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Inventiones mathematicae

Variation of mixed Hodge structure. I

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Introduction

Around 1970, Griffiths introduced the notion of a *variation of Hodge structure* on a complex manifold S (see [17, § 2]). It constitutes the axiomatization of the features possessed by the local systems of cohomology associated to a family of compact Kähler manifolds (esp., smooth projective varieties) parametrized by S. These are, for a variation of Hodge structure $\mathbb{V}$ of weight m:

(0) a locally constant sheaf $\mathbb{V}_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces [or $\mathbb{V}_{\mathbb{R}}$ of real vector spaces],

(1) (Hodge filtration; horizontality) a decreasing filtration $\{\mathscr{F}^p\}$ of $\mathscr{V}=\mathcal{O}_S\otimes_{\mathbb{Q}}\mathbb{V}_{\mathbb{Q}}$ by holomorphic subbundles, such that the evident connection ∇ on $\mathscr{V}$ takes $\mathscr{F}^p$ into $\Omega^1_S\otimes_{\mathcal{O}_S}\mathscr{F}^{p-1}$;

(2) (Hodge decompositions) if $\mathscr{E}_S$ denotes the sheaf of $\mathscr{C}^\infty$ functions on S, and if

$$\mathscr{H}^{p,q}=(\mathscr{E}_S\otimes_{\mathcal{O}_S}\mathscr{F}^p)\cap\overline{(\mathscr{E}_S\otimes_{\mathcal{O}_S}\mathscr{F}^q)}$$

where the bar denotes complex conjugation with respect to $\mathbb{V}_{\mathbb{R}}$, then

$$(\mathscr{E}_S\otimes_{\mathbb{Q}}\mathbb{V}_{\mathbb{Q}})\simeq\bigoplus_{p+q=m}\mathscr{H}^{p,q};$$

and, easiest (and most foolish!) to omit:

(3) (polarizability) there exists a flat pairing

$$\beta\colon \mathbb{V}_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{V}_{\mathbb{Q}}\to\mathbb{Q}\qquad[\text{or } \mathbb{V}_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{V}_{\mathbb{R}}\to\mathbb{R}]$$

such that, when it is extended linearly over $\mathscr{E}_S$, the formula $\beta(Cv,\bar{w})$ defines a positive definite Hermitian form. (Here, C is the so-called Weil operator of the variation; see, e.g., our (3.2)).

In case $\mathbb{V}_{\mathbb{Q}}=R^m f_*\mathbb{Q}$ for some smooth projective [or proper Kähler] morphism $f\colon X\to S$, $\mathscr{F}^p$ is the bundle with fibers

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$$F^p H^m(X_s, \mathbb{C}) = \bigoplus_{r \geq p} H^{r,m-r}(X_s),$$

where we are writing X_s for $f^{-1}(s)$, and F denotes the Hodge filtration of its cohomology; horizontality is a direct consequence of relative de Rham theory; and a polarization can be constructed from cup-product, using the whole Kählerian story: Poincaré duality, hard Lefschetz theorem, primitive decomposition, and Hodge signature theorem. The preceding situation will be referred to as the *geometric case.*

Suppose now that S is a Zariski-open subset of $\bar{S}$, a compact complex manifold. According to Hironaka, we may choose $\bar{S}$ in such a way that $\Sigma = \bar{S} - S$ becomes a divisor with normal crossings. During the course of the 1970's, there were three main developments concerning the behavior of a variation of Hodge structure along Σ and its consequences, almost entirely for the case where S is a curve. In order to state these properly, we find it convenient to first introduce Deligne's notion of the "canonical extension" $\tilde{\mathscr{V}}$ of $\mathscr{V}$ to $\bar{S}$ [2]. At any point of Σ, one has local coordinates in which $j\colon S \hookrightarrow \bar{S}$ is given as the inclusion of the punctured disc in the disc. Then $\tilde{\mathscr{V}} \subset j_* \mathscr{V}$ is characterized by the growth of the coefficients of sections with respect to a (multivalued) frame for $\mathbb{V}$ (see our (3.7)). We interpret the Hodge filtration as giving a filtration of $\tilde{\mathscr{V}}|_S$.

We can now give a summary of the three developments mentioned above.

(A) *Singularities of the period mapping* (Schmid [9]). As can be seen on each punctured disc separately, $\{\mathscr{F}^p\}$ extends to a filtration $\{\tilde{\mathscr{F}}^p\}$ of $\tilde{\mathscr{V}}$ on $\bar{S}$. While this is not so hard to see by other means in the geometric case over a curve, Schmid's proof shows that the filtration of $\mathscr{V}$ that has constant value $\{\tilde{\mathscr{F}}^p(0)\}$ with respect to a standard frame for $\tilde{\mathscr{V}}$ is itself a variation of Hodge structure on some deleted neighborhood of 0; moreover, it carries over to the case of more variables, i.e., to the general local situation of the normalized problem. In the case of one variable again, the general variation of Hodge structure is asymptotic (in a specified way) to a special (locally homogeneous) one associated to a representation of SL_2. This gives rise to asymptotic formulas for the Hodge norms: $\|v\|^2 = \beta(Cv, \bar{v})$. In addition, one obtains a clear picture of the interaction between the filtration $\{\tilde{\mathscr{F}}^p(0)\}$ of $\tilde{\mathscr{V}}(0)$ and the weight filtration M, centered at m, (see our (2.1) and (2.4)) of the logarithm N_0 of the unipotent Jordan factor of the local monodromy transformation (which acts naturally on $\tilde{\mathscr{V}}(0)$). This is the so-called *limit mixed Hodge structure*, and N_0 acts as a morphism of type $(-1, -1)$.

(B) *De Rham theoretic realization of the limit mixed Hodge structure in the geometric case* (Steenbrink [10], Clemens [14]; see also [12]). One considers the local geometric situation

$$\begin{array}{ccc} X^* & \hookrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \Delta^* & \hookrightarrow & \Delta \end{array}$$

with X Kähler, f proper, f' proper and smooth, and $X^* = f^{-1}(\Delta^*)$; as usual, Δ denotes the unit disc and $\Delta^* = \Delta - \{0\}$. Put $Y = f^{-1}(0)$, a divisor with normal crossings. It follows from the construction that

$$\tilde{\mathscr{V}} = \mathbf{R}^m f_* \Omega^{\bullet}_{X/\Delta}(\log Y),$$
$$\tilde{\mathscr{F}}^p = \mathbf{R}^m f_* F^p \Omega^{\bullet}_{X/\Delta}(\log Y),$$

where F denotes the usual Hodge filtration of a holomorphic de Rham complex; and

$$\tilde{\mathscr{V}}(0) = \mathbf{H}^m(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y),$$

on which $\{\tilde{\mathscr{F}}^p(0)\}$ is the filtration induced by F. Assuming for simplicity that Y is reduced (as can always be achieved after a base-change), one constructs a "resolution" $A^{\bullet,\bullet}$ of

$$\Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y$$

(cf. our (5.5)) that admits two filtrations, which are shown to induce the limit mixed Hodge structure. An important consequence of the construction is a proof of the local invariant cycle theorem (posed as a problem in [17, (8.1)]), as $A^{\bullet,\bullet}$ contains a copy of the cohomological mixed Hodge complex of Y (see also Part II, §8).

(C) *Hodge theory with degenerating coefficients* (Zucker [11]). By using the full strength of (A) above, and then using the construction for the case $\Sigma = \emptyset$, given by Deligne, as a model, one can construct a filtration F of the complex (resolving $j_* \mathbf{V}_{\mathbf{C}}$)

$$\tilde{\mathscr{V}} \to \nabla \tilde{\mathscr{V}} \subset \Omega^1_{\bar{S}}(\log \Sigma) \otimes_{\mathcal{O}_{\bar{S}}} \tilde{\mathscr{V}}$$

by setting F^p to be the subcomplex

$$\tilde{\mathscr{F}}^p \to (\nabla \tilde{\mathscr{V}}) \cap (\Omega^1_{\bar{S}}(\log \Sigma) \otimes \tilde{\mathscr{F}}^{p-1}).$$

It is then proved that F induces a functorial Hodge structure of weight $m+i$ on $H^i(\bar{S}, j_* \mathbf{V})$. Moreover, in the global geometric case, the Hodge structures are compatible with those of $H^{\bullet}(X)$, in the sense that the Leray spectral sequence of f becomes a spectral sequence of Hodge structures. To see this, one must first realize that the F here is in a certain precise sense induced by that of $\Omega^{\bullet}_X(\log Y)$, and then make use of many of the results discussed in (B) above. There is a companion mixed Hodge theory for $H^{\bullet}(S, \mathbf{V})$, compatible with $H^{\bullet}(X - Y)$.

It was inevitable that there would be attempts to generalize the preceding to variations of *mixed* Hodge structure, corresponding in geometry to families of varieties that are singular or non-compact. Of course, one must first decide what a variation of mixed Hodge structure is. We take the natural position, as before, that it should axiomatize the features of local systems of cohomology associated to families of varieties. Now, the very first observation is that for any surjective quasi-projective morphism h: $U \to S$, there is a Zariski-dense open subset of S over which h is a stratified fiber bundle, and the weight filtration of Deligne's mixed Hodge structures [4] on the cohomology of the fibers is locally constant. Thus, in the definition of "variation of Hodge structure", one must first add to (0):

$$^{+}(0)\text{: and an increasing filtration } W=\{\mathbb{W}_k\} \text{ of } \mathbb{V}_{\mathbb{Q}} \text{ [or } \mathbb{V}_{\mathbb{R}}]^1,$$

and then each $\mathrm{Gr}_k^W \mathbf{V}$, with the filtration induced by $\mathscr{F}$, should satisfy (1)-(3). It is also quickly seen from relative de Rham theory that one should insist on horizontality for the full filtration $\mathscr{F}$, and not only for each $\mathrm{Gr}_k^W \mathscr{F}$. We call the package of data described above in this paragraph a *graded-polarizable variation of mixed Hodge structure.*[2]

In [5, § 1], Deligne posed the problem of distinguishing a good class of variations of mixed Hodge structure, such that (A) generalizes. This already presupposes, as he made explicit in [5], the existence of a filtration M on $\tilde{\mathscr{V}}(0)$ that induces on each successive quotient $\mathrm{Gr}_k^W \tilde{\mathscr{V}}(0)$ the weight filtration of $\mathrm{Gr}_k^W N_0$. As an abstract linear algebra condition on N_0, it is discovered fairly easily that there is at most one possibility for M (see our (2.8)). When the variation of mixed Hodge structure is geometric, M does, in fact, exist, and is called the *weight filtration of N_0 relative to W.* Existence was shown first by Deligne, in [5, (1.8)], for the analogous assertion for the l-adic cohomology of varieties defined over a finite field, and it then follows over $\mathbb{C}$ by comparison methods. This may seem ungratifying; however, the generalization of (B) provides a proof of the existence of M in the geometric case via characteristic zero techniques, as predicted by Deligne and settled by El Zein [6] and others (see our (5.7)).

We thus add to the assumptions that the filtration M exists. The generalization of (A) would also imply the existence of a limit Hodge filtration $\{\tilde{\mathscr{F}}^p(0)\}$, which, together with M, determines a mixed Hodge structure on $\tilde{\mathscr{V}}(0)$, filtered by W, on which N_0 acts as a morphism of type $(-1, -1)$. In addition, we wish to add the natural and seemingly innocuous, but actually quite strong, assumption that $\{\tilde{\mathscr{F}}^p(0)\}$ induces, on each

$$\mathrm{Gr}_k^W(\tilde{\mathscr{V}}(0)) \simeq (\mathrm{Gr}_k^W \mathscr{V})^{\sim}(0),$$

the limit Hodge filtration of Schmid. These conditions are also satisfied in the geometric case, as follows again from the generalization of (B).

The conditions concerning M, $\{\tilde{\mathscr{F}}^p(0)\}$ and N_0 in the previous paragraph are formulated in § 3 as Properties (3.13). We take them as a minimal list of extra conditions on a graded-polarizable variation of mixed Hodge structure. By some simple examples ((3.15) and (3.16)), we can see that they really do comprise extra hypotheses, and that there is more than one independent condition.

As a justification for our claim that (3.13) already forms a good set of conditions, we prove, as one of the two main results in this paper, that they

[1] Hence also of $\tilde{\mathscr{V}}$

[2] The prefix "graded-" is omitted in the terminology used by others. We will later give some indication why we feel that it is a good idea to include it

allow the generalization of (C). We show, in §4, that $H^{\bullet}(\bar{S}, j_*\mathbb{V})$, $H^{\bullet}(S,\mathbb{V})$ and $H^{\bullet}_c(S,\mathbb{V})$ all admit functorial mixed Hodge structures, filtered by W, and that those of dual cohomology groups are dual mixed Hodge structures. (They reduce to those in [11] when $\mathbb{V}$ is pure.)

It is natural - in fact, essential - to ask whether, when $\mathbb{V}$ comes from a family of varieties $f\colon Z \to S$, the mixed Hodge theory for $H^{\bullet}(S,\mathbb{V})$ is compatible, as before, with that of $H^{\bullet}(Z)$. The answer is "yes", although we found it surprisingly difficult to prove that the weight filtrations are compatible; the general case is more "entangled" than the pure case. The proof will be given in Part II.

In §5, we present our original construction of the (filtered) cohomological mixed Hodge complex $A^{\bullet\cdot\bullet}$ for the generalization of (B) in the case of a family of smooth (esp. non-compact) varieties. It coincides with the one that El Zein gives in [6], but has a somewhat neater formulation. In doing this, we take the opportunity to clarify the treatment in [10] of the underlying $\mathbb{Q}$-structure, which we adapt accordingly to the mixed case.

