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A symplectic fixed point theorem for $\mathbb{C}\mathbb{P}^n$

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Summary. Two symplectic diffeomorphisms ϕ_0, ϕ_1 of a symplectic manifold (X, ω) are said to be homologous if there exists a smooth homotopy ϕ_t, $t \in [0, 1]$ of symplectic diffeomorphisms between them such that the time-dependent vector field ξ_t defined by $d/dt(\phi_t) = \xi_t \circ \phi_t$ is a globally hamiltonian vector field for all t, i.e. there exists a smooth real-valued time-dependent hamiltonian function $h(x, t)$ on $X \times [0,1]$ such that $\xi_t \lrcorner \omega = dh_t$, where $h_t = h(x,t)$.

V.I. Arnold [Ar] conjectured that any symplectic diffeomorphism ϕ of a compact symplectic manifold X, homologous to the identity, has as many fixed-points as a function on X has critical points.

We prove Arnold's conjecture for complex projective spaces, with their standard symplectic structures, i.e. we prove that any symplectic diffeomorphism of $\mathbb{C}\mathbb{P}^n$ homologous to the identity has at least $n+1$ fixed-points.

Introduction

This paper consists of my Ph.D. thesis at the University of California at Berkeley, with Alan Weinstein as my advisor. A joint announcement of these results appears in [FW].

A time-dependent hamiltonian $h(x, t)$ on a symplectic manifold (X, ω) generates a time-dependent hamiltonian vector field ξ_t, defined by $\xi_t \lrcorner \omega = dh_t$, where $h_t(x)$ is the function on X obtained from $h(x, t)$ by fixing t. ξ_t in turn generates a "flow" ϕ_t on X. For fixed t, ϕ_t is a symplectic diffeomorphism of X (homologous to the identity) obtained by following solution curves of $dx/dt = \xi_t(x)$ (Hamilton's equations) for time t.

It follows from Liusternik-Schnirelman theory that a real-valued function f on a compact manifold X has at least $CL(X)+1$ critical points; $CL(X)$, the cup-length of X, is the largest integer k for which there is a non-zero cup-

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product $\alpha_1 \cup \ldots \cup \alpha_k$ of classes $\alpha_i \in H^*(X)$ having non-zero degree. Furthermore from Morse theory it follows that if all the critical points are non-degenerate, then there are at least $SB(X)$ of them, $SB(X)$ being the sum of the Betti numbers of X.

Let $CR(X)$ be the minimum number of critical points that a function on X must have, and $CRN(X)$ the minimum number if all are non-degenerate. Clearly

$$CR(X) \geqq CL(X)+1$$

and

$$CRN(X) \geqq SB(X).$$

During the 1960's V.I. Arnold ([Ar] p. 419) conjectured that for a compact symplectic manifold (X, ω) the symplectic diffeomorphism ϕ_1 induced from a time-dependent hamiltonian has at least as many fixed-points as a function on X has critical points, i.e. at least $CR(X)$ fixed-points in general and at least $CRN(X)$ if all are non-degenerate. x is a non-degenerate fixed-point of ϕ_1 if $T_x\phi_1$ does not have 1 as an eigenvalue.

No substantial progress was made in proving this conjecture until 1983 when Conley and Zehnder [CZ] proved the conjecture for the case when X is the even-dimensional torus, $\mathbf{T}^{2n}$, with the usual symplectic structure induced from $\mathbb{R}^{2n}$.

Shortly after that Alan Weinstein [W 1] extended the result to include all compact symplectic manifolds, provided that $h(x,t)$ is sufficiently C^1-small in x, uniformly for $t \in [0,1]$. $CR(X)$ and $CRN(X)$ were replaced by the weaker estimates $CL(X)+1$ and $SB(X)$ respectively. (For the torus $X=\mathbf{T}^{2n}$ there is no loss: $CRN(X)=CL(X)+1=2n+1$ and $CRN(X)=SB(X)=2^{2n}$.)

We prove Arnold's conjecture for complex projective spaces $X=\mathbb{CP}^n$, with their standard symplectic structure:

Theorem. *Let $h: \mathbb{CP}^n \times I \to \mathbb{R}$ be a C^1 time-dependent hamiltonian on $\mathbb{CP}^n$. Then the time one map ϕ_1 of the hamiltonian vector field X_{h_t} has at least $n+1$ fixed-points.*

For $n=1$ (i.e. S^2) the result holds for every volume element (N. Nikoshin and C.P. Simon). For $\mathbb{CP}^n$, once again $CRN(X)=SB(X)$ and $CRN(X) = CL(X)+1$, but here, unlike the torus all four invariants are equal, to $n+1$.

Actually a fifth invariant, the Euler characteristic $\chi(X)$, also plays a role. A continuous map, homotopic to the identity, from a compact manifold to itself has, by the Lefshetz fixed-point theorem, at least $\chi(X)$ fixed-points, counted with multiplicities. The Lefshetz fixed-point theorem yields no information for the torus since $\chi(\mathbf{T}^{2n})=0$, so Arnold's conjecture is a lot stronger in that case. For $\mathbb{CP}^n$ the Euler characteristic is also $n+1$ so that the non-degenerate part of Arnold's conjecture reduces to the Lefshetz fixed-point theorem.

There are other characterizations of the symplectic diffeomorphisms occurring in Arnold's conjecture. Firstly these are those symplectic diffeomorphisms which are homologous to the identity (see the summary for the definition of homologous). Another is that they belong to the commutator subgroup of the identity component of the group $\mathrm{Diff}(X, \omega)$ [Ba]. Another is that they are precisely those for which the Calabi invariant vanishes [Ca, Ba].

The proof of the conjecture for the torus in [CZ] identified the fixed points of ϕ_1 with the critical points of a function on the space of contractible loops of the torus. $\mathbb{CP}^n$ is more complicated since the loop space of $\mathbb{CP}^n$ is not simply-connected and the resultant function is no longer single-valued. Our proof uses the fact that $\mathbb{CP}^n$ is the reduced symplectic manifold [AM] of $\mathbb{C}^{n+1}$ under the Hopf S^1-action, in order to identify fixed-points of ϕ_1 with critical orbits of a function f, invariant under a natural S^1-action on a submanifold of the loop space of $\mathbb{C}^{n+1}$.

Whe then use a recently refined minimax theory based on the notation of a relative index [BLMR] to prove the existence of countably many critical values of f. However it turns out that each fixed-point of ϕ_1 gives rise to a sequence $\{b | b \equiv b_0 (\mathrm{mod}\, 2\pi)\}$ of critical values of f, so that the existence of countably many critical values is not sufficient for our purposes. To overcome this obstacle we compare the function f with the function that arises from a constant hamiltonian function and show that the critical values of the two functions are close. A combinatorial argument then shows that either two critical values are the same, in which case ϕ_1 has uncountably many fixed-points or else the critical values belong to at least $n+1$ different sequences, which proves Arnold's conjecture.

We denote the unit interval $[0, 1]$ by I.

