Invent. math. 84, 177–194 (1986)

Inventiones mathematicae

Existence and non-existence of homogeneous Einstein metrics

McKenzie Y. Wang[1] and Wolfgang Ziller[2]*

[1] Department of Mathematical Sciences, McMaster University, Hamilton, Ontario, Canada, L8S 4K1

[2] Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA

In this paper we first prove a general existence theorem for homogeneous Einstein metrics and then we exhibit some compact simply connected homogeneous spaces which carry no homogeneous Einstein metric.

A Riemannian metric g is called Einstein if $\mathrm{Ric}(g) = c \cdot g$ for some constant c. If $c > 0$ most known examples are homogeneous, see [Be]. Previous constructions of homogeneous Einstein metrics were usually achieved by more or less explicit calculations on special families of homogeneous spaces, see, e.g. [Je 1,2], [DZ], or [WZ 1].

On the other hand, the Einstein metrics of volume 1 on a compact manifold are precisely the critical points of the total scalar curvature functional $T(g) = \int_M S(g)\, d\,\mathrm{vol}_g$ on the space of Riemannian metrics of volume 1. This suggests a variational approach to finding Einstein metrics, which so far has not been successful.

If G/H is a compact homogeneous space, we can restrict T to the subset of G-invariant metrics of volume 1. The critical points of the restriction of T are again precisely the G-invariant Einstein metrics of volume 1. In this paper we examine when T is bounded from above or below and/or proper. In particular we prove the following

Theorem. *Let G be a connected compact Lie group and H a connected closed subgroup such that G/H is effective. Then the functional T on the set of G-invariant metrics with volume 1 is bounded from above and proper iff H is a maximal connected subgroup of G. For such a G/H, T assumes its global maximum at a G-invariant metric which must be Einstein.*

By contrast, as we will see in § 2, it seldom happens that T is bounded from above but not proper or that T is bounded from below.

* The first author acknowledges partial support from the Mathematical Sciences Research Institute at Berkeley and from the Natural Sciences and Engineering Research Council of Canada. The second author is partially supported by a grant from the Alfred P. Sloan Foundation and a grant from the National Science Foundation

Combined with [Dy 2], Theorem A yields immediately, without explicit calculation, numerous new homogeneous Einstein manifolds. For example, apart from a short finite list of exceptions, every irreducible representation π of a compact simple group H gives rise to a homogeneous Einstein manifold $SO(n)/\pi(H)$, $Sp\left(\frac{n}{2}\right)\Big/\pi(H)$, or $SU(n)/\pi(H)$, depending on whether π is orthogonal, symplectic, or non-self-contragredient.

Next, we exhibit some compact simply connected homogeneous spaces G/H which carry no G-invariant Einstein metric. These G/H have the property that every G-invariant metric is obtained from a fixed Riemannian submersion $K/H \to G/H \to G/K$ by re-scaling the metrics on the fibre and base. Hence the Einstein condition reduces to a quadratic equation in one variable which in some cases has no real roots. It is amusing to note that the trick of re-scaling the fibre in a Riemannian submersion has been frequently used in the past to produce homogeneous Einstein metrics.

Our lowest dimensional non-existence example is the 12-dimensional manifold $SU(4)/SU(2)$ where $SU(2) \subset Sp(2) \subset SU(4)$ and $SU(2)$ is the unique maximal connected subgroup of $Sp(2)$. We will show that no other Lie group acts transitively on $SU(4)/SU(2)$. Hence it carries no homogeneous Einstein metric whatsoever. We do not know if any of these manifolds admit a non-homogeneous Einstein metric.

Recall that R. Hamilton showed [Ha] that if g is a metric of volume 1 on a compact 3-manifold M with $\mathrm{Ric}(g) > 0$, then there is a smooth 1-parameter family of metrics g_t of volume 1 with $g_0 = g$ which is a solution of the natural evolution equation

$$\frac{\partial g_t}{\partial t} = -2\left(\mathrm{Ric}(g_t) - \frac{1}{\dim M}\, T(g_t)\, g_t\right).$$

Moreover, g_t converges as $t \to \infty$ to a smooth Einstein metric, which must have constant sectional curvature since the dimension is 3. Hamilton's equation is a slightly modified version of the equation for the gradient flow of T. g_t is a solution of the gradient flow of T if

$$\frac{\partial g_t}{\partial t} = -2\left(\mathrm{Ric}(g_t) - \frac{1}{\dim M}\, S(g_t)\, g_t\right).$$

If $\dim M > 3$, short-time existence and uniqueness of solutions of Hamilton's evolution equation still hold. However, our non-existence examples show that in general a solution curve g_t will not converge, even if g_0 has positive Ricci curvature and non-negative sectional curvature. This is seen as follows. Since Hamilton's equation is invariant under $\mathrm{Diff}(M)$, by uniqueness, $\mathrm{Isom}(g_0) \subset \mathrm{Isom}(g_t)$ for all $t \geq 0$ for which g_t exists. If $G \subset \mathrm{Isom}(g_0)$ acts transitively on M, then g_t will also be G-invariant and since $S(g_t) = T(g_t)$, the flow of Hamilton's equation on the set of G-invariant metrics agrees with the gradient flow of T. Since the set of G-invariant metrics of volume 1 on M is a smooth finite dimensional manifold, a solution curve g_t has the property that if t_{max} is the maximal time for which g_t exists, then as $t \to t_{max}$ either g_t converges to a G-invariant Einstein metric or g_t goes off to ∞ on the set of G-invariant metrics of volume 1. Hence any solution curve of Hamilton's equation with homogeneous initial metric on one of our non-existence examples will go off to ∞. Moreover, since every compact simply connected homogeneous space carries a normal homogeneous metric with positive Ricci curvature and non-negative sectional curvature, we may choose such a metric as initial metric.

Exactly the same conclusion holds for any differential equation $\frac{\partial g_t}{\partial t}=F(g_t)$ (necessarily invariant under $\mathrm{Diff}(M)$) that satisfies short time existence and uniqueness, and has the property that all stationary points are Einstein metrics.

§ 1. The scalar curvature functional

Let G be a compact connected Lie group and H a closed subgroup such that G acts effectively on $M=G/H$, i.e. H contains no nontrivial normal subgroup of G. The assumption of effectiveness is not necessary but is convenient. Hence in giving actual examples of G/H we will not worry about effectiveness. We will also assume that H is connected, which is automatically the case if G/H is simply connected.

For a Riemannian metric g on M we denote by $S(g)$ its scalar curvature and by $T(g)$ its total scalar curvature $\int_M S(g)\,d\,\mathrm{vol}_g$. Then, on the set $\mathcal{M}$ of Riemannian metrics on M with volume 1, the critical points of T are precisely the Einstein metrics on M, as follows from the first variation formula ([Hi] or [Bg] p. 289)

$$d\,T_g(h)=\int_M \left\langle \frac{S}{n}\,g-\mathrm{Ric}(g),\,h\right\rangle d\,\mathrm{vol}_g,$$

where h is any symmetric 2-tensor on M such that $\int_M (\mathrm{tr}_g h)\,d\,\mathrm{vol}_g=0$ and $n=\dim M$.

Let $\mathcal{M}_G$ be the set of G-invariant metrics of volume 1 on M. Note that on $\mathcal{M}_G$, $T(g)=S(g)$. The set of critical points of $T|\mathcal{M}_G$ are precisely the G-invariant Einstein metrics of volume 1 on M since at a critical point g of $T|\mathcal{M}_G$ we can choose h to be the G-invariant symmetric 2-tensor $\frac{S}{n}\,g-\mathrm{Ric}(g)$.

Let B be the negative of the Killing form of $\mathfrak{g}$. Then $B(X,X)\geqq 0$ with equality iff $X\in\mathfrak{z}(\mathfrak{g})$. We fix once and for all a biinvariant metric Q on $\mathfrak{g}$ such that the induced normal homogeneous metric g_Q on G/H has volume 1.

Next we consider the $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ with $Q(\mathfrak{h},\mathfrak{m})=0$. Then the set of G-invariant metrics on G/H can be identified with the set of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$.

Let $\langle\ ,\ \rangle$ be an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$ and $\{e_\alpha\}$ be a basis of $\mathfrak{m}$ orthonormal with respect to $\langle\ ,\ \rangle$. Then one has the following formula for the scalar curvature of $\langle\ ,\ \rangle$ (see e.g. [Be], (7.39) or [Je 2], p. 1130):

(1.1) $$S=\tfrac{1}{2}\sum_\alpha B(e_\alpha,e_\alpha)-\tfrac{1}{4}\sum_{\alpha,\beta}\langle [e_\alpha,e_\beta]_{\mathfrak{m}},[e_\alpha,e_\beta]_{\mathfrak{m}}\rangle$$

where $[\ ,\ ]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component.

