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Inventiones mathematicae

Maximal and singular integral operators via Fourier transform estimates

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§ 1. Introduction

In the classical theory of singular integral operators $Tf = K * f$, the starting point was a Fourier transform estimate: $\hat{K} \in L^\infty$, from which, the inequality $\|Tf\|_2 \leqq C\|f\|_2$ follows immediately. This fact together with mild regularity assumptions on the kernel yield a weak type (1, 1) (or L^∞-BMO) inequality by a nowadays standard procedure whose basic ingredient is the so-called Calderón-Zygmund decomposition (see [16]). By interpolation and duality, the L^p-boundedness of T is proved for all $1 < p < \infty$.

The same scheme was tried for more singular operators, like

$$Tf(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x - \gamma(t)) \frac{dt}{t} = K * f(x)$$

(Hilbert transform along the curve γ in $\mathbb{R}^n$). Under some hypothesis on the curvature of γ, the fact that $\hat{K} \in L^\infty$ can still be proved, but the kernel K is now a distribution supported in the curve γ (a set of measure zero), which is too singular for the Calderón-Zygmund machinery to be applicable. However, the decay of $\hat{K}(\xi) = m(\xi)$ is such that one can worsen the multiplier m, obtaining certain $m_\alpha \in L^\infty$ if $0 > \mathrm{Re}(\alpha) > -a$, while, for $\mathrm{Re}(\alpha)$ positive, m_α improves, and one actually has $m_\alpha = \hat{K}_\alpha$ with locally integrable kernels K_α which are regular enough to fall under the scope of the classical theory. The L^p inequalities for T are then obtained by analytic interpolation of the family of operators T^α, $(T^\alpha f)^\wedge = \hat{f} m_\alpha$. Similar ideas lead to L^p inequalities for the associated maximal operators

$$Mf(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x - \gamma(t))| \, dt.$$

In this case, the starting point is the majorization: $Mf(x) \leqq f^*(x) + g(f)(x)$, for a suitable quadratic operator $g(\cdot)$. The inequality $\|Mf\|_2 \leqq C\|f\|_2$ is obtained

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by using Plancherel's theorem and Fourier transform estimates for the singular measures arising from a dyadic decomposition of the kernel. L^p inequalities, $1<p<\infty$, result again from the application of analytic interpolation to a suitably defined family of maximal operators. See [18], where the whole process is described in detail, and references to earlier work may be found.

In this paper, we present an alternative approach to obtain the L^p inequalities for a wide class of maximal and singular integral operators, including the ones mentioned above. This approach is based on Littlewood-Paley theory, and its effect is that one has just to look at the decay at ∞ of the Fourier transforms of σ_k and μ_k; here, $\sum_k \sigma_k * f$ is a dyadic decomposition of the singular integral under consideration, and $\sup_k |\mu_k * f|$ is the maximal operator. The use of Littlewood-Paley theory is suggested by the presence in the original approach of g-functions (which were only used to estimate the operators in L^2). Thus, for the maximal operator, we have

$$Mf(x) \leqq f^*(x) + g(f)(x).$$

Plancherel's theorem gives $\|g(f)\|_2 \leqq C\|f\|_2$, and then also $\|Mf\|_2 \leqq C\|f\|_2$; but now, Littlewood-Paley theory is used to show that the last inequality implies $\|g(f)\|_p \leqq C_p\|f\|_p$, $\frac{4}{3}<p<4$, and therefore, the same result holds for M, and so on... This sort of bootstrap argument was already present in the work of Nagel, Stein and Wainger [13] on differentiation in lacunary directions. For singular integral operators, Littlewood-Paley theory is used together with certain vector valued inequalities which follow from the boundedness of M. This is a satisfactory aspect of this approach: The estimates for singular integrals depend, as in the classical Calderón-Zygmund theory, on similar estimates for an associated maximal operator.

Our method gives no contribution to the L^2-boundedness, for which, in each concrete application, we must either rely on known estimates for Fourier transforms or produce the necessary estimates by adapting known arguments (e.g. Van der Corput's lemma). The advantage of the method lies in the fact that, once these Fourier transform estimates are obtained, the L^p inequalities are given for free, due to the neat general principles formulated in Sect. 2. The following sections present a variety of applications. Thus, we study in §3 the lacunary maximal spherical means and some of its variants. In §4, it is shown how the results of [1] for homogeneous singular integrals with no regularity can be proved without using the method of rotations, and we obtain L^p inequalities for some generalizations of this type of singular integrals, as well as A_p-weighted estimates when the kernel satisfies $|K(x)| \leqq C|x|^{-n}$. Section 5 deals with maximal functions and Hilbert transforms along curves; we give simple proofs of the main results in [18] for these operators; we also obtain L^p inequalities, $1<p<\infty$, for convex plane curves under hypothesis allowing flatness of infinite order, sharpening previous results of Nagel and Wainger [14].

We feel that these applications serve to illustrate the power and utility of the method. Apart from being technically and conceptually simpler than previous approaches, it is easy to find variants of it which are well suited to other related problems. In particular, the first named author has been able to extend the whole method to the case of multi-parameter singular integrals and maximal functions. His results, which will be the object of a forthcoming paper, give as applications L^p inequalities for maximal functions and Hilbert transforms along surfaces, etc., as well as for singular integrals of homogeneous type in product domains. Some challenges remain to test the flexibility of the method. In particular, we should like to have a theory of weighted inequalities for these operators (here we have barely initiated the path; see Corollary 4.2), and to be able to deal with operators with variable kernels, i.e., not of convolution type.

§ 2. The general theorems

As a rule, $\{\mu_k\}_{k=-\infty}^{\infty}$ and $\{\sigma_k\}_{k=-\infty}^{\infty}$ will denote two sequences of Borel measures in $\mathbb{R}^n$ such that $\|\mu_k\|=1$ and $\|\sigma_k\|\leqq 1$. However, $\mu_k \geqq 0$ (i.e., the μ_k's are probability measures) while the σ_k's satisfy some cancellation property: $\int d\sigma_k = 0$. We shall denote by $|\sigma_k|$ the total variation of σ_k, which is a positive measure. In all the statements below, $\{a_k\}_{k=-\infty}^{\infty}$ stands for a lacunary sequence of positive numbers:

$$a_k > 0 \quad \text{and} \quad \inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} = a > 1$$

and α will be a fixed constant >0. The maximal theorem is as follows:

Theorem A. *Suppose that* $\mu_k \geqq 0$ *and*

$$|\hat{\mu}_k(\xi) - 1| \leqq C\,|a_{k+1}\,\xi|^{\alpha} \tag{1}$$

$$|\hat{\mu}_k(\xi)| \leqq C\,|a_k\,\xi|^{-\alpha} \tag{1'}$$

for all $k\in\mathbb{Z}$. *Then, the maximal operators* $Mf(x)=\sup_k |\mu_k * f(x)|$ *is bounded in* $L^p(\mathbb{R}^n)$, $1<p\leqq\infty$.

For singular integral operators, the theorem to be applied is

Theorem B. *Suppose that* $\|\sigma_k\|\leqq 1$ *and*

$$|\hat{\sigma}_k(\xi)| \leqq C\,|a_{k+1}\,\xi|^{\alpha} \tag{2}$$

$$|\hat{\sigma}_k(\xi)| \leqq C\,|a_k\,\xi|^{-\alpha} \tag{2'}$$

for all $k\in\mathbb{Z}$, *and suppose also that, for some* $q>1$,

$$\|\sigma^*(f)\|_q \leqq C\,\|f\|_q \tag{3}$$

where σ^* *is the maximal operator:* $\sigma^*(f)=\sup_k \big||\sigma_k| * f\big|$. *Then, both*

$$Tf(x) = \sum_{k=-\infty}^{\infty} \sigma_k * f(x)$$

and

$$g(f)(x)=\left(\sum_{k=-\infty}^{\infty}|\sigma_k * f(x)|^2\right)^{1/2}$$

are bounded operators in $L^p(\mathbb{R}^n)$ *for* $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{2q}$.

Observe that Tf and $g(f)$ are always well defined and bounded in $L^2(\mathbb{R}^n)$ due to (2), (2′) and Plancherel's theorem. In practice, one usually has $|\sigma_k|\leqq$ Const. μ_k, with μ_k satisfying (1) and (1′), so that both theorems together give the boundedness of T in $L^p(\mathbb{R}^n)$ for all $1<p<\infty$.