In §6, we first give a short exposition of El Zein's work on filtered cohomological mixed Hodge complexes. We discovered that an intriguing feature of the cohomological mixed Hodge complexes used in Sections 4 and 5 implies that the complex belongs to both classes of tri-filtered complexes (with nice Hodge theoretic properties) that are discussed in [16]. Specifically, let $(K^{\bullet}, M, F)$ be a cohomological mixed Hodge complex, with a third filtration W such that (say) $\mathrm{Gr}^W_l K^{\bullet}$, with the filtrations induced by M and F, is also a cohomological mixed Hodge complex for all l. The special feature is

$$\mathrm{Gr}^M_k W_l K^{\bullet} \simeq \bigoplus_{j \leq l} \mathrm{Gr}^M_k \mathrm{Gr}^W_j K^{\bullet}.$$

There is an undercurrent that the time is now ripe for the development of the theory of variations of mixed Hodge structure. Eight years ago, there seemed to be only sporadic and casual interest in a general theory, but in the last few years there has been a flurry of activity in the study of degenerations of singular or non-compact varieties. As evidence for this, we cite that Usui uses the concept of a variation of mixed Hodge structure [19] and that, besides El Zein and ourselves, the following mathematicians have produced constructions similar to the one in §5: Du Bois; Guillén, Navarro Aznar and Puerta [13, 18].

No, we haven't forgotten §§1-2; we have only deferred discussing them, as they are to a large degree independent from the geometry. In §1, we present some basic facts about filtered vector spaces, including the useful notion in (1.4) of the *convolution* $L*W$ of two filtrations L and W.

The notion of the weight filtration of a nilpotent endomorphism N of a finite-dimensional vector space V, relative to a filtration W of V (denoted $M(N;W)$), is recalled in §2. The other main result of the paper involves giving a direct linear-algebraic condition (2.20) that is necessary and sufficient for the existence of $M(N;W)$. It is actually a condition on how strict N and its powers must be with respect to adjacent W-filtration levels; this gives the criterion for obtaining $M(N|_{W_k};W)$, given $M(N|_{W_{k-1}};W)$. We say that N is an *admissible* nilpotent transformation if N satisfies the condition of (2.20), i.e., if $M(N;W)$ exists. We list some useful cases:

(i) the following two assertions are equivalent: all powers of N are strictly compatible with W; $M(N;W)=L*W$, where L is the (absolute) weight filtration of N (2.11).

(ii) if $NW_k \subset W_{k-1}$, then $M(N;W)$ exists if and only if $NW_k \subset W_{k-2}$ (and then $M(N;W)=W$) (2.14).

(iii) a necessary condition for the existence of $M(N;W)$ is that

$$N^l W_k \cap W_{k-1} \subset N^l W_{k-1} + W_{k-2}$$

for all k and l; we then say that the powers of N are *quasi-strict* (with respect to W) (2.17).

(iv) (as we have observed earlier) $M(N;W)$ exists whenever N is a nilpotent monodromy logarithm from a geometric variation of mixed Hodge structure and W its weight filtration. (N.B. – N need not be strictly compatible with W; see (2.12).)

In the appendix, based on a letter from Deligne, we describe a completely different approach to the existence of $M(N;W)$. This leads to the observation that in fact Condition (3.13, iii) is a consequence of (3.13, i) and (3.13, ii). Moreover, it follows that if N, N' are admissible endomorphisms of the filtered vector spaces (V,W) and (V',W') respectively, then $N\otimes 1+1\otimes N'$ is an admissible endomorphism of $(V\otimes V', W\otimes W')$.

This finishes the summary of the setting and the contents of the paper. We would like to conclude by posing some unresolved questions and problems related to the results we have just discussed:[3]

1. Given V, W and N, and for each k a weight k Hodge filtration ${}^kF$ on $\mathrm{Gr}_k^W V$, give a characterization of when there exists a filtration F of V that induces all ${}^kF$, such that $NF^p \subset F^{p-1}$.

2. a) Is it possible to eliminate the recursiveness in the necessary and sufficient condition for admissibility given in (2.20)?

b) Give an expression of M_k in closed form.

3. Give a proof of (2.20) using the approach in the appendix.

4. Give a good definition of a polarized mixed Hodge structure. (Note, for instance, that nowhere in the definition of a graded-polarizable variation of mixed Hodge structure is there any statement that relates the polarizations for the different $\mathrm{Gr}_k^W V'$ s.) It is possible that this will be important in:

5. [5, (1.8.15)] Generalize (A) for an appropriate class of variations of mixed Hodge structure.

There are some related recent developments[3] in the theory of variations of (pure) Hodge structure in several variables, i.e., for $S=(\Delta^*)^r$, where $r>1$. (Such a variation of Hodge structure provides variations of mixed Hodge structure whose fibers are the limit mixed Hodge structures in given directions.) In [1], Cattani and Kaplan have proved results on "the uniqueness of the weight filtration" and the weight filtration of one monodromy logarithm relative to the weight filtration of another (see our (3.12) for the precise statement); these are

[3] Also, see notes added in proof

the Hodge theoretic analogues of results of Deligne on the l-adic cohomology of varieties over finite fields [5, (1.9)]. It is trivial that "uniqueness" for $M(N; W)$ holds in the mixed case; we ask (3.18):

6. Is the analogue of assertion (3.12, iii) on relative weight filtrations true in the mixed case?

Finally, we expect that the relations among the various weight filtrations provided by [1] will connect up with the solution to:

7. Carry out the analogue of (B) when the dimension of S is greater than one.

We want to thank Deligne for several helpful conversations, and especially for introducing us to the notion of a relative weight filtration.

The second-named author thanks the University of Leiden and the Z.W.O.-project "Singularity Theory" for their hospitality during June, 1981 and June, 1982.

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§ 1. Generalities on filtered vector spaces

In this chapter, we work over a fixed, though arbitrary, field of scalars. There is no difficulty in seeing that the notions introduced carry over to sheaves of vector spaces.

(1.1) *Definition.* A *vector space of weight* k consists of a vector space V and the assignment of the integer k to V; i.e., it is the pair (V, k). We call V also a *weighted vector space.*

(1.2) *Definition.* i) An *increasing filtration* W on a vector space V is a collection of subspaces $\{W_k\}_{k\in\mathbb{Z}}$, such that

$$W_{k-1} \subset W_k \quad \text{for all } k\in\mathbb{Z};$$

ii) A *decreasing filtration* F of a vector space V is a collection of subspaces $\{F^p\}_{p\in\mathbb{Z}}$, such that

$$F^p \subset F^{p-1}.$$

A decreasing filtration F on V defines also an increasing filtration $\hat{F}$ by the formula

$$\hat{F}_k = F^{-k}.$$

As such, we will consider only increasing filtrations in this chapter.

We call the pair (V, W) a *filtered vector space.*

Given a filtration W on V and $n \in \mathbb{Z}$, one lets $W[n]$ denote the shifted filtration on V given by

$$W[n]_k = W_{k+n}.$$

The *trivial filtration* T of V is determined by

$$T_{-1} = \{0\}, \qquad T_0 = V.$$

The filtration W of V will be called *finite* if $W_k = \{0\}$ if k is sufficiently small, and $W_k = V$ when k is sufficiently large. In practice, the filtrations we will consider are all finite.

As usual, we put

$$\mathrm{Gr}_k^W V = W_k / W_{k-1},$$

which we shall regard as a vector space of weight k.

For any subspace V' of V, there is an induced filtration WV' defined by

$$W_k V' = W_k \cap V'.$$

Likewise, if $V'' \subset V'$, one defines

$$W_k(V'/V'') = W_k V' / W_k V'' = \mathrm{Im}\{W_k V' \to V'/V''\}.$$

In particular, if L is another filtration of V, then W induces a filtration on $\mathrm{Gr}_j^L V$, and one has

(1.3) $\mathrm{Gr}_k^W \mathrm{Gr}_j^L V \simeq \mathrm{Gr}_j^L \mathrm{Gr}_k^W V \simeq (L_j \cap W_k)/[(L_j \cap W_{k-1}) + (L_{j-1} \cap W_k)].$

(1.4) *Definition.* Let L and W be filtrations on V. Then the *convolution* $L * W$ of L and W is the filtration with

$$(L * W)_i = \sum_{j+k=i} (L_j \cap W_k).$$

Clearly, $L * W = W * L$, $T * W = W$, and $(L * W)[n] = L * (W[n])$. Unfortunately, convolution is not an associative operation. The following property of the convolution is the motivation for introducing the notion:

(1.5) **Proposition.** *Let $C = L * W$. Then*

$$C_i \mathrm{Gr}_k^W V \simeq L_{i-k} \mathrm{Gr}_k^W V$$

Proof.

$$\begin{aligned} C_i \cap W_k &= W_k \cap \sum_{j+l=i} (L_j \cap W_l) \\ &= \sum_{\substack{l<k \\ j+l=i}} (L_j \cap W_l) + \Big[W_k \cap \sum_{\substack{l \geq k \\ j+l=i}} (L_j \cap W_l) \Big] \\ &= \Big[\sum_{\substack{l<k \\ j+l=i}} (L_j \cap W_l) \Big] + (L_{i-k} \cap W_k) \qquad \text{(see (1.12, iii))} \\ &= \sum_{\substack{l \leq k \\ j+l=i}} (L_j \cap W_l). \end{aligned}$$

Thus also

$$C_i \cap W_{k-1} = \sum_{\substack{l \leq k-1 \\ j+l=i}} (L_j \cap W_l).$$

Therefore,

$$C_i \operatorname{Gr}_k^W V \simeq (L_{i-k} \cap W_k) \Big/ \Big[(L_{i-k} \cap W_k) \cap \sum_{\substack{l<k \\ j+l=i}} (L_j \cap W_l) \Big].$$

But

$$L_{i-k} \cap W_k \cap \sum_{\substack{l<k \\ j+l=i}} (L_j \cap W_l) = L_{i-k} \cap \sum_{l<k} (L_{i-l} \cap W_l)$$
$$= L_{i-k} \cap W_{k-1},$$

again by (1.12, iii). We now have

$$C_i \operatorname{Gr}_k^W V = (L_{i-k} \cap W_k)/(L_{i-k} \cap W_{k-1})$$
$$= L_{i-k} \operatorname{Gr}_k^W V,$$

as desired. □

(1.6) **Corollary.** $\operatorname{Gr}_i^C \operatorname{Gr}_k^W V \simeq \operatorname{Gr}_{i-k}^L \operatorname{Gr}_k^W V$.

In other words, taking the convolution with W shifts the filtration L on $\operatorname{Gr}_k^W V$ by $-k$:

$$C \operatorname{Gr}_k^W V = L \operatorname{Gr}_k^W V[-k].$$

(1.7) *Definition.* Let V be equipped with filtration W, and V' with filtration W'. The linear mapping $\Phi: V \to V'$ is called a *morphism of filtered vector spaces* if it is compatible with the filtrations:

$$\Phi(W_k) \subset W'_k.$$

We then write $\Phi: (V, W) \to (V', W')$.

If Φ is a morphism of filtered vector spaces, then Φ defines linear mappings

$$\Phi_k: W_k \to W'_k$$
$$\operatorname{Gr}_k \Phi: \operatorname{Gr}_k^W V \to \operatorname{Gr}_k^{W'} V'.$$

The following is elementary and well-known.

(1.8) **Lemma.** *Suppose in the above that W is a finite filtration. If* $\operatorname{Gr}_k \Phi$ *is injective for all k, then Φ is itself injective. Similarly, if W' is finite and* $\operatorname{Gr}_k \Phi$ *is surjective for all k, then Φ is surjective.*

This gives immediately:

(1.9) **Proposition.** *If W and W' are finite filtrations, and* $\operatorname{Gr}_k \Phi$ *is an isomorphism for all k, then Φ is an isomorphism.*

(1.10) *Definition* [3, (1.1.5)]. A morphism of filtered vector spaces as in (1.7) is said to be *strictly compatible with the filtrations* (*strict*, for short) if

$$\Phi(V) \cap W'_k = \Phi(W_k),$$

i.e.,

$$\Phi^{-1}(W'_k) = W_k + \ker \Phi.$$

The above definition can be reformulated as follows. The image of Φ inherits the filtration W'. Via the isomorphism

$$\Phi(V) \simeq V/\ker \Phi,$$

the image of Φ also gets a quotient filtration induced by W, and

$$W_k \operatorname{Im} \Phi \subset W'_k \operatorname{Im} \Phi$$

for all k. To say that Φ is strict is equivalent to asserting that the two filtrations coincide.

(1.11) *Remark.* The composite of strict morphisms need not be strict. In fact, if Φ is an endomorphism of V that is strictly compatible with W, Φ^2 will not in general be strict. For example, let V be a 4-dimensional vector space with basis $\{e_1, e_2, e_3, e_4\}$, and let $V' = \operatorname{Span}\{e_1, e_2\}$. Suppose that V' is the only non-trivial filtration level. Let Φ be the transformation determined by putting $\Phi(e_1) = 0$, $\Phi(e_2) = e_1$, $\Phi(e_3) = 0$, and $\Phi(e_4) = e_1 + e_2 + e_3$. One checks readily that Φ is strict. However, one has that $\Phi^2(e_1) = \Phi^2(e_2) = 0$, while $\Phi^2(e_4) = e_1 \in V'$, and we see that Φ^2 is not strict. It is even easier to see that it is possible for Φ^2 to be strict (e.g., the zero mapping) when Φ is not.