We now examine this formula more closely. Let $\mathfrak{m}=\mathfrak{m}_1\oplus\ldots\oplus\mathfrak{m}_r\oplus\mathfrak{m}_0$ be a Q-orthogonal $\mathrm{Ad}(H)$-invariant decomposition such that $\mathrm{Ad}(H)|\mathfrak{m}_0=id$ and $\mathrm{Ad}(H)|\mathfrak{m}_i$ is irreducible for $i=1,\ldots,r$. Such a decomposition is not unique if some of the representations of $\mathrm{Ad}(H)$ on $\mathfrak{m}_i$ are equivalent to each other. But the subspace $\mathfrak{m}_0$ and the numbers $d_i=\dim\mathfrak{m}_i$ are independent of the chosen decomposition.

The structure of $\mathscr{M}_G$ can be described in terms of a fixed decomposition. By Schur's lemma, $\mathscr{M}_G$ is diffeomorphic to the positive definite elements in

$$\mathbf{R}^r\times\Big[\prod_{1\leq i<j\leq r}\mathrm{Hom}_{\mathrm{Ad}\,H}(\mathfrak{m}_i,\mathfrak{m}_j)\Big]\times S^2(\mathfrak{m}_0)$$

with volume 1. The volume condition is equivalent to requiring the matrix of an element of $\mathscr{M}_G$ (with respect to a Q-orthonormal basis of $\mathfrak{m}$) to have determinant 1. Thus $\mathscr{M}_G$ is diffeomorphic to a smooth hypersurface lying in an open cone in some Euclidean space.

This description of $\mathscr{M}_G$ with respect to a fixed decomposition of $\mathfrak{m}$ is, however, rather complicated to work with. Instead, we shall vary the decomposition of $\mathfrak{m}$ and study $\mathscr{M}_G$ and T this way.

First we decompose $\mathfrak{m}_0$ further into Q-orthogonal 1-dimensional subspaces $\mathfrak{m}_0=\mathfrak{m}_{r+1}\oplus\ldots\oplus\mathfrak{m}_s$. For the rest of this paper we will refer to a Q-orthogonal $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{m}=\mathfrak{m}_1\oplus\ldots\oplus\mathfrak{m}_r\oplus\mathfrak{m}_{r+1}\oplus\ldots\oplus\mathfrak{m}_s$ as above simply as a *decomposition* of $\mathfrak{m}$. For each decomposition there is the family of $\mathrm{Ad}(H)$-invariant "*diagonal*" metrics: namely, those given by

$$\langle\ ,\ \rangle=x_1Q|\mathfrak{m}_1\perp\ldots\perp x_sQ|\mathfrak{m}_s,\qquad x_i>0. \tag{1.2}$$

Every $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$ belongs to the family of $\mathrm{Ad}(H)$-invariant diagonal metrics of *some* decomposition of $\mathfrak{m}$. This is seen as follows. For a given $\mathrm{Ad}(H)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$, we first diagonalize $\langle\ ,\ \rangle$ with respect to Q to obtain a decomposition of $\mathfrak{m}$ into eigenspaces of $\langle\ ,\ \rangle$, which are orthogonal with respect to both Q and $\langle\ ,\ \rangle$. These eigenspaces are $\mathrm{Ad}(H)$-invariant, and so can be further decomposed into irreducible summands which are orthogonal with respect to Q and $\langle\ ,\ \rangle$. Then $\langle\ ,\ \rangle$ has the form (1.2) with respect to this decomposition, where the x_i's are the eigenvalues of $\langle\ ,\ \rangle$ with respect to Q. Note that it can happen that the same $\mathrm{Ad}(H)$-invariant inner product can be diagonal with respect to many different decompositions of $\mathfrak{m}$.

For a fixed Q-orthogonal $\mathrm{Ad}(H)$-invariant decomposition of $\mathfrak{m}$, the scalar curvature of the metrics of the form (1.2) has a nice expression. Let $\{e_\alpha\}$ be a Q-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e., $e_\alpha\in\mathfrak{m}_i$ for some i, and $\alpha<\beta$ if $i<j$. Next set $A^\gamma_{\alpha\beta}=Q([e_\alpha,e_\beta],e_\gamma)$, so that $[e_\alpha,e_\beta]_{\mathfrak{m}}=\sum_\gamma A^\gamma_{\alpha\beta}e_\gamma$, and set $\begin{bmatrix}k\\ ij\end{bmatrix}=\sum(A^\gamma_{\alpha\beta})^2$, where the sum in taken over all indices α,β,γ with $e_\alpha\in\mathfrak{m}_i$, $e_\beta\in\mathfrak{m}_j$, and $e_\gamma\in\mathfrak{m}_k$. Note that $\begin{bmatrix}k\\ ij\end{bmatrix}$ is independent of the Q-orthonormal bases chosen for $\mathfrak{m}_i$, $\mathfrak{m}_j$, and $\mathfrak{m}_k$, but it definitely depends on the choice of the decomposition of $\mathfrak{m}$. $\begin{bmatrix}k\\ ij\end{bmatrix}$ is a continuous function on the space of all Q-

orthogonal ordered decompositions of $\mathfrak{m}$ into $\mathrm{Ad}(H)$-irreducible summands. Notice also that $\begin{bmatrix} k \\ ij \end{bmatrix}$ is symmetric in all 3 indices since $A_{\alpha\beta}^{\gamma}$ is skew-symmetric in all 3 indices.

Let $\langle\ ,\ \rangle$ be an $\mathrm{Ad}(H)$-invariant metric having the form (1.2) with respect to a fixed decomposition of $\mathfrak{m}$. Then $\{e_\alpha/\sqrt{x_i}\,|\,e_\alpha\in\mathfrak{m}_i\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$. By (1.1) the scalar curvature of $\langle\ ,\ \rangle$ is given by

$$S=\frac{1}{2}\sum_{i=1}^{s}\frac{d_i b_i}{x_i}-\frac{1}{4}\sum_{i,j,k}\begin{bmatrix} k \\ ij \end{bmatrix}\frac{x_k}{x_i x_j}, \tag{1.3}$$

where $B|\mathfrak{m}_i=b_i Q|\mathfrak{m}_i$. We have $b_i\geqq 0$ with $b_i=0$ iff $\mathfrak{m}_i\subset\mathfrak{z}(\mathfrak{g})$, and $\begin{bmatrix} k \\ ij \end{bmatrix}\geqq 0$ with $\begin{bmatrix} k \\ ij \end{bmatrix}\equiv 0$ iff $Q([\mathfrak{m}_i,\mathfrak{m}_j],\mathfrak{m}_k)=0$.

(1.4) *Remark.* If for a fixed decomposition of $\mathfrak{m}$ we have $\mathrm{Ric}_g(\mathfrak{m}_i,\mathfrak{m}_j)=0$ whenever $i\neq j$ for all metrics g of the form (1.2) with respect to the decomposition, then the first variation formula for T implies that the critical points for S in (1.3) on the set $\prod_{i=1}^{s} x_i^{d_i}=1$ are Einstein metrics on G/H. But in general if $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are equivalent $\mathrm{Ad}(H)$-representations, then $\mathrm{Ric}(\mathfrak{m}_i,\mathfrak{m}_j)$ may be non-zero, and hence the critical points of (1.3) will not necessarily be Einstein metrics. Nevertheless (1.3) will be sufficient for us to examine the global behavior of S on $\mathscr{M}_G$.

In Sect. 3 we will need the following relationship between $\begin{bmatrix} k \\ ij \end{bmatrix}$ and the Casimir operator $C_{\mathfrak{m}_i,Q|\mathfrak{h}}=-\sum_i \mathrm{ad}\,z_i\circ\mathrm{ad}\,z_i$, where z_i is an orthonormal basis of $\mathfrak{h}$ with respect to $Q|\mathfrak{h}$. Since $\mathfrak{m}_i$ is $\mathrm{Ad}(H)$-irreducible $C_{\mathfrak{m}_i,Q|\mathfrak{h}}=c_i\cdot Id$ with $c_i\geqq 0$ and $c_i=0$ iff $\mathfrak{m}_i\subset\mathfrak{m}_0$.

(1.5) **Lemma.** $\sum_{j,k}\begin{bmatrix} k \\ ij \end{bmatrix}=d_i(b_i-2c_i)$.