Section 1. Lifting to C^{n+1}

We begin with $\mathbb{C}^{n+1}$ with its natural symplectic structure given by $\omega(v, w) = -\mathrm{Im}(v, w)$ where (,) is the canonical hermitian product on $\mathbb{C}^{n+1}$. The Hopf $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ action on $\mathbb{C}^{n+1}$ is given by

$$S_\mu(z_1, \ldots, z_{n+1}) = \exp(i\mu)(z_1, \ldots, z_{n+1}). \tag{1.1}$$

This is a Poisson action in the sense of [Ar], generated by the hamiltonian

$$K(z_1, \ldots, z_{n+1}) = \tfrac{1}{2}(|z_1|^2 + \ldots + |z_{n+1}|^2). \tag{1.2}$$

The reduced manifold [Ar] is $K^{-1}(\frac{1}{2})/S^1 = S^{2n+1}/S^1$ which is isomorphic to $\mathbb{CP}^n$ with a multiple of its standard symplectic structure. Denote the projection from S^{2n+1} to $S^{2n+1}/S^1 = \mathbb{CP}^n$ by π.

Any S^1-invariant hamiltonian H on $\mathbb{C}^{n+1}$ induces an invariant hamiltonian vector field on $\mathbb{C}^{n+1}$, which restricts to an invariant vector field on S^{2n+1} (an invariant manifold) which in turn projects to a vector field on $\mathbb{CP}^n$ called the reduced vector field. By the theory of reduction, the reduced vector field is also a hamiltonian vector field for the hamiltonian h on $\mathbb{CP}^n$, which is uniquely defined by $h \circ \pi = H|_{S^{2n+1}}$. If H is time-dependent then h will be time-dependent as well. We shall use this procedure in reverse.

We are given a smooth time-dependent hamiltonian $h(x, t)$ on $\mathbb{CP}^n$, x being the local co-ordinates on $\mathbb{CP}^n$. Without loss we may assume that h is always positive since the hamiltonian vector field is unchanged when a constant is added to h. Define $H(z, t)$ to be the unique time-dependent hamiltonian on

$\mathbb{C}^{n+1}$ which for fixed t is positive homogeneous of degree two and whose restriction to S^{2n+1} is $h_t \circ \pi$. Clearly H is also positive. H_t is then invariant under the Hopf action generated by X_K hence by Noether's theorem K is a constant of the motion of X_{H_t} so that $K^{-1}(\frac{1}{2}) = S^{2n+1}$ is an invariant manifold for X_{H_t} and furthermore from the remarks above we know that the orbits of X_{H_t} restricted to S^{2n+1} project to orbits of X_{h_t} on $\mathbb{CP}^n$. All orbits of X_{h_t} are obtained in this way. If $\sigma^\dagger$ is an orbit of X_{h_t} on $\mathbb{CP}^n$, then $\sigma^\dagger = \pi \circ \sigma$ for some orbit σ on S^{2n+1}. Then $\sigma^\dagger(0) = \sigma^\dagger(1)$ if and only if $\sigma(0) = S_\mu \sigma(1)$ for some $\mu \in \mathbb{R}/2\pi\mathbb{Z}$.

Consider now the time-dependent vector field $X_{H_t} + \lambda X_K = X_{(H_t + \lambda K)}$ for a suitable $\lambda \in \mathbb{R}$. The reduced vector field for $(X_{H_t} + \lambda X_K)$ is also X_{h_t} since X_K projects to the zero vector field on $\mathbb{CP}^n$. Since H_t is invariant under the S^1-action induced by X_K, the Poisson bracket $\{H_t, K\}$ is zero. Thus $[X_{H_t}, X_K] = X_{\{H_t, K\}} = 0$; i.e. X_K and X_{H_t} commute so we can insure that $\sigma(0) = \sigma(1)$ if we replace H_t by $H_t + \lambda K$ where $\lambda \equiv \mu (\mathrm{mod}\, 2\pi)$.

Thus to each closed integral curve $\sigma^\dagger$ of X_{h_t} on $\mathbb{CP}^n$ there corresponds a family of pairs (σ, λ) where $\lambda \equiv \mu (\mathrm{mod}\, 2\pi)$ and σ is a closed integral curve of $X_{H_t} + \lambda X_K$. If (σ, λ) belongs to the family then so does $(S_\mu \circ \sigma, \lambda)$ for any $\mu \in \mathbb{R}/2\pi\mathbb{Z}$, so that the family is diffeomorphic to $S^1 \times (2\pi\mathbb{Z})$.

To prove the main result we need to show that there are always at least $(n+1)$ distinct such families.

Section 2. Application of Hamilton's variational principle

The proof of our result is based on a variant of Hamilton's principle. The function space we work in, denoted by $H^{\frac{1}{2}}$, is the Hilbert space of loops of Sobolev class $H^{\frac{1}{2}}$ in $\mathbb{C}^{n+1}$. We begin with the space P of C^∞ loops in $\mathbb{C}^{n+1}$, consisting of C^∞ maps $u: I \to \mathbb{C}^{n+1}$ with $u(0) = u(1)$. Such loops can be expanded as Fourier series:

$$u(t) = \sum_{k \in \mathbb{Z}} u_k \exp(2\pi i k t), \qquad \text{where } u_k \in \mathbb{C}^{n+1}.$$

Let $v(t)$ be another loop with Fourier coefficients v_k. The L^2 inner product is given by

$$(u, v) = \sum_{k \in \mathbb{Z}} (u_k, v_k) \tag{2.1}$$

where (u_k, v_k) is the hermitian inner product in $\mathbb{C}^{n+1}$.

The L^2 norm is denoted by $|.|$. The $H^{\frac{1}{2}}$ inner product is given by

$$\langle u, v \rangle = \sum_{k \in \mathbb{Z}} |k| (u_k, v_k) + (u_0, v_0). \tag{2.2}$$

The corresponding $H^{\frac{1}{2}}$ norm will be denoted by $\|.\|$. The space $H^{\frac{1}{2}}$ is the completion of P under the $H^{\frac{1}{2}}$ norm. Similarly for L^2. Gradients ∇ will always be taken with respect to $\langle\ ,\ \rangle$ on $H^{\frac{1}{2}}$. There is a natural splitting of $H^{\frac{1}{2}}$ as

$$H^{\frac{1}{2}} = H^+ \oplus H^0 \oplus H^- \tag{2.3}$$

where H^+ is the direct sum of $\mathbb{C}^{n+1}\exp(2\pi ikt)$, $k>0$

$$H^- \text{ is the direct sum of } \mathbb{C}^{n+1}\exp(2\pi ikt),\ k<0$$

and H^0 is $\mathbb{C}^{n+1}$.

This is an orthogonal decomposition of $H^{\frac{1}{2}}$. Any $u\in H^{\frac{1}{2}}$ splits uniquely as

$$u=u^+ +u^0+u^-. \tag{2.4}$$

For ease of exposition let

$$\begin{aligned} E^- &\equiv H^-\oplus H^0 \\ E^+ &\equiv H^+. \end{aligned} \tag{2.5}$$

Let P_{E^-}, P_{E^+} be the corresponding orthogonal projections. Now define the action function $\mathscr{A}: P\to\mathbb{R}$ by

$$\mathscr{A}(u)=\tfrac{1}{2}\operatorname{Im}\int_0^1 (u(t), du/dt)\,dt \tag{2.6}$$

where (,) is the canonical hermitian inner-product on $\mathbb{C}^{n+1}$. This is just the usual action integral $\int_\sigma p\,dq$.

A simple calculation gives

$$\mathscr{A}(u)=-\pi[\|u^+\|^2-\|u^-\|^2]. \tag{2.7}$$

If we define

$$Lu=-2\pi(u^+-u^-) \tag{2.8}$$

then

$$\mathscr{A}(u)=\tfrac{1}{2}\langle Lu,u\rangle. \tag{2.9}$$

Given a time-dependent Hamiltonian J_t on C^{n+1} define the function $\mathscr{J}: P\to\mathbb{R}$ by

$$\mathscr{J}(u)=\int_0^1 J_t(u(t))\,dt. \tag{2.10}$$

The functions $\mathscr{A}$, $\mathscr{J}$ extend by continuity to $H^{\frac{1}{2}}$. Let $f=\mathscr{A}-\mathscr{J}: H^{\frac{1}{2}}\to\mathbb{R}$. Then Hamilton's principle can be stated as follows.

