$\xrightarrow{\delta} \mathrm{Hom}(\mathbf{H}^{-d-1}(i^*\mathbf{Q}^\bullet), \mathbf{H}^{-d}(i^!\overline{\mathbf{Q}}^\bullet)) \oplus \mathrm{Hom}(\mathbf{H}^{-d}(i^*\mathbf{Q}^\bullet), \mathbf{H}^{-d+1}(i^!\overline{\mathbf{Q}}^\bullet)).$$

We now want an explicit formula for the map δ. To this end let $p\colon j^*\mathbf{Q}^\bullet \to j^*\overline{\mathbf{Q}}^\bullet$. We have

$$\begin{array}{ccc} \delta\colon \mathrm{Hom}(j^*\mathbf{Q}^\bullet, j^*\overline{\mathbf{Q}}^\bullet) & \longrightarrow & \mathrm{Ext}^1(i^*\mathbf{Q}^\bullet, i^!\overline{\mathbf{Q}}^\bullet) \\ \| & & \| \\ \delta\colon \mathrm{Hom}(\mathbf{Q}^\bullet, Rj_*j^*\overline{\mathbf{Q}}^\bullet) & \longrightarrow & \mathrm{Ext}^1(\mathbf{Q}^\bullet, i_*i^!\overline{\mathbf{Q}}^\bullet) \end{array}$$

from which it easily follows that $\delta(p)$ can be given as a composition

$$i_*i^*\mathbf{Q}^\bullet \longrightarrow Rj_*j^*\overline{\mathbf{Q}}^\bullet \xrightarrow{p} Rj_*j^*\overline{\mathbf{Q}}^\bullet \xrightarrow{(1)} i_*i^!\mathbf{Q}^\bullet.$$

So $\delta(p) = (\bar{t} \circ p_* \circ s, \bar{w} \circ p_* \circ v)$ using the notation below.

$$0 \to \mathbf{H}^{-d-1}(i^*\mathbf{Q}^\bullet) \xrightarrow{s} \mathbf{H}^{-d-1}(i^*Rj_*j^*\mathbf{Q}^\bullet) \xrightarrow{t} \mathbf{H}^{-d}(i^!\mathbf{Q}^\bullet) \xrightarrow{u} \mathbf{H}^{-d}(i^*\mathbf{Q}^\bullet) \xrightarrow{v}$$
$$\mathbf{H}^{-d}(i^*Rj_*j^*\mathbf{Q}^\bullet) \xrightarrow{w} \mathbf{H}^{-d+1}(i^!\mathbf{Q}^\bullet) \longrightarrow 0$$
$$0 \to \mathbf{H}^{-d-1}(i^*\bar{\mathbf{Q}}^\bullet) \xrightarrow{\bar{s}} \mathbf{H}^{-d-1}(i^*Rj_*j^*\bar{\mathbf{Q}}^\bullet) \xrightarrow{\bar{t}} \mathbf{H}^{-d}(i^!\bar{\mathbf{Q}}^\bullet) \xrightarrow{\bar{u}} \mathbf{H}^{-d}(i^*\bar{\mathbf{Q}}^\bullet) \xrightarrow{\bar{v}}$$
$$\mathbf{H}^{-d}(i^*Rj_*j^*\bar{\mathbf{Q}}^\bullet) \xrightarrow{\bar{w}} \mathbf{H}^{-d+1}(i^!\bar{\mathbf{Q}}^\bullet) \longrightarrow 0$$

By linear algebra we can see that if $\delta(p) = 0$ then p can be extended to a map in $\mathbf{Z}(X, S)$. This proves Corollary 2.3.

Consider now the composition

$$\mathrm{Hom}(\mathbf{H}^{-d}(i^*\mathbf{Q}^\bullet), \mathbf{H}^{-d}(i^!\bar{\mathbf{Q}}^\bullet)) \hookrightarrow \mathrm{Hom}(\mathbf{Q}^\bullet, \bar{\mathbf{Q}}^\bullet) \xrightarrow{\mathscr{X}} \mathrm{Hom}(\mathscr{X}\mathbf{Q}^\bullet, \mathscr{X}\bar{\mathbf{Q}}^\bullet).$$

Because $\mathrm{Hom}(i^*\mathbf{Q}^\bullet, i^!\bar{\mathbf{Q}}^\bullet) = \mathrm{Hom}(\mathbf{Q}^\bullet, i_* i^!\bar{\mathbf{Q}}^\bullet)$ a map $\theta: i^*\mathbf{Q}^\bullet \to i^!\bar{\mathbf{Q}}^\bullet$ gets sent to the map $\mathbf{Q}^\bullet \to i_* i^!\mathbf{Q}^\bullet \to \bar{\mathbf{Q}}^\bullet$. This yields the last claim of part (ii). It remains to be checked that the map $\mathscr{X}$ is surjective. If we have morphism $\alpha: \mathscr{X}\mathbf{Q}^\bullet \to \mathscr{X}\bar{\mathbf{Q}}^\bullet$ we showed before that there exists a morphism $a: \mathbf{Q}^\bullet \to \bar{\mathbf{Q}}^\bullet$ extending α on $X - S$. Therefore $\alpha - \mathscr{X}_* a: \mathscr{X}\mathbf{Q}^\bullet \to \mathscr{X}\bar{\mathbf{Q}}^\bullet$ whose restriction to $X - S$ is zero. But now by linear algebra there exist a $\theta \in \mathrm{Hom}(\mathbf{H}^{-d}(i^*\mathbf{Q}), \mathbf{H}^{-d}(i^!\bar{\mathbf{Q}}^\bullet))$ such that $\alpha = \mathscr{X}_*(a + \theta)$.

Remark. The proof we gave for part i) of Theorem 2.1 is due to Allen Shepard (in the case that S = point). The observation that one gets a similar theorem for the whole derived category is also due to him. We have carried on further ideas along these lines in Sect. 7.

3. Extensions of perverse sheaves over contractible strata

In this section we construct $\mathbf{P}(X)$ in terms of $\mathbf{P}(X - S)$ for the case that S is a closed, contractible stratum of dimension $2d$.

As part of the structure of X as a Thom-Mather stratified space the stratum S has "control data" consisting of a tubular neighborhood T_S endowed with a fibration $\pi_S: T_S \to S$ and a distance function $\rho_S: T_S \to \mathbb{R}$. If p is any point in S the *link* L° of S is $\pi_S^{-1}(p) \cap \rho_S^{-1}(\varepsilon)$ for small enough $\varepsilon > 0$. Its stratified homeomorphism type is independent of the choice of p and ε. The strata of L°, its intersection with the strata of $X - S$, are odd dimensional. The link is the boundary of a small "normal slice" $\mathbb{D}^\circ$ to S through p which is defined as $\pi_S^{-1}(p) \cap \rho_S^{-1}([0, \varepsilon])$.

Definition 3.1. A perverse link K° of S is a closed subset of the link L° of S such that for all $\mathbf{P}^\bullet \in \mathbf{P}(X - S)$

i) $\mathbb{H}^k(K^\circ, \mathbf{P}^\bullet) = 0 \quad$ for $k \geqq -d$,

ii) $\mathbb{H}^k(L^\circ, K^\circ; \mathbf{P}^\bullet) = 0 \quad$ for $k < -d$.

Perverse links exist. One construction is given in the following lemma and another construction, valid for complex analytic spaces X, is given in Sect. 5.

Lemma 3.2. *Let $\mathcal{T}$ be a triangulation of L° such that each stratum is the union of interiors of simplices. Let K° be the union of all closed simplicies of $\mathcal{T}'$, the barycentric subdivision of $\mathcal{T}$, which intersect each stratum V of L° in a set of dimension $<\frac{1}{2}\dim V$. Then K° is a perverse link.*

Remark. The sets K of the lemma arose in two other contexts. They are the $R^{\bar{m}}_{n/2-1}$ of [GM1] §3.4 and the R of [GM6] §4.

We defer the proof of the lemma to the next section where we prove a more general version of it.

Now we give data for the construction of an abelian category via Proposition 1.1 using the perverse link $K^\circ \subset L^\circ$ as follows: $\mathcal{A}$ is $\mathbf{P}(X-S)$, $\mathcal{B}$ is the category of vector spaces over k, $F(\mathbf{P}^\bullet)$ is $\mathbb{H}^{-d-1}(K^\circ;\mathbf{P}^\bullet)$, $G(\mathbf{P}^\bullet)$ is $\mathbb{H}^{-d}(L^\circ, K^\circ\colon \mathbf{P}^\bullet)$, and T is the coboundary homomorphism in the long exact sequence for the pair (L°, K°). It is obvious from the definition of a perverse link that F is right exact and G is left exact.

Theorem 3.3. *Suppose that S is contractible. Then $\mathbf{P}(X)$ is equivalent to the category $\mathcal{C}(F,G\colon T)$ constructed by using the above data. The equivalence is realized by the functor $\mathcal{C}\colon \mathbf{P}(X)\to\mathcal{C}(F,G\colon T)$ which sends $\mathbf{Q}^\bullet$ to $\mathbf{Q}^\bullet \mid X-S$ together with*

$$\begin{array}{ccc} \mathbb{H}^{-d-1}(K^\circ;\mathbf{Q}^\bullet) & \xrightarrow{\quad T\quad} & \mathbb{H}^{-d}(L^\circ,K^\circ;\mathbf{Q}^\bullet) \\ \searrow & & \nearrow \\ & \mathbb{H}^{-d}(\mathbb{D}^\circ,K^\circ;\mathbf{Q}^\bullet) & \end{array}$$

Remark. It follows from the results of Sect. 7 that this theorem holds as long as $\pi_1(S)=\pi_2(S)=0$.

Our strategy of proving Theorem 3.3 will be to use the results of the previous section. To this end we want to define a functor $\mathcal{V}\colon\mathcal{C}(F,G\colon T)\to\mathbf{Z}(X,S)$ such that we have the following commuting diagram of functors

$$\begin{array}{ccc} & & \mathcal{C}(F,G;T) \\ & \overset{\mathcal{C}}{\nearrow} & \downarrow \mathcal{V} \\ \mathbf{P}(X) & \underset{\mathcal{Z}}{\longrightarrow} & \mathbf{Z}(X,S)\,. \end{array}$$

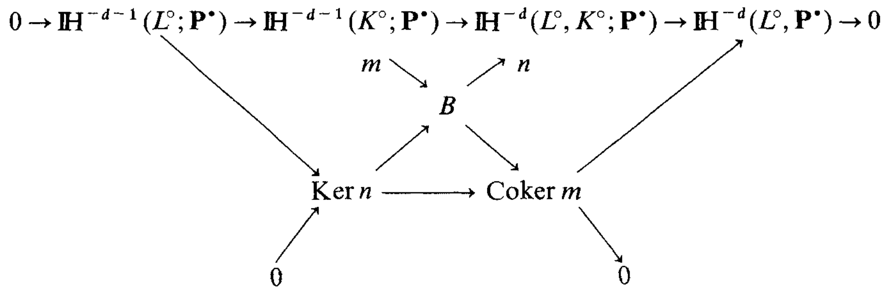
Consider an element of $\mathcal{C}(F,G;T)$: $\mathbf{P}^\bullet\in\mathbf{P}(X-S)$ together with the diagram

(3.1)
$$\begin{array}{ccc} \mathbb{H}^{-d-1}(K^\circ;\mathbf{P}^\bullet) & \xrightarrow{T} & \mathbb{H}^{-d}(L^\circ,K^\circ;\mathbf{P}^\bullet) \\ & {\scriptstyle m}\searrow \quad \nearrow{\scriptstyle n} & \\ & B\,. & \end{array}$$

We define $\mathcal{V}$ of this object as $\mathbf{P}^\bullet\in\mathbf{P}(X-S)$ together with the exact sequence

$$\mathbf{H}^{-d-1}(i^*Rj_*\mathbf{P}^\bullet)\to\operatorname{Ker} n\to\operatorname{Coker} m\to\mathbf{H}^{-d}(i^*Rj_*\mathbf{P}^\bullet)\,.$$

The above sequence arises as the sequence of kernels and cokernels of the diagram (3.1)



by observing that $\mathbf{H}^k(i^* Rj_* \mathbf{P}^\bullet) \cong \mathbb{H}^k(L^\circ;\mathbf{P}^\bullet)$. The fact that $\mathscr{X} = \mathscr{V} \circ \mathscr{C}$ follows because $\mathbf{H}^k(i^*\mathbf{Q}^\bullet) \cong \mathbb{H}^k(\mathbb{D}^\circ;\mathbf{Q}^\bullet) = \operatorname{Coker}(m)$ and $\mathbf{H}^k(i^!\mathbf{Q}^\bullet) \cong \mathbb{H}^k_c(\mathbb{D}^\circ - L^\circ;\mathbf{Q}^\bullet) = \ker n$. For these last two facts we refer to [GM2], [Ba] and [Bo].