Proof.

$$\begin{aligned}
\sum_{j,k}\begin{bmatrix} k \\ ij \end{bmatrix} &= \sum_{e_\alpha\in\mathfrak{m}_i}\sum_{\beta,\gamma} Q([e_\alpha,e_\beta],e_\gamma)^2\\
&= \sum_{e_\alpha\in\mathfrak{m}_i}\sum_{\beta} Q([e_\alpha,e_\beta]_{\mathfrak{m}},[e_\alpha,e_\beta]_{\mathfrak{m}})\\
&= \sum_{e_\alpha\in\mathfrak{m}_i} -\mathrm{tr}_{\mathfrak{m}}(\mathrm{pr}_{\mathfrak{m}}\circ\mathrm{ad}\,e_\alpha)^2\\
&= \sum_{e_\alpha\in\mathfrak{m}_i} (-\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\,e_\alpha)^2+2\,\mathrm{tr}_{\mathfrak{h}}(\mathrm{pr}_{\mathfrak{h}}\,\mathrm{ad}\,e_\alpha\circ\mathrm{pr}_{\mathfrak{m}}\,\mathrm{ad}\,e_\alpha))\\
&= \sum_{e_\alpha\in\mathfrak{m}_i} (B(e_\alpha,e_\alpha)+2\sum_i Q([e_\alpha,[e_\alpha,z_i]],z_i))\\
&= \sum_{e_\alpha\in\mathfrak{m}_i} (B(e_\alpha,e_\alpha)-2Q(C_{\mathfrak{m}_i,Q|\mathfrak{h}}(e_\alpha),e_\alpha))\\
&= d_i(b_i-2c_i). \quad \text{q.e.d.}
\end{aligned}$$

Notice that one can use (1.5) to collect terms in (1.3). The coefficient of $1/x_i$ in (1.3) is equal to

$$\frac{1}{2}\left(d_i b_i + \frac{1}{2}\begin{bmatrix} i \\ i\,i \end{bmatrix} - \sum_j \begin{bmatrix} j \\ i\,j \end{bmatrix}\right) = d_i c_i + \frac{1}{4}\begin{bmatrix} i \\ i\,i \end{bmatrix} + \frac{1}{2}\sum_{j \neq k} \begin{bmatrix} k \\ i\,j \end{bmatrix}$$

which is $\geqq 0$ and $=0$ iff $\mathfrak{m}_i \subset \mathfrak{m}_0$ and $[m_i, m_j] \subset m_j$ for all j.

We also need a formula for the scalar curvature of a Riemannian submersion. Let $\pi\colon M \to B$ be a Riemannian submersion with totally geodesic fibres F. Let g denote the metric on M normalized so that $\mathrm{vol}(g)=1$. Let $S(B)$ and $S(F)$ respectively be the scalar curvature of the base and fibre. On M there is a natural family of metrics: $g_t = t g|_V \perp g|_H$, where V and H are respectively the vertical and horizontal distributions. Then

$$S(g_t) = \frac{1}{t} S(F) + S(B) - t \|A\|^2$$

where $\|A\|$ is the norm of the O'Neill tensor computed with respect to $g=g_1$. (See e.g. [BB], Lemma 14.) If we let $f=\dim F$ and $n=\dim M$, then $\mathrm{vol}(g_t) = t^{f/2}$, so that $\tilde{g}_t = t^{-f/n} g_t$ has volume 1 and

$$S(\tilde{g}_t) = t^{f/n}\left(\frac{1}{t} S(F) + S(B) - t\|A\|^2\right). \tag{1.6}$$

Hence if $S(F)>0$ then $S(\tilde{g}_t) \to +\infty$ as $t \to 0$.

In applications we will consider compact connected intermediate Lie groups K with $H \subset K \subset G$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_b$ and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_f$ be Q-orthogonal decompositions. For any $\mathrm{Ad}(H)$-invariant metric $\langle\ ,\ \rangle$ on $\mathfrak{m}$ such that $\langle \mathfrak{m}_f, \mathfrak{m}_b \rangle = 0$ and such that $\langle\ ,\ \rangle | \mathfrak{m}_b$ is $\mathrm{Ad}(K)$-invariant, the natural projection $G/H \to G/K$ becomes a Riemannian submersion with totally geodesic fibres if the metric on base and fibre are given respectively by $\langle\ ,\ \rangle|\mathfrak{m}_b$ and $\langle\ ,\ \rangle|\mathfrak{m}_f$. (See [BB], Prop. 2.) The induced family of metrics $\tilde{g}_t$ clearly lies in $\mathcal{M}_G$.

§ 2. S bounded on $\mathcal{M}_G$

(2.1) **Theorem.** *S is bounded from below on $\mathcal{M}_G$ iff the universal cover of G/H is a product of several isotropy irreducible homogeneous spaces and a euclidean space $\mathbf{R}^k$, $k \geqq 0$. S is in addition proper iff $k=0$. If $k=0$, S has a unique critical point, which is a product of the unique Einstein metric on each factor, and S is bounded from below by a positive constant. If $k \geqq 1$, S has a critical point iff G/H is a torus.*

Proof. Assume that S is bounded from below on $\mathcal{M}_G$. Fix a decomposition of $\mathfrak{m}$. If $\begin{bmatrix} k \\ i\,j \end{bmatrix} \neq 0$ for some i, j, k with $k \neq i$ and $k \neq j$, then (1.3) implies that $S \to -\infty$ if $x_i \to 0$, $x_j \to 0$ and $x_k \to +\infty$ at suitable rates preserving the volume. Hence, using the symmetry of $[\ ]$ in all 3 indices, it follows that $\begin{bmatrix} k \\ ij \end{bmatrix}=0$ whenever there are two distinct indices. Thus $[\mathfrak{m}_i, \mathfrak{m}_j]=0$ if $i \neq j$ and $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h} \oplus \mathfrak{m}_i$.

If we let $\mathfrak{h}_i=[\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{h}}$, then the biinvariance of Q implies that $\mathfrak{g}_i=\mathfrak{h}_i \oplus \mathfrak{m}_i$ are pairwise Q-orthogonal ideals of $\mathfrak{g}$. The Q-orthogonal complement of $\oplus \mathfrak{g}_i$ in $\mathfrak{g}$ is thus an ideal contained in $\mathfrak{h}$, which is 0 by effectiveness. So $\mathfrak{g}=\oplus \mathfrak{g}_i$, and the universal cover of G/H is the product $\prod_{i=1}^{s} G_i/H_i$. Note that $\mathfrak{m}_0 \subset \mathfrak{z}(\mathfrak{g})$ and if $\mathfrak{m}_i \subset \mathfrak{m}_0$, then $\mathfrak{h}_i=0$. It follows that $\mathfrak{h}=\bigoplus_{i=1}^{r} \mathfrak{h}_i$ and $\mathfrak{h}_i$ acts irreducibly on $\mathfrak{m}_i$, $1 \leqq i \leqq r$. Thus, G_i/H_i is isotropy irreducible if $1 \leqq i \leqq r$, and by effectiveness $\mathfrak{z}(\mathfrak{g}) =\mathfrak{m}_0=\mathbf{R}^k$ for some $k \geqq 0$. The decomposition $\mathfrak{m}=\mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r \oplus \mathfrak{m}_0$ is unique since the $\operatorname{Ad}(H)$-representations $\mathfrak{m}_i$, $i>0$, are pairwise inequivalent. Hence by (1.3) S on $\mathcal{M}_G$ has the form

$$S=\frac{1}{2} \sum_{i=1}^{r} \frac{d_i b_i}{x_i}>0,$$

which is independent of the metric on $\mathfrak{m}_0=\mathfrak{z}(\mathfrak{g})$.

This completes the proof of the first assertion in (2.1). The next two assertions are immediate consequences of the first assertion and the form of S in the preceding paragraph. Finally, if $k \geqq 1$, S has a critical point iff G/H is a torus since Ric is 0 on $\mathfrak{m}_0$ and positive on $\mathfrak{m}_i$, $i>0$. q.e.d.

Remark. The homogeneous spaces occurring in (2.1) are precisely the compact homogeneous spaces of normal type, i.e., every G-invariant metric on G/H is normal homogeneous, see [BB], Lemma 12. Note that Lemma 13 in [BB] is an immediate consequence of (1.3).

(2.2) **Theorem.** *S is bounded from above and proper on $\mathcal{M}_G$ iff H is a maximal connected subgroup of G, or, equivalently, $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$. In this case S has a global maximum, which must be a G-invariant Einstein metric on G/H.*

Proof. Let us first assume that $\mathfrak{h}$ is maximal in $\mathfrak{g}$. Maximality has the following consequence: there is a constant $a>0$ depending only on G/H such that for any non-empty proper subset $I \subset \{1, \ldots, s\}$ and for any decomposition of $\mathfrak{m}$ there exist $i, j \in I$ and $k \notin I$ with $\begin{bmatrix} k \\ ij \end{bmatrix} \geqq a$. To see this, note first that for any fixed decomposition of $\mathfrak{m}$ and for any non-empty proper $I \subset \{1, \ldots, s\}$ there exists an $i, j \in I$, $k \notin I$ such that $\begin{bmatrix} k \\ ij \end{bmatrix}>0$ since otherwise $\mathfrak{h} \oplus \sum_{i \in I} \mathfrak{m}_i$ would be a proper subalgebra of $\mathfrak{g}$ properly containing $\mathfrak{h}$. Now for each non-empty subset $I \subset \{1, \ldots, s\}$ let $a_I=\inf \sum \begin{bmatrix} k \\ ij \end{bmatrix}$ where the infimum is taken over all decompositions of $\mathfrak{m}$ and the sum is taken over all i, j, k with $i, j \in I$ and $k \notin I$. By the compactness of the set of decompositions of $\mathfrak{m}$ and the continuity of the $\begin{bmatrix} k \\ ij \end{bmatrix}$, we see that $a_I>0$, and hence for some $i,j\in I$, $k\notin I$, $\begin{bmatrix}k\\ ij\end{bmatrix}\geqq\frac{a_I}{s^3}$. Since there are only finitely many I's, the existence of a follows.