The estimates (1) and (2) are only relevant when $a_{k+1}|\xi|\leqq 1$, so that it is enough to prove them for $\alpha=1$, and this is, in most cases, a trivial consequence of the fact that μ_k and σ_k are supported in the ball of radius a_{k+1} centered at the origin. Likewise, (1′) and (2′) need to be proved for large $|\xi|$ (namely, $a_k|\xi|\geqq 1$), the usual tool to establish them being Van der Corput's lemma (see [20]). In many applications, the lacunary sequence to be chosen is simply $a_k=a^k$, $-\infty<k<\infty$, for some fixed $a>1$; then (2) and (2′) take a more symmetric form:

$$|\hat{\sigma}_k(\xi)|\leqq C\min(|a^k\xi|, |a^k\xi|^{-1})^{\alpha}$$

and the same remark applies to (1) and (1′). However, in some cases, lacunary sequences, tending to 0 and $+\infty$ faster then a^k must be considered (see 5.3 below).

We shall first prove Theorem B. The hypothesis (3) will be used in the following form:

Lemma. *If* (3) *holds and* $\frac{1}{2q}=\left|\frac{1}{2}-\frac{1}{p_0}\right|$, *then, for arbitrary functions* g_k, *the following vector valued inequality holds:*

$$\|(\sum_k|\sigma_k * g_k|^2)^{1/2}\|_{p_0}\leqq C\,\|(\sum_k|g_k|^2)^{1/2}\|_{p_0}.$$

Proof. It suffices to consider the case $p_0>2$, so that $\left(\frac{p_0}{2}\right)'=q$, and there exists $u\in L^q_+$ of unit norm such that

$$\begin{aligned}\|(\sum_k|\sigma_k * g_k|^2)^{1/2}\|_{p_0}^2&=\int\sum_k|\sigma_k * g_k|^2 u\\ &\leqq\sum_k\int(|\sigma_k| * |g_k|^2)u\leqq\sum_k\int|g_k|^2\sigma^*(u)\\ &\leqq\|(\sum_k|g_k|^2)^{1/2}\|_{p_0}^2\,\|\sigma^*(u)\|_q\leqq C\,\|(\sum_k|g_k|^2)^{1/2}\|_{p_0}^2.\quad\square\end{aligned}$$

Proof of Theorem B. Let $\{\psi_j\}_{-\infty}^{\infty}$ be a smooth partition of the unity in $\mathbb{R}_+=(0,\infty)$ adapted to the intervals $[a_j^{-1}, a_{j-1}^{-1}]$. To be precise, we require the following:

$$\psi_j\in C^1,\quad 0\leqq\psi_j\leqq 1,\quad \sum_j\psi_j(t)^2=1$$

$$\operatorname{supp}(\psi_j)\subset\{t: a_{j+1}^{-1}\leqq t\leqq a_{j-1}^{-1}\}$$

$$|\psi_j'(t)|\leqq\frac{C}{t}$$

$\left(\text{this can be achieved, with } C \sim \frac{1}{a-1}\right)$. Define the multiplier operators S_j in $\mathbb{R}^n$ by $(S_j f)^\wedge(\xi) = \hat{f}(\xi)\,\psi_j(|\xi|)$. We decompose our operator T as follows:

$$Tf = \sum_k \sigma_k * \left(\sum_j S_{j+k} S_{j+k} f\right) = \sum_j \left(\sum_k S_{j+k}(\sigma_k * S_{j+k} f)\right) = \sum_j T_j f$$

$\left(\text{all the sums are } \sum_{-\infty}^{\infty}, \text{ and everything makes sense for Schwartz functions } f\right)$. First we estimate T_j in L^{p_0}, where p_0 is as in the previous lemma

$$\begin{aligned}\|T_j f\|_{p_0} &\leqq C_{p_0} \left\|\left(\sum_k |\sigma_k * S_{j+k} f|^2\right)^{1/2}\right\|_{p_0} \\ &\leqq C_{p_0}\, C \left\|\left(\sum_k |S_{j+k} f|^2\right)^{1/2}\right\|_{p_0} \leqq C_{p_0}^2\, C\, \|f\|_{p_0}.\end{aligned}$$

The first and last inequalities follow from classical Littlewood-Paley theory; see [16]. A more precise inequality is obtained in L^2 by means of Plancherel's theorem and (2), (2′):

$$\|T_j f\|_2^2 \leqq \sum_k \int_{\Delta_{j+k}} |\hat{f}(\xi)|^2\, |\hat{\sigma}_k(\xi)|^2\, d\xi$$

where $\Delta_j = \{\xi : a_{j+1}^{-1} \leqq |\xi| \leqq a_{j-1}^{-1}\}$. If $j<0$ and $\xi \in \Delta_{j+k}$ we have $a_k|\xi| \geqq a^{-j+1}$, and (2′) gives

$$\|T_j f\|_2 \leqq C(a^{-j+1})^{-\alpha}\, \|f\|_2, \qquad j<0.$$

If $j>1$ and $\xi \in \Delta_{j+k}$, we have $a_{k+1}|\xi| \leqq a^{2-j}$, and (2) gives

$$\|T_j f\|_2 \leqq C(a^{2-j})^{\alpha}\, \|f\|_2, \qquad j>1.$$

For $j=0, 1$ we use the simple estimate $|\hat{\sigma}_k(\xi)| \leqq 1$. All together, we have obtained

$$\|T_j f\|_2 \leqq \text{Const.}\, a^{-\alpha|j|}\, \|f\|_2, \qquad j \in \mathbb{Z}$$

where $a>1$ is the lacunarity constant of the sequence $\{a_k\}_{-\infty}^{\infty}$. Now, if $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2q}$, we have $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0}$ for some $0<\theta\leqq 1$, and interpolating both estimates previously obtained we get

$$\|Tf\|_p \leqq \sum_j \|T_j f\|_p \leqq \text{Const.} \sum_j a^{-\theta\alpha|j|}\, \|f\|_p = C_p\, \|f\|_p.$$

The inequality $\|g(f)\|_p \leqq C_p\, \|f\|_p$ can be proved by essentially the same argument. Alternatively, one can observe that, for every choice of $\varepsilon_k = \pm 1$, we have the same result for $T_\varepsilon f = \sum_k \varepsilon_k \sigma_k * f$ (with norm independent of $\{\varepsilon_k\}$) and a randomization argument yields the inequality for $g(f)$. □

Proof of Theorem A. Fix a positive Schwartz function Φ such that $\hat{\Phi}(0)=1$, and define Φ_k by $\hat{\Phi}_k(\xi) = \hat{\Phi}(a_k\xi)$. We can assume that $\alpha \leqq 1$ in (1) and (1′); then, the same estimates are satisfied by the measures $\{\Phi_k(x)\,dx\}_{-\infty}^{\infty}$ (denoted simply

$\{\Phi_k\}_{-\infty}^{\infty}$). Therefore, the measures $\sigma_k = \mu_k - \Phi_k$ satisfy (2) and (2′) and

$$Mf(x) \leqq \sup_k |\Phi_k * f(x)| + \left(\sum_{-\infty}^{\infty} |\sigma_k * f(x)|^2\right)^{1/2} \leqq Cf^*(x) + g(f)(x)$$

where f^* is the Hardy-Littlewood maximal function of f, and $g(f)$ is as in Theorem B. Since $g(\cdot)$ is bounded in L^2, so is M. But $\sigma^*(f) \leqq Mf + Cf^*$, and Theorem B applies with $q=2$ to the effect that $g(\cdot)$ (and therefore, also M) is bounded in L^p for $\frac{4}{3} < p < 4$. A new application of Theorem B gives $\|Mf\|_p \leqq C_p \|f\|_p$ for every $p > \frac{8}{7}$, and the process continues, every $p > 1$ being reached in a finite number of steps. □

In some of the applications, the Fourier transforms of the measures under consideration decay as required by (1), (1′), (2) and (2′) only in a certain subspace of directions. In this case, a variant of Theorems A and B, which we shall now describe, can be used. We decompose $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ with $1 \leqq m < n$, and write every $x \in \mathbb{R}^n$ in the form $x = (x^0, \bar{x})$, $x^0 \in \mathbb{R}^m$, $\bar{x} \in \mathbb{R}^{n-m}$. Given a finite measure μ in $\mathbb{R}^n$, define another measure $\mu^{(0)}$ in $\mathbb{R}^m$ as follows: $\mu^{(0)}(E) = \mu(E \times \mathbb{R}^{n-m})$ for every Borel subset E of $\mathbb{R}^m$; in terms of Fourier transforms, this means $(\mu^{(0)})^\wedge(\xi^0) = \hat{\mu}(\xi^0, 0)$.

Theorem C. *Let $\{\mu_k\}_{k=-\infty}^{\infty}$ be probability measures in $\mathbb{R}^n$ such that*

$$|\hat{\mu}_k(\xi^0, \bar{\xi}) - \hat{\mu}_k(\xi^0, 0)| \leqq C|a_{k+1}\bar{\xi}|^\alpha \tag{4}$$

$$|\hat{\mu}_k(\xi^0, \bar{\xi})| \leqq C|a_k \bar{\xi}|^{-\alpha}. \tag{4′}$$

Suppose that $M^0 g(x^0) = \sup_k |\mu_k^{(0)} * g(x^0)|$ is a bounded operator in $L^p(\mathbb{R}^m)$ for all $p > 1$. Then, $Mf(x) = \sup_k |\mu_k * f(x)|$ is also bounded in $L^p(\mathbb{R}^n)$ for all $p > 1$.