Proposition 3.4. *If S is contractible then*

i) *The functor $\mathscr{V}$ gives rise to a bijection from the isomorphism classes of objects of $\mathscr{C}(F, G; T)$ to the isomorphism classes of objects in $\mathbf{Z}(X, S)$.*

ii) *For any two objects $(\mathbf{P}^\bullet, B)$, $(\overline{\mathbf{P}}^\bullet, \overline{B})$ of $\mathscr{C}(F, G; T)$ we have an exact sequence*

$$0 \to \operatorname{Hom}(\operatorname{Coker} m, \operatorname{Ker} \bar{n}) \to \operatorname{Hom}((\mathbf{P}^\bullet, B), (\overline{\mathbf{P}}^\bullet, \overline{B})) \to G \to 0$$

where $G \subset \operatorname{Hom}(\mathbf{P}^\bullet, \overline{\mathbf{P}}^\bullet)$ is the subgroup of homomorphisms that can be extended to $\mathscr{V}(\mathbf{P}^\bullet, B) \to \mathscr{V}(\overline{\mathbf{P}}^\bullet, \overline{B})$.

Proof. i) Fix $\mathbf{P}^\bullet$ and consider the following object of $\mathscr{C}(F, G; T)$

$$\begin{array}{ccc} F\mathbf{P}^\bullet & \xrightarrow{\;T\;} & G\mathbf{P}^\bullet \\ & {\scriptstyle m}\searrow \quad \nearrow{\scriptstyle n} & \\ & B & \end{array}$$

We write $F\mathbf{P}^\bullet = \operatorname{Ker} T \oplus \operatorname{Coim} T$, $G\mathbf{P}^\bullet = \operatorname{Coker} T \oplus \operatorname{Im} T$ and $B = (\operatorname{Im} m \cap \operatorname{Ker} n) \oplus (\operatorname{Im} m \cap \operatorname{Coim} n) \oplus (\operatorname{Cok} m \cap \operatorname{Cok} n) \oplus (\operatorname{Cok} m \cap \operatorname{Coim} n)$. The only non-trivial maps in these coordinates are the isomorphisms

$$\begin{array}{ccc} \operatorname{Coim} T & \xrightarrow[\cong]{\;T\;} & \operatorname{Im} T \\ & {\scriptstyle m}\searrow{\scriptstyle \cong} \quad {\scriptstyle \cong}\nearrow{\scriptstyle n} & \\ & \operatorname{Im} m \cap \operatorname{Coim} n & \end{array}$$

defined by T, a surjection $\operatorname{Ker} T \to \operatorname{Im} m \cap \operatorname{Ker} n$ and an injection $\operatorname{Cok} m \cap \operatorname{Coim} n \hookrightarrow \operatorname{Cok} T$. Therefore specifying an object in $\mathscr{C}(F, G; T)$ is equivalent to giving $\mathbf{P}^\bullet$ and the surjection and the injection above.

On the other hand we can specify an exact sequence

$$\operatorname{Ker} T \xrightarrow{\;t\;} K \xrightarrow{\;u\;} C \xrightarrow{\;v\;} \operatorname{Cok} T$$

by specifying $\operatorname{Coim} u \cong \operatorname{Im} u$ and a surjection $\operatorname{Ker} T \to \operatorname{Ker} u$ and an injection $\operatorname{Coker} u \hookrightarrow \operatorname{Coker} T$.

Therefore up to isomorphism the data for an object of $\mathscr{C}(F, G; T)$ is equivalent to the data for an object of $\mathbf{Z}(X, S)$.

ii) We consider two objects $(\mathbf{P}^\bullet, B)$, $(\overline{\mathbf{P}}^\bullet, \bar{B})$ of $\mathscr{C}(F, G; T)$ and we fix a morphism p: $\mathbf{P}^\bullet \to \overline{\mathbf{P}}^\bullet$. Assume now that we have extended the pair p_*: $\operatorname{Cok} T(\mathbf{P}^\bullet) \to \operatorname{Cok} T(\overline{\mathbf{P}}^\bullet)$, p_*: $\operatorname{Ker} T(\overline{\mathbf{P}}^\bullet) \to \operatorname{Ker} T(\overline{\mathbf{P}}^\bullet)$ by compatible maps $\operatorname{Ker} n \to \operatorname{Ker} \bar{n}$, $\operatorname{Cok} m \to \operatorname{Cok} \bar{m}$ to a morphism in $\mathbf{Z}(X, S)$. By using the decomposition introduced in i) we see that we can extend p to a morphism in $\mathscr{C}(F, G; T)$. It is clear that all such extensions are parametrized by the affine space $\operatorname{Hom}(\operatorname{Cok} m, \operatorname{Ker} \bar{n})$.

Remark. If $\mathscr{B}$ is abelian then for any $\mathscr{C}(F, G; T)$ there exists a functor $\mathscr{Z}$: $\mathscr{C}(F, G; T) \to Z(F, G; T)$ constructed as above. The above proposition holds in this generality given that the category $\mathscr{B}$ has no non-trivial extensions.

Lemma 3.5. *The functor* $\mathscr{C}: \mathbf{P}(X) \to \mathscr{C}(F, G; T)$ *is exact.*

Proof. It is enough to show that the functor $\mathbf{Q}^\bullet \mapsto \mathbb{H}^{-d}(\mathbb{D}^\circ, K^\circ; \mathbf{Q}^\bullet)$ is exact. Consider the long exact sequences

$$\ldots \to \mathbb{H}^k(\mathbb{D}^\circ, K^\circ; \mathbf{Q}^\bullet) \to \mathbb{H}^k(\mathbb{D}^\circ; \mathbf{Q}^\bullet) \to \mathbb{H}^k(K^\circ; \mathbf{Q}^\bullet) \to \ldots$$
$$\ldots \to \mathbb{H}^k(\mathbb{D}^\circ, L^\circ; \mathbf{Q}^\bullet) \to \mathbb{H}^k(\mathbb{D}^\circ, K^\circ; \mathbf{Q}^\bullet) \to \mathbb{H}^k(L^\circ, K^\circ; \mathbf{Q}^\bullet) \to \ldots$$

We have $\mathbb{H}^k(\mathbb{D}^\circ, \mathbf{Q}^\bullet) \cong 0$ for $k > -d$ because $\mathbf{Q}^\bullet$ is perverse and $\mathbb{H}^k(K^\circ, \mathbf{Q}^\bullet) \cong 0$ for $k \leq -d$ by definition 3.1. Therefore $\mathbb{H}^k(\mathbb{D}^\circ, K^\circ; \mathbf{Q}^\bullet) \cong 0$ for $k > -d$.

Now $\mathbb{H}^k(\mathbb{D}^\circ, L^\circ; \mathbf{Q}^\bullet) \cong \mathbb{H}^k_c(\mathbb{D}^\circ - L^\circ, \mathbf{Q}^\bullet) \cong H^k(i^! \mathbf{Q}^\bullet) \cong 0$ for $k < -d$ because $\mathbf{Q}^\bullet$ is perverse and $\mathbb{H}^k(L^\circ, K^\circ; \mathbf{Q}^\bullet) \cong 0$ for $k < -d$ by definition 3.1. Therefore $\mathbb{H}^k(\mathbb{D}^\circ, K^\circ; \mathbf{Q}^\bullet) \cong 0$ for $k < -d$.

This shows that $\mathbf{Q}^\bullet \mapsto \mathbb{H}^{-d}(\mathbb{D}^\circ, K^\circ; \mathbf{Q}^\bullet)$ is exact and therefore $\mathscr{C}$ is exact.

Proof of Theorem 3.3. Because $\mathscr{C}(\mathbf{Q}^\bullet) = 0 \Rightarrow \mathbf{Q}^\bullet = 0$ and $\mathscr{C}$ is exact (by Lemma 3.5), $\mathscr{C}$ is faithful. By Theorem 2.1 and Proposition 3.1 $\dim \operatorname{Hom}(\mathbf{Q}^\bullet, \overline{\mathbf{Q}}^\bullet) = \dim \operatorname{Hom}(\mathscr{C}(\mathbf{Q}^\bullet), \mathscr{C}(\overline{\mathbf{Q}}^\bullet))$ which implies that $\mathscr{C}$ is also full. Theorem 2.1 and Proposition 3.4 imply further that $\mathscr{C}$ is bijective on the isomorphism classes of objects. Therefore the functor $\mathscr{C}$ is an equivalence of categories (by [ML] p. 91).

Example. Let X be the complex line $\mathbb{C}$ and let the stratification $\mathscr{S}$ be given by $S = \{0\}$ and $\mathbb{C} - \{0\}$. We can choose for L the unit circle and for K the set $\{1\}$. Every element $\mathbf{P}^\bullet \in \mathbf{P}(\mathbb{C} - \{0\})$ is a local system placed in degree -1. We can identify $F(\mathbf{P}^\bullet)$ with the stalk of $\mathbf{P}^\bullet[-1]$ at 1 and by choosing an orientation for $\mathbb{C}$ we can also identify $G(\mathbf{P}^\bullet)$ with the stalk of $\mathbf{P}^\bullet[-1]$ at 1. Under this identification the map T becomes $\mu - 1$ where μ is the monodromy. So a factorization $F(\mathbf{P}^\bullet) \to B \to G(\mathbf{P}^\bullet)$ of T determines μ and hence also determines $\mathbf{P}^\bullet$. Therefore Theorem 3.3 implies that $\mathbf{P}(\mathbb{C})$ is equivalent to the category of pairs of vector spaces $(F(\mathbf{P}^\bullet), B)$ with maps q: $F(\mathbf{P}^\bullet) \to B$ and r: $B \to F(\mathbf{P}^\bullet)$ such that $r \circ q + 1$ is invertible. This is the original result of Deligne [D1].

4. Extensions of perverse sheaves: the topological case

As in the last section we consider a Thom-Mather stratified space X with only even dimensional strata and we focus attention on a closed stratum S of dimension $2d$. We give a construction of $\mathbf{P}(X)$ in terms of $\mathbf{P}(X-S)$. This extends (and makes use of) the result for contractible S of the last section.

Definition 4.1. The *link bundle* $\pi\colon L\to S$ and the *normal slice bundle* $\pi'\colon \mathbb{D}\to S$ are defined as follows: For a small enough positive valued function $\varepsilon\colon S\to\mathbb{R}$,

$$L=\{x\in T_S\,|\,\rho_s(x)=\varepsilon(\pi_s(x))\}$$
$$\mathbb{D}=\{x\in T_S\,|\,\rho_s(x)\leqq\varepsilon(\pi_s(x))\}.$$

The maps π and π' are restrictions of π_s. (See §0 for the meaning of T_S, π_S and ρ_S).

The normal slice bundle is a closed neighborhood of S and the link bundle is its topological boundary. Both are topologically locally trivial stratified fibre bundles [Ma] and are independent of the choice of ε.

Definition 4.2. Let K be a closed subset of L and let $\kappa\colon K\hookrightarrow L$ and $\gamma\colon L-K\hookrightarrow L$ be the inclusions. We call K a *perverse link bundle* if

i) $R^k\pi_*\kappa_*\kappa^*\mathbf{P}^\bullet=0$ for $k\geqq -d$ and for all $\mathbf{P}^\bullet\in\mathbf{P}(X-S)$

ii) $R^k\pi_*\gamma_!\gamma^*\mathbf{P}^\bullet=0$ for $k<-d$ and for all $\mathbf{P}^\bullet\in\mathbf{P}(X-S)$

Remark. The space K is not necessarily a topological fibre bundle on S.