By the compactness of the set of decompositions of $\mathfrak{m}$, we can also find a constant $b>0$ such that for every decomposition $b_i\leqq b$, where b_i is given by $B|\mathfrak{m}_i=b_iQ|\mathfrak{m}_i$. Next we fix constants α_i s.t. $0=\alpha_1<\alpha_2<\ldots<\alpha_s=1$ and $\alpha_{i+1}>(1+\alpha_i)/2$. Let

$$\alpha=\min\{2\alpha_{i+1}-\alpha_i-1,\ 1\leqq i\leqq s-1\}>0.$$

For each fixed decomposition of $\mathfrak{m}$, we consider the family of metrics given by (1.2). If the metric is $\neq Q$, we write $x_i=e^{tv_i}$ with $\sum_{i=1}^{s} v_i^2=1$, $t>0$, and $\sum_{i=1}^{s} d_i v_i=0$ since the volume is assumed to be 1. For a fixed $(v_1,\ldots,v_s)$ let $v_{\min}=\min_i\{v_i\}$. Then there exists a constant $c>0$ which depends only on s and $\{d_i\}$ (and hence not on the decomposition) such that $v_{\min}<-c<0$ for every $(v_1,\ldots,v_s)$ satisfying $\sum_{i=1}^{s} v_i^2=1$ and $\sum_{i=1}^{s} d_i v_i=0$.

Given $(v_1,\ldots,v_s)$ as above, we sub-divide the interval $(v_{\min},0]$ into $s-1$ intervals $(\alpha_{i+1}v_{\min},\alpha_i v_{\min}]$, $i=1,\ldots,s-1$. Since at least one v_i is positive, at least one of these intervals does not contain any v_i's, say $(\alpha_{i_0+1}v_{\min},\alpha_{i_0}v_{\min}]$. Let $I=\{i\,|\,v_i\leqq\alpha_{i_0+1}v_{\min}\}$. By construction I is a non-empty proper subset of $\{1,\ldots,s\}$. Hence there exist i,j,k with $i,j\in I$, $k\notin I$, and $\begin{bmatrix}k\\ ij\end{bmatrix}\geqq a$. Since $v_k>\alpha_{i_0}v_{\min}$, $v_i,v_j\leqq\alpha_{i_0+1}v_{\min}$, and $v_{\min}<-c$, we have, using (1.3),

$$\begin{aligned}S&<\tfrac{1}{2}bn\exp(-tv_{\min})-\tfrac{1}{4}a\exp(t\alpha_{i_0}v_{\min})\exp(-2t\alpha_{i_0+1}v_{\min})\\ &=\tfrac{1}{4}\exp(-tv_{\min})[2bn-a\exp(-tv_{\min}(2\alpha_{i_0+1}-\alpha_{i_0}-1))]\\ &<\tfrac{1}{4}\exp(-tv_{\min})[2bn-a\exp(tc\alpha)].\end{aligned}$$

Hence given $A>0$ we will have $S<-A$ if $t>\frac{1}{c\alpha}\log\left(\frac{2nb+4A}{a}\right)$. We can thus find constants $\beta_1(A)$, $\beta_2(A)$ which depend only on A and G/H but not on the chosen decomposition such that $S>-A$ implies that $\beta_1(A)\leqq x_i\leqq\beta_2(A)$ for all i.

We claim that this uniform estimate implies that $S^{-1}[-A,+\infty)\cap\mathscr{M}_G$ is compact. Indeed, the above inequality says that for a metric $\langle\ ,\ \rangle$ in $\mathscr{M}_G$ with $S\geqq-A$ the eigenvalues of $\langle\ ,\ \rangle$ with respect to Q lie between $\beta_1(A)$ and $\beta_2(A)$. But the set of symmetric matrices with bounded eigenvalues is compact. Hence S is bounded from above and proper on $\mathscr{M}_G$.

Conversely, let G/H be such that S is bounded from above and proper on $\mathscr{M}_G$. If H is not a maximal connected subgroup, then there exists a connected subgroup K with $H\subset K\subset G$. If K is closed, then by (1.6) and boundedness from above, the metric induced by Q on K/H has zero scalar curvature. The same formula shows that $S^{-1}[0,+\infty)$ is then non-compact, which is a contradiction to properness. If K is not closed, then either $\bar{K}\subset G$ or $\bar{K}=G$. We have already treated the first case. If $\bar{K}=G$, then $\mathfrak{g}$ has a non-zero center. Let L be the connected subgroup corresponding to $\mathfrak{h}\oplus\mathfrak{z}(\mathfrak{g})$. Then L is closed and L/H is a torus. If $L \neq G$, then we get a contradiction as before by (1.6). If $L=G$, then G/H is a torus and S is not proper. q.e.d.

(2.3) **Corollary.** *Let H be any closed subgroup of a compact connected Lie group G. If $\mathfrak{h}$ is maximal in $\mathfrak{g}$, then S is bounded from above and proper on $\mathcal{M}_G$. Therefore, G/H has a G-invariant Einstein metric which is a maximum point for S.*

Proof. Let H^0 be the identity component of H. The $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$ form a closed subset of the $\mathrm{Ad}(H^0)$-invariant inner products on $\mathfrak{m}$. On the latter set S is bounded from above and proper by (2.2). Hence the scalar curvature function remains bounded from above and proper when restricted to this closed subset, q.e.d.

Examples. To apply (2.2) to concrete examples, we first describe briefly Dynkin's classification of the maximal connected subgroups of the simple Lie groups (see [Dy 1, 2] and [WZ 2]).

If G/H is isotropy irreducible, then $\mathfrak{h}$ is certainly maximal in $\mathfrak{g}$. But in this case $\mathcal{M}_G$ is a point and (2.2) becomes trivial.

It is easy to see that if $\mathfrak{h}$ is maximal in $\mathfrak{g}$, G/H is effective, and if G is not simple, then $G=H\times H$, where H is embedded diagonally and H is simple. In this case G/H is an irreducible symmetric space. Hence we will assume that G is simple.

If $\operatorname{rank}\mathfrak{h}=\operatorname{rank}\mathfrak{g}$, then the Borel-de Siebenthal classification ([BS] and [Wo, p. 282a, b]) of maximal subalgebras of maximal rank implies that G/H is either isotropy irreducible or $G/H=E_8/\mathrm{SU}(5)\cdot\mathrm{SU}(5)$. In the latter case the biinvariant metric on G already induces an Einstein metric on G/H (see [WZ 1]).

If G is a simple exceptional Lie group and $\operatorname{rank}\mathfrak{h}<\operatorname{rank}\mathfrak{g}$, then $\mathfrak{h}$ is a maximal S-subalgebra in Dynkin's terminology (p. 158 [Dy 1]). [Dy 1], Theorem 14.1 and Tables 14, 15, 24, 35, contain a list of all the maximal S-subalgebras of the exceptional simple Lie algebras together with their isotropy representations. There are 21 maximal S-subalgebras, 11 of which are isotropy irreducible. To the remaining 10 we can apply (2.2), and only in one case does the biinvariant metric on G induce an Einstein metric on G/H (see [WZ 1]).

If G is a classical simple Lie group and H is not simple, Theorems 1.1–1.4 of [Dy 2] imply that G/H is either a Grassmannian (and hence an irreducible symmetric space) or (G, H) is one of the following:

I	$(\mathrm{Sp}(pq), \mathrm{Sp}(p)\,\mathrm{SO}(q))$,	$p\geq 1,\ q\geq 3,\ q\neq 4$; or $p=1,\ q=4$
II	$(\mathrm{SO}(4pq), \mathrm{Sp}(p)\,\mathrm{Sp}(q))$,	$p\geq q\geq 1,\ (p,q)\neq(1,1)$
III	$(\mathrm{SO}(pq), \mathrm{SO}(p)\,\mathrm{SO}(q))$,	$p\geq q\geq 3,\ (p,q)\neq(4,4)$
IV	$(\mathrm{SU}(pq), \mathrm{SU}(p)\,\mathrm{SU}(q))$,	$p\geq q\geq 2,\ (p,q)\neq(2,2)$

where the inclusions are given by the obvious tensor product representations. G/H is isotropy irreducible iff $p=1$ in case I, $q=1$ in case II, or we are in IV. In all other cases the biinvariant metric on G induces an Einstein metric on G/H iff $p=q$ in case II or $p=q$ in case III (see [WZ 1]). In the remaining cases (2.2) yields a new Einstein metric on each G/H.