At the risk of appearing frivolous, we have decided to make explicit some very elementary facts we will make repeated use of; they already appear in the proof of (1.5):

(1.12) *Tautologies.* i) If P, Q and R are sets, then $P \cap Q \subset R$ if and only if $P \cap Q \subset Q \cap R$.

ii) If P, Q and R are subspaces of a vector space, and $P \supset Q$, then $P \cap (Q + R) = Q + (P \cap R)$.

iii) If P, $Q_0, \ldots, Q_n$, $R_0, \ldots, R_n$ are subspaces of a vector space, $P \subset R_0$, and $Q_i \subset Q_0$ for all i, then

$$P \cap \sum_{i=0}^{n} (Q_i \cap R_i) = P \cap Q_0.$$

§2. Relative weight filtrations

Let V be a finite-dimensional vector space, and let N be a nilpotent endomorphism of V. The following is well-known:

(2.1) **Proposition.** *There is a unique filtration $L = L(N)$ of V such that*

(1) $NL_i \subset L_{i-2}$

(2) *N^i induces an isomorphism* $\operatorname{Gr}_i^L V \simeq \operatorname{Gr}_{-i}^L V$.

Suppose that $N^{q+1} = 0$. Then L is given iteratively (as i decreases) by the following formulas:

(i) $L_{-(q+1)} = 0$,

(ii) if $i \geqq 0$, $L_i = \{v \in V \colon N^{i+1} v \in L_{-i-2}\}$,

(iii) if $i > 0$, $L_{-i} = N^i L_i$.

One interprets (2.1(1)) as saying that N is a morphism from (V, L) to $(V, L[-2])$. From (2.1) and (1.8), one obtains the following useful property of L:

(2.2) **Corollary.** *If* $i \leqq l$, *then* $N^l\colon L_i \to L_{i-2l}$ *is surjective,*
ii) *If* $i \geqq -l-1$, *then* $N^l\colon V/L_{i+2l} \to V/L_i$ *is injective*, i.e.,

$$L_{i+2l} = (N^l)^{-1} L_i.$$

(2.3) *Remark.* In fact, one has the following explicit formula for L_i:

$$L_i = \sum_{l \geqq 0, -i} N^l(\ker N^{i+1+2l}).$$

Let now K denote the *kernel filtration* of V:

$$K_j = \ker N^{j+1} \qquad (j \geqq -1)$$

and I the *image filtration*, determined by

$$I_k = N^{-k} V \qquad (k \leqq 0).$$

With the aid of the elementary formula

$$K_j \cap I_k = \ker N^{j+1} \cap N^{-k} V = N^{-k}(\ker N^{j+1-k}),$$

one sees that

$$L = K * I.$$

One calls L the *weight filtration* of N. It is in an obvious sense centered at zero. If V is of weight k, then it is useful to recenter L at k by setting

$$M = M(N) = L[-k]. \tag{2.4}$$

We remark that when N is the logarithm of an unipotent local monodromy transformation of a polarizable variation of Hodge structure, then M is important for mixed Hodge theory (see [9, (6.16)], [11, § 13]).

Let N now be a nilpotent endomorphism of the filtered vector space (V, W), and let ${}^kM$ denote the filtration $M(\mathrm{Gr}_k N)$ on $\mathrm{Gr}_k^W V$.

(2.5) *Definition.* A *weight filtration* of N *relative to* W is a filtration M of V such that

i) $NM_i \subset M_{i-2}$
ii) $M\,\mathrm{Gr}_k^W V = {}^kM$.

(2.6) **Proposition** [5, (1.6.13)]. *There is at most one filtration* M *of* V *satisfying the conditions of* (2.5).

Thus, if such M exists, we call it *the* weight filtration of N relative to W, and denote it $M = M(N; W)$. If N is the logarithm of the unipotent factor of the local monodromy of the l-adic cohomology of an algebraic variety over a field of characteristic p, and W is the weight filtration according to the eigenvalues of Frobenius, the existence of M is proved in [5, (1.8.5)]. It follows by comparison methods that the same holds when N is a local monodromy logarithm for an algebraic family of complex algebraic varieties, and W is the (locally constant) weight filtration of the mixed Hodge structure of the homology or cohomology V of a fiber. We will discuss the geometric construction of M in this latter case, which we henceforth call the *geometric case*, in § 5. It is important to have a filtration on V that induces the weight filtration of each $\mathrm{Gr}_k N$ on $\mathrm{Gr}_k^W V$; because of the possibility of the non-strictness of N (see (2.11), (2.12)), it is not always the case that $L(N)$ induces $L(\mathrm{Gr}_k N)$.

(2.7) *Remark.* Suppose that M exists. Then:

i) $M[n]$ is the weight filtration of N relative to $W[n]$.

ii) For any $k>l$, $M(W_k/W_l)$ is the weight filtration of the induced endomorphism of W_k/W_l relative to the induced filtration.

It follows that whenever W is finite, then M exists if and only if it can be built up successively to higher and higher W_k. Let ${}_{(k)}M$ denote the weight filtration of $N|_{W_k}$ relative to W. In seeking the existence of ${}_{(k)}M$, given that ${}_{(k-1)}M$ exists, *we may*, in view of (2.7), *assume that* $k=0$ *and* $V=W_0$. We then put

$$M'={}_{(-1)}M.$$

Proposition (2.6) is proved by establishing the following recursive formulas for $M={}_{(0)}M$ (cf. (2.1)).

(2.8) **Lemma.** *Suppose that* $W_0=V$, $N^{q+1}=0$. *Then*

i) $i>q$, *then* $M_{-i}=M'_{-i}$,

ii) *if* $i\geqq 0$, $M_i=\{v\in V\colon N^{i+1}v\in M_{-i-2}\}$,

iii) *if* $i>0$, $M_{-i}=N^i M_i+M'_{-i}$.

We can see that $\{M_i\}$, as defined in (2.8), is indeed a filtration of V, and moreover $NM_i\subseteq M_{i-2}$, without any further hypothesis on N, by the following observations:

(2.9) a) If $i\geqq 1$, $N^i M_i\subset M_{-i}$. Therefore $N^{i-1}(NM_i)\subset M_{-i}$, so $NM_i\subset M_{i-2}$ by (2.8, ii).

b) If $i\geqq 1$, then

$$\begin{aligned} NM_{-i}&=N(N^i M_i+M'_{-i})\\ &=N^{i+1}M_i+NM'_{-i}\subset M_{-i-2}\end{aligned}$$

by (2.8, ii) and the properties of M'.

c) $NM_0\subset M_{-2}$ by (2.8, ii) directly.

d) By (b), (2.8, ii), we get for $i\geqq -1$ that

$$M_{-i-4}\subset M_{-i-3}\Rightarrow NM_{-i-2}\subset M_{-i-3}\Rightarrow M_i\subset M_{i+1}.$$

e) By (a), (2.8, iii), and the properties of M', we get for $i\geqq 1$ that

$$M_{i-1}\subset M_i\Rightarrow NM_{i+1}\subset M_i\Rightarrow M_{-i-1}\subset M_{-i}.$$

(2.10) **Corollary.** *The weight filtration of N relative to W exists if and only if the M_i's given in* (2.8) *satisfy:*

a) $M_i W_{-1} = M'_i$

b) $M_i(V/W_{-1}) = {}^0M_i$.

The existence of M places restrictions on what N might be. We are eventually going to give necessary and sufficient conditions for getting M from M'. Before doing so, we would like to present some examples to give the reader some feeling for the nature of the problem.

(2.11) **Proposition.** *Suppose that for every $l>0$, N^l is strictly compatible with W. Then the relative weight filtration exists, and is given by*

$$M = L(N) * W.$$

Conversely, if the above formula for M holds, then all powers of N are strict.

Proof. We claim that if the strictness hypotheses on N hold, then $L(N)(\mathrm{Gr}_k^W V) = L(\mathrm{Gr}_k N)$, from which the desired formula follows by convolution with W. To verify the former, it suffices to see that there is an N-invariant splitting of the filtration W. By induction we may assume that $V = W_0$ and $W_{-2} = 0$. To get a section to the projection π: $V \to \mathrm{Gr}_0^W V$ it is enough to show that whenever $u \in \mathrm{Gr}_0^W V$ and $(\mathrm{Gr}_0 N)^l u = 0$, there exists $v \in V$ with $\pi(v) = u$ and $N^l v = 0$ (for it is a question of lifting the cyclic factors of an N-invariant decomposition of V/W_{-1}). This follows immediately from the strictness assumptions.

Conversely, suppose that $M = L(N) * W$. It is enough to show that if $v \in W_k$ and $N^{l+1} v \in W_{k-1}$ for some $l \geqq 0$, then in fact $v \in W_{k-1} + \ker N^{l+1}$. We may as well put $k=0$, in view of (2.7, i). Then

$$\pi(v) \in \ker(\mathrm{Gr}_0 N)^{l+1} \subset L_l \mathrm{Gr}_0 N = M_l \mathrm{Gr}_0 N.$$

Thus, there exists $u \in M_l$ with

$$w = v - u \in W_{-1}$$

By hypothesis, $u \in L_l(N)$. From (2.3), it follows that we may write

$$u = u' + N u'',$$

with $u' \in \ker N^{l+1}$. We now have

$$v = w + u' + N u''$$
$$N^{l+1} v = N^{l+1} w + N^{l+2} u'',$$

from which we see that $N^{l+2} u'' \in W_{-1}$. Arguing by induction (since $N^j = 0$ is strict for sufficiently large j), we can write

$$N^{l+2} u'' = N^{l+2} w'$$

for some $w' \in W_{-1}$. This yields

$$N^{l+1} v = N^{l+1}(w + N w') \in N^{l+1} W_{-1},$$

or equivalently,

$$v \in W_{-1} + \ker N^{l+1},$$

as desired. □

The following example shows that one does not have this kind of strictness in the general geometric case:

(2.12) *Example.* (Deligne) Let C be a rational curve with one node x_0. Let Δ denote the unit disc in the complex line. Take a parametrization $\sigma\colon \Delta \to C$ of one branch of C through x_0 with $\sigma(0) = x_0$, and put

$$X = (C - \{x_1\} \times \Delta^*) - (\text{graph of } \sigma),$$

where Δ^* is the punctured disc, and $x_1 \in C - \sigma(\Delta)$. The projection $f\colon X \to \Delta^*$ displays X as a topological fiber bundle. For any $t \in \Delta^*$, we take $X_t = f^{-1}(t)$, and put $V = H_1(X_t, \mathbb{Q})$, W its weight filtration. Let

ε_{-2} = homology class of a small loop about $\sigma(t)$,

ε_0 = homology class of a loop in $C - \{x_1, \sigma(t)\}$ that generates $H_1(C, \mathbb{Z})$.

Then ε_{-2} generates W_{-2}, and ε_0 projects to a generator of $\operatorname{Gr}_0^W V$. The Picard-Lefschetz transformation τ is seen to be given by $\tau(\varepsilon_0) = \varepsilon_0 + \varepsilon_{-2}$, $\tau(\varepsilon_{-2}) = \varepsilon_{-2}$. Thus $N = \tau - 1$ satisfies

$$N\varepsilon_0 = \varepsilon_{-2}, \qquad N\varepsilon_{-2} = 0,$$

so is evidently not strict.

(2.13) *Remark.* Example (2.12) was constructed as the geometric realization of the motif

$$\begin{array}{ccccc} \mathbb{Z}(1) & \longrightarrow & H_{\mathbb{Z}} & \longrightarrow & \mathbb{Z} \\ \| & & \big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ \mathbb{Z}(1) & \longrightarrow & \mathbb{C} & \xrightarrow{\ \rho\ } & \mathbb{C}^* \end{array} \qquad \begin{array}{l} \iota(n) = q^n \\ \rho(w) = \exp(2\pi i w). \end{array}$$

We can modify (2.12) by taking C to be of arbitrary genus and still have non-strictness for N. However, these examples are illustrations of the following:

(2.14) **Proposition.** *If* $\operatorname{Gr}_k N = 0$ *for all* k, *then* M *exists if and only if* $NW_k \subseteq W_{k-2}$, *and then* $M = W$.

Proof. Since $L(\operatorname{Gr}_k N)$ is trivial for any k, it is clear that the trivial filtration T of V induces it, so $T * W = W$ induces ${}^{(k)}M$ for all k. The remaining condition defining the relative weight filtration is that $NW_k \subset W_{k-2}$, whence the assertion. □

(2.15) *Remark.* If $NW_k \not\subset W_{k-2}$, the filtration defined by (2.8) will, of course, be something other than W.

The next assertion explains why one cannot find counterexamples to the strictness of N in the local monodromy of families of smooth or complete curves:

(2.16) **Proposition.** *Suppose that W is of length two, i.e., for some k, $V=W_k$ and $W_{k-2}=0$. Then M exists if and only if N^l is strict for all positive integers l.*

Proof. Of course, one direction is contained in (2.11). Suppose, then, that the relative weight filtration M exists. We may, again, assume that $k=0$. Suppose that $N^l v\in W_{-1}$. Let $u=\pi(v)$. Then $N^l u=0$ in $\mathrm{Gr}_0^W V$, so

$$u\in {}^0M_{l-1}.$$

We can therefore find $v'\in M_{l-1}$ with $\pi(v')=u$. Then $\pi(v-v')=0$; that is, $(v-v')\in W_{-1}$. Applying N^1, we see that

$$N^l v'\in M_{-l-1}\cap W_{-1}=M'_{-l-1}=L_{-l}(N_{-1}).$$

By (2.1, iii), we can then write $N^l v'=N^l w$, with $w\in W_{-1}$. We have now

$$N^l v=N^l(v-v')+N^l w\in N^l W_{-1},$$

as desired. □

Applying the above to all length-two quotients W_k/W_{k-2} in the general case, we obtain:

(2.17) **Corollary.** *A necessary condition for M to exist is that for all integers $l>0$ and k,*

$$N^l W_k\cap W_{k-1}\subset N^l W_{k-1}+W_{k-2}.$$

The condition in (2.17) deserves a name:

(2.18) *Definition.* Let Φ: $(V,W)\to(V',W')$ be a morphism of filtered vector spaces. Then Φ is said to be *quasi-strict* if

$$\Phi(W_k)\cap W'_{k-1}\subset\Phi(W_{k-1})+W'_{k-2}.$$

The following example shows that the quasi-strictness of all powers of N in (2.17) is not sufficient to give the existence of M:

(2.19) *Example.* Let V be 3-dimensional, with basis $\{e_1, e_2, e_3\}$, $W_0=V$, $W_{-1}=W_{-2}=\mathrm{Span}\{e_2,e_3\}$, $W_{-3}=0$. Define N by $Ne_1=e_2$, $Ne_2=e_3$, $Ne_3=0$. It is clear that all powers of N are quasi-strict. However, we can see that (2.8) fails to define the relative weight filtration. Retaining our previous notation, we see that $M_{-2}=M'_{-2}$ by (2.8, i), so (2.8, ii) gives

$$M_0=\{v\in V\colon Nv\in M_{-2}\}=W_{-1};$$

whereas $M_0\,\mathrm{Gr}_0^W V$ ought to be equal to $L_0(\mathrm{Gr}_0 N)=\mathrm{Gr}_0^W V$.

