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Inventiones mathematicae

Bounds on the number of non-rational subfields of a function field

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Introduction

Let K be a field, and let C be a smooth, geometrically connected, projective curve defined over K. A classical theorem of de Franchis [5] of 1913, stated in modern language, is the following.

Theorem of de Franchis. *There are only finitely many isomorphism classes of separable, non-constant morphisms $f: C \to C'$, where C' runs over all curves of genus $\geqq 2$.*

The basic idea in *all* the proofs of this theorem (cf. de Franchis [5], Severi [18] and [19], p. 271, Žižcenko [24], Samuel [17], Tamme [20]) is to associate to a given morphism $f: C \to C'$ the invariant

$$\gamma_f^* = f^* \circ f_*: J_C \xrightarrow{f_*} J_{C'} \xrightarrow{f^*} J_C$$

which is the composition of the *direct image map* f_* with the *inverse image map* f^*, both viewed as acting on the respective Jacobian varieties. (Alternately, one could study γ_f^* as a divisor (class) on $C \times C$ or as an endomorphism on the Tate module $T_l(J_C)$, etc.) It is immediate that $\gamma_f^* \in \mathrm{End}_K(J_C)$ depends only on the isomorphism class of f, i.e. on the subfield $E = f^* K(C')$ of F/K; we thus write $\gamma_E^* = \gamma_f^*$.

The de Franchis theorem is thus equivalent to the following two statements:

1) *The number of possible invariants $\gamma_E^* \in \mathrm{End}_K(J_C)$ is finite* (E as above).

2) *There are at most finitely many subfields E of F/K with the same invariant γ_E^*.*

Of these, the first statement can be deduced without much trouble from standard finiteness results in algebraic geometry such as Severi's *Theorem of the Base* or the theory of *Chow coordinates;* this had already been pointed out by Severi [18]. The second statement, on the other hand, is (a special case of) an old theorem of Humbert [8] and Castelnuovo [2] proved in 1893 by

transcendental methods. An algebraic proof of this fact was given by de Franchis [4] in 1903; it is reproduced in Severi [19], p. 284–5 and Samuel [17], p. 68ff.

The purpose of this paper is to present the following *angle theorem* from which, by a "packing argument",[1] a new and *effective* proof of the de Franchis theorem (and related theorems) may be obtained.

Theorem 1 ("Angle Theorem"). *Let E_1 and E_2 be subfields of F/K of genus $g_1 \geqq 1$ and $g_2 \geqq 1$, respectively, and let g_{12} denote the genus of the field $E_1 \cdot E_2$ generated by E_1 and E_2. Then, if $m_i = [E_1 \cdot E_2 : E_i]$ denotes the degree of $E_1 \cdot E_2$ over E_i $(i = 1, 2)$, we have:*

$$\cos(\gamma_{E_1}^*, \gamma_{E_2}^*) \leqq \frac{1}{\sqrt{g_1 g_2}} \left(\left(1 - \frac{1}{m_1}\right) \cdot \left(1 - \frac{1}{m_2}\right) + \frac{g_1}{m_2} + \frac{g_2}{m_1} - \frac{g_{12}}{m_1 m_2} \right), \tag{1}$$

where we view $\mathrm{End}_K(J_C) \otimes \mathbb{R}$ as a Euclidean space whose norm is given by the canonical trace form (associated to the theta-divisor) on $\mathrm{End}_K(J_C)$.

Actually, the above theorem is only a special case (Corollary 1) of a more precise theorem (Theorem 1') proved below in §2 which gives the *exact* value of $\cos(\gamma_{E_1}^*, \gamma_{E_2}^*)$.

A first consequence of the angle theorem is the following *rigidity theorem* which may be viewed as a sharpening of the aforementioned theorem of Castelnuovo and Humbert. For ease of language, let us call a subfield E of F/K *essential* if 1) its genus is positive and 2) it is not properly contained in a subfield of F/K on the same genus.

Theorem 2 ("Rigidity Theorem"). *If E_1 and E_2 are distinct essential subfields of F/K, then their invariants $\gamma_{E_1}^*$ and $\gamma_{E_2}^*$ are $\mathbb{Z}$-linearly independent.*

Remark 1. Strictly speaking, the rigidity theorem does not fully contain the Castelnuovo-Humbert theorem since its statement does not extend to all subfields of F/K of genus $\geqq 1$. However, it is quite easy to deduce the general case from the above theorem. Explicitly, one obtains (cf. §4):

1) *Each subfield E of F/K of genus $g_E \geqq 1$ is contained in a unique essential subfield E_{ess} of the same genus, and one has*

$$\gamma_E^* = n \gamma_{E_{\mathrm{ess}}}^*, \quad \text{where} \quad n = [E_{\mathrm{ess}} : E]. \tag{2}$$

2) *If $g_E \geqq 2$, then E_{ess} is purely inseparable over E, and E is the only subfield of F/K with invariant γ_E^*.*

3) *If $g_E = 1$, then, putting $n = [E_{\mathrm{ess}} : E]$, there are at most $\sigma(n) = \sum_{d \mid n} d$ subfields (all of genus 1) of F/K with the same invariant as E. Moreover, if K is algebraically closed and $\mathrm{char}(K) \nmid n$, then there are exactly $\sigma(n)$ such subfields.*

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[1] The idea of using a "packing argument" to obtain finiteness assertions in diophantine geometry is due to Mumford [12]. Recently, his idea was also taken up by Parshin in connection with Faltings' proof of the Mordell Conjecture. (Cf. L. Szpiro, Seminaire sur les pinceaux arithmétiques: La conjecture de Mordell. Astérisque **127** (1985), Exposé IX)

For the second application, we combine (à la Mumford) the angle theorem with an elementary packing lemma (cf. § 3) to obtain an explicit version of the de Franchis theorem. To state the result, let $g = g_F$ denote the genus of $F = K(C)$, and let $r = \operatorname{rank} \operatorname{End}_K(J_C)$. It is known that $r \leqq r(g)$, where $r(g) = 2g^2$ if $\operatorname{char}(K) = 0$, and $r(g) = 4g^2$, if $\operatorname{char}(K) \neq 0$.