Proposition 2.1. *The closed integral curves $u: I\to\mathbb{C}^{n+1}$ of X_{J_t} are in a one-to-one correspondence with critical points of f.*

The proof of this proposition is standard: the Euler-Lagrange equations for f are just Hamilton's equations; regularity of the critical loops is proved in [BR]. The same principle holds in other function spaces for example in H^1 but we use $H^{\frac{1}{2}}$ so that the Palais-Smale compactness condition holds.

There is also a natural orthogonal S^1-representation T_μ on $H^{\frac{1}{2}}$ preserving this orthogonal splitting, which is induced from the Hopf S^1-action S_μ on $\mathbb{C}^{n+1}$. This is defined first on the space of C^∞ loops by $T_\mu(\sigma)=S_\mu\circ\sigma$ and the extended to $H^{\frac{1}{2}}$ by continuity.

The action function $\mathscr{A}$ is invariant with respect to T_μ and so is $\mathscr{J}$ provided that J is invariant with respect to S_μ. In this case the critical points of f will also be invariant under the action and will therefore consist of a union of S^1-orbits. For our situation $J_t = H_t + \lambda K$. We denote the corresponding function by f_λ.

$$f_\lambda = \mathscr{A} - (\mathscr{H} + \lambda \mathscr{K}). \tag{2.11}$$

Let $S_r L^2$ denote the sphere of radius r in L^2. Treating λ as a Lagrange multiplier, the critical points of $f_\lambda = \mathscr{A} - (\mathscr{H} + \lambda \mathscr{K})$ are in a one-to-one correspondence with critical points of

$$f = f_0 = \mathscr{A} - \mathscr{H} \tag{2.12}$$

constrained to the manifold

$$S = \mathscr{K}^{-1}(1) = S_r L^2 \cap H^{\frac{1}{2}}, \qquad \text{for } r = 2^{\frac{1}{2}}. \tag{2.13}$$

It is easy to check that f is a C^1-function. Since K is continuous, S is closed and therefore complete. S is also invariant with respect to T_μ. In this context the Lagrange multiplier λ is called the non-linear eigenvalue. Since $\mathscr{A}$, $\mathscr{H}$ and $\mathscr{K}$ are all quadratic these non-linear eigenvalues are equal to the critical values of $f|_S$. In fact, using $\langle\ ,\ \rangle$ for the inner product of $H^{\frac{1}{2}}$ and ∇ for the gradient with respect to this inner product we have:

$$\begin{aligned}
0 = \nabla f_\lambda(u) \quad &= \nabla\mathscr{A}(u) - \nabla\mathscr{H}(u) - \lambda \nabla\mathscr{K}(u),\\
0 \quad &= \langle \nabla\mathscr{A}(u), u\rangle - \langle \nabla\mathscr{H}(u), u\rangle - \lambda\langle \nabla\mathscr{K}(u), u\rangle,\\
0 \quad &= 2\mathscr{A}(u) - 2\mathscr{H}(u) - 2\lambda\mathscr{K}(u),\\
0 \quad &= \mathscr{A}(u) - \mathscr{H}(u) - \lambda.
\end{aligned}$$

So $\quad \lambda = \mathscr{A}(u) - \mathscr{H}(u)$

$$\lambda = \text{critical value corresponding to } u. \tag{2.14}$$

For later use we need:

Proposition 2.2. *Let $H(x,t) \geqq 0$ be a hamiltonian on $\mathbb{C}^{n+1}$, which is positively homogeneous of degree 2. Then for $u \in S$*

$$0 \leqq \mathscr{H}(u) \leqq M = 2 \max_{(x,t)\in S^{2n-1}\times I} H(x,t). \tag{2.15}$$

The proof is clear. □

Proposition 2.3. *Let $H(x,t) \geqq 0$ be a hamiltonian on $\mathbb{C}^{n+1}$, which is positively homogeneous of degree 2. Then the map $\nabla\mathscr{H}: H^{\frac{1}{2}} \to H^{\frac{1}{2}}$ is compact. If H is S^1-invariant then $\nabla\mathscr{H}$ is equivariant.*

Proof. By Lemma (3.10) and Proposition (3.12) of [BR] $\mathscr{H}$ is both weakly continuous and uniformly differentiable. (In reading that proof one should substitute 1 for λ, H for H_K and $\mathscr{H}$ for b.) Standard theorems [K p. 73] then imply that $\nabla\mathscr{H}$ takes weakly convergent sequences to strongly convergent sequences and is therefore compact.

If H is invariant then $\mathscr{H}(T_\mu u) = H(u)$, for all $\mu \in S^1$

$$\begin{aligned}\langle \nabla \mathscr{H}(T_\mu u), v\rangle &= (d/d\lambda)|_0 \mathscr{H}(T_\mu u + \lambda v)\\ &= (d/d\lambda)|_0 \mathscr{H}(u + \lambda T_{-\mu} v)\\ &= \langle \nabla \mathscr{H}(u), T_{-\mu}, v\rangle\\ &= \langle T_\mu \nabla \mathscr{H}(u), v\rangle.\end{aligned}$$

Therefore $\nabla \mathscr{H}(T_\mu u) = T_\mu \nabla \mathscr{H}(u)$ so that $\nabla \mathscr{H}$ is equivariant. □

We then see that $f(u)$ is just a compact perturbation of $\mathscr{A}(u)$.

Section 3. The minimax theory

A minimax theory is used to prove existence of critical orbits. The minimax theory consists of various components. Firstly one needs classes of sets over which to minimax. Under suitable hypotheses one ends up with at least one critical orbit for each class of sets.

One way to construct these classes is via an index. We use the relative index theory developed in [BLMR]. This is a refinement over various other index theories appearing in the literature [Bel, Cl, AmR, BR]. The reason that these other index theories do not work for our situation is that $\mathscr{A}(u)$ is a quadratic function on $H^{\frac{1}{2}}$ with infinite index and co-index. The relative index will help to overcome the topological difficulties caused by the infinite co-index and the existence of multiple critical values will be assured.

Definition 3.1. $\mathscr{F} = \{B \mid B \subseteq H^{\frac{1}{2}} \setminus \{0\}.\ B$ closed and invariant.$\}$

Let R_θ be an S^1-representation on $\mathbb{C}^k$. Then we obtain an S^1-representation (T, R) on $E^- \oplus \mathbb{C}^k$ by letting S^1 act on E^- via the T-representation and on $\mathbb{C}^k$ via the R-representation.

Definition 3.2. For $B \in \mathscr{F}$, $\gamma(B)$ is defined as the minimum $k \in \mathbb{N}$ for which there is an S^1-representation R on $\mathbb{C}^k$, with 0 as the only fixed point, and an equivariant continuous map $l: B \to (E^- \oplus \mathbb{C}^k) \setminus \{0, 0)\}$, with respect to the S^1-representations T on B and (T, R) on $E^- \oplus \mathbb{C}^k$ which satisfies

$$P_{E^-} \circ l = P_{E^-} + K \tag{3.1}$$

for some $K: B \to E^-$, which is compact, i.e. maps bounded sets into relatively compact sets in E^-. If no such k exists we define $\gamma(B)$ to be ∞. $\gamma: \mathscr{F} \to \mathbb{N}$ is called the relative index (relative to E^+). Also we define the relative index of the empty set to be 0.