We now address the general problem of determining whether ${}_{(k)}M$ exists, given that ${}_{(k-1)}M$ does. We assert:

(2.20) **Theorem.** *If the weight filtration* $_{(k-1)}M$ *of* $N|_{W_{k-1}}$ *relative to* W *exists, then* $_{(k)}M$ *exists if and only if for all integers* $l>0$

$$N^l W_k \cap W_{k-1} \subset N^l W_{k-1} + {}_{(k-1)}M_{k-l-1}.$$

The remainder of this chapter is devoted to the proof of Theorem (2.20). Of course, by our customary reduction, it suffices to consider the case $k=0$, where our condition reads

$$(2.20)_0^l \qquad N^l W_0 \cap W_{-1} \subset N^l W_{-1} + {}_{(-1)}M_{-l-1}.$$

Since by (2.8, iii), with a shift,

$$_{(-1)}M_{-l-1} = N^l_{(-1)}M_{l-1} + {}_{(-2)}M_{-l-1},$$

we can rewrite $(2.20)_0^l$ as

$$(2.21) \qquad N^l W_0 \cap W_{-1} \subset N^l W_{-1} + {}_{(-2)}M_{-l-1}.$$

This is a condition that visibly lies between strictness and quasi-strictness of all N^l, as we should expect.

By (2.10), the existence of M is equivalent to the following two sequences of assertions about the subspaces defined in (2.8):

$$(2.22) \qquad \begin{aligned} &a_i)\ M_i \cap W_{-1} = M'_i \\ &b_i)\ (M_i + W_{-1})/W_{-1} = {}^0M_i. \end{aligned}$$

The verification of these, and their relation to $(2.20)_0^l$ will go by induction, as in (2.9).

We begin with the easy steps:

(2.23) **Proposition.** *Without any hypothesis on* N,

$$(b_i) \Rightarrow (b_{-i}) \quad \text{when} \quad i>0.$$

Proof. Because

$$\begin{aligned} M_{-i} + W_{-1} &= (N^i M_i + M'_{-i}) + W_{-1} \\ &= N^i M_i + W_{-1}, \end{aligned}$$

we have

$$\begin{aligned} (M_{-1} + W_{-1})/W_{-1} &= (\mathrm{Gr}_0 N)^i\, {}^0M_i && \text{assuming } (b_i) \\ &= {}^0M_{-1} && \text{by (2.1, iii),} \end{aligned}$$

as desired. □

(2.24) **Proposition.** *Also with no additional hypothesis on* N,

$$(a_{-i-2}) \Rightarrow (a_i) \quad \text{when} \quad i \geqq 0.$$

Proof. We have

$$\begin{aligned} M_i \cap W_{-1} &= \{v \in V\colon N^{i+1} v \in M_{-i-2}\} \cap W_{-1} \\ &= \{v \in W_{-1}\colon N^{i+1} v \in M_{-i-2} \cap W_{-1}\} \\ &= \{v \in W_{-1}\colon N^{i+1} v \in M'_{-i-2}\} \quad \text{assuming } (a_{-i-2}) \\ &= \{v \in W_{-1}\colon N^{i+1} v \in N^{i+1} M'_i +_{(-2)} M_{-i-2}\} \end{aligned}$$

by (2.8, iii), with a shift.

We plainly see that $M_i \cap W_{-1} \supset M'_i$. On the other hand:

$$\begin{aligned} M'_i \cap W_1 &\subset \{v \in W_{-1}\colon N^{i+2} v \in N^{i+2} M'_i + N_{(-2)} M_{-i-2}\} \\ &\subset \{v \in W_{-1}\colon N^{i+2} v \in M'_{-i-4}\} \\ &= M'_i \quad \text{by (2.8, ii), with a shift.} \quad \square \end{aligned}$$

For the remaining steps, we will need to impose conditions on N.

(2.25) **Proposition.** *Suppose that*

$(2.25)^i_0$ $$N^i V \cap W_{-1} \subseteq N^i W_{-1} + M'_{-i}$$

(a condition slightly weaker than $(2.20)^i_0$). *Then*

$$(a_i) \Rightarrow (a_{-i})$$

Proof. Statement (a_{-i}) asserts that

$$(N^i M_i + M'_{-i}) \cap W_{-1} = M'_{-i}.$$

By (1.12, ii), we can write this as

$$(N^i M_i \cap W_{-1}) + M'_{-i} = M'_{-i},$$

which is equivalent to the statement

$$N^i M_i \cap W_{-1} \subset M'_{-i}.$$

We compute:

$$\begin{aligned} N^i M_i \cap W_{-1} &= N^i\{v \in V\colon N^{i+1} v \in M_{-i-2}\} \cap W_{-1} \\ &= \{u\colon Nu \in M_{-i-2}\} \cap N^i V \cap W_{-1} \\ &\subset N^{-1} M_{-i-2} \cap (N^i W_{-1} + M'_{-i}) \quad \text{by } (2.25)^i_0 \\ &= (N^{-1} M_{-i-2} \cap N^i W_{-1}) + M'_{-i} \quad \text{by (2.8, iii) and (1.12, ii)} \\ &= N^i\{w \in W_{-1}\colon N^{i+1} w \in M_{-i-2}\} + M'_{-i}. \\ &= N^i(\{v \in V\colon N^{i+1} v \in M_{-i-2}\} \cap W_{-1}) + M'_{-i} \\ &= N^i(M_i \cap W_{-1}) + M'_{-i} \quad \text{by (2.8, ii)} \\ &= N^i M'_i + M'_{-i} \quad \text{assuming } (a_i) \\ &= M'_{-i}, \end{aligned}$$

as desired. $\square$

Finally, the most important ingredient in the proof of (2.20) is the following:

(2.26) **Proposition.** *Let* $i \geqq 0$, *and assume that* (b_{-i-2}) *is true. Then* (b_i) *is true if and only if* $(2.20)_0^{i+1}$ *is satisfied.*

Proof. We can write (b_i) as

$$(N^{i+1})^{-1} M_{-i-2} + W_{-1} = (N^{i+1})^{-1}(M_{-i-2} + W_{-1}).$$

Since the left-hand side is obviously contained in the right-hand side, (b_i) requires

$$(N^{i+1})^{-1}(M_{-i-2} + W_{-1}) \subset (N^{i+1})^{-1} M_{-i-2} + W_{-1}, \tag{2.27}$$

or equivalently, as one readily checks,

$$N^{i+1} V \cap (M_{-i-2} + W_{-1}) \subset (M_{-i-2} + N^{i+1} W_{-1}) \cap N^{i+1} V.$$

We perform some manipulations:

$$\begin{aligned} N^{i+1} V \cap (M_{-i-2} + W_{-1}) &= N^{i+1} V \cap (N^{i+2} M_{i+2} + W_{-1}) \\ &= N^{i+2} M_{i+2} + N^{i+1} V \cap W_{-1}. \end{aligned}$$

From this, we see that (2.27) is equivalent to

$$N^{i+1} V \cap W_{-1} \subset M_{-i-2} + N^{i+1} W_{-1},$$

which is, by (1.12, i), the same as

$$\begin{aligned} N^{i+1} V \cap W_{-1} &\subset (M_{-i-2} + N^{i+1} W_{-1}) \cap W_{-1} \\ &= M_{-i-2} \cap W_{-1} + N^{i+1} W_{-1} \\ &= M'_{-i-2} + N^{i+1} W_{-1} \quad \text{assuming } (b_{-i-2}). \end{aligned}$$

This is precisely $(2.20)_0^{i+1}$, so we are done. □

Putting (2.23)–(2.26) together, we get (2.22) for all i if and only if $(2.20)_0^l$ is satisfied for all l, which gives Theorem (2.20). In the proof, we saw that it was only in the verification of $(2.22, b_i)$ for $i \geqq 0$ that the precise condition on N was needed.

We can view Theorem (2.20) as asserting that the existence of the relative weight filtration imposes severe restrictions on N. If N satisfies the conditions of (2.20), we will say that N is an *admissible* nilpotent endomorphism of V (relative to W). We stress the following:

(2.28) **Corollary.** *If* N *is a nilpotent logarithm of a local monodromy transformation of the cohomology of an algebraic variety, then* N *is admissible.*

§ 3. Variations of mixed Hodge structure

In this chapter, we discuss the definition of a variation of mixed Hodge structure. While certain conditions are evident, it is not fully understood at the present time what constitutes a "good" variation of mixed Hodge structure, as Deligne avers in [5, (1.8.15)].

First we recall the definition of a variation of Hodge structure:

(3.1) *Definition.* Let S be a complex manifold, $\mathbb{F}$ a sub-field of the real numbers. A *variation of Hodge structure of weight k over S, defined over* $\mathbb{F}$, is the collection of data $(\mathbb{V}_{\mathbb{F}}, \mathscr{F})$, where

a) $\mathbb{V}_{\mathbb{F}}$ is a locally constant sheaf (local system) of $\mathbb{F}$-vector spaces[4] on S,

b) $\mathscr{F} = \{\mathscr{F}^p\}$ is a decreasing filtration by holomorphic subbundles of the bundle (more properly, locally free sheaf) $\mathscr{V} = \mathscr{O}_S \otimes_{\mathbb{F}} \mathbb{V}_{\mathbb{F}}$.

c) At each $s \in S$, $\mathscr{F}$ induces the Hodge filtration F_s of a Hodge structure of weight k on the fiber V_s of $\mathscr{V}$:

i) whenever $p+q=k$

$$V_s = F_s^p \oplus \overline{F_s^{q+1}},$$

where the "bar" denotes complex conjugation,

ii) equivalently,

$$V_s = \bigoplus_{p+q=k} H_s^{p,q} \quad \text{where} \quad H_s^{p,q} = F_s^p \cap \overline{F_s^q}.$$

d) Under the flat differentiation ∇ in $\mathscr{V}$,

$$\nabla \mathscr{F}^p \subset \Omega_S^1 \otimes_{\mathscr{O}_S} \mathscr{F}^{p-1} \qquad \text{for all } p.$$

If $f\colon X \to S$ is a smooth, proper holomorphic mapping, and X is a Kähler manifold, then $R^k f_* \mathbb{Q}$ is the underlying local system of a variation of Hodge structure of weight k, defined over $\mathbb{Q}$, in which F_s is the usual Hodge filtration of the cohomology of the fiber: $H^k(X_s, \mathbb{C})$. Such examples provided the motivation for the Definition (3.1). One refers to these, and sometimes also variations of Hodge structure derived from them by standard functorial constructions, as *geometric* variations of Hodge structure.

A variation of Hodge structure defined over $\mathbb{F}$ gives, merely by extending the scalars in $\mathbb{V}_{\mathbb{F}}$, one defined over $\mathbb{E}$ for any $\mathbb{F} \subset \mathbb{E} \subset \mathbb{R}$. One has, also, an obvious notion of a morphism of variations of Hodge structure.

(3.2) *Definition.* A *polarization over* $\mathbb{F}$ of a variation of Hodge structure of weight k over $\mathbb{F}$, is a non-degenerate, flat bilinear pairing:

$$\beta\colon \mathbb{V}_{\mathbb{F}} \times \mathbb{V}_{\mathbb{F}} \to \mathbb{F},$$

such that β is $(-1)^k$-symmetric, and the Hermitian form on each fiber:

$$\beta_s(C_s v, \bar{w})$$

is positive-definite. Here, C_s denotes the Weil operator with respect to F_s, namely the direct sum of multiplications by i^{p-q} on $H_s^{p,q}$. A variation of Hodge structure is said to be *polarizable* (over $\mathbb{F}$) if it admits a polarization (over $\mathbb{F}$).

[4] If $\mathbb{F} = \mathbb{Q}$, it is customary to include a statement about the structure over $\mathbb{Z}$. Such considerations are extraneous for our purposes

By adjusting the cup-product on cohomology by use of the Kähler class and its (flat) primitive decomposition, one obtains a polarization over $\mathbb{R}$ for $R^k f_* \mathbb{R}$ in the geometric case. If X is a family of algebraic varieties, then the polarization is in fact defined over $\mathbb{Q}$.

(3.3) *Remark.* It is immediate that any subvariation of a polarizable variation of Hodge structure is polarizable. In particular, the kernel and image of a functorial morphism of cohomology in the geometric case are polarizable. It is easy to get confused over this point if one worries too much about the source of the polarization!