Theorem D. *Suppose that $\|\sigma_k\| \leqq 1$ and that the measures $\{\sigma_k\}_{k=-\infty}^{\infty}$ satisfy the same estimates (4), (4′) required for $\{\mu_k\}_{k=-\infty}^{\infty}$ above, and also*

$$|\hat{\sigma}_k(\xi^0, 0)| \leqq C|b_{k+1}\xi^0|^\alpha \tag{5}$$

$$|\hat{\sigma}_k(\xi^0, 0)| \leqq C|b_k \xi^0|^{-\alpha} \tag{5′}$$

where $\{b_k\}_{-\infty}^{\infty}$ is another lacunary sequence of positive numbers. If $\sigma^*(f) = \sup_k ||\sigma_k| * f|$ and $\sigma^*_{(0)}(g) = \sup_k ||\sigma_k^{(0)}| * g|$ are bounded in $L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^m)$ respectively, then Tf and $g(f)$ (defined as in Theorem B) are bounded in $L^p(\mathbb{R}^n)$ for $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2q}$.

Sometimes, we have the extra cancellation $\hat{\sigma}_k(\xi^0, 0) = 0$ (for all $k \in \mathbb{Z}$ and $\xi^0 \in \mathbb{R}^m$), which makes things a bit easier. We shall name Theorem D′ the particular case of Theorem D in which this extra cancellation is assumed, i.e., we have instead of (4) and (4′)

$$|\hat{\sigma}_k(\xi^0, \bar{\xi})| \leqq C \min(|a_{k+1}\bar{\xi}|, |a_k \bar{\xi}|^{-1})^\alpha. \tag{6}$$

Proof of Theorems C and D. We begin with Theorem D′, whose proof is but a repetition of that of Theorem B. The only difference is that we use now the decomposition $f=\sum_j S_j S_j f$ with $(S_j f)^\wedge(\xi)=\hat{f}(\xi)\psi_j(|\bar{\xi}|)$ (i.e., we apply Littlewood-Paley theory in the $\bar{\xi}$-variable). For Theorem C, we rely upon Theorem D′; we take a positive Schwartz function Φ in $\mathbb{R}^{n-m}$ with $\hat{\Phi}(0)=1$, and define Φ_k by $\hat{\Phi}_k(\bar{\xi})=\hat{\Phi}(a_k\bar{\xi})$, and

$$\sigma_k=\mu_k-\mu_k^{(0)}\otimes\Phi_k.$$

Then, $\hat{\sigma}_k(\xi^0,\bar{\xi})=\hat{\mu}_k(\xi^0,\bar{\xi})-\hat{\mu}_k(\xi^0,0)\,\hat{\Phi}(a_k\bar{\xi})$ satisfy (6) and we are ready to apply Theorem D′. We only need the boundedness of σ^*, since $\sigma^*_{(0)}g=0$. Now

$$Mf(x)\leqq\sup_k|(\mu_k^{(0)}\otimes\Phi_k)*f(x)|+g(f)(x)$$

and the maximal operator in the right hand side is dominated by the composition of M^0 acting on the x^0-variable and the Hardy-Littlewood operator acting on the $\bar{x}$-variable; thus, it is bounded in $L^p(\mathbb{R}^n)$ for all $p>1$. The rest of the argument is as in Theorem A.

Finally, to prove Theorem D in the general case, consider

$$\sigma_k^{(1)}=\sigma_k^{(0)}\otimes\Phi_k$$

with $\Phi_k\in\mathcal{S}(\mathbb{R}^{n-m})$ as above. Because of (5) and (5′), $(\sigma_k^{(1)})^\wedge(\xi^0,\bar{\xi})=\hat{\sigma}_k(\xi^0,0)\,\hat{\Phi}(a_k\bar{\xi})$ satisfy (6) with the roles of ξ^0 and $\bar{\xi}$ interchanged and with $\{b_k\}$ instead of $\{a_k\}$. The fact that $\sigma^*_{(0)}$ is bounded in $L^q(\mathbb{R}^m)$ implies the same result (in $\mathbb{R}^n$) for $\sup_k|\sigma_k^{(1)}*f|$. Therefore, by Theorem D′

$$T^{(1)}f=\sum_k\sigma_k^{(1)}*f\quad\text{and}\quad g^{(1)}(f)=\Big(\sum_k|\sigma_k^{(1)}*f|^2\Big)^{1/2}$$

are bounded in $L^p(\mathbb{R}^n)$, $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{2q}$. On the other hand, the measures $\sigma_k^{(2)}=\sigma_k-\sigma_k^{(1)}$ satisfy (6), because σ_k satisfy (4) and (4′). Therefore, the operators $T^{(2)}$ and $g^{(2)}$ defined as above with $\sigma_k^{(2)}$ in place of $\sigma_k^{(1)}$ are also bounded in $L^p(\mathbb{R}^n)$ for the same range of p's, and this ends the proof. □

The existence of the singular integrals to be studied below as the pointwise limit almost everywhere of the truncated integrals will be a consequence of the following theorem, where we write $T^*f(x)=\sup_k|T_k f(x)|$ with

$$T_k f(x)=\sum_{j=k}^{\infty}\sigma_j*f(x).$$

Theorem E. *Let σ_k be Borel measures supported in $\{x\in\mathbb{R}^n\colon|x|<a_{k+1}\}$ (resp. $\{x\in\mathbb{R}^n\colon|\bar{x}|<a_{k+1}\}$) verifying the hypotheses of Theorem B (resp. Theorem D) for all $q>1$. Then, T^* is bounded in L^p, $1<p<\infty$.*

Proof. We prove only the part corresponding to Theorem B because the other is essentially similar.

We know from Theorem B that $Tf=\sum_{j=-\infty}^{\infty}\sigma_j*f$ is bounded in L^p, $1<p<\infty$. Take $\varphi\in\mathcal{S}$ such that $\varphi(\xi)=1$ when $|\xi|<1$; write $\varphi_k(\xi)=\varphi(a_k\xi)$ and $\hat{\Phi}_k(\xi)=\varphi(a_k\xi)$. Decompose T_k as

$$T_k f=\Phi_k*\left(Tf-\sum_{j=-\infty}^{k-1}\sigma_j*f\right)+(\delta-\Phi_k)*\sum_{j=k}^{\infty}\sigma_j*f.$$

We have

$$|\Phi_k*Tf(x)|\leqq C(Tf)^*(x)$$

$$\left|\Phi_k*\sum_{j=-\infty}^{k-1}\sigma_j*f(x)\right|\leqq Cf^*(x)$$

the last inequality being a consequence of

$$\left|\Phi_k*\sum_{j=-\infty}^{k-1}\sigma_j(x)\right|\leqq Ca_k^{-n}(1+|a_k^{-1}x|^{n+1})^{-1}.$$

Therefore, the first two terms in the decomposition of T_k are bounded by operators independent of k. We must see that

$$\sup_k\left|(\delta-\Phi_k)*\sum_{j=k}^{\infty}\sigma_j*f\right|$$

is bounded in L^p, $1<p<\infty$. We have

$$\sup_k\left|(\delta-\Phi_k)*\sum_{j=0}^{\infty}\sigma_{j+k}*f\right|\leqq\sum_{j=0}^{\infty}\sup_k|(\delta-\Phi_k)*\sigma_{j+k}*f|.$$

Each one of the terms of the sum is bounded in L^p because of the boundedness of σ^*. Moreover, the j-th term has an L^2-norm of the order $a_j^{-\alpha}$ (by Plancherel's theorem). By interpolating the L^2-norm with an L^{p_0}-norm, $p_0>p$, we get a factor $a_j^{-\varepsilon}$ in the L^p-norm which makes the above sum convergent. This ends the proof of the theorem. □

The preceding theorems can be extended to the case where, instead of the ordinary (isotropic) dilations $\delta_t x=tx$, we have a homogeneous structure in $\mathbb{R}^n$ defined by the group of dilations $\{\delta_t\}_{t>0}$, $\delta_t x=t^A x$, where A is a real $n\times n$ matrix, whose eigenvalues have positive real part. The homogeneous dimension of $\mathbb{R}^n$ under these dilations is $d=$trace of A. Associated with this homogeneous structure, a C^∞ norm function $\|\cdot\|$ can be defined in $\mathbb{R}^n$ satisfying

$$\|x+y\|\leqq C(\|x\|+\|y\|),\qquad \|\delta_t x\|=t\,\|x\|.$$

The "unit sphere" $\Sigma=\{u\in\mathbb{R}^n\colon\ \|u\|=1\}$ is an ellipsoid, each point $x\in\mathbb{R}^n\backslash\{0\}$ has a unique representation in the form $x=\delta_t u$ with $u\in\Sigma$, $t=\|x\|>0$, and then, Lebesgue measure can be written as $dx=t^{d-1}\,d\sigma(u)\,dt$ for a certain Borel measure $d\sigma$ in Σ. We denote by $\|\cdot\|_*$ the norm function associated to the homogeneous structure defined by $A^*=$adjoint of A. In view of the applications we give only a simplified version of the analogues of Theorems A, B and E in this context.