One other index that we use is Benci's index $\Gamma: \mathscr{F} \to \mathbb{N}$ in [Be 1]. This is also defined for closed invariant subsets of a Hilbert space with a S^1-representation but it is not relative to a subspace.

Proposition 3.1. *Properties of the relative index.* For proofs see [BLMR] and [Be 1].

1. *If* $\gamma(B) \geqq k$ *and* E^+ *splits into the orthogonal* sum $E^+ = F_1 + F_2$ *where* F_i, $i = 1, 2$ *are invariant and* $\dim(F_1) < 2k$ *then* $B \cap F_2$ *is not empty.*

2. *Let* $G \subseteq E^+$ *be a* $2k$*-dimensional invariant subspace. Let* $\Sigma_k(r) = \{u \in E^- \oplus G \mid \|u\| = r\}$. *Then* $\gamma(\Sigma_k(r)) = k$.

3. *Let* A, $B \in \mathscr{F}$. *Suppose that there exists a continuous bounded map* l *from* A *onto* B *such that* $P_{E^-} l = P_{E^-} + K$, *for a compact map* $K: A \to E^-$. *Then* $\gamma(A) \leqq \gamma(B)$.

4. *Let* $A, B \in \mathscr{F}$. *Then* $\gamma(A \cup B) \leqq \gamma(A) + \Gamma(B)$.

5. *Let* $A, B \in \mathscr{F}$ *with* $\gamma(B) < \infty$. *Then* $\mathrm{cl}(A \setminus B) \in \mathscr{F}$ *and* $\gamma(\mathrm{cl}(A \setminus B)) \geqq \gamma(A) - \Gamma(B)$.

6. *Let* $K \in \mathscr{F}$ *be compact. Then* $\Gamma(K) < \infty$ *and for small enough* $\delta > 0$, $N_\delta K = \{u \in H^{\frac{1}{2}} \mid \mathrm{dist}(u, K) < \delta\}$ *has the same* Γ*-index.*

7. *If* B *is the union of a finite number of orbits then* $\Gamma(B) \leqq 1$.

8. *Let* M *be an equivariant isomorphism of* $H^{\frac{1}{2}}$ *which leaves* E^- *invariant. Then for* $A \in \mathscr{F}$, $MA \in \mathscr{F}$ *and* $\gamma(MA) = \gamma(A)$.

Definition 3.3. $f^a = \{x \in S \mid f(x) \leqq a\}$.

Definition 3.4. $K_c = \{x \in S \mid f(x) = c, (f|_S)'(x) = 0\}$.

We can now define the classes of sets over which we will minimax. They are given by:

Definition 3.5. $\Gamma_k(S) = \{B \in \mathscr{F} \mid B \subseteq S, \gamma(B) \geqq k\}$.

Now let

$$b_k = \inf_{B \in \Gamma_k(S)} \sup_{u \in B} f(u). \tag{3.2}$$

These are the minimax values. $\Gamma_k(S)$ is non-empty by Proposition 3.1(2) and $b_k < \infty$ by Proposition 3.1(2) and Proposition 2.2. Clearly, for $k_1 > k_2$, $\Gamma_{k_1}(S) \subseteq \Gamma_{k_2}(S)$ so that $b_{k_1} \geqq b_{k_2}$.

We will show that the $b_k > 0$ are critical values of f and furthermore that if $b_k = b_{k+1} = \ldots = b_{k+r-1} = b$ then $\Gamma(K_b) \geqq r$. In particular if $r \geqq 2$ then there are infinitely many critical orbits.

To prove these facts we need the second major ingredient of a minimax theory – the deformation. In order to realize this we need some compactness condition; in this case the Palais-Smale condition holds.

Let S be a Banach manifold (a Finsler manifold). Then there is a norm $\|.\|$ on every tangent space $T_u S$, and by duality a norm, also denoted by $\|.\|$, on each co-tangent space $T_u^* S$. If $f: S \to \mathbb{R}$ is a C^1 function then the Frechet derivative $df(u) = f'(u) \in T_u^* S$.

Definition 3.6. A differentiable function $f: S \to \mathbb{R}$ satisfies the Palais-Smale condition if every sequence $\{u_k\}$ in S for which $\{f(u_k)\}$ is bounded and $\|f'(u_k)\| \to 0$ as $k \to \infty$, has a convergent subsequence.

It then follows that the limit of that subsequence is a critical point for f.

Proposition 3.2. $f(u) = \mathscr{A}(u) - \mathscr{H}(u)$ *satisfies the Palais-Smale condition when restricted to* $S = H^{\frac{1}{2}} \cap \mathscr{H}^{-1}(1)$.

Proof. Let u_k be a sequence in S for which

$$|f(u_k)| \quad \text{is bounded.} \tag{3.3}$$

and

$$\|(f|_S)'(u_k)\| \to 0. \tag{3.4}$$

We must show that u_k has a convergent subsequence. We first show that u_k is bounded. Recall that $\mathscr{K}(u)=\frac{1}{2}|u|^2$. In this case S is a Hilbert manifold and (3.4) is equivalent to

$$\nabla(f|_S)(u_k)\to 0 \quad \text{in } H^{\frac{1}{2}}. \tag{3.5}$$

Since $S=H^{\frac{1}{2}}\cap\mathscr{K}^{-1}(1)$, for $u\in S$, the normal space to T_uS in $T_uH^{\frac{1}{2}}=H^{\frac{1}{2}}$ is spanned by $\nabla\mathscr{K}(u)$, which we denote by $C(u)$. Since $K(u)$ is quadratic C is a linear operator from $H^{\frac{1}{2}}$ to itself. It is easily seen that C respects the splitting of $H^{\frac{1}{2}}=H^+\oplus H^0\oplus H^-$.

For $u\in S$, $\nabla(f|_S)(u)$ is the orthogonal projection P_u of $\nabla f(u)\in H^{\frac{1}{2}}$ onto T_uS. It is given by the formula

$$\begin{aligned}\nabla(f|_S)(u)&=P_u(\nabla f(u))\\ &=\nabla f(u)-\langle\nabla f(u),d(u)\rangle d(u)\\ &=\nabla f(u)-\langle\nabla f(u),Cu\rangle\frac{d(u)}{\|Cu\|},\qquad \text{where } d(u)=Cu/\|Cu\|.\end{aligned}$$

From (2.12) and (2.9) we get

$$\nabla(f|_S)(u)=Lu-\nabla\mathscr{H}(u)-\lambda(u)Cu \tag{3.6}$$

where

$$\lambda(u)=\left(\frac{\langle\nabla f(u),Cu\rangle}{\|Cu\|^2}\right). \tag{3.7}$$

Then (3.5) is equivalent to

$$z_k\equiv Lu_k-\nabla\mathscr{H}(u_k)-\lambda_kCu_k\to 0,\qquad \text{in } H^{\frac{1}{2}} \tag{3.8}$$

where $\lambda_k=\lambda(u_k)$.