If G is a classical simple Lie group and H is a simple maximal connected subgroup, then Dynkin showed that the representation of H induced by the lowest dimensional representation of G must be irreducible except when $G/H = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. Moreover, he proved (see [Dy2], Theorem 1.5) that the converse is essentially true. To be more precise, let H be a simple compact Lie group and π an irreducible representation of H of complex dimension N. Except for a short list of (H, π) (see Table 1, p. 364, [Dy2]), $\pi(H)$ is a maximal connected subgroup of $G=\mathrm{SO}(N)$, $\mathrm{Sp}(\frac{1}{2}N)$ or $\mathrm{SU}(N)$ according to whether π is orthogonal, symplectic, or non-self-contragredient. Moreover, it was observed in [WZ2] that these exceptions have a uniform description in terms of the isotropy representations of symmetric spaces. Hence, apart from these exceptions, every irreducible representation $\pi\colon H \to G$ of a simple compact Lie group H gives rise to a homogeneous Einstein manifold $G/\pi(H)$ by (2.2). Of course, some isotropy irreducible spaces are repeated in this list, but all other examples are new. Moreover, in these new examples, the bi-invariant metric of G never induces an Einstein metric on $G/\pi(H)$ (see [WZ1]).

(2.4) **Theorem.** *S is bounded from above on $\mathcal{M}_G$ but not proper iff $S^1 \cdot H$ is a subgroup of G and $G/H \cdot S^1$ is a compact irreducible hermitian symmetric space other than the hyperquadrics* $\mathrm{SO}(n+2)/\mathrm{SO}(n) \cdot \mathrm{SO}(2)$, $n \geqq 2$.

Proof. Assume that S is bounded from above but not proper. Then H is not a maximal connected subgroup of G by (2.2).

First let $H \subsetneqq K \subsetneqq G$ be any closed connected subgroup. By (1.6) and boundedness from above it follows that Q induces a normal homogeneous metric on K/H whose scalar curvature is 0. But the formula for the sectional curvature of a normal homogeneous metric (see [KN] Theorem X.3.5) implies that Q has 0 scalar curvature on K/H iff $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}$, $Q(\mathfrak{h}, \mathfrak{p}) = 0$, and $[\mathfrak{p}, \mathfrak{p}] = 0$. The bi-invariance of Q implies that $[\mathfrak{h}, \mathfrak{p}] = 0$, so $\mathfrak{p} \subset \mathfrak{m}_0$, and hence $\mathfrak{m}_0$ is a non-zero sub-algebra of $\mathfrak{g}$. If every K with $H \subsetneqq K \subsetneqq G$ is not closed, then $\bar{K} = G$. In this case, $\mathfrak{z}(\mathfrak{k}) \neq 0$ and since $Z(K) \subset Z(G)$ we also have $\mathfrak{z}(\mathfrak{k}) \subset \mathfrak{z}(\mathfrak{g})$. Since K is not closed, $\mathfrak{z}(\mathfrak{k}) \neq \mathfrak{z}(\mathfrak{g})$, and if we let $X \in \mathfrak{z}(\mathfrak{g})$, $X \notin \mathfrak{z}(\mathfrak{k})$, then the projection Y of X into $\mathfrak{m}$ lies in $\mathfrak{m}_0$ and $Y \neq 0$ by effectiveness. Hence again $\mathfrak{m}_0 \neq 0$.

Therefore, let K be the closed connected subgroup with Lie algebra $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_0$. ($K$ is the identity component of the normalizer of H in G.) We have $H \subsetneqq K \subsetneqq G$ (the last inequality by the effectiveness of G). Thus by the above argument, $\mathfrak{m}_0$ is a non-zero abelian ideal of $\mathfrak{k}$.

Let $0 \neq X \in \mathfrak{m}_0$ and $\mathfrak{c}_X$ be the centralizer of X in $\mathfrak{g}$. Note that $\mathfrak{c}_X \supset \mathfrak{k}$ and $\mathfrak{c}_X$ is of maximal rank in $\mathfrak{g}$. If $\mathfrak{c}_X \neq \mathfrak{g}$, then since the connected Lie group corresponding to $\mathfrak{c}_X$ is closed, it follows that $\mathfrak{c}_X = \mathfrak{k}$. We first claim that there exists some $X \in \mathfrak{m}_0$ with $\mathfrak{c}_X \neq \mathfrak{g}$. Indeed if $\mathfrak{c}_X = \mathfrak{g}$ for every $X \in \mathfrak{m}_0$, then $\mathfrak{m}_0 \subset \mathfrak{z}(\mathfrak{g})$ and the effectiveness of G/H implies $\mathfrak{m}_0 = \mathfrak{z}(\mathfrak{g})$. Hence G/H is finitely covered by $(G'/H') \times T^k$, $k \geqq 1$, with G' semisimple. But then $\mathcal{M}_G$ contains the metrics which are products of a normal metric g_1 on G'/H' and an arbitrary flat metric g_2 on T^k such that $\mathrm{vol}(g_1) \cdot \mathrm{vol}(g_2) = 1$. Since $S(g) = S(g_1)$, it follows that S is not bounded from above. Now let $X \in \mathfrak{m}_0$ such that $\mathfrak{k} = \mathfrak{c}_X \neq \mathfrak{g}$. We claim that $\mathfrak{c}_X$ is a maximal subalgebra in $\mathfrak{g}$. Indeed, if $\mathfrak{c}_X \subsetneqq \mathfrak{l} \subsetneqq \mathfrak{g}$, then by the previous arguments, it follows that $\bar{L} = G$. As before this implies $0 \neq \mathfrak{z}(\mathfrak{l}) \subset \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{z}(\mathfrak{l}) \neq \mathfrak{z}(\mathfrak{g})$ which is impossible since $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{c}_X$. Hence K is a closed maximal connected subgroup of G of maximal rank. G/K need not be effective, but since G/H is effective this can only happen if some subspace $\mathfrak{a} \subset \mathfrak{m}_0$ is an ideal in $\mathfrak{g}$. But then a finite cover of G/H is again a product $(G'/H') \times T^k$, $k \geqq 1$, which we already saw, is impossible.

Hence G/K is effective and Theorems 8.10.1 and 8.10.9 of [Wo] show that $\mathfrak{g}$ is simple, $\mathfrak{k} = \mathfrak{h} \oplus \mathbf{R}$, and $(\mathfrak{g}, \mathfrak{h} \oplus \mathbf{R})$ is an irreducible hermitian symmetric pair.

Conversely, assume that $G/H \cdot S^1$ is an irreducible hermitian symmetric space. The isotropy representation of $G/H \cdot S^1$ is $[\phi \otimes \phi_0]_{\mathbf{R}}$ where ϕ is an irreducible complex representation of H and ϕ_0 is the one-dimensional representation of S^1. If ϕ is not self-contragredient, the restriction of $[\phi \otimes \phi_0]_{\mathbf{R}}$ to H is $[\phi]_{\mathbf{R}}$, which is again irreducible. Hence, in this case the isotropy representation of G/H is $\mathfrak{m} = \mathfrak{m}_0 \oplus [\phi]_{\mathbf{R}}$ where $\dim \mathfrak{m}_0 = 1$. Hence any G-invariant metric on G/H is a submersion metric of $G/H \to G/H \cdot S^1$, and $\mathcal{M}_G$ is 1-dimensional. By (1.6), S on $\mathcal{M}_G$ has the form $S = t^{1/n} S(B) - t^{\frac{n+1}{n}} \|A\|^2$, where $S(B) > 0$ and $\|A\| \neq 0$ since G/H is irreducible as a Riemannian manifold (with metric induced by Q) by Corollary X.5.4 in [KN]. Thus S is bounded from above but is not proper.

If ϕ is self-contragredient, then by the classification of irreducible hermitian symmetric spaces (see [Wo], p. 283–284) $G/H \cdot S^1 = \mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$. However, in this case we also have inclusions $\mathrm{SO}(n) \subset \mathrm{SO}(n+1) \subset \mathrm{SO}(n+2)$. Hence, by (1.6), there exists a family of metrics g_t in $\mathcal{M}_G$ on $\mathrm{SO}(n+2)/\mathrm{SO}(n)$ with $S(g_t) \to +\infty$ as $t \to 0$. q.e.d.

Remark 1. If we have a space G/H as in (2.4), then the proof of (2.4) shows that although S is not proper, $S^{-1}[a, \infty] \cap \mathcal{M}_G$ is compact for any $a > 0$. Hence S still has a maximum. Indeed, S has a unique critical point on $\mathcal{M}_G$ at $t = \dfrac{S(B)}{(n+1)\|A\|^2}$. Note that G/H is the canonical circle bundle over the hermitian symmetric space $G/H \cdot S^1$. The unique G-invariant Einstein metric on G/H was discovered by Kobayashi (see [Kob]).

Remark 2. The results in this section clearly have the following consequence. If S is a proper function on $\mathcal{M}_G$, then either G/H is one of the spaces studied in (2.1) and (2.2), or $\mathfrak{m}$ has two inequivalent irreducible copies, $\mathfrak{m}_0 = 0$, and $\mathfrak{h}$ is not maximal in $\mathfrak{g}$. In the latter case we will see that S may fail to have a critical point.