Theorem 3. *If $g > g' \geqq 2$, then the number $N_F(g')$ of separable subfields of F/K of genus g' satisfies*

$$N_F(g') \leqq (c_1 + 1)^{r-1} - (c_1 - 1)^{r-1} \tag{3}$$

with $c_1 = 2\sqrt{\dfrac{(g-g')\,g'}{(g'-1)\,g}}$. In particular, $N_F(g') = 0$ if $r = 1$ and

$$N(g') < 2^{r-1}(2^{r-1} - 1), \quad \text{if } r > 1. \tag{4}$$

Corollary. *A function field*[2] *F/K has at most*

$$(g-1)\,2^{r(g)-2}(2^{r(g)-1} - 1) \tag{5}$$

separable subfields of genus $\geqq 2$.

Remark 2. For $g \geqq 3$ fixed, let

$$M(g) = \max_F N_F,$$

where the maximum extends over all function fields F/K (as above) of genus g, and N_F denotes the number of separable subfields of F/K of genus $\geqq 2$. By the corollary, $M(g) < \infty$ (this had been conjectured by Moh [11]) and satisfies

$$M(g) \leqq c^{g^2}, \tag{5'}$$

for some constant $c > 1$. It is an intriguing problem to determine the exact rate of growth of $M(g)$. One easily sees (cf. § 4) that

$$\sup_{g' \leqq g} M(g') \geqq c^{(\log(g))^2} \tag{6}$$

(provided that K is algebraically closed); in particular, it follows that $M(g)$ cannot be bounded by any polynomial in g.

Note. After submitting this paper for publication[3], I became aware of the paper of A. Howard and A. Sommese, On the theorem of de Franchis, *Ann. Scu. Norm. Sup. Pisa* **10**, 429–436 (1983). In that paper the authors show that the proof given in Samuel [17] (i.e. de Franchis' proof) can be modified so as to yield effective bounds. Explicitly, they obtain[4] (for $K = \mathbb{C}$)

$$N_I \leqq (2\sqrt{6(g-1)+1})^{2+2g^2}\, g^2 (g-1)(\sqrt{2})^{g(g-1)} + 1,$$

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[2] Throughout, "F/K is a function field" means that $F = K(C)$ is the function field of a smooth, geometrically connected, projective curve defined over K

[3] The main results of this paper (in slightly weaker form) were announced in *Mathematisches Forschungsinstitut Oberwolfach-Tagungsbericht* **35/81** (1981), p. 11

[4] It should be pointed out, however, that this is *not* the result they state. In fact, there is a minor error in their Lemma 2 since they disregard the fact that if $f_i: X \to Y_i$ $(i = 1, 2)$ are two morphisms which are isomorphic, then the associated divisors S_{f_i} $(= \Gamma_{f_i}^* - \Delta_X$ in the notation of § 1 below) are *equal*. Explicitly, the statement on line 12$^-$ ff. of p. 432 of their article is not correct, but has to be multiplied by $\#\operatorname{Aut}(Y_i) \leqq 84\left(\dfrac{g-1}{2}\right)$. (The authors count maps rather than isomorphism classes of maps)

which is not as good as (5). Their proof differs from the one presented here in several aspects. Firstly, they prove only a weak form of the rigidity theorem (and by a different method). Secondly, to prove that the number of γ_E^*'s is finite, they use (what amounts to) the fact that the length $\|\gamma_E^*\|$ is bounded and that the γ_E^*'s lie in a lattice; this is not used here. Finally, the angle theorem presented here accomplishes both steps with one stroke.

As a final application of the angle theorem, we give bounds for the "number" of subfields of F/K of genus 1:

Theorem 4. *The number $N_F'(n)_{\text{ess}}$ of essential subfields of F/K of genus 1 of index $\leqq n$ satisfies*

$$N_F'(n)_{\text{ess}} \leqq (c_2 n+1)^{r-1} - (c_2 n-1)^{r-1} \tag{7}$$

where $c_2 = \sqrt{\dfrac{2(g-1)}{g}}$; in particular, we have

$$N_F'(n)_{\text{ess}} \leqq 2^{3r/2-1} n^{r-2}. \tag{8}$$

Remark 3. It has been known for a long time that there exist function fields (e.g. the function field of the modular curve $X(11)$) with infinitely many essential subfields of genus 1. Poincaré [14] (cf. Lange [10] for a modern proof) gave the following characterization of this phenomenon:

A function field F/K has infinitely many essential subfields (of genus 1) if and only if it *has two isomorphic subfields of genus 1* not *contained in a common subfield of genus 1.*

The fact that there are only finitely many (essential) subfields of bounded index seems to have been first stated explicity by Tamme [20].

Corollary. *The number $N_F'(n)$ of all subfields of F/K of genus 1 and index $\leqq n$ satisfies:*

$$N_F'(n) \leqq 2^{3r/2-1} s_r(n) n^{r-2}, \tag{9}$$

where $s_r(n) = \sum_{k=1}^{n} \sigma(k) \cdot k^{2-r}$.[5]

The paper is arranged as follows. For technical reasons we first define in § 1 the invariant $\Gamma_f^* \in \mathrm{Div}(C \times C)$ and prove in § 2 the "angle theorem" for these invariants. In § 4 we explain the connection between these invariants and the invariants γ_f^* defined above, and prove the main theorems, using a packing lemma presented in § 3.