Now take the inner product with u_k.

$$\begin{aligned}\langle z_k,u_k\rangle&=\langle Lu_k,u_k\rangle-\langle\nabla\mathscr{H}(u_k),u_k\rangle-\lambda_k\langle Cu_k,u_k\rangle\\ &=2f(u_k)-2\lambda_k\mathscr{K}(u_k),\qquad \text{by homogeneity}\\ &=2f(u_k)-2\lambda_k.\end{aligned}$$

Hence $|\lambda_k|=|\frac{1}{2}\langle z_k,u_k\rangle-f(u_k)|$
$\leqq\frac{1}{2}\|z_k\|\,\|u_k\|+|f(u_k)|.$

$\|z_k\|$ and $|f(u_k)|$ are bounded by (3.3), (3.8). Thus there exist positive constants c_1 and c_2 such that

$$|\lambda_k|\leqq c_1+c_2\|u_k\|. \tag{3.9}$$

Using 2.8 take the inner product of (3.8) with u_k^+ to obtain

$$\begin{aligned}-2\pi\|u_k^+\|^2 &= \langle Lu_k, u_k^+\rangle \\ &= \langle z_k, u_k^+\rangle + \langle \nabla\mathscr{H}(u_k), u_k^+\rangle + \lambda_k\langle Cu_k, u_k^+\rangle.\end{aligned}$$

Hence

$$2\pi\|u_k^+\|^2 \leqq \|z_k\|\,\|u_k^+\| + |\lambda_k|\,|\langle Cu_k, u_k^+\rangle| + |\langle \nabla\mathscr{H}(u_k), u_k^+\rangle|. \tag{3.10}$$

Now

$$\begin{aligned}\langle Cu_k, u_k^+\rangle &= \langle Cu_k^+, u_k^+\rangle \\ &= 2\mathscr{H}(u_k^+), \quad \text{by homogeneity} \\ &= |u_k^+|^2 \\ &\leqq |u_k|^2.\end{aligned}$$

Since $u_k \in S$

$$\langle Cu_k, u_k^+\rangle \leqq 2. \tag{3.11}$$

Also $|\langle \nabla\mathscr{H}(u_k), u_k^+\rangle| = |\langle\!\langle d\mathscr{H}(u_k), u_k^+\rangle\!\rangle|$, where the Frechet derivative $d\mathscr{H}(u_k) = \mathscr{H}'(u_k)$ belongs to the cotangent space of $H^{\frac{1}{2}}$ at u_k, which can be identified with $(H^{\frac{1}{2}})^* = H^{-\frac{1}{2}}$ via the L^2-pairing $\langle\!\langle\ ,\ \rangle\!\rangle$ between $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$. Then

$$\begin{aligned}|\langle \nabla\mathscr{H}(u_k), u_k^+\rangle| &\leqq \|d\mathscr{H}(u_k)\|_{H^{-\frac{1}{2}}}\,\|u_k^+\|_{H^{\frac{1}{2}}} \\ &\leqq |d\mathscr{H}(u_k)|_{L^2}\,\|u_k^+\|_{H^{\frac{1}{2}}}.\end{aligned} \tag{3.12}$$

It is easy to check that

$$\langle\!\langle d\mathscr{H}(u), v\rangle\!\rangle = \int_0^1 d_u H(u(t), t)\, v(t)\, dt,$$

so that

$$|d\mathscr{H}(u_k)|_{L^2}^2 = \int_0^1 |d_u H(u_k(t), t)|^2\, dt.$$

Let $\Omega_k = \{t \in I \mid u_k(t) \neq 0\}$. $d_z H(z, t)$ is positive homogeneous of degree 1 in z.

Hence

$$\begin{aligned}|d\mathscr{H}(u_k)|^2 &= \int_{\Omega_k} |dH(u_k(t), t)|^2 \\ &= \int_{\Omega_k} |u_k|^2\, |dH(u_k^*(t), t)|^2\, dt\end{aligned}$$

for $u_k^*(t) = (u_k(t)/|u_k(t)|) \in S^{2n-1}$

$$\begin{aligned}|d\mathscr{H}(u_k)|_{L^2}^2 &\leqq N^2 \int_{\Omega_k} |u_k(t)|^2\, dt, \qquad \text{where } N = \max_{(x,t)\in S^{2n-1}\times I} |dH(x, t)| \\ &\leqq N^2 |u_k|^2 = 2N^2.\end{aligned} \tag{3.13}$$

From (3.10), $\|z_k\|$ being bounded, (3.11), (3.9), (3.12) and (3.13) it follows that there are positive constants c_3 and c_4 such that

$$\begin{aligned}\|u_k^+\|^2 &\leqq c_3 + c_4\|u_k\| \\ &\leqq c_3 + c_4(\|u_k^+\| + \|u_k^0\| + \|u_k^-\|).\end{aligned} \tag{3.14}$$

On the other hand (2.12), (2.7), (2.15) and (3.3) then imply that

$$\|u_k^+\|^2 - \|u_k^-\|^2$$

is bounded. It follows that

$$\|u_k^+\| - \|u_k^-\| \quad \text{is bounded} \tag{3.15}$$

Since $u_k \in S$

$$\|u_k^0\|^2 = |u_k^0|^2 \leqq |u_k|^2 = 2. \tag{3.16}$$

(3.14) to (3.16) then show that

$$\|u_k^+\|^2 \leqq c_5 + c_6 \|u_k^+\|,$$

where c_5 and c_6 are positive constants. This proves that u_k^+ is bounded.

Then (3.15) and (3.16) prove u_k to be bounded. By (3.9) $|\lambda_k|$ is bounded so that we can find a convergent subsequence.

Let $L'u = Lu + u^0$.

$$L' = \begin{bmatrix} 2\pi I_{H^-} & & \\ & I_{H^0} & \\ & & -2\pi I_{H^+} \end{bmatrix}$$

where I_{H^+} is the identity transformation on H^+,
I_{H^-} is the identity transformation on H^-,
and I_{H^0} is the identity transformation on H^0.

Clearly L' is an isomorphism. From (3.8)

$$L'u_k = z_k + \nabla\mathscr{H}(u_k) + \lambda_k C u_k + u_k^0.$$

z_k converges zo 0 by (3.8). u_k^0 is bounded in $\mathbb{C}^{n+1}$. By proposition 2.3 both $\nabla\mathscr{H}$ and C are compact operators. Thus there exists a subsequence of u_k for which u_k^0, $\nabla\mathscr{H}(u_k)$ and $C(u_k)$ all converge.

Consequently $L'u_k$ has a convergent subsequence, and since L' is an isomorphism, u_k also has a convergent subsequence. □

The next 3 lemmas are needed for the proof of the deformation theorem.

Lemma 3.1. *Let $\Phi: H^{\frac{1}{2}} \to H^{\frac{1}{2}}$ be bounded* (i.e. takes bounded sets to bounded sets) *and homogeneous of degree* 1. *Then* $\|\Phi(u)\| \leqq M\|u\|$, for some $M>0$. (*We apply this to $\Phi = \nabla\mathscr{H}$.*)

Lemma 3.2. (approximation lemma) [Be2]. *Let $\Phi: H^{\frac{1}{2}} \to H^{\frac{1}{2}}$ be a compact operator. Then given $\rho > 0$, there exists an operator $\phi: H^{\frac{1}{2}} \to H^{\frac{1}{2}}$ which is compact, locally Lipschitz continuous, and satisfies*

$$\|\Phi(u) - \phi(u)\| \leqq \rho.$$

Moreover if Φ is equivariant, then ϕ can also be chosen to be equivariant.