§ 3. Nonexistence

There are many examples of non-compact homogeneous spaces G/H which carry no G-invariant Einstein metric, e.g., if G is nilpotent, see [Mi]. If G is compact and $\pi_1(G/H)$ is infinite, then G/H cannot carry an Einstein metric with positive Einstein constant since by Bonnet-Myers there is not even a metric with positive Ricci curvature. If $\mathrm{Ric} \leqq 0$, Bochner's theorem implies that every Killing vector field is parallel and hence G/H is flat. Therefore, if G is compact, $\pi_1(G/H)$ is infinite, and if G/H is not flat, then G/H carries no G-

invariant Einstein metric. But if G is compact and G/H is simply connected, there is always a metric with Ric>0 and no further obstruction to the existence of an Einstein metric is known. In fact, since Einstein metrics have always been found on the many examples that have been studied, the question was raised whether or not every compact simply connected homogeneous space carries a homogeneous Einstein metric. To settle this question, we examine (in view of (2.2)) the case where there is a unique closed connected subgroup K with $H \subsetneqq K \subsetneqq G$, and find numerous examples that have no G-invariant Einstein metrics. We do not know whether there are non-homogeneous Einstein metric on these examples. At the end of this section we will also give an example of non-existence where $H \subsetneqq K_1 \subsetneqq K_2 \subsetneqq G$.

Let G/H be a homogeneous space and for simplicity we will assume that G is a compact simple Lie group. We choose for the fixed biinvariant metric Q the negative of the Killing form, denoted by B. Then $b_i = 1$ in (1.3). We will examine the situation in which $\mathfrak{m}$ has only two irreducible summands $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Since we are interested in non-existence, we will assume that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are inequivalent representations, since otherwise $B|\mathfrak{m}$ is an Einstein metric on G/H by [WZ 1], (1.1.7). By (2.2) we can assume that H is not maximal in G. Without loss of generality, let $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1$, be a subalgebra of $\mathfrak{g}$ with corresponding closed connected subgroup K. Then every G-invariant metric on G/H is given by $(x_1 B|\mathfrak{m}_1) \perp (x_2 B|\mathfrak{m}_2)$, and can be viewed as a submersion metric for the fibration $G/H \to G/K$. The homogeneous space K/H need not be effective in general. So let K' be the quotient of K acting effectively on K/H. For simplicity we also assume that K' is semisimple and $B_{\mathfrak{k}'} = \alpha B|\mathfrak{k}'$ for some $\alpha > 0$, where $B_{\mathfrak{k}'}$ is the negative of the Killing form of $\mathfrak{k}'$. Note that if $\mathfrak{k}'$ is simple, this last condition is automatically satisfied.

Now since $\mathfrak{k}$ is a subalgebra, we have $\begin{bmatrix} 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \end{bmatrix} = 0$. We can assume that $\begin{bmatrix} 2 \\ 12 \end{bmatrix} \neq 0$ since otherwise we are in the situation of (2.1). By (1.5), $d_1 - \begin{bmatrix} 1 \\ 11 \end{bmatrix} - \begin{bmatrix} 2 \\ 12 \end{bmatrix} = 2 d_1 c_1$. We can also apply (1.5) to G/K to obtain $d_2 - \begin{bmatrix} 2 \\ 22 \end{bmatrix} = 2 d_2 c_2^*$, where c_2^* is the Casimir constant of the $\mathfrak{k}$ representation $\mathfrak{m}_2$ with respect to $B|\mathfrak{k}$. Notice that the $\mathfrak{k}$ representation $\mathfrak{m}_2$ and the $\mathfrak{h}$ representation $\mathfrak{m}_2$ have different Casimir constants. If c_1^* is the Casimir constant of $\mathfrak{m}_1$ with respect to $B_{\mathfrak{k}'}|\mathfrak{h}$ (instead of $B|\mathfrak{h}$!), we have $c_1 = \alpha \cdot c_1^*$. We also observe that $\begin{bmatrix} 2 \\ 12 \end{bmatrix} = d_1(1-\alpha)$ since

$$\begin{aligned} \begin{bmatrix} 2 \\ 12 \end{bmatrix} &= \sum_{e_i \in \mathfrak{m}_1} -tr_{\mathfrak{m}_2}(pr_{\mathfrak{m}_2} \operatorname{ad} e_i)^2 \\ &= \sum_{e_i \in \mathfrak{m}_1} (-tr_{\mathfrak{g}}(\operatorname{ad} e_i)^2 + tr_{\mathfrak{k}}(\operatorname{ad} e_i)^2) \\ &= \sum_{e_i \in \mathfrak{m}_1} (B(e_i, e_i) - B_{\mathfrak{k}}(e_i, e_i)) = d_1(1-\alpha). \end{aligned}$$

Combining the above with (1.3), we get

$$S = \frac{1}{2x_1} d_1 \alpha \left(c_1^* + \frac{1}{2}\right) + \frac{1}{2x_2} d_2 \left(c_2^* + \frac{1}{2}\right) - \frac{1}{4} \frac{x_1}{x_2^2} d_1 (1-\alpha).$$

The critical points of S subject to $x_1^{d_1} x_2^{d_2} = v$, where v^{-1} is the volume of the metric g_B on G/H, are given by the solutions of the quadratic equation

$$\alpha\left(c_1^* + \frac{1}{2}\right) t^2 - \left(c_2^* + \frac{1}{2}\right) t - \frac{1}{2}(1-\alpha)\left(1 + \frac{2d_1}{d_2}\right) = 0,$$

where $t = x_2/x_1$. Note that $c_i^* = \frac{1}{2}$ iff the corresponding space is (locally) symmetric (see Corollary I.1.6 in [WZ 1]). Therefore, we have

(3.1) **Theorem.** *Assume that G is a compact simple Lie group, $\mathfrak{m}$ decomposes into two inequivalent irreducible summands, and that $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1$ is a subalgebra with $B_{\mathfrak{k}'} = \alpha B|_{\mathfrak{k}'}$, $\alpha > 0$. Then there exists no G-invariant Einstein metric on G/H iff*

$$D = \left(c_2^* + \frac{1}{2}\right)^2 - \left(1 + \frac{2d_1}{d_2}\right) \alpha(1-\alpha)(2c_1^* + 1) < 0,$$

where c_1^*, c_2^* are the Casimir constants of K'/H' and G/K with respect to $B_{\mathfrak{k}'}|\mathfrak{h}'$ and $B|_{\mathfrak{k}}$. If G/K and K'/H' are both symmetric, then

$$D = 1 - 2\alpha(1-\alpha)\left(1 + \frac{2d_1}{d_2}\right).$$

If $D > 0$ (resp. $D = 0$) *there exist precisely* 2 (resp. 1) *G-invariant Einstein metric on G/H.*

We now give a number of examples without G-invariant Einstein metrics. Observe first that under the conditions in (3.1) both base G/K and fibre K'/H' are isotropy irreducible homogeneous spaces. Conversely, if in $G/H \to G/K$ the base and fibre are isotropy irreducible *and* if the irreducible $\mathfrak{k}$ representation $\mathfrak{m}_2$ restricts to an irreducible representation of $\mathfrak{h}$, then G/H is a candidate for (3.1), where we only have to verify the additional conditions that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are inequivalent and that $B_{\mathfrak{k}'} = \alpha B|_{\mathfrak{k}'}$. Now, if $\mathfrak{k}$ is simple, then the requirement that there exists some subalgebra $\mathfrak{h}$ for which the $\mathfrak{k}$-representation $\mathfrak{m}_2$ remains irreducible, is a very strong one. In fact, Dynkin ([Dy 1], Table 40, [Dy 2], Table 5) classified those irreducible representations of simple Lie groups which restrict to an irreducible representation of some subgroup. Using this classification, one can easily compile a complete list of all homogeneous spaces G/H for which $\mathfrak{m}$ has only two irreducible summands. However, we will not do this here. Instead, we will give a number of simple examples which show that $D < 0$ occurs quite frequently. For the computation of the constants α and for other notation we refer the reader to [WZ 1].

Example 1. $G = \mathrm{SO}(2n)$, $K = U(n)$, $H = \mathrm{SO}(n) \cdot U(1)$, $\bar{K} = \mathrm{SU}(n)$, and $n \geqq 3$. The base and fibre are symmetric spaces and $\mathfrak{m}_1 = (S^2 \rho_n - \mathrm{Id}) \otimes \mathrm{Id}$. The $\mathfrak{k}$ representation $\mathfrak{m}_2$ is $[\Lambda^2 \mu_n]_R$, which restricts to $[\Lambda^2 \rho_n \otimes \phi]_{\mathbf{R}}$. So all the conditions in (3.1) are satisfied. We get $d_1 = \frac{1}{2}(n+2)(n-1)$, $d_2 = n(n-1)$, $\alpha = \dfrac{n}{2(n-1)}$. Hence $D = (3-n)/(n-1)^2$, and so $D < 0$ if $n \geqq 4$, $D = 0$ if $n = 3$.