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[5] The constant $s_r(n)$ is easily estimated in terms of values of the Riemann ζ-function. By e.g. Hardy and Wright [7], p. 266 and Polya and Szegö [15], p. 127, one has:

$$s_2(n) < \frac{\zeta(2)}{2} n^2 + \frac{n}{2}(\log(n)+1)$$

$$s_3(n) < \zeta(2) n$$

$$s_4(n) < \zeta(2)(\log(n)+1)$$

$$s_r(n) < \zeta(r-2)\zeta(r-3), \quad \text{if } r \geqq 5$$

§ 1. The basic invariant

Let K be an arbitrary field, and C a smooth, geometrically connected, projective curve defined over K. Its function field $F=K(C)$ is then a finitely generated, regular extension of K of transcendence degree one with the property that its genus g is invariant under constant field extensions. Conversely, any field extension F/K with these properties is the function field of a smooth, geometrically connected, projective curve defined over K.

Suppose $f: C \to C'$ is a non-constant morphism to a normal curve C', where both C' and F are defined over K. Then C' is also a smooth, geometrically connected, projective curve[6], and f is surjective and hence induces an embedding $f^*: F' \hookrightarrow F$ of the function field $F'=K(C')$ of C' into that of C. Conversely, every subfield $E \subset F$ (with $E \supsetneq K$) arises in this fashion: given E, there exists a (smooth) curve C', unique up to isomorphism, such that $K(C') \simeq E$ and a surjective morphism $f: C \to C'$ such that the induced map $f^*: K(C') \hookrightarrow K(C)=F$ coincides with the inclusion $E \hookrightarrow F$. Note that once we have fixed the curve C' and an identification $K(C') \simeq E$, the morphism $f: C \to C'$ is uniquely determined by the inclusion $E \subset F$; in particular, we see that two morphisms $f_i: C \to C'_i$ $(i=1,2)$ determine the same subfield if and only if they are *isomorphic*, i.e. $f_2=\varphi \circ f_1$ for some isomorphism $\varphi: C'_1 \xrightarrow{\sim} C'_2$.

Following Castelnuovo [1], p. 11, and Samuel [17], p. 64, we attach to each surjective morphism $f: C \to C'$ the divisor

$$\Gamma_f^* = (f \times f)^*(\Delta_{C'}) \in \mathrm{Div}(C \times C) \tag{1}$$

on the product surface $C \times C$; here, $\Delta_{C'} \in \mathrm{Div}(C' \times C')$ denotes the diagonal divisor on $C' \times C'$. Note that we have

$$\Gamma_f^* = (\mathrm{id} \times f)^*(\Gamma_f), \tag{1'}$$

where $\Gamma_f \in \mathrm{Div}(C \times C')$ denotes the *graph* of the morphism f; this motivates the notation "Γ_f^*". We observe that the divisor Γ_f^* depends only on the subfield $E = f^* K(C') \subset F$, for if $\alpha \in \mathrm{Aut}(C')$ is an automorphism of C', then $\Gamma_{f\circ\alpha}^* = \Gamma_f^*$ and so we may write

$$\Gamma_E^* = \Gamma_f^*, \quad \text{if } E = f^* K(C'). \tag{2}$$

Remark 4. One can show, conversely, that the divisor Γ_E^* determines E (cf. Samuel [17], p. 72) but we do not need this fact. Classically, the divisors Γ_E^* (or its associated algebraic family $\{f^*(f_*(P))\}_{P \in C}$) were called *involutions*.

§ 2. The angle theorem

If C_1 and C_2 are two (smooth etc.) curves defined over a field K, then the divisor group $\mathrm{Div}(C_1 \times C_2)$ of the product surface $C_1 \times C_2$ is endowed with a

[6] To see that C' is smooth, note that f is flat and apply EGA IV. 17.7.7. (Alternately, one can easily see that the genus of C' is invariant under base change)

canonical bilinear form $\sigma=\sigma_{C_1\times C_2}$, called the *Severi-Weil Metric*, which is defined by

$$\sigma(D_1, D_2)=n(D_1)\cdot v(D_2)+n(D_2)\cdot v(D_1)-(D_1\cdot D_2), \tag{1}$$

where, as usual, $(\cdot)$ denotes the intersection number of two divisors and for $i=1, 2$

$$n(D_i)=(D_i\cdot A_1\times C_2)/\deg(A_1),$$
for any divisor $A_1\in\mathrm{Div}(C_1)$ with $\deg(A_1)\neq 0$

$$v(D_i)=(D_i\cdot C_1\times A_2)/\deg(A_2),$$
for any divisor $A_2\in\mathrm{Div}(C_2)$ with $\deg(A_2)\neq 0$.

In case that $D_1=D_2=D$ is effective, we can re-write Eq. (1) as

$$\sigma(D, D)=2((n(D)-1)\cdot(v(D)-1)+n(D)\,g_1+v(D)\,g_2-p_a(D)), \tag{2}$$

if g_1 resp. g_2 denotes the genus of C_1 resp. C_2 and $p_a(D)$ the *arithmetic genus* of D (cf. Hartshorne [6], p. 366); this follows immediately from the *adjunction formula* (cf. Hartshorne [6], p. 366) and the fact that $\omega_{C_1}\times C_2+C_1\times\omega_{C_2}$ is a canonical divisor on $C_1\times C_2$, if ω_{C_i} is a canonical divisor on C_i.

In particular, if $C_1=C_2=C$, and $D=\Delta_C$ is the diagonal divisor, then (2) gives

$$\sigma(\Delta_C, \Delta_C)=2g_C \tag{3}$$

because $n(\Delta_C)=v(\Delta_C)=1$ and $p_a(\Delta_C)=g_C$. Furthermore, if $f_i\colon C_i\to C_i'$ $(i=1, 2)$ are two surjective morphisms, then we have the *projection formula*

$$\sigma_{C_1\times C_2}((f_1\times f_2)^*D_1', D_2)=\sigma_{C_1'\times C_2'}(D_1', (f_1\times f_2)_*D_2) \tag{4}$$

for divisors $D_1'\in\mathrm{Div}(C_1'\times C_2')$ and $D_2\in\mathrm{Div}(C_1\times C_2)$; in particular, we have

$$\sigma_{C_1\times C_2}((f_1\times f_2)^*(D_1'), (f_1\times f_2)^*(D_2'))$$
$$=(\deg f_1)(\deg f_2)\,\sigma_{C_1'\times C_2'}(D_1', D_2') \tag{4'}$$

for divisors $D_1', D_2'\in\mathrm{Div}(C_1'\times C_2')$.