Lemma 3.3. [Be 2]. *Let $X(u) = -Lu + \phi(u)$ be a vector field on $H^{\frac{1}{2}} \setminus \{0\}$, where L is linear and $\phi(u)$ is compact. Assume that a flow $\eta_t(u)$ exists for this vector field* *such that* η_t *maps bounded sets to bounded sets, for all* t. *Then* $\eta_t(u)$ *has the form*:

$$\eta_t(u) = \exp(-tL)u + K(u, t)$$

with K *compact for every fixed* t.

Theorem 3.1. (The deformation theorem.) *Let* $f\colon H^{\frac{1}{2}} \to \mathbb{R}$ *be defined by*

$$f(u) = \tfrac{1}{2}\langle Lu, u\rangle - \mathcal{H}(u).$$

Given $\delta > 0$, $c \in \mathbb{R}$. *Then there exists an* $\varepsilon > 0$ *and* $\eta\colon H^{\frac{1}{2}}\setminus\{0\} \to H^{\frac{1}{2}}\setminus\{0\}$ *with the following properties:*

a) η *is an equivariant homeomorphism, which restricts to an equivariant homeomorphism on* S.

b) $\eta[S \cap f^{c+\varepsilon}\setminus N_\delta K_c] \subseteq [S \cap f^{c-\varepsilon}]$,

c) $\eta(u) = Mu + K(u)$

where $M\colon H^{\frac{1}{2}} \to H^{\frac{1}{2}}$ *is an equivariant linear isomorphism of the form* $\exp(-tL)$, *for some* $t > 0$ *and* $K\colon H^{\frac{1}{2}}\setminus\{0\} \to \mathbb{R}$ *is compact.*

Proof. η will be constructed as the flow at a fixed time t of an approximate gradient vector field $X(u)$ on $H^{\frac{1}{2}}$. We cannot take an extension of $-\nabla(f|_S)$ as this vector field since it may not even be locally Lipschitz.

Step 1. By Proposition 2.3 $\nabla\mathcal{H}$ is a compact equivariant nonlinear operator on $H^{\frac{1}{2}}$. Hence by Lemma 3.2, for any ρ, to be chosen later, there exists an operator $\phi\colon H^{\frac{1}{2}} \to H^{\frac{1}{2}}$ which is compact, equivariant, locally Lipschitz continuous, and such that

$$\|\phi(u) - \nabla\mathcal{H}(u)\| \leqq \rho, \qquad \text{for all } u \in H^{\frac{1}{2}}. \tag{3.17}$$

Let $d(u) = Cu/\|Cu\|$. So

$$\begin{aligned} -\nabla(f|_S) &= -P_u(\nabla f(u)) \\ &= -Lu + \nabla\mathcal{H}(u) + \langle Lu - \nabla\mathcal{H}(u), d(u)\rangle d(u). \end{aligned}$$

We replace this by the vector field, X, on S given by

$$X(u) = P_u(-Lu + \phi(u)). \tag{3.18}$$

Extend $X(u)$ to a vector field on $H^{\frac{1}{2}}\setminus\{0\}$ given by the same formula. $X(u)$ is a continuous equivariant vector field on $H^{\frac{1}{2}}\setminus\{0\}$ tangent to the homothetic images λS, $\lambda > 0$, of S. We show that $X(u)$ is locally Lipschitz, so that an equivariant local flow exists. A global flow will be established in step 2. Since X is tangent to λS, $\lambda > 0$, the flow will restrict to λS, $\lambda > 0$.

$$\begin{aligned} \|X(u) - X(v)\| &= \|P_u(-Lu + \phi(u)) - P_v(-Lv + \phi(v)\| \\ &\leqq \|P_u[(-Lu + \phi(u)) - (-Lv + \phi(v))]\| + \|(P_u - P_v)(-Lv + \phi(v))\| \\ &\leqq \|P_u\|[\|L\|\,\|u - v\| + \|\phi(u) - \phi(v)\|] + \|(P_u - P_v)(-Lv + \phi(v))\|. \end{aligned}$$

Since $\|P_u\|=1$, $\|L\|=2\pi$ and ϕ is locally Lipschitz, the first term is bounded by a constant times $\|u-v\|$, whenever v is sufficiently close to u.

$$\begin{aligned}(P_u-P_v)(-Lv+\phi(v)) &= -Lv+\phi(v)-\langle -Lv+\phi(v), d(u)\rangle d(u)\\ &\quad -[-Lv+\phi(v)-\langle -Lv+\phi(v), d(v)\rangle d(v)\\ &=\langle -Lv+\phi(v), Cv\rangle \frac{d(v)}{\|Cv\|}-\langle -Lv+\phi(v), Cu\rangle \frac{d(u)}{\|Cu\|}.\end{aligned}$$

So

$$\begin{aligned}\|(P_u-P_v)(-Lv+\phi(v))\| &\leqq |\langle -Lv+\phi(v), Cv-Cu\rangle| \frac{d(v)}{\|Cv\|}\\ &\quad +|\langle -Lv+\phi(v), Cu\rangle| \left\| \left[\frac{d(v)}{\|Cv\|}-\frac{d(u)}{\|Cu\|}\right]\right\|\\ &\leqq \|-Lv+\phi(v)\| \frac{\|C\|}{\|Cv\|}\|v-u\|\\ &\quad +\frac{\|-Lv+\phi(v)\|\,\|Cu\|}{\|Cv\|^2\|Cu\|^2}\|[Cv\|Cu\|^2-Cu\|Cv\|^2]\|.\end{aligned}$$

C is bounded and if v is sufficiently close to u then $\|-Lv+\phi(v)\| \dfrac{\|C\|}{\|Cv\|}$ is bounded, so that the first term on the right hand has been taken care of. If v is sufficiently close to u then $\dfrac{\|-Lv+\phi(v)\|\,\|Cu\|}{\|Cu\|^2\|Cv\|^2}$ is bounded.

Also

$$\begin{aligned}&\|[Cv\|Cu\|^2-\|Cv\|^2 Cu]\|\\ &=\|[(Cv-Cu)\|Cu\|^2+Cu(\|Cu\|^2-\|Cv\|^2)]\|\\ &\leqq \|C\|\,\|Cu\|^2\|v-u\|+\|Cu\|(\|Cu\|+\|Cv\|)\|C\|\,\|v-u\|\\ &\leqq \text{constant}\ \|v-u\|,\end{aligned}$$

provided that v is sufficiently close to u. We have therefore shown that $X(u)$ is locally Lipschitz.

Step 2. Global flow.

$$\begin{aligned}\|X(u)\| &= \|P_u(-Lu+\phi(u))\|\\ &\leqq \|-Lu+\phi(u)\|\\ &\leqq \|-Lu+\nabla\mathcal{H}(u)\|+\|\nabla\mathcal{H}(u)-\phi(u)\|\\ &\leqq \|L\|\,\|u\|+\|\nabla\mathcal{H}(u)\|+\rho.\end{aligned}$$

By Lemma 3.1 $\|\nabla\mathcal{H}(u)\| \leqq \text{constant}\ \|u\|$. Hence

$$\|X(u)\| \leqq \rho+c\|u\|, \quad \text{for a constant } c>0. \tag{3.19}$$

Also since $X(u)$ is tangent to λS for every $\lambda>0$, the integral curves never leave $H^{\frac{1}{2}}\setminus\{0\}$. Therefore the flow $\eta_t(u)$ of X is globally defined, equivariant and takes bounded sets to bounded sets. Since X is tangent to S it restricts to such a flow on S. By Lemma 3.3, η will be of the form:

$$\eta(t,u)=\exp(-tL)u+K(u,t),$$

where $K(u,t)$ is compact for each fixed t, provided we can show that $X(u)$ is of the form:

$$X(u)=-Lu+K'(u),$$

with K' compact. This is shown in step 3.