Perverse link bundles exist. A construction is given in the following lemma.

Lemma 4.3. *Let $\mathcal{T}$ be a triangulation of L such that $\pi\colon L\to S$ is simplicial and the strata of L are unions of open simpleces. Let $\mathcal{T}'$ be the barycentric subdivision of $\mathcal{T}$ constructed with respect to barycenters whose projections to S are in general position.*

Then the set K which is the union of all closed simpleces Δ of $\mathcal{T}'$ such that $\dim(\Delta\cap V\cap\pi^{-1}(s))<\frac{1}{2}\dim(V\cap\pi^{-1}(s))$ *for all strata V of L and all $s\in S$ is a perverse link bundle.*

Proof. By applying the lemma in the appendix together with the long exact sequence associated to the triangle

$$\begin{array}{ccc} & & R\pi_*\kappa_*\kappa^*\mathbf{IC}_X^\bullet \\ R\pi_*\mathbf{IC}_X^\bullet|L & \nearrow\nwarrow & \Big\downarrow{\scriptstyle(1)} \\ & & R\pi_*\gamma_!\gamma^*\mathbf{IC}_X^\bullet \end{array}$$

we see that the statement holds for $\mathbf{P}^\bullet=\mathbf{IC}_X^\bullet$ where we take the intersection homology with any coefficient system on the generic part of X.

If $Y\subset X$ is a stratified subvariety of X containing S then $L_Y=Y\cap L$ and $K_Y=Y\cap K$ where L_Y is the link bundle of Y over S and K_Y is the K of the lemma associated to L_Y. This implies that the lemma holds for $\mathbf{P}^\bullet=\mathbf{IC}_Y^\bullet$ for any stratified subvariety Y of X. The general case now follows by induction because an arbitrary $\mathbf{P}^\bullet\in\mathbf{P}(X-S)$ has a composition series whose quotients are intersection homology complexes.

We fix a perverse link bundle K. We assume that there is a stratification $\mathscr{S}'$ subordinate to $\mathscr{S}$ such that the map $\pi\colon K \to S$ is stratified with respect to $\mathscr{S}'$. Such perverse link bundles exist by the previous lemma. We now give data for the construction of an abelian category by Proposition 1.1 as follows: $\mathscr{A} = \mathbf{P}(X-S)$, $\mathscr{B}$ = sheaves of k vector spaces on S, F is the functor which sends $\mathbf{P}^\bullet$ to $R^{-d-1}\pi_*\kappa_*\kappa^*\mathbf{P}^\bullet$, G is the functor which sends $\mathbf{P}^\bullet$ to $R^{-d}\pi_*\gamma_!\gamma^*\mathbf{P}^\bullet$ and T is induced by the canonical map $\delta\colon \kappa_*\kappa^*\mathbf{P}^\bullet \to \gamma_!\gamma^*\mathbf{P}^\bullet[1]$, in the triangle

$$\begin{array}{ccc} & & \kappa_*\kappa^*\mathbf{P}^\bullet \\ & \nearrow & \\ \mathbf{P}^\bullet & & \Big\downarrow (1) \\ & \nwarrow & \\ & & \gamma_!\gamma^*\mathbf{P}^\bullet. \end{array}$$

Lemma 4.4. i) *The functor F is right exact and the functor G is left exact.*

ii) *For every $\mathbf{P}^\bullet \in \mathbf{P}(X-S)$ we have the following exact sequence of sheaves on S*

$$0 \to R^{-d-1}\pi_*\mathbf{P}^\bullet \to F\mathbf{P}^\bullet \to G\mathbf{P}^\bullet \to R^{-d}\pi_*\mathbf{P}^\bullet \to 0$$

where the first and last terms are local systems on S.

Proof. i) Immediate from Definition 4.2.

ii) By applying $R\pi_*$ to the triangle

$$\begin{array}{ccc} & & \kappa_*\kappa^*\mathbf{P}^\bullet \\ & \nearrow & \\ \mathbf{P}^\bullet & & \Big\downarrow (1) \\ & \nwarrow & \\ & & \gamma_!\gamma^*\mathbf{P}^\bullet \end{array}$$

and using Definition 4.2 we get the exact sequence. The first and last terms are local systems because $\pi\colon L \to S$ is a stratified fibre bundle.

Theorem 4.5. *The category $\mathbf{P}(X)$ is equivalent to the full subcategory $\tilde{\mathscr{C}}$ of the category $\mathscr{C}(F, G; T)$ whose objects satisfy the condition that in the factorization $F(\mathbf{P}^\bullet) \xrightarrow{m} B \xrightarrow{n} G(\mathbf{P}^\bullet)$* ker n *and* coker m *are local systems on S. The equivalence of categories is explicitly given by sending $\mathbf{Q}^\bullet$ to $\mathbf{Q}^\bullet \mid X-S$ together with*

$$\begin{array}{ccc} F(\mathbf{Q}^\bullet) & \longrightarrow & G(\mathbf{Q}^\bullet) \\ & \searrow \quad \nearrow & \\ & R^{-d}\pi_*\varphi_!\varphi^*\mathbf{Q}^\bullet & \end{array}$$

where $\varphi\colon \mathbb{D} - K \hookrightarrow \mathbb{D}$ is the inclusion.

Example 4.6. For $\mathbf{P}^\bullet \in \mathbf{P}(X-S)$ there are three functorial ways ${}^p j_! \mathbf{P}^\bullet$, ${}^p j_{!*}\mathbf{P}^\bullet$ and ${}^p j_*\mathbf{P}^\bullet$ to extend $\mathbf{P}^\bullet$ to an object in $\mathbf{P}(X)$ [BBD]. In our language these functors correspond to the following factorizations of $T\colon F\mathbf{P}^\bullet \to G\mathbf{P}^\bullet$:

${}^p j_! \mathbf{P}^\bullet$	$F(\mathbf{P}^\bullet) \xrightarrow{\text{id}} F(\mathbf{P}^\bullet) \xrightarrow{T} G(\mathbf{P}^\bullet)$
${}^p j_{!*} \mathbf{P}^\bullet$	$F(\mathbf{P}^\bullet) \twoheadrightarrow \operatorname{Im} T \hookrightarrow G(\mathbf{P}^\bullet)$
${}^p j_* \mathbf{P}^\bullet$	$F(\mathbf{P}^\bullet) \xrightarrow{T} G(\mathbf{P}^\bullet) \xrightarrow{id} G(\mathbf{P}^\bullet)$.

If E is a local system on S then $E[d]$ is a perverse object on X and it is represented by

$$E[d] \qquad F(0)=0\to E\to 0=G(0)$$

via Theorem 4.5.

We want to reduce the proof of Theorem 4.5 to Theorem 3.3. To do so we need to use the theory of stacks. For the convenience of the reader we recal the definition.

Let X be a topological space. Consider a transformation P which to each open set U of X associates an abelian category $P(U)$ and to each inclusion of open sets $V\subset U$ associates a functor P_{VU}: $P(U)\to P(V)$.

Definition 4.7. The transformation P is called a *stack of abelian categories* if

i) The association $U\mapsto \operatorname{Mor} P(U)$ is a sheaf.

ii) Given an open cover $\{U_i\}_{i\in I}$ of U, $A_i\in P(U_i)$ and g_{ji}: $A_i\,|\,U_i\cap U_j \xrightarrow{\cong} A_j\,|\,U_i\cap U_j$ for any pair i, j such that $g_{ii}=id$ and such that for any triple i, j, k we have $g_{kj}\circ g_{ji}=g_{ki}$ on $U_i\cap U_j\cap U_k$, then there exists a unique $A\in P(U)$ and g_i: $A\,|\,U_i \xrightarrow{\cong} A_i$ such that $g_j=g_{ji}\,g_i$ for all i and j.

Lemma 4.8. *Let P and P' be two stacks of abelian categories and $N: P\to P'$ a natural transformation. If $\{U_i\}$ is an open cover of X such that $N_{U_i}: P(U_i)\to P'(U_i)$ and $N_{U_i\cap U_j}: P(U_i\cap U_j)\to P'(U_i\cap U_j)$ are equivalences of categories for all* i and j *then N_X: $P(X)\to P'(X)$ is an equivalence of categories.*

Proof. By the condition i) in Definition 4.6 we get that N_X is fully faithful because all the N_{U_i}'s are. It remains to be shown that for any A' in $P'(X)$ there exists an $A\in P(X)$ such that $N_X A\cong A'$. But now for any i there exists an $A_i\in P(U_i)$ such that g_i: $N_X A_i \xrightarrow{\cong} A'\,|\,U_i$. The maps $g_{ji}=g_j^{-1}\circ g_i$ now satisfy the gluing condition in ii) as do the maps f_{ji}: $A_i|U_i\cap U \xrightarrow{\cong} A_j\,|\,U_i\cap U_j$ which are defined uniquely by the condition that $N_{U_i\cap U_j}(f_{ji})=g_{ji}$. Therefore there exists an object $A\in P(X)$ s.t. $N_X A\cong A'$.

Lemma 4.9. *The category* $\mathbf{P}(X)$ *and the* $\mathscr{C}(F,G;T)$ *and* $\tilde{\mathscr{C}}$ *of Theorem* 4.5 *form stacks.*

Proof. For the proof that $\mathbf{P}(X)$ is a stack we refer to [BBD]. The category $\mathscr{B}=\mathbf{P}(X-S)$ is a stack. So $\mathscr{C}(F,G;T)$ is a stack on $X-S$. It is a stack along S because $\mathscr{B}$ is. The category $\tilde{\mathscr{C}}$ is a stack because $\mathscr{C}$ is and the condition of belonging to $\tilde{\mathscr{C}}$ is local.

Proof of Theorem 4.5. Observe first that the sheaf B is constructible with respect to $\mathscr{S}'$ because $F(\mathbf{P}^\bullet)$, ker m and coker m are. Choose a triangulation $\mathscr{T}$ subordinate to $\mathscr{S}'$ and consider the cover $\mathscr{U}=\{St(\Delta)\}_{\Delta\in\mathscr{T}}$ of S. This cover has the property that if U and U' belong to $\mathscr{U}$ then $U\cap U'$ belongs to $\mathscr{U}$. For every $\Delta\in\mathscr{T}$ choose a point x in the interior of Δ and denote the open set $St(\Delta)$ by U_x. For a sheaf $\mathscr{F}$ constructible with respect to $\mathscr{T}$ we have $\mathscr{F}_x=\Gamma(U_x,\mathscr{F})$. Therefore we get a canonical map s: $\mathscr{F}_x\to\mathscr{F}\,|\,U_x$.

Denote by K_x and L_x the fibres of K and L over x. Let F_x be the functor sending $\mathbf{P}^\bullet\in\mathbf{P}(X-S)$ to $\mathbb{H}^{-d-1}(K_x,\mathbf{P}^\bullet)$, G_x the functor sending $\mathbf{P}^\bullet\in\mathbf{P}(X-S)$ to $\mathbb{H}^d(L_x,K_x;\mathbf{P}^\bullet)$ and $T_x\mathbf{P}^\bullet$ the coboundary map δ: $F_x\mathbf{P}^\bullet\to G_x\mathbf{P}^\bullet$. The canonical map

s induces natural transformations $f\colon F_x \to F$ and $g\colon G_x \to G$. The induced maps $\operatorname{Ker} T_x = \mathbb{H}^{-d-1}(L_x, \mathbf{P}^\bullet) \to R^{-d-1}\pi_* \mathbf{P}^\bullet | U_x = \operatorname{Ker}(T) | U_x$ and $\operatorname{Cok} T_x = \mathbb{H}^{-d}(L_x, \mathbf{P}^\bullet) \to R^{-d}\pi_* \mathbf{P}^\bullet | U_x = \operatorname{Cok}(T) | U_x$ are isomorphisms because $\pi_1(U_x) = 0$. Therefore Lemma 1.2 implies that $\mathscr{C}(F_x, G_x; T_x) \cong \mathscr{C}(F, G; T) | X - (S - U_x)$. Because U_x is contractible Theorem 3.3 implies that $\mathbf{P}(X) | X - (S - U_x) \cong \mathscr{C}(F_x, G_x; T_x)$. Thus applying lemmas 4.8 and 4.9 concludes the proof.