Let us now return to the situation of §1 and suppose that we have a surjective morphism $f\colon C\to C'$ of degree n to a curve C' of genus g'. We then obtain by (3) and (4'), and by (3) and (4), respectively:

$$\sigma_{C\times C}(\Gamma_f^*, \Gamma_f^*)=2n^2g', \tag{5}$$

$$\sigma_{C\times C}(\Delta_C, \Gamma_f^*)=2ng'. \tag{6}$$

Next, suppose that we have two surjective morphisms $f_i\colon C\to C_i$, where $\deg(f_i)=n_i$ and $g_{C_i}=g_i$ $(i=1, 2)$; we are interested in computing $\sigma_{C\times C}(\Gamma_{f_1}^*, \Gamma_{f_2}^*)$. For this, let $f_{12}=f_1\times f_2\circ\Delta_C\colon C\to C_1\times C_2$ denote the composition of $f_1\times f_2\colon C\times C\to C_1\times C_2$ with the diagonal morphism. Moreover, let $C_{12}=f_{12}(C)\subset C_1\times C_2$ denote the image scheme on $C_1\times C_2$ which is an (irreducible) projective curve (but which may have singularities), and let $p_a=p_a(C_{12})$ denote its arithmetic genus. Finally, let $n_{12}=\deg(f_{12})$ denote the degree of the (finite) morphism $f_{12}\colon C\to C_{12}$.

Theorem 1′ ("Angle Theorem"). *With the above notations we have:*

$$\sigma_{C\times C}(\Gamma_{f_1}^*, \Gamma_{f_2}^*) = n_{12}^2 \cdot \sigma_{C_1\times C_2}(C_{12}, C_{12}), \tag{7}$$

and hence

$$\tfrac{1}{2}\sigma(\Gamma_{f_1}^*, \Gamma_{f_2}^*) = (n_1 - n_{12})(n_2 - n_{12}) + n_1 n_{12} g_1 + n_2 n_{12} g_2 - n_{12}^2 p_a. \tag{8}$$

Note that since $n(C_{12}) = n_1/n_{12}$ and $v(C_{12}) = n_2/n_{12}$, formula (8) is an immediate consequence of (7) and (2).

Before presenting the proof of the angle theorem, let us deduce the following corollaries. Anticipating language that will be justified in § 4, let us put, for subfields E_1 and E_2 of F/K:

$$\cos(\Gamma_{E_1}^*, \Gamma_{E_2}^*) = \frac{\sigma(\Gamma_{E_1}^*, \Gamma_{E_2}^*)}{\sigma(\Gamma_{E_1}^*, \Gamma_{E_1}^*)^{1/2} \cdot \sigma(\Gamma_{E_2}^*, \Gamma_{E_2}^*)^{1/2}}, \tag{9}$$

which, by (5), is defined whenever E_1 and E_2 have positive genus.

Corollary 1. *If E_1 and E_2 are two subfields of F/K of positive genus g_1 and g_2, respectively, and if the compositum $E_1 \cdot E_2 \subset F$ has genus g_{12} and degree m_i over E_i $(i=1,2)$ then*

$$0 \leqq \cos(\Gamma_{E_1}^*, \Gamma_{E_2}^*) \leqq \left(1 + \frac{g_1 - 1}{m_2} + \frac{g_2 - 1}{m_1} - \frac{g_{12} - 1}{m_1 m_2}\right) \Big/ \sqrt{g_1 g_2}. \tag{10}$$

Proof. Put $E_i = f_i^* K(C_i)$; then $K(C_{12}) \simeq E_1 \cdot E_2$ and so we have $p_a(C_{12}) \geqq g_{12}$ (cf. Hartshorne [6], p. 272), from which, together with (5) and (8), the second inequality in (10) follows. (Note that $n_i = n_{12} m_i$, $i=1,2$). For the other inequality we combine (7) with Castelnuovo's inequality $\sigma(C_{12}, C_{12}) \geqq 0$ (cf. Hartshorne [6], p. 368, Castelnuovo [3], Weil [21] or [9]) to obtain the result.

Corollary 2. *If, in addition, $g_1 \leqq g_2$ and $E_1 \cdot E_2$ is not purely inseparable over E_2, then*

$$\cos(\Gamma_{E_1}^*, \Gamma_{E_2}^*) \leqq \frac{g_1 + 1}{2\sqrt{g_1 g_2}}. \tag{11}$$

Proof. Assume first that $E_1 \cdot E_2$ is separable over E_2 (and $E_1 \cdot E_2 \neq E_2$). Then by Riemann-Hurwitz, $g_{12} - 1 \geqq m_2(g_2 - 1)$ and so from (10) we obtain $\cos(\Gamma_{E_1}^*, \Gamma_{E_2}^*) \leqq \left(1 + \frac{g_1 - 1}{m_2}\right) \Big/ \sqrt{g_1 g_2}$. Since $m_2 \geqq 2$, (11) follows.

If $E_1 \cdot E_2$ is not separable over E_2, say $[E_1 \cdot E_2 : E_2]_{\text{ins}} = p^r$ (where $p = \text{char}(K) \neq 0$), put $E_1' = E_1^{p^r} \cdot K$. Then $\Gamma_{E_1'}^* = p^r \Gamma_{E_1}^*$ and so $\cos(\Gamma_{E_1'}^*, \Gamma_{E_2}^*) = \cos(\Gamma_{E_1}^*, \Gamma_{E_2}^*)$. Since $E_1' \cdot E_2$ is separable over E_2 and $E_1 \cdot E_2 \neq E_2$ by hypothesis, (11) follows as before.