Step 3.

$$\begin{aligned} K'(u)&=X(u)+Lu \\ &=\phi(u)+\langle Lu-\phi(u),d(u)\rangle d(u). \end{aligned}$$

$C: H^{\frac{1}{2}}\to H^{\frac{1}{2}}$ is a compact linear operator by Proposition 2.3. Hence $u\to d(u)$ is a non-linear compact operator on $H^{\frac{1}{2}}\setminus\{0\}$. If u is bounded then so is

$$|\langle Lu-\phi(u),d(u)\rangle|$$

so that $K'(u)$ is a compact non-linear operator on $H^{\frac{1}{2}}$.

Step 4. (This follows Benci [Be 2].) Let $\eta(u)=\eta(T,u)$ for an appropriate T to be chosen later. η is an equivariant homeomorphism which restricts to S, proving (a). K' is compact so (c) is also proved.

Since f satisfies the Palais-Smale condition K_c is a compact set, so that $N_\delta(K_c)$ is bounded. Let $R>0$ be a constant such that $K_c\subseteq B_R(0)$. From $P-S$ one also deduces the existence of an $\varepsilon>0$, such that

$$\|\nabla(f|_S)\|\geqq(4\varepsilon/T)^{\frac{1}{2}},\quad \text{for } u\in\mathcal{T}_\varepsilon \tag{3.20}$$

where

$$\mathcal{T}_\varepsilon=S\cap[f^{-1}[c-\varepsilon,c+\varepsilon]\setminus N_{\delta/2}(K_c)]. \tag{3.21}$$

Otherwise there exists a sequence $u_n\in\mathcal{T}_{1/n}$ with $\|\nabla(f|_S)(u_n)\|<(4\varepsilon/T)^{\frac{1}{2}}$. Then $f(u_n)\to c$ and $\nabla(f|_S)(u_n)\to 0$. By $P-S$ there is a convergent subsequence u_{n_k} with $u_{n_k}\to u\in K_c$. But u_n is bounded away from K_c, which gives a contradiction, so the existence of ε satisfying (3.20) is proved. If we make δ smaller (b) becomes stronger, so without loss of generality we may assume that

$$N_{\delta/2}(K_c)\cap f^{-1}(c-\varepsilon)=\emptyset. \tag{3.22}$$

We now show that at a point $u\in\mathcal{T}_\varepsilon$, f is decreasing sufficiently rapidly along the flow of $X(u)$.

$$\begin{aligned} (d/dt)|_0 f(\eta_t(u))&=\langle\nabla f(u),X(u)\rangle \\ &=\langle\nabla(f|_S)(u),X(u)\rangle \\ &=-\langle\nabla f|_S)(u),\nabla(f|_S)(u)-\phi(u)+\nabla\mathcal{H}(u)\rangle \\ &=-\|\nabla(f|_S)(u)\|^2-\langle\nabla(f|_S)(u),\,-\phi(u)+\nabla\mathcal{H}(u)\rangle \\ &\leqq-\|\nabla(f|_S)(u)\|^2+\|\nabla(f|_S)(u)\|\rho,\quad \text{by (3.17).} \end{aligned}$$

Let $y=\|\nabla(f|_S)(u)\|$. The function $y\to -y^2+\rho y$ is decreasing for $y>\rho/2$. We still have freedom to choose ρ; choose $\rho=(\varepsilon/4T)^{\frac{1}{2}}$. ρ depends upon T, which is still to be chosen.

By (3.20) $y\geqq(4\varepsilon/T)^{\frac{1}{2}}=4\rho>\rho/2$.

Therefore

$$\begin{aligned}-y^2+\rho y&\leqq-(4\rho)^2+\rho(4\rho)\\&=-12\rho^2\\&=-3\varepsilon/T.\end{aligned}$$

I.e.

$$(d/dt)|_0 f(\eta_t(u))\leqq -3\varepsilon/T\leqq -2\varepsilon/T, \qquad \text{for } u\in\mathcal{T}_\varepsilon. \tag{3.23}$$

Therefore if $\eta(t,u)\in\mathcal{T}_\varepsilon$ for $t\in[0,T]$ we have

$$\begin{aligned}f(\eta(T,u))&=f(u)+\int_0^T (d/dt)f(\eta(t,u))\,dt\\&\leqq f(u)-2\varepsilon.\end{aligned} \tag{3.24}$$

In particular if $u\in f^{c+\varepsilon}$ and $\eta(t,u)\in\mathcal{T}_\varepsilon$ for $t\in[0,1]$ then

$$f(\eta(u))=f(\eta(T,u))\leqq c-\varepsilon. \tag{3.25}$$

We now prove that for T small enough

$$u\in S\cap f^{-1}[c-\varepsilon,c+\varepsilon]\backslash N_\delta(K_c)\Rightarrow \eta(t,u)\notin N_{\delta/2}K_c, \qquad \text{for } t\in[0,T]. \tag{3.26}$$

Assume to the contrary. Then there exist t_0,t_1 with $0\leqq t_0<t_1\leqq T$ such that

$$\eta(t_0,u)\in\partial N_\delta(K_c),$$
$$\eta(t_1,u)\in\partial N_{\delta/2}(K_c)$$

and

$$\eta(t,u)\in\mathrm{cl}\,[N_\delta(K_c)\backslash N_{\delta/2}(K_c)], \qquad \text{for } t\in[t_0,t_1].$$

Then

$$\begin{aligned}\delta/2&\leqq\|\eta(t_1,u)-\eta(t_0,u)\|\\&=\left\|\int_{t_0}^{t_1}(d/dt)\eta(t,u)\,dt\right\|\\&=\left\|\int_{t_0}^{t_1}X(\eta(t,u)\,dt\right\|\\&\leqq\int_{t_0}^{t_1}\|X(\eta(t,u))\|\,dt\\&\leqq\int_{t_0}^{t_1}(\rho+c\|\eta(t,u)\|)dt, \qquad \text{by 4.19.}\\&\leqq(t_1-t_0)(\rho+cR), \qquad \text{since } \eta(t,u)\in N_\delta(K_c)\subseteq B_R.\\&\leqq T(\rho+cR).\end{aligned}$$

For T small enough we get a contradiction. Now fix T for which (3.26) does hold. The flow $\eta(t,u)$ for $u\in f^{c-\varepsilon}$ cannot enter the region $f^{-1}(c-\varepsilon,\,c+\varepsilon]$ since by (3.21), (3.22) it would then have entered the region $\mathcal{T}_\varepsilon$ which is impossible since by (3.23) f is decreasing along the flow in $\mathcal{T}_\varepsilon$.