4.10. Construction of an inverse

We will conclude this section by constructing an explicit inverse to the functor $\mathbf{P}(X) \to \tilde{\mathscr{C}}$ of Theorem 4.5. Given an element in $\mathscr{C}(F, G\colon T)$, i.e. $\mathbf{P}^\bullet \in \mathbf{P}(X - S)$ together with a factorisation

$$\begin{array}{ccc} F\mathbf{P}^\bullet & \xrightarrow{T} & G\mathbf{P}^\bullet \\ & {}^{m}\searrow \quad \nearrow^{n} & \\ & B & \end{array}$$

we construct explicitly a complex of sheaves $\mathbf{Q}^\bullet$ in $\mathbf{P}(X)$ corresponding to the element in $\mathscr{C}(F, G\colon T)$. The complex $\mathbf{Q}^\bullet$ will be a complex of cellular sheaves, given that $\mathbf{P}^\bullet$ is cellular. By a cellular sheaf, we mean a sheaf which is constant on open cells, and which has finite dimensional stalks. For a detailed treatment of cellular sheaves see [S]. Notice that this will allow us in principle to iterate the construction of Theorem 4.5 since all the sheaf theoretic functors involved in the definition of F and G ($R^k\pi_*$, κ_*, κ^*, $\tilde{\gamma}^!$, and γ^*) have combinatorial constructions in the category of cellular sheaves [S].

Let $K \subset L$ be a perverse link bundle. Choose a regular CW complex structure C on X subordinate to the stratification such that K and L are subcomplexes of X and $\pi\colon L \to S$ is a cellular map. Give X the following cellular structure. Outside of D take the original cellulation of X. On L and S take the barycentric subdivision of the original induced cellulation and cellulate the rest of D by mapping cylinders of π with respect to the barycentric subdivision cellulation of L and S. It is important to note that we do not require the closures of our cells to be compact so that we can give an open subset the induced cellulation.

We will give some constructions in the cellular category of sheaves in the context that we need them.

If σ is a cell we denote by $[\sigma]$ the sheaf that takes the constant value k on $\bar{\sigma}$. These sheaves $[\sigma]$ are injective in the category of cellular sheaves and all injectives are direct sums of these. To any complex of sheaves we can in a functorial manner construct an injective complex of cellular sheaves called its injectivization [S].

Let $\tilde{K} = [c\sigma \,|\, \sigma \in L$ and $\bar{\sigma} \cap K \neq \emptyset\}$, where c denotes the mapping cylinder of $\pi\colon \sigma \to \pi(\sigma)$ and denote by $\tilde{\gamma}\colon X - (\tilde{K} \cup S) \subset X - S$ the inclusion. Assume that the complex $\mathbf{P}^\bullet$ has already been injectivized. We can now construct a subcomplex $\tilde{\gamma}_* \tilde{\gamma}^! \mathbf{P}^\bullet \hookrightarrow \mathbf{P}^\bullet$ which is still injective. The complex $\mathbf{Q}^\bullet$ can now be constructed as follows.

$$
\begin{array}{lcl}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathbf{Q}^k & = & j_*\mathbf{P}^k \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathbf{Q}^{-d-2} & = & j_*\mathbf{P}^{-d-2} \\
\downarrow & & \downarrow \\
\mathbf{Q}^{-d-1} & = & \delta^{-1}(j_*\tilde{\gamma}_*\tilde{\gamma}^!\mathbf{P}^{-d}), \quad \text{where } \delta: j_*\mathbf{P}^{-d-1} \to j_*\mathbf{P}^{-d} \\
\downarrow & & {}^{c}\swarrow \ \searrow^{\delta} \\
\mathbf{Q}^{-d} & = & i_*B \oplus \ker\tilde{\delta}, \quad \text{where } \tilde{\delta}: j_*\tilde{\gamma}_*\tilde{\gamma}^!\mathbf{P}^{-d} \to j_*\tilde{\gamma}_*\tilde{\gamma}^!\mathbf{P}^{-d+1} \\
\downarrow & & {}^{m}\searrow \ \swarrow^{-\tilde{c}} \\
\mathbf{Q}^{-d+1} & = & i_*G(\mathbf{P}^\bullet). \\
\downarrow & & \\
0 & &
\end{array}
$$

Note that j_*, $\tilde{\gamma}_*$, κ_* and κ^* take injective complexes to injective complexes. The same is true for a $\tilde{\gamma}^!$ if we choose the following model for it:

$$
\tilde{\gamma}^![\sigma] = \begin{cases} [\sigma] & \text{if } \bar{\sigma} \cap \tilde{K} = \emptyset \\ 0 & \text{otherwise}. \end{cases}
$$

To define the cycle maps c and $\tilde{c}$ we proceed as follows. Move the link L towards S so that it intersects the cellulation of X along the mapping cylinders. Then for any cellular sheaf $\mathbf{A}$ we have $i^*j_*\mathbf{A} = \pi_*(\mathbf{A}|L)$. The cycle map c can now be defined by the following formula:

$$c: \mathbf{Q}^{-d-1} \to i_*\pi_*(\mathbf{Q}^{-d-1}|L) \to i_*\pi_*\kappa_*\kappa^*\mathbf{Q}^{-d-1} \to \operatorname{Ker}\delta \to i_*R^{-d-1}\pi_*\kappa_*\kappa^*\mathbf{P}^\bullet \to B,$$

where δ: $i_*\pi_*\kappa_*\kappa^*\mathbf{P}^{-d-1} \to i_*\pi_*\kappa_*\kappa^*\mathbf{P}^{-d}$ and the first map exists by what was said above and the last map is the quotient map in computing $F(\mathbf{P}^\bullet)$ from a resolution. The map $\tilde{c}$ is defined similarly as the composition

$$\tilde{c}\colon \operatorname{Ker}\tilde{\delta} \to i_*\pi_*(\ker(\tilde{\delta})|L) \to i_*R^{-d}\pi_*(\tilde{\gamma}_*\tilde{\gamma}^!\mathbf{P}^\bullet|L) \cong i_*G(\mathbf{P}^\bullet)$$

where the first two maps exist for the same reason as above and the last isomorphism can be easily verified. It is important to note that the map $\tilde{c}$ is a surjection.

5. Extensions of perverse sheaves: the complex analytic case

In this section, we consider a complex manifold X with an analytic Whitney stratification $\mathscr{S}$. For any stratum $S \in \mathscr{S}$, let

$\Lambda_S = T_S^*X$ and $\tilde{\Lambda}_S = T_S^*X - \bigcup_{\substack{R \in \mathscr{S} \\ R \neq S}} \overline{T_R^*X}$ where $\overline{T_R^*X}$ is the closure of T_R^*X in T^*X. We say $\mathscr{S}$ satisfies *condition C* if for all $S \in \mathscr{S}$ $\tilde{\Lambda}_S$ is a topological fiber bundle over S by the natural projection. (Every Whitney stratification of a complex manifold can be refined to one satisfying condition C. This is proved by adding condition C to the inductive step of the usual proof of the existence of Whitney stratification [GM 3].) Many natural stratifications, for example, that of a flag manifold by Shubert cells, satisfy condition C.

We assume that our stratification $\mathscr{S}$ satisfies condition C.

We consider a closed stratum S of complex dimension d. We give a construction of $\mathbf{P}(X)$ in terms of $\mathbf{P}(X-S)$ which does not involve a choice of a perverse link bundle K, but instead uses conormal geometry of X near S. We will define functors ψ and ψ_c: $\mathbf{P}(X-S) \to \{\text{local systems on } \tilde{\Lambda}_S\}$ (called the "nearby cycles" and the "nearby cycles with compact support"), a functor Φ: $\mathbf{P}(X) \to \{\text{local systems on } \tilde{\Lambda}_S\}$ (called the "vanishing cycles"), and a natural transformation var: $\psi \to \psi_c$ (called the "variation"). We give a topological construction of these.

There is a fiber bundle π: $D \to \tilde{\Lambda}_S$, a subbundle $L \subset D$, a further subbundle $\mathscr{L} \subset L$, and a map r: $L \to X-S$ with the following property: If ξ is a covector in $\tilde{\Lambda}_S$ over a point s in S, the r identifies the fiber D_ξ with a normal disk to S at s, the fiber L_ξ with the link ∂D_δ of S at s, and $\mathscr{L}_\xi$ with the complex link [GM 3] in the direction ξ. To construct these, we use "control data" T_S, π_S, and ρ_S as in section 0. T_S identifies with a neighborhood of the zero section in the normal bundle to S in X by an identification which is complex analytic on the fibers, π_S identifies with the projection of the normal bundle to S, and ρ_S identifies with the square of a fiberwise hermitiam norm in the normal bundle. For each ξ in $\tilde{\Lambda}_S$ projecting to s in S, since $\pi_S^{-1}(s)$ identifies with a set of normal vectors to S at s, ξ gives a complex valued function of $\pi_S^{-1}(s)$. Then there is a set J_ξ of pairs of real numbers (δ, ε) with $0 < \delta \ll \varepsilon \ll 1$ so that for all $\varepsilon' \leqq \varepsilon$ there is a $\delta_0(\varepsilon') > 0$ so that 1) for all $x \in \pi^{-1}(s)$ with $\rho(x) = \varepsilon'$ and $0 < |\xi(x)| \leqq \delta_0(\varepsilon')$, $d(Re\,\xi)(x)$ and $d\rho(x)$ are linearly independent in cotangent space of the stratum containing x, and 2) $\delta \leqq \delta_0(\varepsilon)$ (see [GM 3]). It is shown, in [GM 3], that there are smooth functions $\delta(\xi)$ and $\varepsilon(\xi)$ on $\tilde{\Lambda}_S$ so that for all ξ, $(\delta(\xi), \varepsilon(\xi))$ is in J_ξ. Finally, D is the subspace of the fiber product $\tilde{\Lambda}_S \times_S T_S$ consisting of pairs such that $\inf(|\xi(x)| - \delta(\xi), \rho(x) - \varepsilon(\xi)) \leqq 0$, L is defined similarly with $\leqq$ replaced by $=$, and $\mathscr{L}$ is the subspace consisting of pairs such that $\xi(x) = \delta(\xi)$ and $\rho(x) \leqq \varepsilon(\xi)$. (See [GM 3] for more details.)

Now let κ: $\mathscr{L} \hookrightarrow L$ and γ: $L - \mathscr{L} \hookrightarrow L$ and φ: $D - \mathscr{L} \hookrightarrow D$ be the inclusions. Then

$$\psi \mathbf{P}^\bullet = R^{-d-1} \pi_* \kappa_* \kappa^* \mathbf{P}^\bullet$$

$$\psi_c \mathbf{P}^\bullet = R^{-d} \pi_* \gamma_! \gamma^* \mathbf{P}^\bullet$$

T is induced by the canonical map δ: $\kappa_* \kappa^* \mathbf{P}^\bullet \to \gamma_! \gamma^* \mathbf{P}^\bullet [1]$ and

$$\Phi(\mathbf{P}^\bullet) = R^{-d} \pi_* \varphi_! \varphi^* \mathbf{P}^\bullet$$

Remark. 1) Let $\xi \in \tilde{\Lambda}_S$ and $x = p(\xi)$. Then we have $\psi(\mathbf{P}^\bullet)_\xi = (R^{-d-1} \psi \mathbf{P}^\cdot)_x$ and $\Phi(\mathbf{P}^\bullet)_\xi = (R^{-d-1} \Phi \mathbf{P}^\bullet)_x$ where the functors $R\psi$ and $R\Phi$ defined by Deligne in [D2] are taken with respect to a function f: $U \to \mathbb{C}$ defined in a neighborhood U of x such that $(df)_x = \xi$ and $f \mid S \cap U = 0$.

2) The local system $\Phi(\mathbf{Q}^\bullet)$ can also be defined using $\mathscr{D}$-modules. Take $\mathscr{M}$ a holonomic, regular singularities $\mathscr{D}$-module such that $R\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}, \mathscr{O}_X)[n] = \mathbf{Q}^\bullet$.