Corollary 3. *If E_1 and E_2 are subfields of F/K of genus $g_1 = g_2 = 1$ and index n_1 and n_2, respectively, which are not contained in a common subfield of genus 1, then*

$$\cos(\Gamma_{E_1}^*, \Gamma_{E_2}^*) \leqq 1 - \frac{1}{n_1 n_2}. \tag{12}$$

Proof. By hypothesis, $g_{12} \geqq 2$ and so we obtain by (8)

$$\cos(\Gamma_{E_1}^*, \Gamma_{E_2}^*) \leqq 1 - \frac{1}{m_1 m_2} \leqq 1 - \frac{1}{n_1 n_2}.$$

The key point in the proof of the angle theorem is the following fact.

Proposition 1. *With the above notation, we have*

$$(\mathrm{id}_C \times f_2)_*(\Gamma_{f_1}^*) = n_{12}(f_1 \times \mathrm{id}_{C_2})^*(C_{12}). \tag{13}$$

Before proving this proposition, let us first see how the angle theorem follows from it.

Proof of Theorem 1'. As already remarked, it is enough to prove (7). For this we observe that

$$(f_1 \times \mathrm{id}_{C_2})_*(\Gamma_{f_2}) = n_{12}\, C_{12} \tag{14}$$

(because $(\mathrm{id}_C \times f_2)_*(\Delta_C) = \Gamma_{f_2}$ and hence $(f_1 \times \mathrm{id}_{C_2})_*(\Gamma_{f_2}) = (f_1 \times f_2)_*(\Delta_C) = n_{12}\, C_{12}$), and so by the projection formula and Proposition 1 we obtain

$$\begin{aligned} n_{12}^2\, \sigma_{C_1 \times C_2}(C_{12}, C_{12}) &= n_{12}\, \sigma_{C \times C_2}((f_1 \times \mathrm{id}_{C_2})^*(C_{12}), \Gamma_{f_2}) \\ &= \sigma_{C \times C_2}((\mathrm{id}_C \times f_2)_*(\Gamma_{f_1}^*), \Gamma_{f_2}) = \sigma_{C \times C}(\Gamma_{f_1}^*, \Gamma_{f_2}^*). \end{aligned}$$

Proof of Proposition 1. For brevity write $f = f_1$ and $g = f_2$. Then:

$$(\mathrm{id}_C \times g)_*(\Delta_C) \overset{(a)}{=} \Gamma_g \overset{(b)}{=} (g \times \mathrm{id}_{C_2})^*(\Delta_{C_2}). \tag{15}$$

Moreover, if $\Gamma_f' \subset C_1 \times C$ denotes the "dual graph" of f (i.e. $\mathrm{supp}(\Gamma_f') = \{(f(x), x) : x \in C\}$), then (cf. (14))

$$(\mathrm{id}_{C_1} \times g)_*(\Gamma_f') = n_{12}\, C_{12}, \tag{16}$$

$$(f \times \mathrm{id}_C)^*(\Gamma_f') = \Gamma_f^*. \tag{17}$$

If $Y_1, \ldots, Y_4$ are curves over K, we let

$$\mathrm{pr}_{ij}^{Y_1 \times Y_2 \times Y_3 \times Y_4} : Y_1 \times Y_2 \times Y_3 \times Y_4 \to Y_i \times Y_j$$

denote the projection on the (i, j)-th factor. Then

$$\Gamma_f^* = \mathrm{pr}_{14}^{C \times C_1 \times C \times C}{}_*((\Gamma_f \times \Delta_C) \cdot (\mathrm{pr}_{23}^{C \times C_1 \times C \times C})^*(\Gamma_f'))$$

(by (17) and Hartshorne [6], p. 426) and hence

$$\begin{aligned} &(\mathrm{id}_C \times g)_*(\Gamma_f^*) \\ &\quad = \mathrm{pr}_{14}^{C \times C_1 \times C \times C_2}{}_*((\mathrm{id}_C \times \mathrm{id}_{C_1} \times \mathrm{id}_C \times g)_*((\Gamma_f \times \Delta_C) \cdot (\mathrm{pr}_{23}^{C \times C_1 \times C \times C})^*(\Gamma_f'))) \\ &\quad \overset{(15a)}{=} \mathrm{pr}_{14}^{C \times C_1 \times C \times C_2}{}_*((\Gamma_f \times \Gamma_g) \cdot (\mathrm{pr}_{23}^{C \times C_1 \times C \times C_2})^*(\Gamma_f')) \\ &\quad = \mathrm{pr}_{14}^{C \times C_1 \times C_2 \times C_2}{}_*((\mathrm{id}_C \times \mathrm{id}_{C_1} \times g \times \mathrm{id}_{C_2})_*((\Gamma_f \times \Gamma_g) \cdot (\mathrm{pr}_{23}^{C \times C_1 \times C \times C_2})^*(\Gamma_f'))) \\ &\quad \overset{(15b)}{=} \mathrm{pr}_{14}^{C \times C_1 \times C_2 \times C_2}{}_*((\Gamma_f \times \Delta_{C_2}) \cdot (\mathrm{pr}_{23}^{C \times C_1 \times C_2 \times C_2})^*((\mathrm{id}_{C_1} \times g)_*(\Gamma_f'))) \\ &\quad \overset{(16)}{=} \mathrm{pr}_{14}^{C \times C_1 \times C_2 \times C_2}{}_*((\Gamma_f \times \Delta_{C_2}) \cdot (\mathrm{pr}_{23}^{C \times C_1 \times C_2 \times C_2})^*(n_{12}\, C_{12})) \\ &\quad = n_{12}(f \times \mathrm{id}_{C_2})^*(C_{12}), \end{aligned}$$

as claimed. (Note that above we have made repeated use of the projection formula; cf. Hartshorne [6], p. 426.)