Theorem F. *Let $\{\mu_k\}_{k=-\infty}^{\infty}$ and $\{\sigma_k\}_{k=-\infty}^{\infty}$ be Borel measures in $\mathbb{R}^n$ such that, for a certain constant $\alpha>0$,*

$$|\hat{\mu}_k(\xi)-\hat{\mu}_k(0)|\leqq C(2^k\|\xi\|_*)^\alpha, \quad k\in\mathbb{Z} \tag{7}$$

$$|\hat{\mu}_k(\xi)|\leqq C(2^k\|\xi\|_*)^{-\alpha}, \quad k\in\mathbb{Z} \tag{7'}$$

and the same conditions are satisfied by $\{\sigma_k\}$. We assume that $|\sigma_k|\leqq\mu_k$, $\hat{\sigma}_k(0)=0$ and $\sup_k\|\mu_k\|<\infty$. Then, the operators $Mf(x)=\sup_k|\mu_k*f(x)|$, $g(f)=(\sum_k|\sigma_k*f(x)|^2)^{1/2}$ and $Tf(x)=\sum_k\sigma_k*f(x)$ are bounded in $L^p(\mathbb{R}^n)$, $1<p<\infty$. Moreover, if $\operatorname{supp}\sigma_k\subset\{x\colon\|x\|<2^{k+1}\}$, T^* is also bounded in L^p, $1<p<\infty$.

The proof consists in a repetition of previous proofs, taking into account that the analogues of the Hardy-Littlewood maximal operator and Littlewood-Paley theory are still available in this context (see [5] and [15]).

§ 3. Lacunary maximal functions

We shall present here some immediate applications of Theorem A based on known estimates for Fourier transforms of singular measures.

Corollary 3.1. *Let Γ denote a C^∞, compact, $(n-1)$-dimensional manifold without boundary in $\mathbb{R}^n$ such that, at each point of Γ, at least one of the principal curvatures is $\neq 0$. Let Ω be a function $\geqq 0$ in Γ belonging to the Sobolev space $L^1_\alpha(\Gamma)$ for some $\alpha>0$. If $d\sigma$ denotes Lebesgue measure in Γ, then*

$$Mf(x)=\sup_k\left|\int f(x-2^k y)\,\Omega(y)\,d\sigma(y)\right|$$

is a bounded operator in $L^p(\mathbb{R}^n)$, $1<p<\infty$.

Proof. Consider the Borel measure in $\mathbb{R}^n$ (supported in Γ) $d\mu(y)=\Omega(y)\,d\sigma(y)$. Then $Mf(x)=\sup_k|\mu_k*f(x)|$ where μ_k is the 2^k-dilate of μ, so that $\hat{\mu}_k(\xi)=\hat{\mu}_k(2^k\xi)$. Without loss of generality we can assume that $\mu(\Gamma)=1$ and then $|\hat{\mu}(\xi)-1|\leqq C|\xi|$ because $\hat{\mu}$ is C^∞. By Theorem A it suffices to show that $|\hat{\mu}(\xi)|\leqq C|\xi|^{-\beta}$ for some $\beta>0$, but this follows by interpolating the inequalities

$$|(\Omega(y)\,d\sigma(y))^\wedge(\xi)|\leqq C\|\Omega\|_{L^1(\Gamma)} \quad \text{if } \Omega\in L^1(\Gamma),$$

$$|(\Omega(y)\,d\sigma(y))^\wedge(\xi)|\leqq C\|\Omega\|_{L^1_N(\Gamma)}|\xi|^{-1/n} \quad \text{if } \Omega\in L^1_N(\Gamma).$$

The second one holds for N large enough by the Sobolev embedding theorem $L^1_N(\Gamma)\hookrightarrow C^m(\Gamma)$ $(m<N-n+1)$ and the results of Littman [12]. □

This result was essentially known. When $\Gamma=S^{n-1}$ and $\Omega\equiv 1$, it was proved by C. Calderón [2] and Coifman-Weiss [6]. Their methods can be modified to yield the more general statement given above. It is noteworthy that results of the same type hold for certain singular measures in $\mathbb{R}$. In fact, if $0<\alpha<1/2$, there are singular measures μ in $\mathbb{R}$ which are compactly supported and satisfy $\hat{\mu}(\xi)=O(|\xi|^{-\alpha})$ (see [20], Chap. XII, 10.12). Therefore, we have

Corollary 3.2. *There are compactly supported singular measures $\mu \geqq 0$ in $\mathbb{R}$, with $\|\mu\| = 1$, for which we have*

$$\lim_{k \to \infty} \int f(x - 2^{-k} y)\, d\mu(y) = f(x) \qquad \text{a.e.}$$

for every $f \in L^p_{\mathrm{loc}}(\mathbb{R})$, $p > 1$.

§ 4. Singular integral operators

Even though more general results will be proved later, we begin by showing how simply the L^p-estimates for homogeneous singular integral operators can be obtained, without using the method of rotations (see [1]). Let

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} f(x - y)\, \Omega(y)\, |y|^{-n}\, dy = \lim_{\varepsilon \to 0} T_\varepsilon f(x)$$

where Ω is homogeneous of degree 0, with mean value zero over the unit sphere S^{n-1}, and $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Then $Tf = \sum_{-\infty}^{\infty} \sigma_k * f$, where $d\sigma_k(x) = \Omega(x)\, |x|^{-n}\, \chi_{[2^k, 2^{k+1}]}(|x|)\, dx$, and we have $\hat{\sigma}_k(\xi) = \hat{\sigma}_0(2^k \xi)$. Thus, in order to apply Theorem B, it suffices to prove that

$$|\hat{\sigma}_0(\xi)| \leqq C\, |\xi|, \qquad |\hat{\sigma}_0(\xi)| \leqq C\, |\xi|^{-\alpha}.$$

The first one is obvious, because σ_0 has compact support and $\hat{\sigma}_0(0) = 0$. For the second one, we have

$$\hat{\sigma}_0(\xi) = \int_{S^{n-1}} \Omega(\theta) \left\{ \int_1^2 e^{-2\pi i r \theta \cdot \xi} \frac{dr}{r} \right\} d\theta.$$

The integral in curly brackets is dominated by $|\theta \cdot \xi|^{-1}$ and also by $\log 2$. Thus, it is also majorized by $C\, |\theta \cdot \xi|^{-\alpha}$ for any $0 < \alpha < 1$, and we take α so that $\alpha q' < 1$. Then

$$|\hat{\sigma}_0(\xi)| \leqq |\xi|^{-\alpha}\, \|\Omega\|_{L^q(S^{n-1})}\, \| |\theta \cdot \xi'|^{-\alpha} \|_{L^{q'}(S^{n-1})} \leqq C\, |\xi|^{-\alpha}$$

and we have proved that T is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The fact, needed in Theorem B, that $\sup_k \| |\sigma_k| * g|$ is bounded in L^p for all $p > 1$ is a consequence of Theorem A, since we also have $|(|\sigma_0|)^\wedge(\xi)| \leqq C\, |\xi|^{-\alpha}$ (it suffices to substitute Ω by $|\Omega|$ in the above argument).

Since $\operatorname{supp} \sigma_k \subset \{x : |x| < 2^{k+1}\}$, we can also apply Theorem E and, taking into account that

$$|T_\varepsilon f(x)| \leqq \left| \sum_{j=k}^{\infty} \sigma_j * f(x) \right| + \sigma^* f(x)$$

(if $2^{k-1} \leqq \varepsilon < 2^k$) we obtain that $T^* f(x) = \sup_{\varepsilon \to 0} |T_\varepsilon f(x)|$ is also bounded in L^p for all $1 < p < \infty$. This implies that $Tf(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x)$ a.e. for all $f \in L^p$.

Singular integral operators defined by kernels which are products of a homogeneous function $\Omega(x')$ (with some regularity) and a bounded radial function $h(r)$ were first considered by R. Fefferman [7]. His results can be extended in the following way:

Corollary 4.1. *Consider the kernel* $K(x)=h(|x|)\,\Omega\left(\frac{x}{|x|}\right)|x|^{-n}$ *in* $\mathbb{R}^n$, $n>1$, *where*

a) $\int\limits_{S^{n-1}} \Omega(x')\,dx'=0$ *and* $\Omega\in L^q(S^{n-1})$ *for some* $q>1$

b) $\int\limits_0^R |h(t)|^2\,dt \leqq CR$ *for all* $R>0$.