Then $\Phi(\mathbf{Q}^\bullet) = R\mathrm{Hom}_{\mathscr{E}_X}(\mathscr{E}_X \underset{p^{-1}\mathscr{D}_X}{\otimes} p^{-1}\mathscr{M}, \mathscr{C}_{X|S}^{\mathbb{R}})$, where p: $\tilde{\Lambda}_S \to S$ is the projection and $\mathscr{C}_{X|S}^{\mathbb{R}}$ is the sheaf of microfunctions on Λ_s [K], [KK], [BMV], [LM].

3) For a $\xi \in \tilde{\Lambda}_S$ we have $\psi_c(\mathbf{P}^\bullet)_\xi = \mathbb{H}_c^{-d-1}(\mathring{\mathscr{L}}_\xi, \mathbf{P}^\bullet)$ where $\mathring{\mathscr{L}}_\xi$ is the interior of $\mathscr{L}_\xi$ [GM4].

Proposition 5.1. *The complex link $\mathscr{L}_\xi$ is a perverse link. The functor ψ is right exact, and ψ_c is left exact.*

Proof. That $\mathbb{H}^k(\mathscr{L}, \mathbf{IC}^\bullet) = 0$ for $k \leqq -d$ and $\mathbb{H}^k(L, \mathscr{L} : \mathbf{IC}^\bullet) = 0$ for $k < -d$ is proved in [GM4], wher $\mathbf{IC}^\bullet$ is an intersection chain complex of a stratum. The general case follows by induction because an arbitrary perverse sheaf has a composition series whose quotients are intersection chain complexes of strata. From this it follows that $R^k \pi_* \kappa_* \kappa^* r^* \mathbf{P}^\bullet = 0$ for $k \leqq -d$ and $R^k \pi_* \gamma_! \gamma^* r^* \mathbf{P}^\bullet = 0$ for $k < -d$ for all $\mathbf{P}^\bullet$ in $\mathbf{P}(X-S)$ which shows that ψ is right exact and ψ_c is left exact.

We now want to give data for the construction of an abelian category $\mathscr{C}(F, G; T)$ via Lemma 1.1. Let $\mathscr{A} = \mathbf{P}(X-S)$, $\mathscr{B}$ = local systems on $\tilde{\Lambda}_S$, $F = \psi$, $G = \psi_c$ and $T = \mathrm{var}$. Let $p\colon \tilde{\Lambda}_S \to S$ be the projection. The following proposition which is due to Gabber and Malgrange [M] is used to define a subcategory $\tilde{\mathscr{C}} \subset \mathscr{C}(F, G; T)$.

Proposition 5.2. *If S is contractible then there are unique natural transformations $I_\alpha\colon \psi_c \to \psi$ for any $\alpha \in \pi_1(\tilde{\Lambda}_S)$ such that for any $\mathbf{Q}^\bullet \in \mathbf{P}(X)$ the local system structure on $\Phi(\mathbf{Q}^\bullet)$ is given by $\mu_\alpha - 1 = m \circ I_\alpha \circ n$, where μ_α is the monodromy of α on $\Phi(\mathbf{Q}^\bullet)$.*

Proof. Let $\mathbf{P}^\bullet \in \mathbf{P}(X-S)$. By Theorem 3.3 we can construct $\mathbf{Q}^\bullet \in \mathbf{P}(X)$ via the commutative diagram

$$\begin{array}{ccc} \psi(\mathbf{P}^\bullet)_\xi & \xrightarrow{\ \mathrm{var}\ } & \psi_c(\mathbf{P}^\bullet)_\xi \\ & {}_{m}\searrow \quad \nearrow_{n} & \\ & B & \end{array}$$

Where B is minimal such that m is injective and n is surjective and $\xi \in \tilde{\Lambda}_S$. Using the descriptions of ${}^p j_! \mathbf{P}^\bullet$ and ${}^p j_* \mathbf{P}^\bullet$ in example 4.6 we obtain the diagram

$$\begin{array}{ccc} {}^p j_! \mathbf{P}^\bullet & \xrightarrow{\ c\ } & {}^p j_* \mathbf{P}^\bullet \\ & {}_{\tilde{m}}\searrow \quad \nearrow_{\tilde{n}} & \\ & \mathbf{Q}^\bullet & \end{array} \tag{5.1}$$

in $\mathbf{P}(X)$. Also $\Phi({}^p j_! \mathbf{Q}^\bullet) = \psi(\mathbf{P}^\bullet)$, $\Phi({}^p j_* \mathbf{P}^\bullet) = = \psi_c(\mathbf{P}^\bullet)$ and $\Phi(c) = \mathrm{var}$. By taking kernels and cokernels in diagram (5.1) and by applying the functor Φ we get

$$\begin{array}{ccccccc} \Phi(\ker c) & \longrightarrow & \psi(\mathbf{P}^\bullet) & \xrightarrow{\ \mathrm{var}\ } & \psi_c(\mathbf{P}^\bullet) & \longrightarrow & \Phi(\mathrm{Cok}\, c) \\ & \searrow & & {}_{m}\searrow \quad \nearrow_{n} & & \nearrow & \\ & & & \Phi(\mathbf{Q}^\bullet) & & & \\ & & \nearrow & & \searrow & & \\ & & \Phi(\ker \tilde{n}) & \longrightarrow & \Phi(\mathrm{Cok}\, \tilde{m}) & & \end{array}$$

Because $\operatorname{Ker} c$, $\operatorname{Cok} c$, $\operatorname{Ker} \tilde{n}$, $\operatorname{Cok} \tilde{m}$ are supported on S the functor Φ applied to them gives a trivial local system. Let $x \in \psi_c(\mathbf{P}^\bullet)$. Choose $z \in \Phi(\mathbf{Q}^\bullet)$ such that $n(z) = x$ and consider $\mu_\alpha(z) - z$. Because the monodromy is trivial on $\Phi(\operatorname{Cok} \tilde{m})$ $\mu_\alpha(z) - z$ gets mapped to zero in $\Phi(\operatorname{Cok} \tilde{m})$ and therefore there is an element $y \in \psi(\mathbf{P}^\bullet)$ such that $m(y) = \mu_\alpha(z) - z$. We define $I_\alpha(x) = y$. It is now easy to verify that I_α is well-defined and has the desired properties.

Theorem 5.3. *The category* $\mathbf{P}(X)$ *is equivalent to the subcategory* $\tilde{\mathscr{C}}$ *of the category* $\mathscr{C}(F, G; T)$ *whose objects satisfy the condition that in the factorization* $F\mathbf{P}^\bullet \to B \to G\mathbf{P}^\bullet$ *the monodromy* μ_α *on* B *for any* $\alpha \in \pi_1(p^{-1}x)$, $x \in S$ *is given by* $\mu_\alpha - 1 = m \circ I_\alpha \circ n$.

Proof. We want to reduce the proof using stacks to the case where S is contractible. We know by [BBD] that $\mathbf{P}(X)$ forms a stack. It is also immediate that both $\mathscr{C}(F, G; T)$ and $\tilde{\mathscr{C}}$ form a stack. By Theorem 3.3 and Proposition 5.1 the theorem is true for contractible S. The result now follows by Lemma 4.7 because any manifold S can be covered by contractible open sets whose intersections are contractible. (We could take for examples stars of simplices in some triangulation.)

6. Examples and Applications

In this section we give some examples and applications both of the analytic and the topological versions of our results.

Suppose $\mathbf{P}^\bullet \in \mathbf{P}(X - S)$. An extension $\mathbf{Q}^\bullet \in \mathbf{P}(X)$ of $\mathbf{P}^\bullet$ is called *indecomposable with respect to* $\mathbf{P}^\bullet$ if $\mathbf{Q}^\bullet$ does not have a direct summand concentrated on S. An extension $\mathbf{Q}^\bullet$ of $\mathbf{P}^\bullet$ is called *maximal indecomposable* if every extension of it is decomposable.

Proposition 6.1. *Assume that* $\pi_1(S) = \pi_2(S) = 0$. *Then every* $\mathbf{P}^\bullet \in \mathbf{P}(X - S)$ *has a maximal indecomposable extension* $\mathbf{Q}^\bullet$ *which is unique up to a (non-canonical) isomorphism. If* $U \subset X$ *is open such that* $\pi_1(U) = 0$ *then* $\mathbf{Q}^\bullet \mid U$ *is the maximal indecomposable extension of* $\mathbf{P}^\bullet \mid U - S \cap U$.

Proof. We assume first that X is a complex manifold. We claim that $\mathscr{C}(F, G: T)$ is equivalent to the category of triangles of vector spaces

$$\begin{array}{ccc} \psi(\mathbf{P}^\bullet)_\xi & \xrightarrow{\text{var}} & \psi_c(\mathbf{P}^\bullet)_\xi \\ & \searrow \quad \nearrow & \\ & B_\xi & \end{array}$$

for some ξ in $\tilde{\Lambda}_S$. This follows from Theorem 5.4 by observing that $\pi_1(S) = \pi_2(S) = 0$ implies that $\pi_1(\tilde{\Lambda}_S) \cong \pi_1(p^{-1}(x))$ for $x \in S$ and therefore the $\pi_1(\tilde{\Lambda}_S)$ action is completely determined by the maps I_α. Then, the maximal indecomposable extension is given by choosing B minimally such that

$$\begin{array}{ccc} \psi(\mathbf{P}^\bullet)_\xi & \xrightarrow{\text{var}} & \psi_c(\mathbf{P}^\bullet)_\xi \\ & \searrow \quad \nearrow & \\ & B_\xi & \end{array} \tag{6.1}$$

If X is not complex analytic then the techniques of Sect. 7 show that we can extend Theorem 3.3 to the case when $\pi_1(S) = \pi_2(S) = 0$. We can now define the maximal indecomposable extension via diagram (6.1) just as above.

Next we present the following two examples. The first one shows that if $\pi_1(S) \neq 0$ then the maximal indecomposable extension of Proposition 6.1 does not need to be unique and the functor $\mathcal{Z}: \mathbf{P}(X) \to \mathbf{Z}(X, S)$ need not be injective on the isomorphism classes of objects. The second example shows that if $\pi_2(S) \neq 0$ then a global extension which localizes to maximal extensions on all U open in S does not necessarily exist and that the functor $\mathcal{Z}: \mathbf{P}(X) \to \mathbf{Z}(X, S)$ need not be surjective on the isomorphism classes of objects.

Example 6.2. Let $X = \mathbb{C}^2 - (\{0\} \times \mathbb{C})$ stratified by $S = \mathbb{C} \times \{0\} - \{(0,0)\}$ and its complement. We claim that $\mathbf{P}(X)$ is equivalent to the category of pairs of finite dimensional vector spaces (V, W) together with the diagram

(6.2)
$$\begin{array}{ccc} \beta \circlearrowleft V & \xrightarrow{\alpha - 1} & V \circlearrowright \beta \\ & {\scriptstyle m} \searrow \quad \nearrow {\scriptstyle n} & \\ & W & \\ & \circlearrowleft & \\ & \gamma & \end{array}$$

where α and β are automorphisms of V, γ is an automorphism of W and $\beta \circ \alpha = \alpha \circ \beta$, $m\beta = \gamma m$ and $n\gamma = \beta n$. To see this by Theorem 4.5 choose $L = \{(x, y) \in \mathbb{C}^2 \mid |y| = 1, x \in S\}$ and $K = \{(x, 1) \in \mathbb{C}^2 \mid x \in S\} \subset L$. Choose the generators $a: t \mapsto (1, e^{2\pi i t})$ and $b: t \mapsto (e^{2\pi i t}, 1)$ for $\pi_1(X - S) \cong \mathbb{Z} \oplus \mathbb{Z}$ (with base point $(1,1)$) and $c: t \mapsto (e^{2\pi i t}, 0)$ for $\pi_1(S) \cong \mathbb{Z}$ (with base point $(1,0)$).

An element $\mathbf{P}^\bullet \in \mathbf{P}(X - S)$ is just a local system and it can be interpreted as a vector space V with commuting automorphism α and β induced by a and b. A calculation yields that $\mathbf{F}(\mathbf{P}^\bullet) = V$ with c acting via β, $G(\mathbf{P}^\bullet) = V$ with c acting via β and $T = \alpha - 1: V \to V$. Theorem 4.5 now implies that the data of diagram (6.2) is equivalent to $P(X)$. The action γ on W is given by c.