Example 2. $G = \mathrm{SU}(n+m)$, $K = S(U(n)\,U(m))$, $H = S(\mathrm{SO}(n)\,U(1)\,U(m))$, $\bar{K} = \mathrm{SU}(n)$ and $n \geqq 3$, $m \geqq 1$. The base and fibre are again symmetric spaces and $\mathfrak{m}_1 = (S^2 \rho_n$

$-\mathrm{Id})\otimes\mathrm{Id}\otimes\mathrm{Id}$. The $\mathfrak{k}$ representation $\mathfrak{m}_2$ is $[\mu_n\otimes\mu_m^*]_{\mathbf{R}}$, which restricts to $[\rho_n\otimes\phi\otimes\mu_m^*]_{\mathbf{R}}$. We obtain $d_1=\frac{1}{2}(n+2)(n-1)$, $d_2=2mn$, $\alpha=\dfrac{n}{m+n}$, and hence $D=(m^2-n+2)/(m+n)^2$. Thus $D<0$ if $n>m^2+2$, $D=0$ if $n=m^2+2$, and $D>0$ if $n<m^2+2$.

Example 3. A straightforward but lengthy computation shows that the only other examples for which both fibre and base are symmetric spaces and $D<0$ are:

G	K	H
E_6	$\mathrm{Spin}(10)\cdot\mathrm{SO}(2)$	$\mathrm{Spin}(3)\,\mathrm{Spin}(7)\,\mathrm{SO}(2)$
E_6	$\mathrm{Spin}(10)\cdot\mathrm{SO}(2)$	$\mathrm{Spin}(5)\,\mathrm{Spin}(5)\,\mathrm{SO}(2)$
E_7	$E_6\cdot\mathrm{SO}(2)$	$\mathrm{Sp}(4)\cdot\mathrm{SO}(2)$
E_7	$\mathrm{Spin}(12)\cdot\mathrm{Sp}(1)$	$\mathrm{Spin}(5)\,\mathrm{Spin}(7)\,\mathrm{Sp}(1)$

Example 4. We now describe some large families of examples of non-existence where the base is symmetric and the fibre is isotropy irreducible but non-symmetric. Since the $\mathfrak{k}$ representation $\mathfrak{m}_2$ must restrict to an irreducible representation of $\mathfrak{h}$, this mostly occurs when G/K is a Grassmannian. First, let $G/K=\mathrm{SO}(n+m)/\mathrm{SO}(n)\cdot\mathrm{SO}(m)$, $n\geqq m\geqq 1$, and $H=L\cdot\mathrm{SO}(m)$ such that $\mathrm{SO}(n)/L$ is isotropy irreducible but not symmetric. Then $K'=\mathrm{SO}(n)$ with $n\geqq 5$ and $\mathfrak{m}_1=(\text{isotropy representation of }\mathrm{SO}(n)/L)\otimes Id$. The $\mathfrak{k}$ representation $\mathfrak{m}_2$ is $\rho_n\otimes\rho_m$, which restricts to $\pi\otimes\rho_m$ where π is the inclusion $L\to\mathrm{SO}(n)$. The quotients $\mathrm{SO}(n)/L$ can be uniformly described in terms of symmetric spaces, see [WZ 2]. One knows that π is irreducible, $\Lambda^2\pi=\mathrm{ad}_L\oplus\mathfrak{m}_1$, and if L is simple and $L\neq G_2$, then $c_1^*=\dfrac{2\dim L}{n(n-2)}$ (see [WZ 1], Table 5). Note that here we have to adjust for the fact that in [WZ 1] the Casimir constant was defined with respect to the normalized Killing form and hence the constants in Table 5 have to be divided by $\alpha_{\mathrm{SO}(n)}=2(n-2)$. Since $\alpha=\dfrac{n-2}{n+m-2}$, we get

$$D=1-(mn+n(n-1)-2\dim L)(4\dim L+n(n-2))/n^2(m+n-2)^2.$$

If $m=1$, one easily shows that $D<0$ by using the inequalities $\dfrac{4\dim L}{n(n-2)}<1$ and $\dfrac{n}{2\dim L}<1$, which follows respectively from $c_1^*<\frac{1}{2}$ and from the fact that $\mathrm{SO}(n)/L$ is constructed from a symmetric space and the quotient of the Killing forms in the symmetric space must be positive (see [WZ 1], II.3(C)). Similarly one checks that if L is not simple, i.e., $L=\mathrm{Sp}(1)\cdot\mathrm{Sp}(k)$ with $k\geqq 2$, and if $m=1$ we have $D<0$. Hence we conclude that $\mathrm{SO}(n+1)/L$ *has no* $\mathrm{SO}(n+1)$*-invariant Einstein metric if* $\mathrm{SO}(n)/L$ *is isotropy irreducible but not symmetric and* $L\neq G_2$. Explicit calculations for each L show that for some values of $m>1$, one still has $D<0$, even for infinite families.

Similarly, if $G/K = \mathrm{SU}(n+1)/S(U(n)U(1))$ and $H = S(L \cdot U(1) \cdot U(1))$ where $\mathrm{SU}(n)/L$ is isotropy irreducible but not symmetric, then G/H has no G-invariant Einstein metric. Finally, if $G/K = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ and $H = L \cdot \mathrm{Sp}(1)$ where $\mathrm{Sp}(n)/L$ is isotropy irreducible but not symmetric, then G/H has no G-invariant Einstein metric unless either $\mathrm{Sp}(n)/L = \mathrm{Sp}(n)/\mathrm{Sp}(1)\,\mathrm{SO}(n)$ with $3 \leqq n \leqq 6$ or $\mathrm{Sp}(n)/L = \mathrm{Sp}(2)/\mathrm{SU}(2)$.

Example 5. The lowest-dimensional example that we obtain from (3.1) is the following 12-dimensional manifold. Let $G/K = \mathrm{SU}(4)/\mathrm{Sp}(2)$, which is a symmetric space isometric to S^5, and let $H = \mathrm{SU}(2)$ where $\mathrm{Sp}(2)/\mathrm{SU}(2)$ is isotropy irreducible ($\mathrm{Sp}(2)/\mathrm{SU}(2)$ is a rational 7-sphere). Then $\mathfrak{m}_1 = \overset{6}{\bigcirc}$ and $\mathfrak{m}_2 = \bullet\!=\!\overset{1}{\bigcirc}$, so that $\mathfrak{m}_2|_{\mathfrak{h}} = \overset{4}{\bigcirc}$. One easily obtains $c_2^* = \frac{1}{2}$, $c_1^* = \frac{2}{5}$, $\alpha = \frac{3}{4}$ and hence $D < 0$. Thus $M^{12} = \mathrm{SU}(4)/\mathrm{SU}(2)$, which is an $\mathrm{Sp}(2)/\mathrm{SU}(2)$ bundle over S^5, carries no $\mathrm{SU}(4)$-invariant Einstein metric.

For this particular example we will show that $\mathrm{SU}(4)$ is the only compact connected Lie group acting transitively on M^{12}. Thus, *M^{12} carries no G-invariant Einstein metric for any Lie group G acting transitively on it.* (Although this will most likely be true in most of the above examples, it is usually quite difficult to verify that no other Lie group acts transitively on a given homogeneous space.)

We give only an outline of the proof here since the arguments needed for the first half of the proof can be essentially found in [On]. Assume that G' acts transitively on M^{12}. If G' is simple, then it follows from [On], Theorem 7, that $G' = \mathrm{SU}(4)$ and $H' = \mathrm{SU}(2)$. If $G' = G_1 \times G_2$, then it follows easily from [On], Theorem 11, Lemma 10, and Lemma 11 that this is only possible if $M^{12} = G'/H' = G_1/H_1 \times G_2/H_2$.

The index of $\mathrm{SU}(2)$ in $\mathrm{Sp}(2)$ is 10, hence $\pi_3(\mathrm{SU}(2)) \to \pi_3(\mathrm{Sp}(2))$ is multiplication by 10, and so $\pi_3(\mathrm{Sp}(2)/\mathrm{SU}(2)) = \pi_3(M^{12}) = \mathbf{Z}/10$. It follows that $H^k(\mathrm{Sp}(2)/\mathrm{SU}(2), \mathbf{Z}) = \mathbf{Z}$ if $k = 0$ or 7, $\mathbf{Z}/10$ if $k = 4$, and 0 otherwise. The Serre spectral sequence for $\mathrm{Sp}(2)/\mathrm{SU}(2) \to M^{12} \to S^5$ then implies

$$\begin{aligned} H^*(M^{12}, \mathbf{Z}) &= H^*(\mathrm{Sp}(2)/\mathrm{SU}(2), \mathbf{Z}) \otimes H^*(S^5, \mathbf{Z}) \\ &= H^*(\mathrm{Sp}(2)/\mathrm{SU}(2) \times S^5, \mathbf{Z}) \end{aligned}$$

with Poincaré polynomial $(1+t^7)(1+t^5)$. Hence if M^{12} is a product, it follows from [On], Table 2 that this is possible only if $M^{12} = (\mathrm{Sp}(2)/\mathrm{SU}(2)) \times S^5$. Now let y_3 be the generator in $H^3(M^{12}, \mathbf{Z}/2)$. Then the Steenrod square $Sq^2(y_3) = 0$ if M^{12} is equal to this product. On the other hand, by considering the Gysin sequence of the fibration $p\colon \mathrm{SU}(4) \to \mathrm{SU}(4)/\mathrm{SU}(2) = M^{12}$ with fibre $\mathrm{SU}(2) \approx S^3$, it follows that $p^*(y_3) = u_3$ is the unique generator of $H^3(\mathrm{SU}(4), \mathbf{Z}/2)$. Finally, since it is well-known that $Sq^2(u_3) \neq 0$, (see e.g. [Wh], p. 410), it follows that $Sq^2(y_3) \neq 0$, showing that $M^{12} \neq (\mathrm{Sp}(2)/\mathrm{SU}(2)) \times S^5$.