One can now formulate the following:

(3.4) *Definition.* A *variation of mixed Hodge structure* defined over $\mathbb{F}$ on the complex manifold S is the collection of data $\mathbb{V} = (\mathbb{V}_{\mathbb{F}}, W, \mathscr{F})$, where

a) $\mathbb{V}_{\mathbb{F}}$ is a local system of $\mathbb{F}$-vector spaces on S,

b) $W = \{\mathbb{W}_k\}$ is an increasing filtration of $\mathbb{V}_{\mathbb{F}}$ by local subsystems,

c) $\mathscr{F} = \{\mathscr{F}^p\}$ is a decreasing filtration by holomorphic subbundles of $\mathscr{V} = \mathscr{O}_S \otimes_{\mathbb{F}} \mathbb{V}_{\mathbb{F}}$,

d) $\nabla \mathscr{F}^p \subseteq \Omega^1_S \otimes \mathscr{F}^{p-1}$,

e) with $\mathscr{W}_k$ denoting $\mathscr{O}_S \otimes_{\mathbb{F}} \mathbb{W}_k$:

i) the data $(\mathrm{Gr}^W_k \mathbb{V}_{\mathbb{F}}, \mathscr{F}(\mathscr{W}_k/\mathscr{W}_{k-1}))$ is a variation of Hodge structure of weight k, defined over $\mathbb{F}$;

ii) equivalently, on the fiber, (V_s, W_s, F_s) is a mixed Hodge structure, defined over $\mathbb{F}$.

(3.5) *Definition.* A variation of mixed Hodge structure will be called *graded-polarizable* if the induced collection of variations of Hodge structure (3.4, e, i) are all polarizable.

(3.6) *Example.* If $f: X \to S$ is a proper flat morphism such that there exists a hyperresolution $X_\bullet$ of X ([4] § 6.2) such that all X_i are proper and smooth over S, then $R^i f_* \mathbb{Q}_X$ is the underlying local system of a natural, graded-polarizable variation of mixed Hodge structure [13, 18]. The same holds if f is not necessarily proper but its fibres can be compactified in a sufficiently equisingular way.

Next, we recall a little from the theory of degeneration of Hodge bundles.

Suppose that the complex manifold S is embedded in $\bar{S}$, via the mapping j, such that $\Sigma = \bar{S} - S$ is a divisor with normal crossings. Let $\mathbb{V}$ be any local system of complex vector spaces on S, and $\mathscr{V}$ the corresponding vector bundle. Then $\mathscr{V}$ admits a "canonical" prolongation $\tilde{\mathscr{V}}$ to $\bar{S}$, as defined in [2, p. 91]. It is easy enough to describe $\tilde{\mathscr{V}}$ as a subsheaf of $j_* \mathscr{V}$ when the local monodromy is unipotent. The local picture of $S \subset \bar{S}$ is $(\Delta^*)^r \times \Delta^{n-r} \subset \Delta^n$. We let $t_1, \ldots, t_r$ denote the variables on the punctured disc factors, and $N_1, \ldots, N_r$ the (commuting) nilpotent logarithms of the associated monodromy transformations of the fiber. For $z_1, \ldots, z_r$ in the upper half-plane, the universal covering mapping for $(\Delta^*)^r$ is given by

$$t_j = \exp(2\pi i z_j) \quad j = 1, \ldots, r.$$

Then, as v ranges over the multi-valued sections of $\mathbb{V}$, the formula

$$\tilde{v} = \exp\left(-\sum_{j=1}^{r} z_j N_j\right) v \tag{3.7}$$

determines sections of $\mathscr{V}$ over $(\Delta^*)^r \times \Delta^{n-r}$, and these are, by definition, the generators of $\tilde{\mathscr{V}}$ over Δ^n. By construction,

$$\nabla \tilde{\mathscr{V}} \subseteq \Omega^1_{\bar{S}}(\log \Sigma) \otimes \tilde{\mathscr{V}}. \tag{3.8}$$

Now, let $\mathbb{V}_{\mathbb{C}} = \mathbb{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ underlie a variation of Hodge structure of weight k over S. We construe the data of $\mathscr{F}$ as describing a holomorphic mapping

$$\Psi: S \to D(\mathscr{V}),$$

where D denotes the appropriate flag bundle of $\mathscr{V}$. Of course, (3.1, c, d) imposes some conditions on Ψ. One of the important results from [9] can be expressed as:

(3.9) **Nilpotent Orbit Theorem.** [9, (4.12)] *For polarizable variations of Hodge structure,*

i) the mapping Ψ extends to

$$\tilde{\Psi}: \bar{S} \to D(\tilde{\mathscr{V}}),$$

i.e., *the filtration $\mathscr{F}$ of $\mathscr{V}$ extends to a filtration $\tilde{\mathscr{F}}$ of $\tilde{\mathscr{V}}$;*

ii) in terms of local coordinates "at infinity", $(\Delta^*)^r \times \Delta^{n-r}$, and the trivialization of $\tilde{\mathscr{V}}$ via (3.7), the "constant" mapping $\tilde{\Psi}(0)$ defines a variation of Hodge structure (of the same weight) on some open set of the form

$$\prod_{j=1}^{r} \left|\log|t_j|\right| > C, \qquad |t_j| < 1 - \varepsilon.$$

(3.10) *Remark.* In the case of a geometric variation of Hodge structure, the regularity Theorem [7] already implies the existence of $\tilde{\Psi}(0)$ when S is a curve.

In the case of one variable, i.e., $S = \Delta^*$, one can say more about the filtration $\tilde{\Psi}(0)$. We regard the fiber V_0 of $\tilde{\mathscr{V}}$ at the origin as a vector space of weight k. The unipotent monodromy logarithm N_0 becomes an endomorphism of V_0, namely the residue of the connection at the origin, and we then also have its shifted weight filtration $M(N_0)$ of (2.4). From his SL_2-orbit theorem, Schmid deduced:

(3.11) **Theorem.** [9, (6.25)] *$(V_0, M(N_0), \tilde{\Psi}(0))$ is a mixed Hodge structure. A polarization of the variation of Hodge structure determines*[5] *a graded-polarization of this mixed Hodge structure.*

Some good information about the situation in the several-variable case is provided by:

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[5] By a well-known procedure: see [9, (6.4)] or [1, (1.14)]

(3.12) **Theorem.** [1, § 3] *For a polarizable variation of Hodge structure on* $(\Delta^*)^r$ *with unipotent monodromy, let, for any non-empty subset* $J \subset \{1, \ldots, r\}$,

$$\tau_J = \{N = \sum_{j \in J} \lambda_j N_j \colon \lambda_j > 0\}.$$

Then:

i) for any fixed J, $L(N)$ *(or* $M(N)$*) is the same filtration for all* $N \in \tau_J$,

ii) if $J = \{1, \ldots, r\}$, *then for* $N \in \tau_J$, $(V_0, M(N), \tilde{\Psi}(0))$ *is a mixed Hodge structure,*

iii) if $N \in \tau_J$ *and* $N' \in \tau_{J'}$, *then if* $N'' \in \tau_{J \cup J'}$, $L(N'')$ *is the weight filtration of* N *relative to* $L(N')$*; thus,*

$$M(N; M(N')) = M(N'').$$ [6]

In [5, (1.8.15)], Deligne poses the problem of defining a notion of a good variation of mixed Hodge structure, even in one variable, so that (3.9), (3.11) and (3.12) will have suitable generalizations. We will look briefly at the case $S = \Delta^*$.

For a variation of mixed Hodge structure over Δ^*, we have a morphism

$$\Psi \colon \Delta^* \to D(\mathscr{V}),$$

inducing for each k

$$\Psi_k \colon \Delta^* \to D(\mathscr{W}_k),$$
$$\mathrm{Gr}_k \Psi \colon \Delta^* \to D(\mathrm{Gr}_k^{\mathscr{W}} \mathscr{V}),$$

where D again stands for a suitable flag bundle. The image of Ψ necessarily lies in a certain subset of $D(\mathscr{V})$, namely the set of flags which induce flags of the right type on the $\mathrm{Gr}_k^{\mathscr{W}} \mathscr{V}$'s, which we call $D_G(\mathscr{V})$. We again have the prolongation $\tilde{\mathscr{V}}$, but now it comes equipped with the filtration $\{\tilde{\mathscr{W}}_k\}$ by sub-bundles. The fiber V_0 at the origin is naturally filtered by W. One wants the following to hold:

(3.13) *Properties.* i) The unipotent local monodromy logarithm N_0 is admissible (i.e., the weight filtration M of N_0 relative to W exists),

ii) Ψ extends to give

$$\tilde{\Psi} \colon \Delta \to D(\tilde{\mathscr{V}}),$$

such that $\tilde{\Psi}(0) \in D_G(\tilde{\mathscr{V}})$.

iii) For each k, $(W_k, M, \tilde{\Psi}_k(0))$ is a mixed Hodge structure, and N_0 gives a morphism of type $(-1, -1)$.

We will see in § 4 that the Properties (3.13) are sufficient to produce a mixed Hodge theory with degenerating coefficients, generalizing [11]. In the geometric case of (3.6), that they are satisfied will follow from the construction in § 5.

(3.14) *Remark.* i) From (3.13, ii), it follows that $\tilde{\Psi}$ induces

$$\tilde{\Psi}_k \colon \Delta \to D(\tilde{\mathscr{W}}_k),$$
$$\mathrm{Gr}_k \tilde{\Psi} \colon \Delta \to D(\widetilde{\mathrm{Gr}_k^{\mathscr{W}} \mathscr{V}}).$$

[6] This assertion is slightly misstated in [1]

ii) It will be shown in the appendix that (3.13, iii) is a consequence of (3.13, i) and (3.13, ii), given (3.11). This is not at all trivial: one must be careful when working with three filtrations. Although

$$\begin{aligned}\mathrm{Gr}_k^W \mathrm{Gr}_l^W V &\simeq (W_k \cap M_l)/[(W_k \cap M_{l-1}) + (W_{k-1} \cap M_l)]\\ &\simeq [(W_k \cap M_l) + (W_{k-1} + M_{l-1})]/(W_{k-1} + M_{l-1})\end{aligned}$$

is symmetric in W and M, the filtration induced by F may depend on the order the quotient is formed. For instance, in the order given above $F^p \mathrm{Gr}_k^W \mathrm{Gr}_l^W V$ is the image of

$$(F^p \cap M_l + M_{l-1}) \cap (W_k \cap M_l + M_{l-1}).$$

iii) Note that (3.13, iii) contains the assertion that N_0 is strictly compatible with M, in the sense that $N_0 V \cap M_{k-2} = N_0 M_k$. This is not, however, implied by the admissibility of N_0. (It is always true in the pure case.) There are even counterexamples of type (2.14).

The following two examples show that the conditions in (3.13) are, in a certain sense, independent.

(3.15) *Example.* Let V be 4-dimensional, with basis $\{e_0, e_1, e_2, e_3\}$, $W_1 = V$, $W_0 = W_{-1} = \mathrm{Span}\{e_2, e_3\}$, $W_{-2} = 0$; and define N by $Ne_0 = 0$, $Ne_1 = e_2$, $Ne_2 = e_3$, $Ne_3 = 0$. We have merely added a trivial one-dimensional summand to Example (2.19) and shifted W, so the weight filtration of N relative to W does not exist. With this defining the underlying local system, it is possible to define a graded-polarizable variation of mixed Hodge structure on Δ^*, with Hodge numbers $h^{1,0} = h^{0,-1} = 1$, such that the limit filtration exists, and behaves well under passage to Gr^W. In fact, we can take a "nilpotent orbit" as the variation:

$$\begin{aligned}\mathscr{F}^2 &= 0,\\ \mathscr{F}^1 &= \mathrm{Span}\{\lambda e_0 + \tilde{e}_1 + \mu \tilde{e}_2\}, \qquad \lambda, \mu \notin \mathbb{R}\\ \mathscr{F}^0 &= \mathrm{Span}\{e_0, \tilde{e}_1, \tilde{e}_2 + \mu e_3\},\\ \mathscr{F}^{-1} &= \mathscr{V}.\end{aligned}$$

(One readily verifies that the conditions of (3.4) and (3.5) are satisfied.)

(3.16) *Example.* Let V be as in (2.12), and take

$$F_t^0 = \mathrm{Span}\{\tilde{\varepsilon}_0 + f(t)\, \varepsilon_{-2}\}.$$

This defines a graded-polarizable variation of mixed Hodge structure for any analytic function f on Δ^*. We distinguish three cases:

i) f extends analytically across the origin. Then (V_0, W, F_0) is a mixed Hodge structure (recall that $M = W$ here).

ii) f has a pole at the origin. Then F_0 behaves poorly with respect to Gr^W; (V_0, W, F_0) fails to be a mixed Hodge structure.

iii) f has an essential singularity at the origin. Then the limit filtration F_0 does not even exist.

We feel that the assumption that a variation of mixed Hodge structure be graded-polarizable is too weak. In fact, those of geometric origin ((3.6), or variations on (3.11)) have the property that the polarizations of the $\mathrm{Gr}_k^W \mathbb{V}$'s are of a common source, i.e., are related. An axiomatic understanding of this point may be the missing ingredient in the theory of variations of mixed Hodge structure.

The following embarrassingly simple observation contains a touch of the above complaint, so we mention it now:

(3.17) **Proposition.** *Let V be a complex vector space with real structure with an increasing filtration W defined over $\mathbb{R}$ and a decreasing filtration F. If F induces Hodge structures of the same weight on each $\mathrm{Gr}_k^W V$, then (V, F) is itself a Hodge structure.*

The proof of (3.17) is left to the reader. Although one could regard it as a clever trick that might be useful in constructing Hodge structures, it seems to us that it should be possible, even desirable, to avoid using (3.17). (We add for comparison the well-known fact that a filtration of a polarizable Hodge structure necessarily splits.)

We conclude this chapter with some remarks on the mixed Hodge theoretic variants of (3.12). Given a graded-polarizable variation of mixed Hodge structure on $(\Delta^*)^r$, we again have monodromy logarithms $N_1, \ldots, N_r$. We define τ_J as before. Suppose that for each $N \in \tau_J$, $M(N; W)$ exists. Since $M(N; W)$ induces $M(\mathrm{Gr}_k^W N)$, we obtain, from (3.12, i) and the uniqueness of M, that $M(N; W)$ is constant on τ_J.