If we take $V = k$, $\alpha = \beta = \text{id}$, $W = k^n$ and

$$\gamma = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & . & & & . \\ & & & . & . & 0 \\ & & & & & 1 \\ 0 & & & & 0 & 1 \end{bmatrix}$$

we see that this object is indecomposable. This shows that no maximal indecomposable extension can exist. Also the functor $\mathcal{Z}$ is not injective on the isomorphism classes of objects because the choice $\gamma = \text{id}$ leads to the same Zig-Zag diagram.

Example 6.3. Let $X = \mathbb{C}P^2$ stratified by $S = \mathbb{C}P^1$ and its complement $X - S = \mathbb{C}^2$. We claim that $\mathbf{P}(X)$ is equivalent to the category of pairs of vector spaces (V, W)

together with maps m and n $V \underset{m}{\overset{n}{\rightleftarrows}} W$ such that $m \circ n = 0$ and $n \circ m = 0$. To verify this claim we want to apply Theorem 5.4. Because $\pi_1(\mathbb{C}^2) = 0$ an ekment in $\mathbf{P}(X-S)$ is just a vector space V. Because $\tilde{\Lambda}_S \cong \mathbb{C}^2 - \{0,0\}$ we have $\pi_1(\tilde{\Lambda}_S) = 0$. However for $x \in S$ we have $p^{-1}(s) = \mathbb{C}^* \subset \tilde{\Lambda}_S$ and for the canonical generator γ of $\pi_1(p^{-1}(s))$ the Gabber-Malgrange map $I_\gamma = \mathrm{id}$. We also see that the variation map var $= 0$. Therefore by Theorem 5.4 the category $\mathbf{P}(X)$ is equivalent to fixing two vector space V and W together with a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{0} & V \\ & {\scriptstyle m}\searrow \quad \nearrow{\scriptstyle n} & \\ & W & \end{array}$$

such that $m \circ I_\gamma \circ n = 0$, i.e., $m \circ n = 0$. This proves our claim.

Observe that maximal indecomposable extensions do exist but they are not unique. For example $V = k$ has two maximal indecomposable extensions $k \underset{\mathrm{id}}{\overset{0}{\rightleftarrows}} k$ and $k \underset{0}{\overset{\mathrm{id}}{\rightleftarrows}} k$. Note further that the diagram

$$\begin{array}{ccc} k & \xrightarrow{0} & k \\ & {\scriptstyle i_1}\searrow \quad \nearrow{\scriptstyle pr_2} & \\ & k^2 & \end{array} \tag{6.3}$$

is not allowed. This example also shows that the functor $\mathscr{Z}$ of Sect. 2 is not onto on isomorphism classes of objects if $\pi_2(S) \neq 0$ because the Zig-Zag diagram corresponding to the diagram (6.3) is not in the image of $\mathscr{Z}$.

Next we consider the following practical example.

Example 6.4. Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be given by $f(z_0, \ldots, z_n) = z_0^2 + \ldots + z_n^2$ and let $X = f^{-1}(0)$. We can now form the sheaf $R\psi\,\mathbb{Q}$ on X which comes with a canonical automorphism $M\colon R\psi\,\mathbb{Q} \to R\psi\,\mathbb{Q}$ given by monodromy in $\mathbb{C} - \{0\}$. The sheaf $R\psi\,\mathbb{Q}[n]$ is perverse [GM4]. We want to use Theorem 3.3 to describe this sheaf and the map M.

Clearly $R\psi\,\mathbb{Q}[n] \,|\, X - \{0\} = \mathbb{Q}_{X-\{0\}}[n]$. It is well known that $L = \{$unit tangent vectors of $S^n\}$ (see e.g. [L]). We can take $K = \{$one fibre of $L \to S^n)$. A calculation shows that the diagram

$$\begin{array}{ccc} H^{n-1}(K; \mathbb{Q}) & \xrightarrow{T} & H^n(L, K; \mathbb{Q}) \\ & \searrow \quad \nearrow & \\ & \mathbb{H}^n(B, K; R\psi\,\mathbb{Q}) & \end{array}$$

is for n even

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{T = \mathrm{id}} & \mathbb{Q} \\ & {\scriptstyle i_1}\searrow \quad \nearrow{\scriptstyle pr_1} & \\ & \mathbb{Q} \oplus \mathbb{Q} & \end{array}$$

and $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. For n odd we get

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{T=0} & \mathbb{Q} \\ & {\scriptstyle i_1}\searrow \quad \nearrow{\scriptstyle pr_2} & \\ & \mathbb{Q} \oplus \mathbb{Q} & \end{array}$$

and $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

So we see that for n even $R\psi\,\mathbb{Q}[n]$ splits as a direct sum $R\psi\,\mathbb{Q}[n] = \mathbb{Q}_X[n] \oplus \mathbb{Q}_{\{0\}}$. For n odd we get that $R\psi\,\mathbb{Q}[n]$ is the maximal indecomposable extension of Proposition 6.1. It is also interesting to observe that the map M_*: $\mathbf{H}^q(R\psi\,\mathbb{Q}) \to \mathbf{H}^q(R\psi\,\mathbb{Q})$ is identity for all q but $M \neq \mathrm{Id}$ in $\mathbf{P}(X)$.

Next we want to give a topological proof of the following theorem of Kashiwara and Kawai [KK] established initially by micro-local techniques.

Proposition 6.5. *Let X be a complex manifold and S a closed stratum in the stratification $\mathscr{S}$ such that*

$$T_S^* X \cap \bigcup_{R \neq S} \overline{T_R^* X}$$

is of complex codimension at least two in $T_S^* X$. Then $\psi(\mathbf{P}^\bullet) = \psi_c(\mathbf{P}^\bullet) = 0$ for all $\mathbf{P}^\bullet \in \mathbf{P}(X-S)$ and so $\mathbf{P}(X) \cong \mathbf{P}(X-S) \oplus \mathbf{P}(S)$.

The proof will use the following.

Proposition 6.6. *In the situation of Proposition 6.5 the complex link $\mathscr{L}$ of S is homeomorphic to $\mathscr{L}' \times D$ in a stratum preserving way, where $\mathscr{L}'$ is a complex link of one lower dimension and D is a disk.*

Proof of Proposition 6.5. Let $\mathbf{P}^\bullet \in \mathbf{P}(X-S)$. The stalk of $\psi(\mathbf{P}^\bullet)$ at $\xi \in \tilde{\Lambda}_S$ is isomorphic to $\mathbb{H}^{-d-1}(\mathscr{L}, \mathbf{P}^\bullet)$, where $\mathscr{L}$ is a complex link to the direction ξ. By the Künneth theorem and Proposition 6.6 we have $\mathbb{H}^{-d-1}(\mathscr{L}, \mathbf{P}^\bullet) \cong \mathbb{H}^{-d-1}(\mathscr{L}', \mathbf{P}^\bullet)$. Because $\mathscr{L}'$ is a complex link $\mathbb{H}^{-d-1}(\mathscr{L}', \mathbf{P}^\bullet) = 0$ ([GM4]). Therefore $\psi(\mathbf{P}^\bullet) = 0$. A similar argument shows that $\psi_c(\mathbf{P}^\bullet) = 0$.

Proof of Proposition 6.6. This is proved in detail in [GM3]. The following is a sketch. By cutting through x with a normal slice to S we can assume that $X = B_\delta \subset \mathbb{C}^n$ is a δ-disk $x = 0$ and $S = \{0\}$. Using the hypothesis, we choose linearly independent vectors $\xi_1, \xi_2 \in T_{\{0\}}^* X$ such that $\mathrm{span}(\xi_1, \xi_2) \cap \bigcup_{R \neq S} \overline{T_R^* X} = \{0\}$. Let f: $\mathbb{C}^n \to \mathbb{C}^2$ be the projection such that $df_0 = (\xi_1, \xi_2)$. Then by choosing δ smaller if necessary the map f: $B \to \tilde{D} \subset \mathbb{C}^2$ is a stratified submersion outside of the exceptional fibre $f^{-1}(0)$.

Now consider a projection p: $\tilde{D} \to \mathbb{C}$, and denote the composition $p \circ f$ by g. If we now choose appropriate $0 < \varepsilon \ll \eta \ll \delta \ll 1$ using the technique of moving the wall [GM3], we can establish a homeomorphism between $\mathscr{L}$ and $g^{-1}(\varepsilon) \cap B_\delta \cap f^{-1}(D_\eta)$ where D_η is the η-disk. Now, projection by f of $\mathscr{L}$ to D_η is a stratified fibration and the lemma follows since D_η is contractible.

Next we will give a generalization of Proposition 6.5 which answers a question of *W.* Borho. Let X be a complex manifold and let $\mathbf{Q}^\bullet \in \mathbf{P}(X)$. We can associate to $\mathbf{Q}^{\bullet}$ its characteristic variety $\mathrm{Char}(\mathbf{Q}^{\bullet}) \subset T^*X$ consisting of unions of closures of conormal bundles $\overline{T_S^* X}$ such that $\Phi(\mathbf{Q}^{\bullet}) \neq 0$ generically on $T_S^* X$. (If $\mathcal{M}$ is a regular singularities holonomic $\mathcal{D}_X$-module such that $DR(\mathcal{M})[\dim X] = \mathbf{Q}^{\bullet}$ then $\mathrm{Char}(\mathcal{M}) = \mathrm{Char}(\mathbf{Q}^{\bullet})$.)

Let $\Lambda = \bigcup_{S \in \mathcal{S}'} \overline{T_S^* X} \subset T^*X$ be a conical Lagrangian variety such that $\mathcal{S}' \subset \mathcal{S}$. We define $\mathbf{P}_\Lambda(X) \subset \mathbf{P}(X)$ to be the full subcategory of $\mathbf{Q}^{\bullet} \in \mathbf{P}(X)$ such that $\mathrm{Char}(\mathbf{Q}^{\bullet}) \subset \Lambda$. Define the following relation on the components of Λ:

$$\overline{T_S^* X} \sim \overline{T_R^* X} \text{ if there exist } S_1, \ldots, S_h \in \mathcal{S}' \text{ such that } S_1 = S,\ S_h = R$$

and $\overline{T_{S_j}^* X} \cap \overline{T_{S_{j+1}}^* X}$ is of codimension 1 in $\overline{T_{S_j}^* X}$.

This equivalence relation splits Λ into equivalence classes $\Lambda_1, \ldots, \Lambda_k$ such that $\Lambda = \bigcup_{i=1}^{k} \Lambda_i$.

Theorem 6.7. *We have a direct sum decomposition of categories* $\mathbf{P}_\Lambda(X) = \bigoplus_{i=1}^{k} \mathbf{P}_{\Lambda_i}(X)$.

Proof. We prove this by induction on the number of components of Λ (which we assume to be finite). If we have only one component the theorem obviously holds. Let $\mathbf{Q}^{\bullet} \in \mathbf{P}_\Lambda(X)$, let $S \in \mathcal{S}'$ be of minimal dimension and assume that $\overline{T_S^* X} \subset \Lambda_r$. By induction $\mathbf{Q}^{\bullet} | X - \bar{S} = \bigoplus_{i=1}^{k} \mathbf{P}_i^{\bullet}$ where $\mathbf{P}_i^{\bullet} \in \mathbf{P}_{\Lambda_i}(X)$.