Example 6. (3.1) can also be used to produce many new examples of homogeneous Einstein metrics on G/H's not covered by the theorems in §2. One interesting one is obtained as follows. One easily sees that on the symmetric space $\mathrm{SO}(2n)/U(n)$ the subgroup $\mathrm{SO}(2n-1) \subset \mathrm{SO}(2n)$ acts transitively with isotropy group $U(n-1)$. We can then apply (3.1) to the groups $G=\mathrm{SO}(2n-1)$, $K=\mathrm{SO}(2n-2)$, and $H=U(n-1)$. Notice that $K/H \to G/H \to G/K$ is a fibration whose base and fibre are symmetric spaces and G/H is diffeomorphic to $\mathrm{SO}(2n)/U(n)$. We have $\mathfrak{m}_2=\rho_{2n-2}$ and so $\mathfrak{m}_2|_{\mathfrak{h}}=[\mu_{n-1}]_{\mathbf{R}}$. Furthermore, $d_1=(n-1)(n-2)$, $d_2=2(n-1)$, $\alpha=\dfrac{2n-4}{2n-3}$, and hence $D=1/(2n-3)^2>0$. By (3.1) we obtain two Einstein metrics on $\mathrm{SO}(2n-1)/U(n-1)$ one of which is isometric to the symmetric metric. The other one is a new Einstein metric on G/H.

Example 7. We finally construct some examples of G/H which have no G-invariant Einstein metrics and which have 3 irreducible summands in $\mathfrak{m}$. Let $H \subset K_1 \subset K_2 \subset G$ be connected closed subgroups and write $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3$, where $\mathfrak{k}_1=\mathfrak{h}\oplus\mathfrak{m}_1$, $\mathfrak{k}_2=\mathfrak{h}\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2$. We assume that $\mathfrak{m}_i$ are mutually inequivalent irreducible H-representations. Therefore,

$$\begin{bmatrix}2\\11\end{bmatrix}=\begin{bmatrix}3\\11\end{bmatrix}=\begin{bmatrix}3\\12\end{bmatrix}=\begin{bmatrix}3\\22\end{bmatrix}=0.$$

By (1.3), a G-invariant metric of volume 1 has scalar curvature

$$S=\frac{1}{2}\left(\frac{A_1}{x_1}+\frac{A_2}{x_2}+\frac{A_3}{x_3}\right)-\frac{1}{4}\left(\begin{bmatrix}2\\12\end{bmatrix}\frac{x_1}{x_2^2}+\begin{bmatrix}3\\13\end{bmatrix}\frac{x_1}{x_3^2}+\begin{bmatrix}3\\23\end{bmatrix}\frac{x_2}{x_3^2}\right)$$

with $x_1^{d_1}x_2^{d_2}x_3^{d_3}=v$ and

$$A_1=d_1-\frac{1}{2}\begin{bmatrix}1\\11\end{bmatrix}-\begin{bmatrix}2\\12\end{bmatrix}-\begin{bmatrix}3\\13\end{bmatrix},$$

$$A_2=d_2-\frac{1}{2}\begin{bmatrix}2\\22\end{bmatrix}-\begin{bmatrix}3\\23\end{bmatrix}, \quad \text{and} \quad A_3=d_3-\frac{1}{2}\begin{bmatrix}3\\33\end{bmatrix}.$$

Let us assume further that $\begin{bmatrix}2\\12\end{bmatrix}=0$, which is the case if $H=H_1\times H_2$, $H_1\subset L_1$, $H_2\subset L_2$, $K_1=L_1\times H_2$, $K_2=L_1\times L_2$. Suppose that G is semisimple and that $B_{\mathfrak{l}'_1}=\alpha_1 B|\mathfrak{l}'_1$, $B_{\mathfrak{l}'_2}=\alpha_2 B|\mathfrak{l}'_2$, $\alpha_i>0$, where L'_i/H'_i is the effective homogeneous space corresponding to L_i/H_i. By (1.5) we get

$$A_1=\left(c_1+\frac{1}{2}\right)d_1-\frac{1}{2}\begin{bmatrix}3\\13\end{bmatrix}, \quad A_2=\left(c_2+\frac{1}{2}\right)d_2-\frac{1}{2}\begin{bmatrix}3\\23\end{bmatrix},$$

$$A_3=\left(c_3+\frac{1}{2}\right)d_3+\begin{bmatrix}3\\13\end{bmatrix}+\begin{bmatrix}3\\23\end{bmatrix}.$$

As in the discussion before (3.1), we have $\begin{bmatrix}3\\13\end{bmatrix}=d_1(1-\alpha_1)$, $\begin{bmatrix}3\\23\end{bmatrix}=d_2(1-\alpha_2)$.

Let c_3^* be the Casimir constant of the $\mathfrak{k}_2$ representation $\mathfrak{m}_3$ with respect to $B|\mathfrak{k}_2$ and c_2^* be the Casimir constant of the $\mathfrak{k}_1$ representation $\mathfrak{m}_2$ with respect to $B_{\mathfrak{l}'_2}|\mathfrak{h}_2$. Clearly, we have

$$c_2=\alpha_2 c_2^* \quad \text{and} \quad 1-2c_3=\frac{1}{d_3}\left(\begin{bmatrix}3\\13\end{bmatrix}+\begin{bmatrix}3\\23\end{bmatrix}\right)=\frac{1}{d_3}(d_1(1-\alpha_1)+d_2(1-\alpha_2)).$$

Let $t_1=\frac{x_1}{x_3}$, $t_2=\frac{x_2}{x_3}$. Then a routine computation shows that $t_1>0$, $t_2>0$ corresponds to a critical point of S on $x_1^{d_1}x_2^{d_2}x_3^{d_3}=v$ iff

$$\frac{d_2}{d_3}(1-\alpha_2)t_1t_2=-d_1(1-\alpha_1)\left(\frac{1}{d_3}+\frac{1}{2d_1}\right)t_1^2+\frac{A_3}{d_3}t_1-\frac{A_1}{d_1}$$

$$\frac{d_1}{d_3}(1-\alpha_1)t_1t_2=-d_2(1-\alpha_2)\left(\frac{1}{d_3}+\frac{1}{2d_2}\right)t_2^2+\frac{A_3}{d_3}t_2-\frac{A_2}{d_2}.$$

Since the left hand side must be positive, a necessary condition for the existence of a G-invariant Einstein metric is

$$\left(\frac{A_3}{d_3}\right)^2-4A_2(1-\alpha_2)\left(\frac{1}{d_3}+\frac{1}{2d_2}\right)\geqq 0.$$

In other words, $(c_3^*+\frac{1}{2})^2-(1-\alpha_2)\left(1+\frac{2d_2}{d_3}\right)\alpha_2(1+2c_2^*)\geqq 0$. Comparing this with the expression for D in (3.1), we see that if G/H has a G-invariant Einstein metric, then so does $G/(L_1\times H_2)$. So whenever $G/(L_1\times H_2)$ does not have a G-invariant Einstein metric and L_1/H_1 is isotropy irreducible, we obtain a new example of non-existence.

As a concrete example, let $G=\mathrm{SU}(m+n)$, $K_2=S(U(m)\times U(n))$, $K_1=S(U(m)\times \mathrm{SO}(n)\times U(1))$ as in example 2. Let $H=S(R\times U(1)\times \mathrm{SO}(n)\times U(1))$ where $\mathrm{SU}(m)/R$ is isotropy irreducible (possibly symmetric). Now $\mathfrak{m}_3=[\mu_m^*\otimes\mu_n]_{\mathbf{R}}$ as a $\mathfrak{k}_2$-representation. Upon restriction to $\mathfrak{h}$, $\mathfrak{m}_3=[\pi^*\otimes\phi\otimes\rho_n\otimes\psi]_{\mathbf{R}}$ where $\phi\colon U(1)\to U(m)$, $\psi\colon U(1)\to U(n)$ are the inclusions of the centers, and $\pi\colon R\to\mathrm{SU}(m)$ is the embedding of R into $\mathrm{SU}(m)$. By the above argument, G/H has no G-invariant Einstein metric and $\mathfrak{m}$ consists of three irreducible representations of H.

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Oblatum 26-III-1985 & 26-VIII-1985