Then, $Tf(x)=p.v.\int K(y)f(x-y)\,dy$ *exists a.e. and both* Tf *and* $T^*f(x) = \sup\limits_{\varepsilon>0}|T_\varepsilon f(x)|$ *are bounded operators in* $L^p(\mathbb{R}^n)$, $1<p<\infty$.

Proof. We write S instead of S^{n-1}. As before, we have $Tf=\sum\limits_{-\infty}^{\infty}\sigma_k*f$, where

$$\hat\sigma_k(\xi)=\int\limits_{2^k}^{2^{k+1}} h(r)\{\int\limits_S \Omega(\theta)\,e^{-2\pi i r\xi\cdot\theta}\,d\theta\}\,\frac{dr}{r}.$$

Denote by $I_r(\xi)$ the integral in curly brackets. Then

$$\begin{aligned}|\hat\sigma_k(\xi)|^2 &\leqq \left(\int\limits_{2^k}^{2^{k+1}} |h(r)|^2\,\frac{dr}{r}\right)\left(\int\limits_{2^k}^{2^{k+1}} |I_r(\xi)|^2\,\frac{dr}{r}\right)\\ &\leqq 2C\int\limits_{2^k}^{2^{k+1}} |I_r\xi)|^2\,\frac{dr}{r}.\end{aligned}$$

But

$$|I_r(\xi)|^2=\iint\limits_{S\times S}\Omega(\theta)\,\overline{\Omega(\omega)}\,e^{-2\pi i r\xi\cdot(\theta-\omega)}\,d\theta\,d\omega$$

and

$$\begin{aligned}\left|\int\limits_{2^k}^{2^{k+1}} e^{2\pi i r\xi\cdot(\theta-\omega)}\,\frac{dr}{r}\right| &\leqq C\min(1,|2^k\xi\cdot(\theta-\omega)|^{-1})\\ &\leqq C|2^k\xi|^{-\alpha}|\xi'\cdot(\theta-\omega)|^{-\alpha}\end{aligned}$$

where $\xi'=\xi/|\xi|$, and $0<\alpha<1$. We choose α so that $\alpha q'<1$, and, collecting everything, we obtain

$$\begin{aligned}|\hat\sigma_k(\xi)| &\leqq \mathrm{Const.}\,|2^k\xi|^{-\alpha/2}\{\iint\limits_{S\times S}|\Omega(\theta)\,\Omega(\omega)|\,|\xi'\cdot(\theta-\omega)|^{-\alpha}\,d\theta\,d\omega\}^{1/2}\\ &\leqq \mathrm{Const.}\,|2^k\xi|^{-\alpha/2}\,\|\Omega\|_q\left\{\iint\limits_{S\times S}\frac{d\theta\,d\omega}{|\theta_1-\omega_1|^{\alpha q'}}\right\}^{1/2q'}.\end{aligned}$$

The last integral is finite because $\alpha q'<1$. Thus, we have

$$|\hat\sigma_k(\xi)|\leqq C|2^k\xi|^{-\alpha/2}.$$

Also, since $\|\sigma_k\| = \|\Omega\|_1 \int_{2^k}^{2^{k+1}} |h(r)| \frac{dr}{r} \leqq \text{Const.}$, σ_k is supported in the ball $\{x: |x| \leqq 2^{k+1}\}$ and $\hat{\sigma}_k(0) = 0$,

$$|\hat{\sigma}_k(\xi)| \leqq C|2^k \xi|.$$

Finally, if $\mu_k = |\sigma_k|$, the same arguments apply to yield

$$|\hat{\mu}_k(\xi)| \leqq C|2^k \xi|^{-\alpha/2}, \qquad |\hat{\mu}_k(\xi) - \hat{\mu}_k(0)| \leqq C|2^k \xi|$$

(and $\hat{\mu}_k(0) = \|\sigma_k\|$ is a bounded sequence). By Theorem A,

$$Mf(x) = \sup_k |\mu_k * f(x)| \sim \sup_{R>0} R^{-n} \Big| \int_{|y|<R} f(x-y)\, \Omega(y')\, h(|y|)\, dy \Big|$$

is a bounded operator in $L^p(\mathbb{R}^n)$ if $p > 1$. By Theorems B and E, the same result holds for Tf and T^*f. □

Results of this type have also been obtained by L.K. Chen [3]. In the particular case where both Ω and h are bounded functions without any regularity, weighted estimates can also been obtained:

Corollary 4.2. *Let $K(rx') = r^{-n} h(r)\, \Omega(x')$ be a kernel in $\mathbb{R}^n$, $n > 1$ (where $r > 0$ and $|x'| = 1$), and assume that $h \in L^\infty(\mathbb{R}_+)$, $\Omega \in L^\infty(S^{n-1})$ and $\int \Omega(x')\, dx' = 0$. Then, the operator $Tf(x) = p.v.\ K * f(x)$ is bounded in $L^p(w)$ for every weight $w \in A_p$ and $1 < p < \infty$ and the same holds for T^*.*

Proof. By the extrapolation theorem for A_p weights (see [8]) it suffices to prove that T and T^* are bounded in $L^2(w)$ for every $w \in A_2$. From the preceding proof, we know that

$$Tf = \sum_{-\infty}^{\infty} \sigma_k * f \quad \text{ with } |\hat{\sigma}_k(\xi)| \leqq C \min(|2^k \xi|, |2^k \xi|^{-1})^\alpha$$

with $\alpha > 0$. We decompose T as in the proof of Theorem B

$$Tf = \sum_{j=-\infty}^{\infty} T_j f \quad \text{ with } T_j f = \sum_k S_{j+k}(\sigma_k * S_{j+k} f)$$

for a suitable Littlewood-Paley decomposition. If $w \in A_2$, we have (see [11])

$$\begin{aligned} \|T_j f\|_{L^2(w)}^2 &\leqq C_1 \sum_k \|\sigma_k * S_{j+k}\|_{L^2(w)}^2 \\ &\leqq C_1\, C_2 \sum_k \|S_{j+k} f\|_{L^2(w)}^2 \leqq C_1\, C_2\, C_3 \|f\|_{L^2(w)}^2. \end{aligned}$$

The second inequality follows from: $|\sigma_k * f(x)| \leqq C f^*(x)$ (f^* = Hardy-Littlewood maximal function) which holds due to the size of the kernel: $|K(x)| \leqq C|x|^{-n}$. By the reverse Hölder's inequality, we also have

$$\|T_j f\|_{L^2(w^{1+\varepsilon})} \leqq C \|f\|_{L^2(w^{1+\varepsilon})}, \qquad j \in \mathbb{Z}$$

for some $\varepsilon>0$ (see [8]). From the proof of Theorem B we recall that

$$\|T_j f\|_{L^2} \leqq C2^{-\alpha|j|} \|f\|_{L^2}, \quad j\in\mathbb{Z}$$

and interpolating with change of measure we obtain

$$\|T_j f\|_{L^2(w)} \leqq C2^{-\alpha\theta|j|} \|f\|_{L^2(w)}, \quad j\in\mathbb{Z}$$

$\left(\theta=\frac{\varepsilon}{1+\varepsilon}>0\right)$. This proves the result for T. By adapting the proof of Theorem E and using the same arguments as above we obtain the corresponding result for T^*. □

This result was unknown even in the case of a homogeneous kernel $K(x) =|x|^{-n}\Omega\left(\frac{x}{|x|}\right)$ with $\Omega\in L^\infty$. The same arguments can be applied (with slight modifications) to deal with more general kernels. For instance, Corollaries 4.1 and 4.2 are still true if $K(rx')=r^{-n}h(r)\Omega_r(x')$ provided that Ω_r satisfies the required conditions uniformly in r and $\Omega_r(x')$ has, as a function of r, uniformly bounded variation over each dyadic interval $[2^k, 2^{k+1}]$. Rather than pursuing this kind of generalizations, we shall try to clarify the role of the unit sphere in the proof of Corollary 4.1. We observe that, the only thing necessary for the whole proof to carry over with a more general (compact) manifold Γ instead of S^{n-1} is that

$$\iint_{\Gamma\times\Gamma} |(\theta-\omega)\cdot u|^{-\varepsilon}\, d\sigma(\theta)\, d\sigma(\omega) \leqq C<\infty$$

for some $\varepsilon>0$ and for all unit vectors u, where $d\sigma$ denotes Lebesgue measure on Γ. A sufficient condition for this is

$$\sigma(\{\theta\in\Gamma\colon \theta\cdot u\in I\}) \leqq C|I|^\delta, \quad \delta>\varepsilon$$

for every interval $I\subset\mathbb{R}$, i.e., the part of Γ lying between two close parallel hyperplanes has an "area" of the order of the distance between the hyperplanes raised to somme fixed positive power. This is certainly the case if Γ has a contact of order $\leqq k$ with every hyperplane $\left(\text{we can then take } \delta=\frac{1}{k+1}\right)$. We are in position to formulate the following extension of Corollary 4.1:

Let Γ be a compact, C^1, m-dimensional manifold in $\mathbb{R}^n$, $1\leqq m\leqq n-1$. Suppose that $0\notin\Gamma$ and that no halfline from the origin intersects Γ in more than one point. Define the "cone" $C(\Gamma)=\{r\theta\colon r>0,\ \theta\in\Gamma\}$, which is an $(m+1)$-dimensional manifold with Lebesgue measure $ds(r\theta)=r^m\,dr\,d\sigma(\theta)$. We consider a locally integrable function in $C(\Gamma)$ of the form

$$K(r\theta)=r^{-m-1}\,h(r)\,\Omega(\theta).$$

Corollary 4.3. *With the preceding conditions and notation, we assume that*

a) $\int_\Gamma \Omega(\theta)\,d\sigma(\theta)=0$, $\Omega\in L^q(\Gamma, d\sigma)$ *for some* $q>1$

b) $\int_0^R |h(t)|^2\, dt \leqq CR$ *for all* $R>0$

c) Γ *has a contact of finite order with every hyperplane. Then, the maximal operator*

$$Mf(x)=\sup_{R>0} \int_{\substack{y\in C(\Gamma)\\ R\leqq |y|\leqq 2R}} |f(x-y)\,K(y)|\,ds(y)$$

and the singular integral operator

$$\begin{aligned} Tf(x)&=\text{p.v.} \int_{C(\Gamma)} K(y)\,f(x-y)\,ds(y)\\ &=\lim_{\varepsilon\to 0} \int_{\substack{y\in C(\Gamma)\\ |y|>\varepsilon}} K(y)\,f(x-y)\,ds(y)\end{aligned}$$

(both defined a priori for Schwartz functions) have bounded extensions to $L^p(\mathbb{R}^n)$, $1<p<\infty$. *Moreover, the principal value integral defining* $Tf(x)$ *exists a.e.*

If Γ is symmetric with respect to the origin and $K(r\theta)=r^{-m-1}\Omega(\theta)$ with Ω odd, then the condition $\Omega\in L^1(\Gamma, d\sigma)$ is sufficient. This can be proved by the method of rotations, but the usual argument to deal with even kernels in no longer available. Singular integrals of the type considered in the last corollary, with "variable kernels", are partially studied in [4].

Another result for singular integrals of the type considered in [7] is

Corollary 4.4. *Let* $K(rx')=r^{-n}\Omega_r(x')$, $r>0$, $x'\in S^{n-1}$, *be a kernel in* $\mathbb{R}^n$. *Assume that, for some* $\alpha>0$, *each* Ω_r *belongs to the Sobolev space* $L^1_\alpha(S^{n-1})$, *and*

a) $\int_0^\infty \left| \int_{S^{n-1}} \Omega_r(x')\,dx'\right| \frac{dr}{r}<\infty$

b) $\int_0^R \|\Omega_r\|_{L^1_\alpha}\,dr\leqq CR$ *for all* $R>0$.

Then, $Tf(x)=p.v. \int_{\mathbb{R}^n} K(y)\,f(x-y)\,dy$ *is a bounded operator in* $L^p(\mathbb{R}^n)$, $1<p<\infty$ *and the principal value integral exists a.e.*

Proof. Let $g(r)=\frac{1}{|S^{n-1}|}\int_{S^{n-1}} \Omega_r(x')\,dx'$ and $G(y)=g(|y|)\,|y|^{-n}$, $y\in\mathbb{R}^n$. Then, a) means that $G\in L^1(\mathbb{R}^n)$, and

$$Tf(x)=G*f(x)+\sum_{-\infty}^{\infty} \sigma_k*f(x)$$

where

$$\hat\sigma_k(\xi)=\int_{2^k}^{2^{k+1}} \left\{\int_{S^{n-1}} e^{-2\pi i x'\cdot r\xi}[\Omega_r(x')-g(r)]\,dx'\right\}\frac{dr}{r}.$$

The term $G*f$ is obviously bounded. Since $\hat\sigma_k(0)=0$, we have $|\hat\sigma_k(\xi)|\leqq C|2^k\xi|$. On the other hand, from the proof of Corollary 3.1 (see estimate for $\hat\mu$ in that proof) we know that the inner integral in the above expression for $\hat\sigma_k$ is majorized by $C(\|\Omega_r\|_{L^1_\alpha}|r\xi|^{-\varepsilon}+|g(r)|\,|r\xi|^{-(n-1)/2})$. Thus, the hypothesis a) and b) together imply: $|\hat\sigma_k(\xi)|\leqq \text{Const.}\,|2^k\xi|^{-\beta}$, and Theorem B applies (observe that $\sup_k |\sigma_k|*f$ was proved to be bounded in L^p in Corollary 3.1). $\square$

When comparing this corollary with the first one in this section, we observe that the kernels considered in 4.1 are products of radial and angular functions, while the form $K(rx') = r^{-n}\Omega_r(x')$ is satisfied by any function in $\mathbb{R}^n$. However, the hypothesis in 4.4, $\Omega_r \in L^1_\alpha$, is more restrictive than $\Omega \in L^q$ for some $q > 1$. Thus, no corollary implies the other. A variant of Corollary 4.4 is the following: Suppose that $\Omega \in L^1_\alpha(S^{n-1})$ for some $\alpha > 0$, and $\int \Omega(x')\,dx' = 0$. Then

$$Tf(x) = \sum_k \int_{S^{n-1}} f(x - 2^k y')\,\Omega(y')\,dy'$$

is a bounded operator in $L^p(\mathbb{R}^n)$, $1 < p < \infty$ $(n \geqq 2)$. This is a discrete analogue of a homogeneous singular integral operator, and it is clear that the discrete version implies the continuous one:

$$\left\| \int_0^\infty \frac{dr}{r} \int_{S^{n-1}} f(x - ry')\,\Omega(y')\,dy' \right\|_p \leqq C_p \|f\|_p$$

but not conversely. It must be noticed that no such result for discrete analogues of the Hilbert transform holds in $L^p(\mathbb{R})$.

Non-isotropic analogues of the preceding results can be obtained by using Theorem F. We limit ourselves to state the result corresponding to Corollary 4.1 in this more general setting. The notation is as in § 2.

Corollary 4.5. *Consider a kernel* $K(\delta_t u) = t^{-d} h(t)\,\Omega(u)$, $u \in \Sigma$, $t > 0$, *such that*

a) $\int_\Sigma \Omega(u)\,d\sigma(u) = 0$ *and* $\Omega \in L^q(\Sigma, d\sigma)$ *for some* $q > 1$

b) $\int_0^R |h(t)|^2\,dt \leqq CR$ *for all* $R > 0$

then, $Tf(x) = \text{p.v.} \int K(y) f(x-y)\,dy$ *exists a.e. for every* $f \in L^p(\mathbb{R}^n)$ *and defines a bounded operator in* L^p, $1 < p < \infty$.

§ 5. Operators with kernels supported in curves

Given a continuous curve in $\mathbb{R}^n$ $\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$, $t \in \mathbb{R}$, such that $\gamma(0) = 0$, the maximal function and the Hilbert transform along γ are defined, at least for Schwartz functions f, as follows

$$M_\gamma f(x) = \sup_{h>0} \frac{1}{2h} \left| \int_{-h}^{h} f(x - \gamma(t))\,dt \right|$$

$$H_\gamma f(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x - \gamma(t))\,\frac{dt}{t}.$$

Let $\{\mu_k\}_{k=-\infty}^{\infty}$ and $\{\sigma_k\}_{k=-\infty}^{\infty}$ be given by

$$\int g\,d\mu_k = 2^{-k-1} \int_{2^k \leqq |t| \leqq 2^{k+1}} g(\gamma(t))\,dt$$

$$\int g\,d\sigma_k = \int_{2^k \leqq |t| \leqq 2^{k+1}} g(\gamma(t))\,\frac{dt}{t}.$$

They are all finite Borel measures. Moreover, $\mu_k \geqq 0$, $\int d\mu_k = 1$, $\int d\sigma_k = 0$ and $|\sigma_k| \leqq 2\mu_k$. It is also obvious that

$$M_\gamma f(x) \leqq 2 \sup_{k \in \mathbb{Z}} \mu_k * f(x) \qquad (f \geqq 0)$$

$$H_\gamma f(x) = \sum_{-\infty}^{\infty} \sigma_k * f(x).$$

Thus, this is the sort of operators considered in §2. In some cases, when the conditions on γ only control its behaviour near the origin, local versions of M_γ and H_γ must be considered, namely

$$\bar{M}_\gamma f(x) = \sup_{0 < h \leqq 1} \frac{1}{2h} \left| \int_{-h}^{h} f(x - \gamma(t))\, dt \right| \sim \sup_{k < 0} |\mu_k * f(x)|$$

$$\bar{H}_\gamma f(x) = \text{p.v.} \int_{-1}^{1} f(x - \gamma(t)) \frac{dt}{t} = \sum_{-\infty}^{-1} \sigma_k * f(x).$$