Using theorem 5.3 $\mathbf{Q}^{\bullet} | X - (\bar{S} - S)$ corresponds in $\mathscr{C}(F, G{:}\,T)$ to $\mathbf{Q}^{\bullet} | X - \bar{S}$ together with

$$\begin{array}{ccc} \psi(\mathbf{Q}^{\bullet} | X - \bar{S}) & \longrightarrow & \psi_c(\mathbf{Q}^{\bullet} | X - \bar{S}) \\ & \searrow \quad \nearrow & \\ & \Phi(\mathbf{Q}^{\bullet}) & \end{array} \tag{6.2}$$

But $\mathbf{Q}^{\bullet} | X - \bar{S} = \mathbf{P}_1^{\bullet} \oplus \ldots \oplus \mathbf{P}_k^{\bullet}$ and by Proposition 6.5 $\psi(\mathbf{Q}^{\bullet} | X - \bar{S}) = \psi(\mathbf{P}_r^{\bullet})$ and $\psi_c(\mathbf{Q}^{\bullet} | X - \bar{S}) = \psi_c(\mathbf{P}_r^{\bullet})$. If we now define $\tilde{\mathbf{Q}}_r^{\bullet}$ to be $\mathbf{P}_r^{\bullet}$ together with (6.2) and $\tilde{\mathbf{Q}}_k^{\bullet}$ for $k \neq r$ to be $\mathbf{P}_k^{\bullet}$ together with

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ & \searrow \quad \nearrow & \\ & 0 & \end{array}$$

we get that $\mathbf{Q}^{\bullet} | X - (\bar{S} - S) = \tilde{\mathbf{Q}}_1^{\bullet} \oplus \ldots \oplus \tilde{\mathbf{Q}}_k^{\bullet}$. But for all the strata of $\mathcal{S}$ in $\bar{S} - S$ we have $\Phi(\mathbf{Q}^{\bullet}) = 0$ and so all the $\tilde{\mathbf{Q}}_i^{\bullet}$ have a unique extension $\mathbf{Q}_i^{\bullet}$ to all of X such that for these strata in $\bar{S} - S$ we have $\Phi(\mathbf{Q}_i^{\bullet}) = 0$. It now follows easily that $\mathbf{Q}^{\bullet} = \mathbf{Q}_1^{\bullet} \oplus \ldots \oplus \mathbf{Q}_k^{\bullet}$.

Open Problems.

1. Given a stratified space X, find a procedure to construct a "quiver" of linear spaces and maps satisfying commutation relations, in the spirit of examples 6.2, 6.3, and 6.4. such that $\mathbf{P}(X)$ is equivalent to the category of realizations of the quiver. This problem is particularly important for the following special case: For a reductive complex algebraic Lie group G, the space X should be the variety of Borel

subgroups of G stratified by the orbits of some subgroup K of G. In this case, the category of K-equivariant objects of $\mathbf{P}(X)$ is equivalent to the category of $(\mathfrak{g}, K)$ Harish-Chandra modules where $\mathfrak{g}$ is the Lie algebra of G (see [BB]). To give a "quiver" description of the category of $(\mathfrak{g}, K)$ Harish-Chandra modules is a problem of Gelfand [G].

2. In Sect. 5 we give a construction of $\mathbf{P}(X)$ as a category of objects which consist of local systems on the $\tilde{\Lambda}_S$ for each of the strata held together by some "glue". It is an open problem of Beilinson to do this with "glue" which is local on T^*X. This is important because a solution should give a topological description of the microlocal category of $\mathscr{E}$-modules.

3. Given our data for an object of $\mathbf{P}(X)$, construct in an explicit way the $\mathscr{D}$-module which corresponds to it by the Riemann-Hilbert correspondence. This is probably difficult because even for the irreducible perverse sheaves of intersection homology chains, the corresponding $\mathscr{D}$-module has only been constructed explicitly for strata whose closures are divisors with isolated singularities [Vi].

7. Extensions of Sheaves in the derived category

In this section we study the problem of extending objects of $\mathbf{D}^b_{\mathscr{S}}(X-S)$ to $\mathbf{D}^b_{\mathscr{S}}(X)$. The results obtained in this section are a generalization of the results in Sect. 2. In particular we see that Theorem 2.1 is valid under the assumption that $\pi_1(S) = \pi_2(S) = 0$, which implies that Theorem 3.3 is true under the same hypothesis.

Let S be a closed stratum of dimension $2d$ of stratification $\mathscr{S}$ of a stratified topological pseudo-manifold X [GM 2]. We fix an element $\mathbf{P}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(X-S)$ and we want to describe all extensions $\mathbf{Q}^\bullet$ of $\mathbf{P}^\bullet$, i.e. all elements $\mathbf{Q}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(X)$ such that $\mathbf{Q}^\bullet \mid X-S = \mathbf{P}^\bullet$. Let $i\colon S \hookrightarrow X$ and $j\colon X-S \hookrightarrow X$ denote the inclusions.

Definition. A complex of sheaves $\mathbf{Q}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(X)$ belongs to $\mathbf{D}^b_{\mathscr{S}}(X, S, r)$ if

i) $\mathbf{H}^k(i^*\mathbf{Q}^\bullet) = 0 \quad$ for $k \geqq -d+r$

ii) $\mathbf{H}^k(i^!\mathbf{Q}^\bullet) = 0 \quad$ for $k \leqq -d-r$.

Remark. If $r=1$ and $\mathbf{Q}^\bullet \mid X-S$ is perverse then i) and ii) are the conditions for $\mathbf{Q}^\bullet$ to be perverse. If $r=0$ and $\mathbf{Q}^\bullet \mid X-S$ is an intersection homology sheaf then i) and ii) are the condition for $\mathbf{Q}^\bullet$ to be an intersection homology sheaf ([GM 2]).

We define a category $\mathbf{Z}(X,S)$ as follows: An object of $\mathbf{Z}(X,S)$ is an object $\mathbf{P}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(X-S)$ together with a long exact sequence of local systems on S

$$\ldots \to \mathbf{H}^q(i^*Rj_*\mathbf{P}^\bullet) \to K^{q+1} \to C^{q+1} \to \mathbf{H}^{q+1}(i^*Rj_*\mathbf{P}^\bullet) \to \ldots$$

A morphism in $\mathbf{Z}(X,S)$ is a morphism $p\colon \mathbf{P}^\bullet \to \tilde{\mathbf{P}}^\bullet$ together with a commutative diagram

$$\begin{array}{ccccccccc} \ldots \to & \mathbf{H}^q(i^*Rj_*\mathbf{P}^\bullet) & \to & K^{q+1} & \to & C^{q+1} & \to & \mathbf{H}^{q+1}(i^*Rj_*\mathbf{P}^\bullet) & \to \ldots \\ & \downarrow{\scriptstyle p_*} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p_*} & \\ \ldots \to & \mathbf{H}^q(i^*Rj_*\bar{\mathbf{P}}^\bullet) & \to & \bar{K}^{q+1} & \to & \bar{C}^{q+1} & \to & \mathbf{H}^{q+1}(i^*Rj_*\bar{\mathbf{P}}^\bullet) & \to \ldots \end{array}$$

We denote by $\mathbf{Z}(X,S,r)$ the full subcategory of $\mathbf{Z}(X,S)$, where $C^q=0$ for $q \geqq -d+r$ and $K^q=0$ for $q \leqq -d-r$.

We define a functor $\mathscr{Z}\colon \mathbf{D}^b_{\mathscr{S}}(X) \to \mathbf{Z}(X,S)$ by sending an object $\dot{\mathbf{Q}}^\bullet$ of $\mathbf{D}^b_{\mathscr{S}}(X)$ to $j^*\mathbf{Q}^\bullet$ together with the long exact sequence associated to the triangle

$$\begin{array}{ccc} & & i^*Rj_*j^*\mathbf{Q}^\bullet \\ & \nearrow & \\ i^*\mathbf{Q}^\bullet & & \downarrow{\scriptstyle (1)} \\ & \nwarrow & \\ & & i^!\mathbf{Q}^\bullet \end{array}$$

By definition $\mathscr{Z}\colon \mathbf{D}^b_{\mathscr{S}}(X,S,r) \to \mathbf{Z}(X,S,r)$.

Theorem 7.1. *If $\pi_1(S) = \ldots = \pi_{2r}(S) = 0$ then the functor $\mathscr{Z}$ is a bijection on the isomorphism classes of objects.*

Remark. If $r=0$ this theorem gives the existence and uniqueness of the intersection homology extension of [GM2]. If $r=1$ we are in the situation of perverse objects and this theorem gives a generalization of Theorem 2.1 i) to the case that $\pi_1(S)=\pi_2(S)=0$. Theorem 2.1 ii) can also be generalized to the situation of Theorem 7.1. The proof of this theorem is essentially the same as the proof of Theorem 2.1.

We will first prove a sequence of lemmas.

Lemma 7.2. *If $\pi_1(S)=\pi_2(S)= \ldots = \pi_k(S)=0$ and $\mathbf{A}^\bullet$ is a complex of sheaves on S such that $\mathbf{H}^i(\mathbf{A}^\bullet)$ is (locally) constant and $\mathbf{H}^i(\mathbf{A}^\bullet)=0$ for $i<0$ and $i \geqq k$ then $\mathbf{A}^\bullet \cong \mathbf{H}^i(\mathbf{A}^\bullet)[-i]$ in $\mathbf{D}(S)$.*

Proof. The same as the proof of Lemma 2.4.

Lemma 7.3. *Let $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(X)$ and assume that $\mathbf{H}^k(\mathbf{A}^\bullet)=0$ for $k>0$ and $\mathbf{H}^k(\mathbf{B}^\bullet)=0$ for $k \leqq 0$ then $\mathrm{Ext}^p(\mathbf{A}^\bullet, \mathbf{B}^\bullet)=0$ for $p \leqq 0$.*

Proof. We have $\mathrm{Ext}^p(\mathbf{A}^\bullet,\mathbf{B}^\bullet) = \mathrm{Hom}_{K(X)}(\mathbf{A}^\bullet, \mathbf{I}^\bullet[p])$ where $\mathbf{I}^\bullet$ is an injective resolution of $\mathbf{B}^\bullet$ s.t. $\mathbf{I}^k=0$ for $k \leqq 0$ and $\mathbf{K}(X)$ is the homotopy category of chain complexes. Clearly $\mathrm{Hom}_{\mathbf{K}(X)}(\mathbf{A}^\bullet, \mathbf{I}^\bullet[p])=0$ for $p \leqq 0$.

Remark. The validity of the previous lemma does not depend on the existence of injectives. As easy a proof could be given by using spectral sequences.

Lemma 7.4. *Assume that $\pi_1(S)= \ldots = \pi_k(S)=0$ and let $\mathbf{A}^\bullet, \mathbf{B}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(S)$ be such that $\mathbf{H}^q(\mathbf{A}^\bullet)=0$ for $q \geqq k$ and $\mathbf{H}^q(\mathbf{B}^\bullet)=0$ for $q<0$. Then* $\mathrm{Hom}(\mathbf{A}^\bullet,\mathbf{B}^\bullet) = \bigoplus_{q=0}^{k-1} \mathrm{Hom}(\mathbf{H}^q(\mathbf{A}^\bullet), \mathbf{H}^q(\mathbf{B}^\bullet))$.

Proof. Consider the triangle

$$\begin{array}{ccc} & & \tau_{\geqq 0}\mathbf{A}^\bullet \\ & \nearrow & \\ \mathbf{A}^\bullet & & \downarrow{\scriptstyle (1)} \\ & \nwarrow & \\ & & \tau_{<0}\mathbf{A}^\bullet \end{array}.$$

By applying $R\mathrm{Hom}(-,\mathbf{B}^\bullet)$ to this triangle we get the long exact sequence

$$\to \mathrm{Ext}^{-1}(\tau_{<0}\mathbf{A}^\bullet, \mathbf{B}^\bullet) \to \mathrm{Hom}(\tau_{\geqq 0}\mathbf{A}^\bullet, \mathbf{B}^\bullet) \to \mathrm{Hom}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \to \mathrm{Hom}(\tau_{<0}\mathbf{A}^\bullet, \mathbf{B}^\bullet) \to \ldots.$$

By Lemma 7.3 $\mathrm{Ext}^{-1}(\tau_{<0}\mathbf{A}^\bullet, \mathbf{B}^\bullet) = \mathrm{Hom}(\tau_{<0}\mathbf{A}^\bullet, \mathbf{B}^\bullet) = 0$ and therefore $\mathrm{Hom}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \cong \mathrm{Hom}(\tau_{\geqq 0}\mathbf{A}^\bullet, \mathbf{B}^\bullet)$. By a similar argument we get that

$$\mathrm{Hom}(\tau_{\geqq 0}\mathbf{A}^\bullet, \mathbf{B}^\bullet) \cong \mathrm{Hom}(\tau_{\geqq 0}\mathbf{A}^\bullet, \tau_{\leqq k-1}\mathbf{B}^\bullet).$$

By Lemma 7.2 $\tau_{\geqq 0}\mathbf{A}^\bullet \cong \bigoplus_{i=0}^{k-1} \mathbf{H}^i(\mathbf{A}^\bullet)[-i]$ and $\tau_{\leqq k-1}\mathbf{B}^\bullet \cong \bigoplus_{i=0}^{k-1} \mathbf{H}^i(\mathbf{B}^\bullet)[-i]$. Therefore we have

$$\mathrm{Hom}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) = \bigoplus_{j=0}^{k-1} \bigoplus_{i=0}^{k-1-j} \mathrm{Ext}^j(\mathbf{H}^{i+j}(\mathbf{A}^\bullet), \mathbf{H}^i(\mathbf{B}^\bullet)),$$

But for $0<j<k$ $\mathrm{Ext}^j(\mathbf{H}^{i+j}(\mathbf{A}^\bullet), \mathbf{H}^i(\mathbf{B}^\bullet)) \cong \mathbf{H}^j(\mathbf{H}^{i+j}(\mathbf{A}^\bullet)^* \otimes \mathbf{H}^i(\mathbf{B}^\bullet)) = 0$ and the conclusion of the lemma follows.