Remark 5. Note that the formula (13) of Proposition 1 is equivalent to the formula

$$n_{12}(f_1 \times f_2)^*(C_{12}) = (\mathrm{id}_C \times f_2)^*(\mathrm{id}_C \times f_2)_*(\Gamma_{f_1}^*) \tag{18}$$

which may also be written in the form

$$n_{12}(f_1 \times f_2)^*(C_{12}) = \Gamma_{f_2}^* \circ \Gamma_{f_1}^*, \tag{18'}$$

where the symbol $\circ$ on the right denotes composition of correspondences (i.e. the divisor defined by the subscheme $\Gamma_{f_1}^* \times_C \Gamma_{f_2}^*$ of $C \times C$, where the fibre product is taken with respect to the morphisms $\mathrm{pr}_2 : \Gamma_{f_1}^* \to C$ and $\mathrm{pr}_1 : \Gamma_{f_2}^* \to C$). Now formula (18) may also be verified directly (by viewing both sides as correspondences); this therefore gives another proof of Proposition 1.[7]

§ 3. A packing lemma

Let V be a real vector space of dimension d with norm $\| \ \|$ (i.e. $(V, \| \ \|)$ is a euclidean space). The "packing lemma" in question is the following simple fact.

Lemma. *Suppose $v_1, \ldots, v_N$ is a finite sequence of vectors of V which are all of the same length $R > 0$ and which satisfy*

$$\|v_i - v_j\| \geqq 2r, \quad \text{for } i \neq j, \tag{1}$$

for some r with $0 < r \leqq R$. Then

$$N \leqq (R/r + 1)^d - (R/r - 1)^d. \tag{2}$$

Proof. Let $B_r(v_i) = \{v \in V : \|v - v_i\| < r\}$ denote the open ball of radius r centered at v_i. By the triangle inequality we obviously have

$$B_r(v_i) \subset B_{R+r}(0) \setminus B_{R-r}(0)$$

and by (1),

$$B_r(v_i) \cap B_r(v_j) = \emptyset, \quad \text{if } i \neq j.$$

Thus

$$\dot{\bigcup_{i=1}^{N}} B_r(v_i) \subset B_{R+r}(0) \setminus B_{R-r}(0),$$

and so, taking volumes, we obtain

$$N \cdot (c \cdot r^d) \leqq c \cdot (R+r)^d - c \cdot (R-r)^d$$

where $c = \mathrm{vol}(B_1(0))$.

For our purposes the following variant of the packing lemma will be useful.

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[7] In the case that $n_{12} = 1$ and f_1 is separable (which suffices for the proof of (7)), still another proof can be given by showing that $\mathrm{supp}((\mathrm{id}_C \times f_2)_*(\Gamma_{f_1}^*)) = \mathrm{supp}((f_1 \times \mathrm{id}_{C_2})^*(C_{12})$ and noticing that both divisors are reduced

Corollary. *Suppose a and b are real numbers with $b<1$ and $2(1-a^2)\geqq 1-b$, and let $v_0, v_1, \ldots, v_N$ be a finite sequence of non-zero vectors of V satisfying*[8]

$$\begin{aligned} &\text{a) } \cos(v_0, v_i)=a, \text{ for } 1\leqq i\leqq N \\ &\text{b) } \cos(v_i, v_j)\leqq b, \text{ for } 1\leqq i<j\leqq N. \end{aligned} \tag{3}$$

Then, putting $c=\sqrt{\dfrac{2(1-a^2)}{1-b}}$, *we have*

$$N\leqq (c+1)^{d-1}-(c-1)^{d-1}. \tag{4}$$

Proof. By replacing each v_i by $v_i/\|v_i\|$, we may (without loss of generality) assume that $\|v_i\|=1$, for $0\leqq i\leqq N$. Put $v_i'=v_i-av_0$. Then $(v_i', v_0)=0$ for $1\leqq i\leqq N$ and so $v_1', \ldots, v_N'$ all lie in the hyperplane $W=\langle v_0\rangle^{\perp}$. Moreover, by (3a) and (3b), we obtain $\|v_i'\|=\sqrt{1-a^2}$ for $1\leqq i\leqq N$ and $\|v_i'-v_j'\|\geqq\sqrt{2(1-b)}$, for $1\leqq i<j\leqq N$, and so, applying the packing lemma to W, $v_1', \ldots, v_N'$ with $R=\sqrt{1-a^2}$ and $r=\sqrt{(1-b)/2}$, we obtain the desired result.

Remark 6. Note that

$$\begin{aligned} M(c,d) &\overset{\text{def}}{=} (c+1)^{d-1}-(c-1)^{d-1} \\ &= 2\left[\binom{d-1}{1}c^{d-2}+\binom{d-1}{3}c^{d-4}+\ldots\right]; \end{aligned} \tag{5}$$

thus, for $c\geqq 1$ fixed (resp. for $d\geqq 1$ fixed), $M(c,d)$ is an increasing function of $d\geqq 1$ (resp. of $c\geqq 0$).

We also observe that we have the bound

$$M(c,d)\leqq 2^{d-1}c^{d-2}, \quad \text{if } c\geqq 1, \tag{6}$$

because $M(c,d)\leqq 2\left[\binom{d-1}{1}c^{d-2}+\binom{d-1}{3}c^{d-2}+\ldots\right]=2^{d-1}c^{d-2}$.