One can raise the obvious next question:

(3.18) *Is it true that if $N \in \tau_J$ and $N' \in \tau_{J'}$, that for $N'' \in \tau_{J \cup J'}$*

$$M(N; M(N'; W)) = M(N''; W)?$$

This is asserting that for all i and k

$$\mathrm{Gr}_{k+i}^{M''} \mathrm{Gr}_k^{M'} V \xrightarrow{N^i} \mathrm{Gr}_{k-i}^{M''} \mathrm{Gr}_k^{M'} V$$

is an isomorphism, where we are writing $M' = M(N'; W)$, etc. By (3.12, iii), we have that

$$\mathrm{Gr}_{k+i}^{W''} \mathrm{Gr}_k^{M'} \mathrm{Gr}_l^{W} \xrightarrow{N^i} \mathrm{Gr}_{k-i}^{M''} \mathrm{Gr}_k^{M'} \mathrm{Gr}_l^{W} V$$

is an isomorphism; moreover, it would suffice to know that

$$\mathrm{Gr}_l^{W} \mathrm{Gr}_{k+i}^{M''} \mathrm{Gr}_k^{M'} V \xrightarrow{N^i} \mathrm{Gr}_l^{W} \mathrm{Gr}_{k-i}^{M''} \mathrm{Gr}_k^{M'} V$$

is an isomorphism, but we face a problem similar to the one in (3.14, ii).

§4. Mixed Hodge theory with degenerating coefficients

Let $\bar{S}$ be a compact Riemann surface, $\Sigma \subset \bar{S}$ a finite set of points, $S = \bar{S} - \Sigma$, and $j: S \to \bar{S}$ the inclusion mapping. Let $(\mathbb{V}_{\mathbb{F}}, W, \mathscr{F})$ be a variation of mixed Hodge structure on S, defined over $\mathbb{F} \subset \mathbb{R}$, and put $\mathbb{V}_{\mathbb{C}} = \mathbb{V}_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{C}$. This chapter will be devoted to the proof of the following theorem, which generalizes the results in [11]:

(4.1) **Theorem.** *Suppose that* $(\mathbb{V}_{\mathbb{F}}, W, \mathscr{F})$ *is graded-polarizable and satisfies the Properties* (3.13). *Then the spaces* $H^i(S, \mathbb{V})$ *and* $H^i(\bar{S}, R^k j_* \mathbb{V})$, *for all* i, $k \geqq 0$, *carry natural mixed Hodge structures defined over* $\mathbb{F}$, *such that the Leray spectral sequence for* j *becomes a spectral sequence of mixed Hodge structures. Moreover, the mixed Hodge structure is functorial in both* $\mathbb{V}$ *and* $\bar{S}$.

We will have to study several extensions to $\bar{S}$ of the connection

$$\nabla: \mathscr{A} \to \Omega_S^1 \otimes_{\mathcal{O}_S} \mathscr{A}, \tag{4.2}$$

where $\mathscr{A} = \mathcal{O}_S \otimes_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}$, for subquotients $\mathbb{A}$ of $\mathbb{V}$. We recall that the canonical extension $\tilde{\mathscr{A}}$ of $\mathscr{A}$ is characterized by the properties that (4.2) extends to

$$\nabla: \tilde{\mathscr{A}} \to \Omega_{\bar{S}}^1(\log \Sigma) \otimes \tilde{\mathscr{A}}, \tag{4.3}$$

and for each $\sigma \in \Sigma$, the eigenvalues of the residue $N(\sigma)$, of (4.3) on the fiber $A(\sigma)$ of $\tilde{\mathscr{A}}$ at σ, lie in the interval $[0, 1)$. The extensions of (4.2) that we will consider are complexes of the form

$$\tilde{\mathscr{A}} \xrightarrow{\nabla} \mathscr{B}, \tag{4.4}$$

where $\nabla \tilde{\mathscr{A}} \subset \mathscr{B} \subset \Omega_{\bar{S}}^1(\log \Sigma) \otimes \tilde{\mathscr{A}}$. Such an extension is characterized by ($\mathbb{A}$ and) the image B of $\mathscr{B}$ in

$$A = \bigoplus_{\sigma \in \Sigma} A(\sigma)$$

via the residue mappings; we have

$$B = \bigoplus_{\sigma \in \Sigma} B(\sigma),$$

with $N(\sigma) A(\sigma) \subset B(\sigma) \subset A(\sigma)$. *We let* $\{\mathbb{A}, B\}$ *denote the complex* (4.4).

(4.5) *Remark.* If one puts $B = NA$, where $N = \oplus N(\sigma)$, then $\{\mathbb{A}, B\}$ is a resolution of $j_* \mathbb{A}_{\mathbb{C}}$. At the other extreme, if $B = A$, then the complex $\{\mathbb{A}, B\}$ represents the object $Rj_* \mathbb{A}_{\mathbb{C}}$ in the derived category $D^+(\bar{S}, \mathbb{C})$.

It is clear that $B \subset A$ is determined by its image $\bar{B}$ in

$$H^0(\bar{S}, R^1 j_* \mathbb{A}_{\mathbb{C}}) \simeq A/NA.$$

We can then see how to make a parallel construction to that in (4.5) for $\mathbb{A}_{\mathbb{F}}$ and $D^+(\bar{S}, \mathbb{F})$. Let $C^\cdot(\mathbb{A}_{\mathbb{F}})$ be the canonical resolution of $\mathbb{A}_{\mathbb{F}}$ (by discontinuous sections). We claim that the complex

$$0 \to j_* C^0(\mathbb{A}_{\mathbb{F}}) \to \ker\{j_* C^1(\mathbb{A}_{\mathbb{F}}) \to j_* C^2(\mathbb{A}_{\mathbb{F}})\} \to 0 \tag{4.6}$$

represents $Rj_* \mathbb{A}_{\mathbb{F}}$. Indeed, on S the sequence

$$0 \to \mathbb{A}_{\mathbb{F}} \to C^\cdot(\mathbb{A}_{\mathbb{F}})$$

is exact, and $C^{\cdot}(\mathbb{A}_{\mathbb{F}})$ is a complex of flasque sheaves. Hence, $R^k j_* C^i(\mathbb{A}_{\mathbb{F}})=0$ for $k>0$. Thus

$$\mathscr{H}^k(j_*(C^{\cdot}(\mathbb{A}_{\mathbb{F}}))\simeq R^k j_* \mathbb{A}_{\mathbb{F}}=0$$

for $k>1$, so we may cut off the complex $j_* C^{\cdot}(\mathbb{A}_{\mathbb{F}})$ to obtain (4.6). This shows that a complex $\{\mathbb{A}, B\}$ should be said to be *defined over* $\mathbb{F}$ if the image $\bar{B}$ of B in $R^1 j_* \mathbb{A}_{\mathbb{C}}$ is defined over $\mathbb{F}$.

The quotient of two complexes of type (4.4) need not be of the same type. However, we have the following:

(4.7) **Proposition.** *Let $\mathbb{A}' \subset \mathbb{A}$ be local systems on S, and suppose that*

$$A' \supset B' \supset NA' \subset NA \subset B \subset A,$$

so that $\{\mathbb{A}', B'\} \subset \{\mathbb{A}, B\}$. Then there is an exact sequence of complexes

$$0 \to K^{\cdot} \to \{\mathbb{A}, B\}/\{\mathbb{A}', B'\} \to \{\mathbb{A}/\mathbb{A}', (B+A')/A'\} \to 0,$$

where $K^{\cdot}=(B \cap A')/B'[-1]$. Moreover, if $(NA \cap A') \subset B'$ and $B \subset (NA+A')$, the sequence has a canonical splitting, which is defined over $\mathbb{F}$ if $\{\mathbb{A}, B\}$ and $\{\mathbb{A}', B'\}$ are.

Proof. First note that $\tilde{\mathscr{A}}/\tilde{\mathscr{A}}'$ is the canonical extension of $\mathscr{A}/\mathscr{A}'$. Moreover, K^1 is naturally identified with the torsion subsheaf of $\mathscr{B}/\mathscr{B}'$, for the latter is the kernel of the natural mapping

$$\mathscr{B}/\mathscr{B}' \to \Omega^1_{\bar{S}}(\log \Sigma) \otimes \tilde{\mathscr{A}}/\tilde{\mathscr{A}}'$$

(whose image contains $\Omega^1_{\bar{S}} \otimes \tilde{\mathscr{A}}/\tilde{\mathscr{A}}$, so is locally free of the same rank as $\mathscr{B}/\mathscr{B}'$). This image is then characterized by its residue in A/A', namely

$$B/B' \to (B+A')/A'.$$

The condition $NA \cap A' \subset B'$ means that $\nabla(\tilde{\mathscr{A}}/\tilde{\mathscr{A}}') \cap K^1=\{0\}$, and $B \subset NA+A'$ says that $\nabla(\tilde{\mathscr{A}}/\tilde{\mathscr{A}}')+K^1=\mathscr{B}/\mathscr{B}'$. We see that under both assumptions, $\nabla(\tilde{\mathscr{A}}/\tilde{\mathscr{A}}')$ is a complementary submodule to K^1 in $\mathscr{B}/\mathscr{B}'$, which gives the desired splitting of $\{\mathbb{A}, B\}/\{\mathbb{A}', B'\}$. The proof of the rationality of the splitting over $\mathbb{F}$ uses (4.6) and is left to the reader. □

To prove (4.1), we define filtrations $\mathfrak{M}$ and F on the complex $\{\mathbb{V}, V\}$, with $\mathfrak{M}$ defined over $\mathbb{F}$, such that we obtain a cohomological mixed Hodge complex on $\bar{S}$. In analogy with [11, §13], we put

(4.8)
$$\begin{aligned} Z_k(\sigma) &= N(\sigma)W_k(\sigma)+M_{k-1}(\sigma)W_{k-1}(\sigma) \\ &= N(\sigma)W_k(\sigma)+M_{k-1}(\sigma)W_k(\sigma) \qquad \text{(cf. (2.21))}, \\ Z_k &= \bigoplus_{\sigma\in\Sigma} Z_k(\sigma). \end{aligned}$$

Here, $W_k(\sigma)$ is the fiber of the canonical extension $\tilde{\mathscr{W}}_k$ of $\mathscr{W}_k$, and $M(\sigma)$ is the weight filtration of $N(\sigma)_0$ relative to $W(\sigma)$. Recall that $N(\sigma)_0$ is the nilpotent part of $N(\sigma)$ in its Jordan decomposition. We put

(4.9)
$$\mathfrak{M}_k\{\mathbb{V}, V\}=\{\mathbb{W}_k, Z_k\},$$

and let $F^p\{\mathbb{V}, V\}$ be the complex

$$0 \to \tilde{\mathscr{F}}^p \xrightarrow{V} \Omega^1_{\bar{S}}(\log \Sigma) \otimes \tilde{\mathscr{F}}^{p-1} \to 0, \tag{4.10}$$

where $\tilde{\mathscr{F}}^p = \tilde{\mathscr{V}} \cap j_* \mathscr{F}^p$ is locally free on $\bar{S}$ by (3.13, ii). Moreover, $\tilde{\mathscr{F}}$ is assumed to be inducing on each $V(\sigma)$ a filtration $\tilde{F}(\sigma)$ of the right type with respect to the filtration determined by the $\tilde{\mathscr{W}}_k$'s. Because $N(\sigma)$, and also the splitting of $N(\sigma)$ into nilpotent and semi-simple summands, is defined over $\mathbb{F}$, the filtration $\mathfrak{M}$ is also defined over $\mathbb{F}$.

We compute $\mathrm{Gr}_k^{\mathfrak{M}} = \mathfrak{M}_k/\mathfrak{M}_{k-1}$. First, we observe that $Z_k \subset N W_k + W_{k-1}$, and also $N W_k \cap W_{k-1} \subset Z_{k-1}$ by Theorem (2.20). Hence we obtain by Proposition (4.7) a splitting:

$$\mathrm{Gr}_k^{\mathfrak{M}} \simeq \{\mathbb{W}_k/\mathbb{W}_{k-1}, (Z_k + W_{k-1})/W_{k-1})\} \oplus K_k^{\bullet}. \tag{4.11}$$

Since

$$\begin{aligned}(Z_k + W_{k-1})/W_{k-1} &= (N W_k + W_{k-1})/W_{k-1}\\ &= \mathrm{Gr}_k N(\mathrm{Gr}_k^W V),\end{aligned}$$

the first summand in (4.11) is just the complex resolving $j_*(\mathbb{W}_k/\mathbb{W}_{k-1})$ that was studied in [11, §7]. The filtration on it, induced by F, makes it into a cohomological Hodge complex of weight k.