Lemma 7.5. *There is a unique functor* $j_*^r : \mathbf{D}^b_{\mathscr{S}}(X-S) \to \mathbf{D}^b_{\mathscr{S}}(X,S,r)$ *such that for any* $\mathbf{Q}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(X,S,r)$ *there is a unique morphism* $\mathbf{Q}^\bullet \to j_*^r j^* \mathbf{Q}^\bullet$ *restricting to the identity map on* $X-S$.

Proof. The uniqueness of this functor is clear because it satisfies a universal property. To prove the existence take

$$j_*^r \mathbf{Q}^\bullet = \tau^s_{<-d+r} Rj_* \mathbf{Q}^\bullet .$$

Applying the functor $R\mathrm{Hom}(\mathbf{Q}^\bullet, -)$ to the triangle

$$\begin{array}{ccc} & & i_*\tau_{\geqq r-d} i^* Rj_* j^* \mathbf{Q}^\bullet \\ & \nearrow & \\ Rj_* j^* \mathbf{Q}^\bullet & & \Big\downarrow {\scriptstyle (1)} \\ & \nwarrow & \\ & & j_*^r j^* \mathbf{Q}^\bullet \end{array}$$

and applying the adjunction formulas we get the following long exact sequence

$$\ldots \to \mathrm{Ext}^{-1}(i^*\mathbf{Q}^\bullet, \tau_{\geqq r-d} i^* Rj_* j^* \mathbf{Q}^\bullet) \to \mathrm{Hom}(\mathbf{Q}^\bullet, j_*^r j^* \mathbf{Q}^\bullet) \to \mathrm{Hom}(j^*\mathbf{Q}^\bullet, j^*\mathbf{Q}^\bullet) \to$$
$$\to \mathrm{Hom}(i^*\mathbf{Q}^\bullet, \tau_{\geqq r-d} i^* Rj_* j^* \mathbf{Q}^\bullet) \to \ldots .$$

By Lemma 7.3 the first and the fourth terms vanish and we get that

$$\mathrm{Hom}(\mathbf{Q}^\bullet, j_*^r j^* \mathbf{Q}^\bullet) \cong \mathrm{Hom}(j^*\mathbf{Q}^\bullet, j^*\mathbf{Q}^\bullet)$$

which proves the existence and the uniqueness of a morphism $\mathbf{Q}^\bullet \to j_*^r j^* \mathbf{Q}^\bullet$ extending the identity on $X-S$.

Proof of Theorem 7.1. Let's fix $\mathbf{P}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(X-S)$ and let $\mathbf{Q}^\bullet \in \mathbf{D}^b_{\mathscr{S}}(X,S,r)$ such that $j^*\mathbf{Q}^\bullet = \mathbf{P}^\bullet$. Consider the triangle

$$\begin{array}{ccc} & & j_*^r \mathbf{P}^\bullet \\ & \nearrow & \\ \mathbf{Q}^\bullet & & \Big\downarrow {\scriptstyle (1)} \\ & \nwarrow & \\ & & i_* \mathbf{N}^\bullet \end{array}$$

where $\mathbf{N}^\bullet$ is a complex supported on S satisfying $\mathbf{H}^k(\mathbf{N}^\bullet) = 0$ for $k \leqq -d-r$ and $k > -d+r$. The first vanishing result follows because $\tau_{\leqq r-d}\mathbf{N}^\bullet = \tau_{\leqq r-d} i^! \mathbf{Q}^\bullet$ and the second one is obvious from the long exact sequence of the above triangle.

Because the morphism $\mathbf{Q}^\bullet \to j^r_* \mathbf{P}^\bullet$ is canonical the isomorphic classes of the extensions of $\mathbf{P}^\bullet$ correspond to a choice of $\mathbf{N}^\bullet$ and an element in $\mathrm{Ext}^1(j^r_* \mathbf{P}^\bullet, i_* \mathbf{N}^\bullet) = \mathrm{Ext}^1(i^{*r} j_* \mathbf{P}^\bullet, \mathbf{N}^\bullet)$. By Lemma 7.2 $\mathbf{N}^\bullet \cong \bigoplus_{k=-d-r+1}^{r-d} \mathbf{H}^k(\mathbf{N}^\bullet)[-k]$ and by Lemma 7.5

$$\mathrm{Ext}^1(i^* j^r_* \mathbf{P}^\bullet, \mathbf{N}^\bullet) = \bigoplus_{k=-d-r+1}^{r-d} \mathrm{Hom}(\mathbf{H}^{k-1}(i^* j_* \mathbf{P}^\bullet), \mathbf{H}^k(\mathbf{N}^\bullet)).$$

The above data corresponds up to an isomorphism to a unique element in $\mathbf{Z}(X, S, r)$.

Appendix

Simplicial Intersection Homology
(Mark Goresky and Robert MacPherson)

Let X be a stratified pseudomanifold with a fixed triangulation T, let X_k denote the union of its strata of codimension k, and let $\bar{p} = (p_1, p_2, \ldots)$ be a perversity. We assume that the triangulation and the stratification are compatible, i.e. that the closures of the X_k are subcomplexes of T. If i is an integer, a subspace $Y \subset X$ is called $(\bar{p}, i)$-allowable if $\dim(Y \cap X_k) \leqq i - k + p_k$ for all $k \geqq 2$. An i-chain ξ in X is called $\bar{p}$-allowable if $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$-allowable. (Here $|\ |$ denotes the support of a chain.) The intersection homology groups $\mathbf{IH}^{\bar{p}}_*(X)$ were originally defined to be the homology groups of the complex $\mathbf{IC}^{\bar{p}}_*(X)$ of $\bar{p}$-allowable piecewise linear geometric chains in X.

Proposition. *The intersection homology groups $\mathbf{IH}^{\bar{p}}_*(X)$ are the homology groups of the chain complex $\mathbf{IS}^{\bar{p}}_*(X)$ of $\bar{p}$-allowable chains on X that are simplicial with respect to T', the barycentric subdivision of T. In fact the inclusion of chain complexes*

$$i\colon \mathbf{IS}^{\bar{p}}_*(X) \hookrightarrow \mathbf{IC}^{\bar{p}}_*(X)$$

induces an isomorphism on homology groups.

Remark 1. Suppose that X is n dimensional and that L is a local system on $X - X_2$. Then the result holds for intersection homology with coefficients in L. Note that the chain complex $\mathbf{IS}^{\bar{p}}_*(X)$ makes sense in this case because by the $\bar{p}$-allowability conditions, all simplices with nonzero coefficients have their interiors in $X - X_2$.

Remark 2. The result is not true without taking the barycentric subdivision of T. (A counterexample is provided by an n-manifold stratified by its skeleta of dimension n, $n-2$, $n-3$, ..., $1, 0$.) The minimal condition needed on a triangulation T' for the proposition to work is that it should be flaglike with respect to the stratification: for any simplex Δ, the intersection of Δ with the closure of an X_k is a single face of Δ.

Remark 3. Let $R^{\bar{p}}_i$ be the set of all i-simplices of T' which are $(\bar{p}, i)$-allowable. It follows from the above proposition that the $R^{\bar{p}}_i$ are basic sets, i.e. that

$$\mathbf{IH}_i^{\bar{p}}(X) = \operatorname{Image}(H_i(R_i^{\bar{p}}) \to H_i(R_{i+1}^{\bar{p}}))$$

This was stated without proof in [GM1], Sect. 3.4.

Proof. We shall construct a homomorphism

$$\alpha_*: \mathbf{IC}_i^{\bar{p}}(S) \to \mathbf{IS}_i^{\bar{p}}(X)$$

such that

$$\alpha_* \partial = \partial \alpha_*$$
$$\alpha_* i = I$$

(where I is the identity) and for every P. L. geometric i-cycle ξ, we shall construct an $i+1$ chain β so that

$$\partial \beta = (-1)^i (\alpha_* i \xi - \xi)$$

The proposition then follows directly.

For any P. L. geometric chain $\xi \in \mathbf{IC}_i^{\bar{p}}(X)$, choose a triangulation of $|\xi|$ such that each simplex of $|\xi|$ is contained in a unique simplex of T'. Triangulate $|\xi| \times [0,1]$ as in [Gr] p. 46 with exactly two vertices $(v,0)$ and $(v,1)$ for each vertex v in $|\xi|$.

Now every vertex v in $|\xi|$ lies in the *interior* of a unique simplex $\sigma \in T'$. Say σ is spanned by the barycentres $\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_m$ where $\tau_0 < \tau_1 < \ldots < \tau_m$ are simplices of T. Define $\alpha(v) = \hat{\tau}_m$ and extend α linearly over the simplices of $|\xi|$. The chain $\alpha_*(\xi)$ is simplicial with respect to T', is $(\bar{p}, i)$-allowable, and is independent of the triangulation of $|\xi|$ which was chosen. Clearly $\alpha_* i = I$.

Similarly, define $f\colon |\xi| \times [0,1] \to X$ by setting $f(v,0) = v$, $f(v,1) = \alpha(v)$ for each vertex $v \in |\xi|$ and extending linearly over the simplices of $|\xi| \times [0,1]$. Orienting $|\xi| \times [0,1]$ with the product orientation defines the chain

$$f_*(\xi \times [0,1]) = \beta \in \mathbf{IC}_{i+1}^{\bar{p}}(X).$$

Clearly

$$\partial \beta = f_*(-1)^i (\xi \times \{1\} - \xi \times \{0\})) = (-1)^i (\alpha(\xi) - \xi)$$

as desired.

Lemma. *In the notation of Lemma 4.3, we have $R^i \pi_* \kappa_* \kappa^* \mathbf{IC}_L^{\bullet} = 0$ for $i \geqq -d$ and the map $R^i \pi_* \mathbf{IC}_L^{\bullet} \to R^i \pi_* \kappa_* \kappa^* \mathbf{IC}_L^{\bullet}$ is an isomorphism for $i \leqq -d-2$ and an inclusion for $i = -d-1$.*

Proof. Let s be the dimension of S (so that $\dim L = s + 2d - 1$). Using intersection homology-intersection cohomology duality and taking account of shifts in numbering, the lemma is equivalent to the statement that for all open $U \subset S$, $\mathbf{IH}_i^{\bar{m}}(\pi^{-1} U)_K = 0$ for $i \geqq s+d$ and $\mathbf{IH}_i^{\bar{m}}(\pi^{-1} U)_K$ is the homology of the complex of geometric intersection chains supported in $\pi^{-1} U \cap K$. But K is just $R_{s+d-1}^{\bar{l}}$ where $\bar{l}$ is the logarithmic perversity $\bar{l}(2k) = k$. The vanishing statement is true becaue $\dim K = s + d - 1$. The other statements result from the proof of the proposition since $R_i^{\bar{p}}$ is monotonic in i and $\bar{p}$.

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