Now let $u \in S \cap [f^{c+\varepsilon} \setminus N_\delta K_c]$. To complete the proof of the theorem we must show that $\eta(T,u) \in f^{c-\varepsilon}$. From the last paragraph we know that if $u \in f^{c-\varepsilon}$ then $\eta(T,u)$ also belongs to $f^{c-\varepsilon}$. If $u \notin f^{c-\varepsilon}$ then $u \in \mathcal{T}_\varepsilon$. $f(\eta(t,u))$ is then decreasing by (3.23) so that $\eta(t,u)$ remains in $\mathcal{T}_\varepsilon \setminus f^{c-\varepsilon}$, enters $N_{\delta/2} K_c$, or enters $f^{c-\varepsilon}$. The first two possibilities are not possible by (3.25) and (3.26). Therefore it must enter $f^{c-\varepsilon}$ and by the previous paragraph it will remain there. This proves (b) and completes the proof of the deformation theorem. □

Proposition 3.3. *The map $\eta: S \to S$ takes $\Gamma_k(S)$ to $\Gamma_k(S)$.*

Proof. Suppose that $B \in \Gamma_k(S)$, i.e. $B \subseteq S$ and $\gamma(B) \geqq k$. Since η is an equivariant homeomorphism $\eta(B)$ is also closed and invariant subset of S. We need to show that $\gamma(\eta(B)) \geqq k$. By Theorem 3.1 $\eta(u) = Mu + K(u)$, where M is an equivariant isomorphism of $H^{\frac{1}{2}}$ which leaves E^- invariant and K is compact. Hence η is a composition of the maps M and $I+K$ and the result follows directly from Proposition 3.1(3) and 3.1(8). □

Theorem 3.2. *The minimax values b_k defined by* (4.2) *are critical values of f. Furthermore if $b_k = b_{k+1} = \ldots = b_{k+r-1} = b$, then $\Gamma(K_b) \geqq r$.*

Proof. This is standard minimax theory. If $\Gamma(K_b) \leqq r-1$, then there exists a $\delta > 0$ such that $\gamma(N_\delta K_b) \leqq p-1$. Let $N = \operatorname{int}(N_\delta K_b)$; then $\operatorname{cl}(N) = N_\delta K_b$. Hence there exists an ε such that $\eta(f^{b+\varepsilon} \setminus N) \equiv Q \subseteq f^{b-\varepsilon}$. Since $b = b_{k+r-1} = \inf\{a \in \mathbb{R} \mid \gamma(f^a) \geqq k+r-1\}$, $\gamma(f^{b+\varepsilon}) \geqq k+r-1$. Hence by Proposition 3.1(5)

$$\gamma(f^{b+\varepsilon} \setminus N) = \gamma(\operatorname{cl}(f^{b+\varepsilon} \setminus \operatorname{cl}(N))) \geqq \gamma(f^{b+\varepsilon}) - \Gamma(\operatorname{cl}(N)) \geqq (k+r-1) - (r-1) = k,$$

by Proposition 3.1(16).

Thus by the previous proposition $\gamma(Q) \geqq k$; i.e. $Q \in \Gamma_k(S)$. Therefore

$$b_k = \inf_{A \in \Gamma_k(S)} \sup_{u \in A} f(u) \leqq \sup_{u \in Q} f(u) \leqq b - u = b_k - \varepsilon.$$

This gives the contradiction. □

Note. In particular we see that if $r > 1$, there are an infinite number of distinct periodic orbits. The follows from Proposition 3.1(7). The crucial factor here is that the S^1 action is free.

We now have a minimax theory that yields a countable number of distinct critical orbits for the function f on S. This is not sufficient to prove the existence of multiple fixed points for ϕ_1, however, since we know that a single fixed point of ϕ_1 corresponds to an arithmetic sequence of critical values of f.

The way to proceed is to examine the special case in which H is constant. For that case the non-linear spectrum is degenerate.

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{n+1} = 2\pi: \quad K_{2\pi} = \mathbb{C}^{n+1}$$
$$\lambda_{n+2} = \lambda_{n+3} = \ldots = \lambda_{2n+2} = 4\pi: \quad K_{4\pi} = \mathbb{C}^{n+1} \exp(2\pi i k t)$$

and so on, as will be seen in the next proposition.

Note. By (2.14) the critical values are equal to the non-linear eigenvalues.

Also only the positive spectrum appears here, because we are using the relative index to minimax.

Proposition 3.4 [BLMR]. *When $H \equiv$ constant, the minimax values b_k are just the positive eigenvalues, with multiplicity, of the linear eigenvalue problem $Lu = \nabla f(u) = \lambda Cu$.* □

These eigenvalues are easily calculated. One obtains the positive multiples of 2π, each one with multiplicity $n+1$.

We know from Proposition 2.2 that $\mathscr{H}$ assumes values in the interval $[0, M]$, where

$$M = 2 \max_{(u,t) \in S^{2n-1} \times I} H(u, t).$$

From this it follows that

Proposition 3.5. *Let $f_0 = \frac{1}{2}\langle Lu, u\rangle = f(u) + \mathscr{H}(u)$. Let $a_k = \inf_{A \in \Gamma_k(S)} \sup_{u \in A} f$.*

Let $b_k = \inf_{A \in \Gamma_k(S)} \sup_{u \in A} f_0$.

Then $b_k - M \leqq a_k \leqq b_k$.

Proof. Let A be an arbitrary element of $\Gamma_k(S)$. Then

$$\begin{aligned} a_k &= \inf_{B \in \Gamma_k(S)} \sup_{u \in B} f(u) \\ &\leqq \sup_{u \in A} f(u) \\ &\leqq \sup_{u \in A} (f(u) + \mathscr{H}(u)). \end{aligned}$$

This is true for all A in $\Gamma_k(S)$.

Hence

$$a_k \leqq \inf_{A \in \Gamma_k(S)} \sup_{u \in A} (f(u) + \mathscr{H}(u)) = b_k.$$

For any $\varepsilon > 0$ there exists an A_ε in $\Gamma_k(S)$ such that

$$\sup_{u \in A_\varepsilon} f(u) \leqq a_k + \varepsilon.$$

Now

$$\begin{aligned} b_k &= \inf_{A \in \Gamma_k(S)} \sup_{u \in A} (f(u) + \mathscr{H}(u)) \\ &\leqq \sup_{u \in A_\varepsilon} (f(u) + \mathscr{H}(u)) \\ &\leqq \sup_{u \in A_\varepsilon} f(u) + \sup_{u \in A_\varepsilon} \mathscr{H}(u) \\ &\leqq a_k + \varepsilon + M. \end{aligned}$$

This is true for all ε, so that $b_k \leqq a_k + M$, which proves the proposition. □

Finally we have the following combinatorial result.

Proposition 3.6. *Either* 1) *The a_k are not all distinct,*

or 2) $W \equiv \#\{a_k (\mathrm{mod}\, 2\pi) \mid k \in \mathbb{N}\} \geqq n+1.$

Proof. Let $t \geqq 1$ be an integer such that $M < 2t\pi$. Let $s > t$ be an integer. We count how many of the a_k's must lie in the interval $I_s = (2\pi, 2s\pi]$. If $b_k \in [2\pi(1+t), 2s\pi]$ then $a_k \leqq b_k \leqq 2s\pi$ and $a_k \geqq b_k - M > b_k - 2t\pi > 2(1+t)\pi - 2t\pi = 2\pi$, so that $b_k \in I_s$. There are $(s-t)(n+1)$ such b_k's; hence there are at least $(s-t)(n+1)$ a_k's in I_s. Assume that all are distinct.

Let $W_s = \# [\{a_k | k \in \mathbb{N}\} \cap I_s] \pmod{2\pi}$.

Then the number of a_k's in I_s is less than $W_s(s-1)$, so that

$$(s-t)(n+1) \leqq W_s(s-1),$$
$$W_s \geqq (s-t)(n+1)/(s-1) = (n+1) - (t-1)(n+1)/(s-1).$$

Take s large enough such that $(t-1)(n+1)/(s-1) < 1$. W_s is an integer so that $W \geqq W_s \geqq n+1$. □

This concludes the proof that ϕ_1 has at least $n+1$ fixed points.

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