§ 4. Proof of the main theorems

As before, let $F=K(C)$ be the function field of a smooth, projective, geometrically connected curve C of genus g defined over a field K, and let $\sigma=\sigma_{C\times C}$ denote the Severi-Weil metric on $\mathrm{Div}(C\times C)$. By a theorem of Castelnuovo [3] (cf. also Weil [21] or [9]), σ is positive - definite and non-degenerate[9] on $M=\mathrm{Div}(C\times C)/V(C\times C)$, where $V(C\times C)$ denotes the subgroup of divisors "of

[8] Here, as usual, $\cos(v_i, v_j)$ is defined by

$$\cos(v_i, v_j)=\frac{(v_i, v_j)}{\|v_i\|\cdot\|v_j\|},$$

where $(\ ,\ )$ denotes the inner product associated to the norm $\|\ \|$

[9] Whereas the positive definiteness of σ is an easy consequence of the Hodge index theorem (cf. Hartshorne [6], p. 368), its non-degeneracy (on M) requires more careful analysis (cf. [9] for a discussion)

valence 0" (i.e. divisors $D \in \mathrm{Div}(C \times C)$ linearly equivalent to divisors of the form $A \times C + C \times B$, with $A, B \in \mathrm{Div}(C)$). By Severi's *Theorem of the Base*, M is a (free) $\mathbb{Z}$-module of finite rank; in fact, we have

$$r \overset{\text{def}}{=} \operatorname{rank} M = \rho - 2, \tag{1}$$

where $\rho = \operatorname{rank} NS(C \times C)$ is the *Picard number* of the surface $C \times C$. Thus, if we put $V = M \otimes_{\mathbb{Z}} \mathbb{R}$, then σ extends to a positive - definite, non-degenerate bilinear form on V and so V is a euclidean space (of dimension r), whose norm we denote by $\| \ \|$.

Remark 7. Note that the r defined above differs slightly from the one defined in the introduction. To establish the connection, recall that any divisor $D \in \mathrm{Div}(C \times C)$ defines a *correspondence* on C and hence a K-rational endomorphism $\alpha_D \in \mathrm{End}_K(J_C)$. Since by the "see-saw lemma", $\alpha_D = 0 \Leftrightarrow D \in V(C \times C)$, we obtain an injection[10] $\alpha: M \to \mathrm{End}_K J_C$ and so $r \leqq \operatorname{rank} \mathrm{End}_K(J_C) \leqq r(g)$, the latter inequality by Mumford [13], p. 178 (and p. 182, for $\mathrm{char}(K) = 0$). In particular, this shows that r (or ρ) is finite[11]. Moreover, it is known that the trace form on $\mathrm{End}(J_C)$ coincides, via α, with σ on M; cf. Weil [22].

Suppose now that E_1 and E_2 are subfields of F/K. If we denote by $\gamma^*_{E_i}$ the image of the divisor $\Gamma^*_{E_i}$ in M (or in $\mathrm{End}_K(J_C)$) then, by definition, the quantity $\cos(\Gamma^*_{E_1}, \Gamma^*_{E_2})$ defined by (9) of § 2 is nothing but the cosine of the angle between $\gamma^*_{E_1}$ and $\gamma^*_{E_2}$; this therefore justifies the earlier notation. Moreover, we see that Theorem 1 of the introduction (= § 0) is nothing but a re-statement of Corollary 1 of Theorem 1'.

Proof of Theorem 2. It is enough to show that $\cos(\gamma^*_{E_1}, \gamma^*_{E_2}) < 1$. Let g_i denote the genus of E_i, and assume $g_1 \leqq g_2$. Since E_2 is essential, $g_{E_1 \cdot E_2} > g_2$ and so $E_1 \cdot E_2$ is not purely inseparable over E_2. Thus, by Corollary 2 of the angle theorem we have $\cos(\gamma^*_{E_1}, \gamma^*_{E_2}) \leqq \dfrac{g_1 + 1}{2\sqrt{g_1 g_2}} < 1$ unless $g_1 = g_2 = 1$. In that case, however, we can apply Corollary 3 of the angle theorem to obtain $\cos(\gamma^*_{E_1}, \gamma^*_{E_2}) \leqq 1 - \dfrac{1}{n_1 n_2} < 1$.

Proof of Remark 1. Let E be a subfield of F/K of genus $g' \geqq 1$, and let $E' \supset E$ be a subfield of F/K with the same genus g'.

a) If $g' \geqq 2$, then by Riemann-Hurwitz, E' is purely inseparable over E and so E_{ess} is the maximal subfield of F which is purely inseparable over E (which is unique). Moreover, $\gamma^*_{E'} = [E' : E]\, \gamma^*_E$, so (2) of § 0 holds.

b) If $g' = 1$, then the separable closure E's of E in E' is unramified over E and so E_{ess} is the subfield of F generated by all subfields of genus 1 containing E (which is unique and of genus 1). To prove (2) of § 0, note that by formulae (5) and (6) of § 2 (applied to E_{ess} in place of F) we obtain

$$\cos(\gamma^*_{E_{\mathrm{ess}}}, \gamma^*_E) = \sqrt{\frac{g_E}{g_{E_{\mathrm{ess}}}}} = 1,$$

[10] If K is algebraically closed, then α is known to be a bijection (Weil [22] or [9])

[11] For other proofs, cf. discussion in Zariski [23], p. 122

i.e. $\gamma^*_{E_{\text{ess}}} = r\gamma^*_E$ for some $r \in \mathbb{R}$, $r \neq 0$. On the other hand, from (6) of §2 it follows that $r = [E_{\text{ess}}: E]$, and so (2) of §0 holds.

We have thus verified statement 1) of Remark 1. To verify statements 2) and 3), let $E_1 = E, \ldots, E_N$ be subfields with invariant γ^*_E. Then by the rigidity theorem and (2) of §0 we have that $(E_i)_{\text{ess}} = E_{\text{ess}}$ and $[E_{\text{ess}}: E_i] = n$, $1 \leqq i \leqq N$. Thus, if $g' \geqq 2$ then $N = 1$ since each E_{ess}/E_i is purely inseparable, and if $g' = 1$ then $N \leqq \sigma(n)$ by Proposition 2 below.