It remains to study the torsion summand $K_k^{\bullet}$. At this point, we use Hypothesis (3.13, iii): for every $\sigma \in \Sigma$, the filtrations $M(\sigma)$ and $\tilde{F}(\sigma)$ define a mixed Hodge structure on $V(\sigma)$, filtered by $W(\sigma)$. (We may ignore the non-unipotent summand, as it plays a trivial role here.) We have:

$$\begin{aligned}\mathbf{H}^1(\bar{S}, K_k^{\bullet}) &= H^0(\bar{S}, K_k^1) = (Z_k \cap W_{k-1})/Z_{k-1}\\ &= (N W_k + (M_{k-1} \cap W_{k-1})) \cap W_{k-1}/(N W_{k-1} + (M_{k-2} \cap W_{k-1}))\\ &= (N W_{k-1} + (M_{k-1} \cap W_{k-1}))/(N W_{k-1} + (M_{k-2} \cap W_{k-1})) \quad \text{by (2.20)};\end{aligned}$$

that is,

$$(Z_k \cap W_{k-1})/Z_{k-1} \simeq \mathrm{Gr}_{k-1}^M (W_{k-1}/N W_{k-1}). \tag{4.12}$$

(4.13) **Lemma.** *F induces on $\mathrm{Gr}_{k-1}^M(W_{k-1}/N W_{k-1})$ a Hodge structure of weight $k-1$.*

Proof. $N_0: W_{k-1} \to W_{k-1}$ is a morphism of mixed Hodge structures of type $(-1, -1)$, as it maps M_i to M_{i-2} and $\tilde{F}^p$ to $\tilde{F}^{p-1}$. Hence, M and $\tilde{F}$ induce on $W_{k-1}/N W_{k-1}$ a mixed Hodge structure. □

One can easily check that the isomorphism (4.12) is strictly compatible with $\tilde{F}$. Taking into account the shift in the Hodge filtration under the residue mapping, we see from (4.13) that as a Hodge structure,

$$\mathbf{H}^1(\bar{S}. K_k^{\bullet}) \simeq \mathrm{Gr}_{k-1}^M (W_{k-1}/N W_{k-1})(-1),$$

where (-1) denotes the tensor product with the Tate Hodge structure $\mathbb{Q}(-1)$, which is of pure type $(1, 1)$. Of course, $\mathbf{H}^i(\bar{S}, K_k^{\bullet}) = 0$ for $i \neq 1$. Summarizing, we obtain that for all $i \geqq 0$,

$$\mathbf{H}^i(\bar{S}, \mathrm{Gr}_k^{\mathfrak{M}}) \simeq H^i(\bar{S}, \mathbb{W}_k/\mathbb{W}_{k-1}) \oplus \mathbf{H}^i(\bar{S}, K_k^\bullet)$$

carries a Hodge structure of weight $i+k$, i.e., $\mathfrak{M}$ and F give $\{\mathbb{V}, V\}$ the structure of a cohomological mixed Hodge complex.

The complex $\{\mathbb{V}, NV\}$ is a subcomplex of $\{\mathbb{V}, V\}$, and is quasi-isomorphic to $j_* \mathbb{V}_{\mathbb{C}}$. On it, we have the induced filtrations $\mathfrak{M}$ and F. One computes $\mathrm{Gr}_k^{\mathfrak{M}}\{\mathbb{V}, NV\}$ in a way similar to the above. Again, we have a splitting

$$(4.14) \quad \mathrm{Gr}_k^{\mathfrak{M}}\{\mathbb{V}, NV\} \simeq \{\mathbb{W}_k/\mathbb{W}_{k-1}, ((Z_k \cap NV) + W_{k-1})/W_{k-1}\} \oplus S_k^1[-1]$$

where

$$S_k^1 \simeq (Z_k \cap NV \cap W_{k-1})/(Z_{k-1} \cap NV).$$

Since

$$\begin{aligned} Z_k \cap NV &= (NW_k + (M_{k-1} \cap W_{k-1})) \cap NV \\ &= NW_k + (M_{k-1} \cap W_{k-1} \cap NV), \end{aligned}$$

we can rewrite (4.14) as

$$\begin{aligned} (4.15) \quad & \mathrm{Gr}_k^{\mathfrak{M}}\{\mathbb{V}, NV\} \\ & \simeq \{\mathbb{W}_k/\mathbb{W}_{k-1}, \mathrm{Gr}_k N(\mathrm{Gr}_k^W V)\} \oplus \mathrm{Gr}_{k-1}^M((NV \cap W_{k-1})/NW_{k-1})(-1)[-1]. \end{aligned}$$

We see that $\mathrm{Gr}_k^{\mathfrak{M}}\{\mathbb{V}, NV\}$ is a cohomological Hodge subcomplex of $\mathrm{Gr}_k^{\mathfrak{M}}\{\mathbb{V}, V\}$. Thus, $\mathfrak{M}$ and F also give $\{\mathbb{V}, NV\}$ the structure of a cohomological mixed Hodge complex.

The long exact sequence of hypercohomology of the exact sequence

$$0 \to \{\mathbb{V}, NV\} \to \{\mathbb{V}, V\} \to Q^\bullet \to 0,$$

where $Q^\bullet$ is the quotient (concentrated on Σ), is identified with the Leray spectral sequence for the mapping j and the sheaf $\mathbb{V}$, which thereby becomes a spectral sequence of mixed Hodge structures.

For the functoriality of our construction with respect to $\bar{S}$, one argues as in [11, (8.2)]. Proving the functoriality with respect to $\mathbb{V}$ comes down to showing the following:

(4.16) **Proposition.** *Let N be a nilpotent endomorphism of the filtered vector space (V, W), and $\hat{N}$ be one of $(\hat{V}, \hat{W})$. Assume that the relative weight filtrations $M = M(N; W)$ and $\hat{M} = M(\hat{N}; \hat{W})$ both exist. Suppose that $\Phi: (V, W) \to (\hat{V}, \hat{W})$ is a morphism of filtered vector spaces such that $\Phi N = \hat{N} \Phi$. Then*

$$\Phi(M_k) \subset \hat{M}_k.$$

Proof. One can argue recursively by a straight-forward double induction, using (2.8). Details are omitted. □

(4.17) **Corollary.** *With Z_k and $\hat{Z}_k$ defined as in* (4.8), $\Phi(Z_k) \subset \hat{Z}_k$.

It is easy to see that this completes the proof of Theorem (4.1).

(4.18) *Remark.* i) For any $k>l$, $\mathbb{W}_k/\mathbb{W}_l$ underlies a graded-polarizable variation of mixed Hodge structure in its own right, and these subquotients are covered by the functoriality assertion. One might ask whether the filtration $\mathfrak{M}$ defined for $\mathbb{V}$ (4.9) induces the one correspondingly defined for $\mathbb{W}_k/\mathbb{W}_l$, e.g., the filtration of [11, §13] for $\mathbb{W}_k/\mathbb{W}_{k-1}$. This doesn't seem to be the case; it involves verifying relations such as

$$(N W_k+(M_{k-1}\cap W_{k-1}))\cap W_q=N W_q+(M_{k-1}\cap W_q)$$

if $q<k$, and this is clear only for $q=k-1$. (The inclusion $\supset$, which is obvious, is all one needs in this instance for functoriality.) Note also the corresponding problem with (4.15).

ii) We can see that (4.11) gives

$$\mathrm{Gr}_k^{\mathfrak{M}}\{\mathbb{V}, V\}\simeq\bigoplus_l \mathrm{Gr}_k^{\mathfrak{M}}\,\mathrm{Gr}_l^W\{\mathbb{V}, V\}$$

as a filtered complex. This is because $K_k^{\bullet}$, being supported on Σ, is comprised of a polarizable Hodge structure at each point σ; W induces a filtration by Hodge substructures, hence is automatically split. The above decomposition of $\mathrm{Gr}_k^{\mathfrak{M}}$ seems to be a typical phenomenon in the theory (compare (5.24)).

iii) It is not hard to see that Theorem (4.1) generalizes to the case where $\bar{S}$ is a compact Kähler manifold of any dimension and Σ is a smooth hypersurface, if we state (3.13, ii, iii) accordingly with parameters along Σ. In particular, it holds when $\Sigma=\emptyset$, in which case (4.9) reduces to

$$\mathfrak{M}_k=\Omega_{\bar{S}}^{\bullet}(\mathbb{W}_k).$$

Let $s\in S$. From (4.9) and (4.10), we see immediately (cf. [11, (8.4)]):

(4.19) **Proposition.** *The evaluation mapping*

$$H^0(S, \mathbb{V})\to V(s)$$

is a morphism of mixed Hodge structures. In other words, the monodromy invariant subspace of $V(s)$ *is a mixed Hodge substructure* (*independent of* s).

As in [9, (7.24)], we obtain the following rigidity theorem, asserting that a good variation of mixed Hodge structure is determined by the Hodge filtration at one point and the monodromy representation.

(4.20) **Theorem.** *Let* $\mathbb{V}_{\mathbb{F}}$ *be a local system on* S, *with filtration* W. *If* $s_0\in S$, *and* F *is a filtration of* $V(s_0)$, *there is at most one filtration* $\mathscr{F}$ *of* $\mathscr{V}$ *such that*

i) $\mathscr{F}$ *gives* F *at* s_0,

ii) $\mathbb{V}=(\mathbb{V}_{\mathbb{F}}, W, \mathscr{F})$ *is a graded-polarizable variation of mixed Hodge structure satisfying Properties* (3.13).

Proof. Suppose we had two: $\mathbb{V}_1$ with $\mathscr{F}_1$ and $\mathbb{V}_2$ with $\mathscr{F}_2$. The identity mapping on $\mathbb{V}_{\mathbb{F}}$ defines an element

$$e\in W_0 H^0(S, \mathrm{Hom}(\mathbb{V}_{\mathbb{F}}, \mathbb{V}_{\mathbb{F}})).$$

Using $\mathscr{F}_1$ on the domain and $\mathscr{F}_2$ on the range, we obtain a graded-polarizable variation of mixed Hodge structure $\mathrm{Hom}(\mathbb{V}, \mathbb{V})$; it satisfies (3.13) by (A.10). By (i), $e(s_0) \in F^0 \mathrm{Hom}(V(s_0), V(s_0))$. By (4.19), it follows that $e(s) \in F^0(s)$ for all s, i.e. e defines an isomorphism between $\mathbb{V}_1$ and $\mathbb{V}_2$ (as variations of mixed Hodge structure) as desired.

(4.21) *Remark.* It is easy to see that rigidity fails to hold in the absence of (3.13). We can even see counterexamples in (3.16), if we let $t \in \mathbb{C}$ and allow t to approach infinity. The set of all such variations of mixed Hodge structure is parametrized by the set of all entire functions, but only the constant functions provide variations that satisfy (3.13).

(4.22) If $\mathbb{V}$ is any local system on S, by Poincaré duality the dual of $H^i(S, \mathbb{V})$ with respect to $\mathbb{Q}(-1)$ is identified with $H_c^{2-i}(S, \mathbb{V}^*)$, where $\mathbb{V}^*$ is the dual local system of $\mathbb{V}$. Of course, one may use this fact to put a mixed Hodge structure on $H_c^{2-i}(S, \mathbb{V})$, but it is more satisfactory to dispose of a construction as above.

The cohomology groups with compact supports $H_c^i(S, \mathbb{V})$ can be computed as $H^i(\bar{S}, j_! \mathbb{V})$, where $j_! \mathbb{V}$ is the extension of $\mathbb{V}$ by zero over Σ. One has an exact sequence

$$0 \to j_! \mathbb{V} \to j_* \mathbb{V} \to Q \to 0,$$

where Q is the skyscraper sheaf supported on Σ with $Q_\sigma = \ker(N_\sigma)$ as fibers. We will put a natural mixed Hodge structure on these cohomology groups in such a way that Poincaré duality becomes a duality between mixed Hodge structures.

We construct first a cohomological mixed Hodge complex on $\bar{S}$ that is quasi-isomorphic to $j_! \mathbb{V}$. We will not bother about the rational structure here; it is treated in a similar way as for $j_* \mathbb{V}$. We consider the single complex of sheaves associated to the double complex

(4.23)
$$\begin{array}{ccc} \tilde{\mathscr{V}} & \longrightarrow & \bigoplus_\sigma V_\sigma = V \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle N} \\ \nabla \tilde{\mathscr{V}} & \longrightarrow & \bigoplus_\sigma N_\sigma V_\sigma = NV. \end{array}$$

Because the first column is quasi-isomorphic to $j_* \mathbb{V}$, and the second to $\mathrm{Ker}(N)$, the complex as a whole is quasi-isomorphic to $j_! \mathbb{V}$. We let F^p denote the subcomplex

(4.24)
$$\begin{array}{ccc} \tilde{\mathscr{F}}^p & \longrightarrow & F^p V \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle N} \\ (\tilde{\mathscr{F}}^{p-1} \otimes \Omega^1_{\bar{S}} \log \Sigma)) \cap \nabla \tilde{\mathscr{V}} & \longrightarrow & F^{p-1} V \cap NV, \end{array}$$

and $\mathfrak{M}_k$ the subcomplex

$$
(4.25)\qquad
\begin{array}{ccc}
\tilde{\mathcal{W}}_k & \longrightarrow & W_k + M_{k+1} \cap N^{-1} W_{k-1} =: U_k \\
\Big\downarrow{\scriptstyle \nabla} & & \Big\downarrow{\scriptstyle N} \\
\mathcal{Z}'_k & \longrightarrow & N W_k + M_{k-1} \cap W_{k-1} \cap N V =: Z'_k,
\end{array}
$$

where $\mathcal{Z}'_k = \mathrm{Res}^{-1}(Z'_k) \cap (\Omega^1_{\bar{S}}(\log \Sigma) \otimes \tilde{\mathcal{W}}_k)$.