Corollary 5.1 (Stein and Wainger [18]). *Let*

$$\gamma(t) = \begin{cases} (p_1 t^{b_1}, p_2 t^{b_2}, \ldots, p_n t^{b_n}) & \textit{if } t \geqq 0 \\ (q_1 |t|^{b_1}, q_2 |t|^{b_2}, \ldots, q_n |t|^{b_n}) & \textit{if } t \leqq 0 \end{cases}$$

for some positive numbers $0 < b_1 < b_2 < \ldots < b_n$ *and points* $p = \gamma(1)$ *and* $q = \gamma(-1)$ *in* $\mathbb{R}^n \setminus \{0\}$. *Then,* M_γ *and* H_γ *are bounded in* $L^p(\mathbb{R}^n)$, $1 < p < \infty$ *and the principal value integral defining* H_γ *exists a.e.*

Proof. We shall prove the maximal theorem by induction on n (the dimension of the underlying space). For $n=1$, M_γ is controlled by the Hardy-Littlewood maximal operator. We assume the result to be true for curves in $\mathbb{R}^{n-1}$ of the type considered, and decompose the space in the form $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, so that $\bar{\xi} = \xi_n$ and $\xi^0 = (\xi_1, \xi_2, \ldots, \xi_{n-1})$. The crucial estimate for $\hat{\mu}_0(\xi)$, which is obtained by means of Van der Corput's lemma is

$$|\hat{\mu}_0(\xi)| = \left| \int_{1 \leqq |t| \leqq 2} e^{-2\pi i \xi \cdot \gamma(t)}\, dt \right| \leqq C |\xi|^{-1/n}$$

(this is elementary with exponent $-1/b_n$ if $b_1, \ldots, b_n$ are natural numbers and if they are rational a change of variables reduces this case to that of $b_j \in \mathbb{N}$; for the general case, see [18]).

Since $\hat{\mu}_k(\xi) = \hat{\mu}_0(2^{kb_1} \xi_1, 2^{kb_2} \xi_2, \ldots, 2^{kb_n} \xi_n)$, we have in particular

$$|\hat{\mu}_k(\xi)| \leqq C |2^{kb_n} \xi_n|^{-1/n}$$

and also

$$|\hat{\mu}_k(\xi^0, \xi_n) - \hat{\mu}_k(\xi^0, 0)| \leqq \int_{1 \leqq |t| \leqq 2} |\exp(2\pi i \xi_n \gamma_n(2^k t) - 1|\, dt \leqq C |2^{kb_n} \xi_n|.$$

These are the estimates (4) and (4′) needed to apply Theorem C $\left(\text{with } \alpha = \frac{1}{n};\ a_k = 2^{kb_n}\right)$. Since M^0 is bounded in $L^p(\mathbb{R}^{n-1})$ for all $p > 1$ by the induction hypothesis, the boundedness of M_γ is completely proved.

In order to estimate $H_\gamma f=\sum_k \sigma_k * f$, we decompose the operator into n essentially one-dimensional operators. For each $0 \leqq j \leqq n$, let $\{\sigma_k^{(j)}\}_{k=-\infty}^{\infty}$ be defined in terms of Fourier transforms as

$$\widehat{\sigma_k^{(j)}}(\xi)=\hat{\sigma}_k(\xi_1, \xi_2, \ldots, \xi_j, 0, \ldots, 0)\, \hat{\Phi}(2^{kb_{j+1}} \xi_{j+1}) \ldots \hat{\Phi}(2^{kb_n} \xi_n)$$

where $\Phi \in \mathscr{S}(\mathbb{R})$ and $\hat{\Phi}(0)=1$; in particular, $\sigma_k^{(n)}=\sigma_k$ and $\sigma_k^{(0)}=0$, because $\hat{\sigma}_k(0)=0$, so that

$$\sigma_k=\sum_{j=1}^{n} [\sigma_k^{(j)}-\sigma_k^{(j-1)}]$$

and

$$H_\gamma f=\sum_{j=1}^{n} H^{(j)} f \quad \text{with} \quad H^{(j)} f=\sum_{k=-\infty}^{\infty} (\sigma_k^{(j)}-\sigma_k^{(j-1)}) * f.$$

Now, it is also known (and this is a simple consequence of the corresponding estimate for μ_0) that

$$|\hat{\sigma}_0(\xi)|=\left|\int_{1 \leqq |t| \leqq 2} e^{-2\pi i \xi \cdot \gamma(t)} \frac{dt}{t}\right| \leqq C|\xi|^{-1/n}.$$

Therefore, $|\hat{\sigma}_k(\xi)| \leqq C|2^{kb_j} \xi_j|^{-1/n}$. Since $\hat{\sigma}_k$ and $\hat{\Phi}$ are uniformly bounded, and $\hat{\Phi}(t) \leqq \frac{C}{|t|}$, we have

$$|[\sigma_k^{(j)}-\sigma_k^{(j-1)}]^\wedge(\xi)| \leqq C|2^{kb_j} \xi_j|^{-1/n}.$$

Also

$$|[\sigma_k^{(j)}-\sigma_k^{(j-1)}]^\wedge(\xi)| \leqq C|2^{kb_j} \xi_j|$$

and both estimates together are equivalent to (6) for an appropriate decomposition of $\mathbb{R}^n$. If $\{\mu_k^{(j)}\}_{k=-\infty}^{\infty}$ is defined as $\sigma_k^{(j)}$ above, the maximal theorem for the curve $(\gamma_1(t), \gamma_2(t), \ldots, \gamma_j(t))$ in $\mathbb{R}^j$ implies that $\sup_k |\mu_k^{(j)} * f(x)|$ is a bounded operator in $L^p(\mathbb{R}^n)$ for all $p>1$. Since $|\sigma_k^{(j)}| \leqq 2\mu_k^{(j)}$, Theorem D (actually, the weaker version, Theorem D') can be applied to yield $\|H^{(j)} f\|_p \leqq C_p \|f\|_p$, $1<p<\infty$, and this finishes the proof for H_γ. The a.e. convergence follows from Theorem E if we take $\operatorname{supp} \Phi \subset [-1, 1]$. □

The preceding proof has been given in order to illustrate the applicability of Theorems C and D. Observe that no use was made of the non-isotropic dilations naturally attached to the curve γ. However, by using these dilations and Theorem F, a more direct proof can be given for a more general class of curves: $\gamma(t)$ is a *two-sided homogeneous curve* if

$$\gamma(t)=\delta_t p \ \text{ if } t>0, \qquad \gamma(t)=\delta_{-t} q \ \text{ if } t<0, \qquad \gamma(0)=0$$

for certain $p, q \in \mathbb{R}^n \setminus \{0\}$ and for some group of generalized dilations $\{\delta_t\}_{t>0}$ and if $\{\gamma(t)\}_{t>0}$, $\{\gamma(t)\}_{t<0}$ span the same linear space. Then, we have the following result, which is also proved in [18] by more complicated methods:

Corollary 5.2. *Let $\gamma(t)$ be a two-sided homogeneous curve. Then, M_γ, H_γ and H_γ^* (the maximal Hilbert transform along γ) are bounded in $L^p(\mathbb{R}^n)$, $1<p<\infty$.*

The estimates $|\hat{\mu}_k(\xi)|+|\hat{\sigma}_k(\xi)| \leqq C(2^k \|\xi\|_*)^{-\beta}$ needed in the proof of Corollary 5.2 are also known to hold (for $k<k_0$) if the curve γ is *approximately*

homogeneous (see [19] for the definition and the estimates). Therefore, for this class of curves, $\bar{M}_\gamma$, $\bar{H}_\gamma$ and $\bar{H}^*_\gamma$ are bounded in L^p, $1<p<\infty$. In [19] this result is only obtained for $p=2$. A particular case of approximately homogeneous curves for which the whole range $1<p<\infty$ was known (see [18]) consists of the C^∞ curves $\gamma(t)$ such that $\gamma(0)=0$ and the derivatives $\{\gamma^{(j)}(0)\}_{j\in\mathbb{N}}$ span $\mathbb{R}^n$. We shall now give a positive result for a class of curves in $\mathbb{R}^2$ which may have a contact of infinite order with their tangent at the origin (after a rotation, this means that $\gamma(t)=(t,\varphi(t))$ with $\varphi^{(j)}(0)=0$ for all j, so that $\{\gamma^{(j)}(0)\}=\{(1,0)\}$ does not span $\mathbb{R}^2$).