Proof of Theorem 3 and its Corollary. Fix $g' \geqq 2$, and let $E_1, \ldots, E_N$ be distinct separable subfields of F/K, each of genus g'. (We may assume that there exists at least one subfield of genus g', for else the inequalities (3) and (4) of Theorem 3 hold trivially.) Put

$$a = \sqrt{g'/g}, \qquad b = (g'+1)/(2g')$$

and

$$c = [(2(1-a^2))/(1-b)]^{1/2} = 2[((g-g')\,g')/(g(g'-1))]^{1/2}.$$

We observe:

$$\sqrt{2} \leqq c \leqq 2\sqrt{2} < 3. \tag{2}$$

(The upper bound is clear, for $\dfrac{(g-g')\,g'}{(g'-1)\,g} \leqq \dfrac{g'}{g'-1} \leqq 2$. For the lower bound, note that by Riemann-Hurwitz we have $g \geqq 2g'-1$ and so $\dfrac{(g-g')\,g'}{(g'-1)\,g} \geqq \dfrac{g'}{2g'-1} \geqq \dfrac{1}{2}$, as claimed.) Then, by (5) and (6) of §2 we have $\cos(\gamma^*_F, \gamma^*_{E_i}) = a$ and, by Corollary 1 of the angle theorem, $\cos(\gamma^*_{E_i}, \gamma^*_{E_j}) \leqq b$ for $i \neq j$. Thus, since $b < 1$ and $c \geqq 1$, we can apply the (variant of the) packing lemma to V and $v_0 = \gamma^*_F$, $v_1 = \gamma^*_{E_1}, \ldots, v_N = \gamma^*_{E_N}$ with the above values of a, b and c to obtain the desired inequality (3) of §0, from which (4) of §0 follows by (2) above. Finally, the Corollary of Theorem 3 is an immediate consequence of (4) of §0 and the fact that (by Riemann-Hurwitz) we have $2 \leqq g' \leqq \dfrac{g+1}{2}$.

Proof of Remark 2. To prove (6) of §0 it is enough to find a sequence $F_1, F_2, \ldots, F_n, \ldots$ of function fields over K of genus $g_{F_1} < g_{F_2} < \ldots < g_{F_n} < \ldots$, such that

$$N_{F_n} \geqq \exp(c(\log(g_{F_n}))^2) \tag{3}$$

for some constant $c > 0$ (independent of n).

To construct such a sequence, choose first a sequence $E_1, E_2, \ldots, E_n, \ldots$ of function fields $/K$ with genus $g_{E_n} = n$. Next, choose a prime $p \neq \text{char}(K)$ and let F_n be the maximal unramified extension of E_n which is abelian of exponent p. Then $\text{Gal}(F_n/E_n) \simeq (\mathbb{Z}/p\mathbb{Z})^{2n}$ and hence

$$g_{F_n} = p^{2n}(n-1) + 1 < p^{3n}. \tag{4}$$

By Redei [16], p. 367 the number of subgroups of order p^n of $(\mathbb{Z}/p\mathbb{Z})^{2n}$ is

$$\begin{bmatrix} 2n \\ n \end{bmatrix} = \frac{(p^{2n}-1)(p^{2n-1}-1)\ldots(p^{n+1}-1)}{(p-1)(p^2-1)\ldots(p^n-1)} \geqq \frac{p^{2n-1}\ldots p^n}{p\ldots p^n} = p^{n(n-1)}.$$

Thus, by galois theory we have for $n \geqq 2$:

$$N_{F_n} > p^{n(n-1)} \geqq p^{\frac{1}{2}n^2} \overset{(4)}{>} \exp((18\log(p))^{-1}(\log(g_{F_n}))^2).$$

Proof of Theorem 4. Fix $n \geqq 2$, and let $E_1, \ldots, E_N$ be distinct essential subfields of F/K of genus 1 with index $n_i = [F: E_i] \leqq n$, for $1 \leqq i \leqq N$. Then by Corollary 3 of the angle theorem we have

$$\cos(\gamma^*_{E_i}, \gamma^*_{E_j}) \leqq 1 - \frac{1}{n_i n_j} \leqq 1 - \frac{1}{n^2}, \quad \text{if } i \neq j.$$

Thus, if we put $a = \sqrt{1/g}$, $b = 1 - \frac{1}{n^2}$, then $b < 1$ and $c = n\sqrt{\frac{2(g-1)}{g}} > 1$, so we can apply the packing lemma to $v_0 = \gamma^*_F$, $v_1 = \gamma^*_{E_1}, \ldots, v_N = \gamma^*_{E_N}$ with these values of a and b to obtain the inequalities (7) and (8) of §0, the latter in view of inequality (6) of §3.

The proof of the Corollary to Theorem 4 depends on the following well-known fact:[12]

Proposition 2. *Let F/K be a function field of genus 1 over an algebraically closed field K, and let $n \geqq 2$ be an integer.*

a) *If $\mathrm{char}(K) \nmid n$, then F/K has exactly $\sigma(n) = \sum_{d \mid n} d$ (separable) subfields of genus 1 and index n.*

b) *If $\mathrm{char}(K) = p \neq 0$, and $p^r \| n$ with $r > 0$, then F/K has*

either: (i) *$(r+1)\,\sigma(n/p^r)$ subfields of genus 1 and index n, of which $\sigma(n/p^r)$ are separable (F/K is "ordinary")*

or: (ii) *$\sigma(n/p^r)$ subfields of genus 1 and index n, all of which are inseparable (F/K is "supersingular").*

Proof of the corollary to Theorem 4. Since each subfield of genus 1 is contained in a *unique* essential subfield (and since distinct subfields of F/K stay distinct after constant field extension) we obtain by Proposition 2:

$$N'_F(n) \leqq \sum_{k=1}^{n} \sigma(k)\, N'_F\left(\left[\frac{n}{k}\right]\right)_{\mathrm{ess}}$$

(with equality holding if K is algebraically closed and $\mathrm{char}(K) \nmid n$). From this and formula (8) of Theorem 4 the Corollary follows immediately.

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[12] There does not seem to exist a convenient reference. However, the number of *cyclic* subfields of F/K of genus 1 and index n can be found in any book on modular functions (since it equals the degree of the covering $j: X_0(n) \to X(1) = \mathbb{P}^1$), and the general case can be proven similarly (using the fact that if $m \mid n$ and $(m, n/m) = 1$ then $\mathbb{Z}/n \times \mathbb{Z}/m$ has exactly $\sigma(m)$ subgroups of order n)

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