Corollary 5.3. *Let $\gamma(t)=(t,\varphi(t))$ be a C^1 curve in $\mathbb{R}^2$ such that $\varphi(0)=\varphi'(0)=0$. Suppose that φ is either even or odd and that $\varphi'(t)$ is a convex increasing function for $t>0$. Then, M_γ, H_γ and H^*_γ are bounded in $L^p(\mathbb{R}^2)$, $1<p<\infty$.*

The Fourier transform estimates needed for the proof are contained in the following

Lemma. *Let $h_k(t)=2^k t\xi_1+\varphi(2^k t)\xi_2$, with φ as above and $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$, $k\in\mathbb{Z}$. Then, for $1\leqq R\leqq 2$, we have*

$$\left|\int_1^R e^{-2\pi i h_k(t)}\,dt\right|\leqq C|2^k\xi_1|^{-1/2}$$

where C is a constant independent of ξ, k and R.

Proof of Corollary 5.3. We take momentarily the lemma for granted. By the symmetry of $\gamma(t)$, we have

$$|\hat{\mu}_k(\xi)|\leqq\left|\int_1^2 e^{-2\pi i h_k(t)}\,dt\right|+\left|\int_1^2 e^{-2\pi i h_k(-t)}\,dt\right|\leqq C|2^k\xi_1|^{-1/2}.$$

On the other hand

$$|\hat{\mu}_k(\xi_1,\xi_2)-\hat{\mu}_k(0,\xi_2)|\leqq\int_1^2|\exp(-2\pi i 2^k t\xi_1)-1|\,dt\leqq C|2^k\xi_1|.$$

We are now in position to apply Theorem C (with the order of ξ^0 and $\bar{\xi}$ reversed: $\xi^0=\xi_2$, $\bar{\xi}=\xi_1$). It only remains to check the boundedness in $L^p(\mathbb{R})$ for all $p>1$ of the maximal operator $\sup_k|\mu_k^{(0)}*g(x)|$, which is equivalent to

$$\sup_{h>0}\frac{1}{2h}\left|\int_{-h}^{h} g(x-\varphi(t))\,dt\right|.$$

But this is controlled by the Hardy-Littlewood operator, since, for $g\geqq 0$

$$\frac{1}{h}\int_0^h g(x-\varphi(t))\,dt=\frac{1}{h}\int_0^{\varphi(h)} g(x-u)\frac{du}{\varphi'(\varphi^{-1}(u))}\leqq g^*(x)$$

because $\dfrac{1}{h\varphi'(\varphi^{-1}(u))}$ is decreasing and its integral over $[0,\varphi(h)]$ is equal to 1. This completes the proof of the result for M_γ.

The L^p-boundedness of H_γ and H_γ^* will be a consequence of Theorems D and E (again, with $\xi^0=\xi_2$ and $\bar{\xi}=\xi_1$). By the previous lemma and integration by parts:

$$\left|\int_1^2 e^{-2\pi i h_k(t)}\frac{dt}{t}\right|\leqq C|2^k\xi_1|^{-1/2}$$

and the same is true for $h_k(-t)$. Therefore, $|\hat{\sigma}_k(\xi)|\leqq C|2^k\xi_1|^{-1/2}$, and also $|\hat{\sigma}_k(\xi_1,\xi_2)-\hat{\sigma}_k(0,\xi_2)|\leqq C|2^k\xi_1|$ (the proof is as for the μ_k's). We have the control of σ^* and $\sigma^*_{(0)}$, because $|\sigma_k|\leqq 2\mu_k$, $|\sigma_k^{(0)}|\leqq 2\mu_k^{(0)}$. Now, if $\varphi(t)$ is even, $\hat{\sigma}_k(0,\xi_2)=0$ and the result is proved as a consequence of Theorem D′ (the weaker version of Theorem D). If $\varphi(t)$ is odd, we only have to prove the estimates (5), (5′) for $\hat{\sigma}_k(0,\xi_2)$. This we shall do with the following choice of the lacunary sequence $\{b_k\}_{k=-\infty}^{\infty}$: $b_k=\varphi(2^k)$. Since φ is convex and increasing in $(0,\infty)$, $\dfrac{\varphi(t)}{t}$ is also increasing for $t>0$, and this proves the lacunarity: $b_{k+1}/b_k\geqq 2$ for all $k\in\mathbb{Z}$. Let us prove (5):

$$\begin{aligned}|\hat{\sigma}_k(0,\xi_2)|&\leqq \int_{1\leqq|t|\leqq 2}|\exp(-2\pi i\varphi(2^k t)\xi_2)-1|\frac{dt}{|t|}\\&\leqq C|\varphi(2^{k+1})\xi_2|=C|b_{k+1}\xi_2|.\end{aligned}$$

On the other hand,

$$\left|\frac{d}{dt}(\varphi(2^k t)\xi_2)\right|=|2^k\varphi'(2^k t)\xi_2|\geqq\left|\frac{\varphi(2^k t)}{t}\xi_2\right|\geqq\frac{1}{2}|b_k\xi_2|$$

if $1\leqq|t|\leqq 2$, and Van der Corput's lemma gives: $|\hat{\sigma}_k(0,\xi_2)|\leqq C|b_k\xi_2|^{-1}$, finishing the proof. □

Proof of the lemma. We shall only use the fact that $\dfrac{\varphi'(t)}{t}$ is increasing for $t>0$, and the constant C will be independent of the particular function φ. Thus, we can substitute $2^k\xi_1$ by ξ_1 and $\varphi(2^k t)$ by $\varphi(t)$, which amounts to assuming $k=0$. We can also assume that $\xi_2>0$. If $\xi_1\geqq 0$, we have

$$h'(t)=\xi_1+\varphi'(t)\xi_2\geqq|\xi_1|$$

and Van der Corput's lemma gives the result. If $\xi_1<0$, there is a unique $t_0>0$ such that

$$h'(t_0)=\xi_1+\varphi'(t_0)\xi_2=0.$$

We take $t_1=\min(t_0,2)$, $\delta=|\xi_1|^{-1/2}$, and decompose

$$\int_1^R e^{-2\pi i h(t)}dt=\int_{I_1}+\int_{I_2}+\int_{I_3}$$

where $I_1=[1,R]\cap[t_1-\delta,t_1+\delta]$, $I_2=[1,t_1-\delta]$ and $I_3=[t_1+\delta,R]$. Then, trivially, $\left|\int_{I_1}\right|\leqq 2\delta=2|\xi_1|^{-1/2}$. We shall prove that $|h'(t)|\geqq\frac{\delta}{2}|\xi_1|=\frac{1}{2}|\xi_1|^{1/2}$ if $t\in I_2\cup I_3$; since $h'(t)$ is monotone, this will prove the lemma by a new appeal to

Van der Corput's result. First, if $t \in I_2$

$$h'(t) \leqq \xi_1 + \varphi'(t_1) \frac{t}{t_1} \xi_2 \leqq \left(1 - \frac{t}{t_1}\right) \xi_1 \leqq \frac{-\delta}{2} |\xi_1|.$$

On the other hand, $I_3 = \emptyset$ unless $t_1 = t_0 \leqq 2$, and in this case, if $t \in I_3$

$$h'(t) \geqq \xi_1 + \varphi'(t_0) \frac{t}{t_0} \xi_2 = \xi_1 \left(1 - \frac{t}{t_0}\right) \geqq \frac{\delta}{2} |\xi_1|$$

and the proof is ended. □

If $a > 0$, the function $\varphi(t)$ defined by $\varphi(t) = e^{-1/|t|^a}$ for $0 \leqq t \leqq \varepsilon$ (ε small enough) and suitably continuated for $t > \varepsilon$ satisfies the hypothesis of Corollary 5.3, and the curve $(t, \varphi(t))$ has a contact of infinite order with the line $(t, 0)$ at the point $(0, 0)$. We remark that, for M_γ to be bounded, there is no need of having any relation between $\varphi(t)$ and $\varphi(-t)$. However, for H_γ to be bounded, even in L^2, some cancellation between $\gamma_+ = \{(t, \varphi(t))\}_{t>0}$ and $\gamma_- = \{(t, \varphi(t)\}_{t<0}$ must exist. The preceding proof can be carried over under the assumption $\varphi(-t) \sim \varphi(t)$ or $\varphi(-t) \sim -\varphi(t)$ (instead or evenness or oddness). Under a slightly more restrictive hypothesis (γ was assumed to be C^2) the boundedness of M_γ in L^2 was established in [18], and the fact that H_γ is bounded in L^p, $\frac{5}{3} < p < \frac{5}{2}$, was proved in [14]. We have also been informed that M. Christ has recently obtained a result quite similar to our Corollary 5.3.

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