(4.26) **Theorem.** *Suppose that* $\mathbb{V}$ *satisfies the conditions of* (3.13). *Then the bifiltered complex defined above is a cohomological mixed Hodge complex. Moreover, we have*

$$\mathrm{Gr}_k^{\mathfrak{M}} j_! \mathbb{V} \simeq \{\mathrm{Gr}_k^W \mathbb{V}, \mathrm{Gr}_k N(\mathrm{Gr}_k^W V)\} \oplus \mathrm{Gr}_{k+1}^M(\ker N;\, V/W_k)[-1].$$

Proof. By (4.15), $\mathcal{Z}'_k/\mathcal{Z}'_{k-1} \simeq \nabla(\tilde{\mathcal{W}}_k/\tilde{\mathcal{W}}_{k-1}) \oplus \mathrm{Gr}_{k-1}^M((NV \cap W_{k-1}/N W_{k-1}))$, and the second summand injects into Z'_k/Z'_{k-1} with quotient $\mathrm{Gr}_k N(\mathrm{Gr}_k^W V)$. Hence in $\mathrm{Gr}_k^{\mathfrak{M}} j_! \mathbb{V}$ we discover the acyclic subcomplex

$$0 \to \mathrm{Gr}_{k-1}^M((NV \cap W_{k-1})/N W_{k-1})) \to \mathrm{Gr}_{k-1}^M((NV \cap W_{k-1})/N W_{k-1})).$$

Factoring this out, we obtain the complex

$$
\begin{array}{ccc}
\tilde{\mathcal{W}}_k/\tilde{\mathcal{W}}_{k-1} & \longrightarrow & U_k/U_{k-1} \\
\Big\downarrow{\scriptstyle \nabla} & & \Big\downarrow{\scriptstyle N} \\
\nabla(\tilde{\mathcal{W}}_k/\tilde{\mathcal{W}}_{k-1}) & \longrightarrow & \mathrm{Gr}_k N(Gr_k^W V).
\end{array}
$$

Note that

$$(4.27)\qquad M_k \cap N^{-1}(W_{k-1}) + W_{k-1} = M_k \cap N^{-1}(W_{k-2}) + W_{k-1}.$$

For let $x \in M_k$ be such that $Nx \in W_{k-1}$. Then $Nx \in W_{k-1} \cap M_{k-2}$. Since the latter equals $N(W_{k-1} \cap M_k) + W_{k-2} \cap M_{k-2}$, we can write $Nx = Nx' + y$ with $x' \in W_{k-1} \cap M_k$ and $y \in W_{k-2}$. Then $x = x' + (x - x')$, where $x' \in W_{k-1}$ and $x - x' \in M_k \cap N^{-1}(W_{k-2})$. Next, observe that

$$W_k \cap M_{k+1} \cap N^{-1} W_{k-1} \subset U_{k-1},$$

so U_k/U_{k-1} is the direct sum of

$$(W_k + U_{k-1})/U_{k-1}$$

(which is mapped isomorphically by N onto $\mathrm{Gr}_k N(\mathrm{Gr}_k^W V)$) and

$$(M_{k+1} \cap N^{-1}(W_{k-1}) + U_{k-1})/U_{k-1} =: T.$$

This means that after taking the quotient by an acyclic subcomplex again, we obtain a quasi-isomorphism

$$\mathrm{Gr}_k^{\mathfrak{M}} j_! \mathbb{V} \simeq \{\mathrm{Gr}_k^W \mathbb{V}, \mathrm{Gr}_k N(\mathrm{Gr}_k^W V)\} \oplus T[-1].$$

We must therefore show that

$$T \simeq \mathrm{Gr}^M_{k+1}(\mathrm{Ker}\, N;\; V/W_k).$$

We have

$$\begin{aligned} T &= (M_{k+1} \cap N^{-1}(W_{k-1}) + W_{k-1})/(M_k \cap N^{-1}(W_{k-2}) + W_{k-1}) \\ &= (M_{k+1} \cap N^{-1}(W_{k-1}) + W_{k-1})/(M_k \cap N^{-1}(W_{k-1}) + W_{k-1}) \quad \text{by (4.27)} \\ &= \mathrm{Gr}^M_{k+1}(N^{-1}(W_{k-1})/W_{k-1}) = \mathrm{Gr}^M_{k+1}(\mathrm{Ker}\, N;\; V/W_{k-1}). \end{aligned}$$

We wish to replace the W_{k-1} by W_k in this expression. We have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to \mathrm{Gr}^M_{k+1}\mathrm{Gr}^W_k V & \longrightarrow & \mathrm{Gr}^M_{k+1}(V/W_{k-1}) & \longrightarrow & \mathrm{Gr}^M_{k+1}(V/W_k) \to 0 \\ \downarrow N & & \downarrow N & & \downarrow N \\ 0 \to \mathrm{Gr}^M_{k-1}\mathrm{Gr}^W_k V & \longrightarrow & \mathrm{Gr}^M_{k-1}(V/W_{k-1}) & \longrightarrow & \mathrm{Gr}^M_{k-1}(V/W_k) \to 0, \end{array}$$

in which the left vertical N is an isomorphism. Hence (by strictness of morphisms of Hodge structures),

$$\begin{aligned} \mathrm{Gr}^M_{k+1}(\mathrm{Ker}\, N;\; V/W_{k-1}) &= \mathrm{Ker}(N\colon \mathrm{Gr}^M_{k+1}(V/W_{k-1}) \to \mathrm{Gr}^M_{k-1}(V/W_{k-1})) \\ &= \mathrm{Gr}^M_{k+1}(\mathrm{Ker}\, N;\; V/W_k) \end{aligned}$$

by the snake lemma. □

(4.28) *Remark.* Analogous to (4.19), we have for any $s \in S$ a natural mapping

$$V(s) \to H^2_c(S, \mathbb{V}),$$

which is a morphism of mixed Hodge structures of type (1, 1) and identifies $H^2_c(S, \mathbb{V})$ with the largest quotient space of $V(s)(-1)$ on which $\pi_1(S, s)$ acts trivially.

(4.29) *Example.* Let $\mathbb{V} = \mathbb{Q}_S$. Then the resulting Hodge structure on $H^2_c(S, \mathbb{V})$ is just the Tate Hodge structure $\mathbb{Q}(-1)$, purely of type (1, 1).

(4.30) **Theorem.** *Let $\mathbb{V}$ be a variation of mixed Hodge structure as in* (4.1) *with quasi-unipotent monodromy* (this is the case whenever $\mathbb{V}$ can be defined over a number field, e.g. in the geometric case). *Then for all i, the mixed Hodge structures on $H^i_c(S, \mathbb{V})$ and $H^{2-i}(S, \mathbb{V}^*)(1)$ are dual to each other.*

The proof of this theorem will occupy the rest of this section. We may as well assume that all local monodromy transformations of $\mathbb{V}$ are unipotent. For if this is not the case, we can find a Zariski-dense open subset S' of S and a finite unramified covering $\pi\colon \tilde{S} \to S'$ such that the local monodromy of $\tilde{\mathbb{V}} = \pi^*(\mathbb{V}|_{S'})$ is unipotent everywhere, and obtain morphisms of mixed Hodge structures

$$\text{(4.31)} \qquad \begin{aligned} H^\cdot(S, \mathbb{V}) &\hookrightarrow H^\cdot(\tilde{S}, \tilde{\mathbb{V}}), \\ H^\cdot_c(S, \mathbb{V}) &\twoheadleftarrow H^\cdot_c(\tilde{S}, \tilde{\mathbb{V}}). \end{aligned}$$

This reduces us to the unipotent case. (It should be possible to avoid this reduction step and give a direct argument along the lines which follow, which would work even in the case of general monodromy.)

We observe that the connection $\nabla: \mathscr{V} \to \Omega_S^1 \otimes \mathscr{V}$ has a dual connection on $\mathscr{V}^* = \mathrm{Hom}_{\mathcal{O}_S}(\mathscr{V}, \mathcal{O}_S)$ given by

$$\langle v, \nabla^* v^* \rangle + \langle \nabla v, v^* \rangle = d\langle v, v^* \rangle$$

as sections of Ω_S^1. The canonical extensions of $\mathscr{V}$ and $\mathscr{V}^*$ are related by

$$(\mathscr{V}^*)^{\sim} = \mathrm{Hom}_{\mathcal{O}_{\bar{S}}}(\tilde{\mathscr{V}}, \mathcal{O}_{\bar{S}}).$$

If $\mathbb{V}$ is a variation of mixed Hodge structure, the same holds for $\mathbb{V}^*$ and the dual Hodge filtration on $\mathscr{V}^*$ is given by $\mathscr{F}^{*p} = (\mathscr{F}^{-p-1})^{\perp}$. Moreover, if $\mathbb{V}$ satisfies the conditions of (3.13), the same is true for $\mathbb{V}^*$. The relative weight filtration on V^* is just the dual of the relative weight filtration on V.

We concentrate on the duality between $H_c^1(S, \mathbb{V})$ and $H^1(S, \mathbb{V}^*)$ as the remaining cases are easy consequences of (4.19) and (4.28). We must show that under the pairing

$$H_c^1(S, \mathbb{V}) \times H^1(S, \mathbb{V}^*) \to H_c^2(S, \mathbb{F}) = \mathbb{F}(-1)$$

we have for all p, k:

(4.32) $$F^p H_c^1(S, \mathbb{V}_{\mathbb{C}}) = (F^{-p} H^1(S, \mathbb{V}_{\mathbb{C}}^*))^{\perp};$$

(4.33) $$\mathfrak{M}_k H_c^1(S, \mathbb{V}_{\mathbb{C}}) = (\mathfrak{M}_{1-k} H^1(S, \mathbb{V}_{\mathbb{C}}^*))^{\perp}.$$

To do this we first give an explicit description of this pairing. Observe that $H^1(S, \mathbb{V}_{\mathbb{C}}^*)$ can be computed as the cohomology of the single complex associated to the double complex

$$\bigoplus_{p,q} C^q(\mathfrak{U}, (\mathscr{V}^*)^{\sim} \otimes \Omega_{\bar{S}}^p(\log \Sigma))$$

for an affine open covering $\mathfrak{U}$ of $\bar{S}$. A similar result holds for $F^p H^1(S, \mathbb{V}_{\mathbb{C}}^*)$. Thus a class $[v^*] \in F^p H^1(S, \mathbb{V}_{\mathbb{C}}^*)$ is represented by a pair

$$(v_1^*, v_2^*) \in C^1(\mathfrak{U}, \mathscr{F}^p(\mathscr{V}^*)^{\sim}) \oplus C^0(\mathfrak{U}, \mathscr{F}^{p-1}(\mathscr{V}^*)^{\sim} \otimes \Omega_{\bar{S}}^1(\log \Sigma)).$$

For cohomology with compact supports we observe that the double complex (4.23) is quasi-isomorphic to the complex

$$\mathscr{I}_\Sigma \tilde{\mathscr{V}} \xrightarrow{\nabla} \tilde{\mathscr{V}} \otimes \Omega_{\bar{S}}^1$$

(take cohomology sheaves in the horizontal direction). Similarly the filtration level $\mathscr{F}^q j_! \mathbb{V}_{\mathbb{C}}$ is quasi-isomorphic to the complex

$$\mathscr{I}_\Sigma \tilde{\mathscr{F}}^q \to \tilde{\mathscr{F}}^{q-1} \otimes \Omega_{\bar{S}}^1$$

hence a class $[v] \in F^q H_c^1(S, \mathbb{V}_{\mathbb{C}})$ is represented by a pair

$$(v_1, v_2) \in C^1(\mathfrak{U}, \mathscr{I}_\Sigma \tilde{\mathscr{F}}^q) \oplus C^0(\mathfrak{U}, \tilde{\mathscr{F}}^{q-1} \otimes \Omega_{\bar{S}}^1).$$

The element $[v]\wedge[v^*]\in H_c^2(S,\mathbb{C})\simeq H^1(\bar{S},\Omega_{\bar{S}}^1)$ is then represented by

$$\langle v_1, v_2^*\rangle + \langle v_2, v_1^*\rangle \in C^1(\mathfrak{U},\Omega_{\bar{S}}^1).$$

By the relation between $\mathscr{F}$ and $\mathscr{F}^*$ and this formula we can conclude that $F^p H_c^1(S,\mathbb{V}_{\mathbb{C}})\subset(F^{-p}H^1(S,\mathbb{V}_{\mathbb{C}}^*))^{\perp}$. To verify (4.32) it remains to be checked that the induced pairing

$$\mathrm{Gr}_F^p H_c^1(S,\mathbb{V}_{\mathbb{C}})\times \mathrm{Gr}_F^q H^1(S,\mathbb{V}_{\mathbb{C}}^*)\to\mathbb{C}$$

is nonsingular for $p+q=1$. But these two spaces are the hypercohomology groups of the $\mathcal{O}_{\bar{S}}$-linear complex $\mathrm{Gr}_{\mathscr{F}}^p j_! \mathbb{V}_{\mathbb{C}}$ and its dual complex

$$\mathrm{Hom}_{\mathcal{O}_{\bar{S}}}(\mathrm{Gr}_{\mathscr{F}}^p j_! \mathbb{V}_{\mathbb{C}}, \Omega_{\bar{S}}^1),$$

so the required duality follows from Serre duality.

By a similar argument, we obtain that the pairing

$$H^1(\bar{S},\mathfrak{M}_k j_! \mathbb{V}_{\mathbb{C}})\times H^1(\bar{S},\mathfrak{M}_l\{\mathbb{V}^*, V^*\})\to\mathbb{C}$$

has zero image if $k+l<0$. By passing to subquotients we also get that

$$\mathfrak{M}_k H_c^1(S,\mathbb{V}_{\mathbb{C}})\times\mathfrak{M}_l H^1(S,\mathbb{V}_{\mathbb{C}}^*)\to\mathbb{C}$$

is the zero pairing for $k+l<2$. We are led to show that the induced pairings

$$\mathrm{Gr}_{k+1}^{\mathfrak{M}} H_c^1(S,\mathbb{V}_{\mathbb{C}})\times\mathrm{Gr}_{-k+1}^{\mathfrak{M}} H^1(S,\mathbb{V}_{\mathbb{C}}^*)\to\mathbb{C}$$

are nonsingular. We recall that these are the E_2-terms of the spectral sequences

(4.34) $$E_1^{-k,q+k}\simeq\mathbf{H}^q(\bar{S},\mathrm{Gr}_k^{\mathfrak{M}} j_! \mathbb{V}_{\mathbb{C}})\Rightarrow H_c^q(S,\mathbb{V}_{\mathbb{C}});$$

(4.35) $$^*E_1^{-k,q+k}\simeq\mathbf{H}^q(\bar{S},\mathrm{Gr}_k^{\mathfrak{M}}\{\mathbb{V}^*, V^*\})\Rightarrow H^q(S,\mathbb{V}_{\mathbb{C}}^*).$$

By (4.26) we have

$$E_1^{-k,1+k}\simeq H^1(\bar{S},j_*\mathrm{Gr}_k^W\mathbb{V}_{\mathbb{C}})\oplus\mathrm{Gr}_{k+1}^M(\mathrm{Ker}\,N;\,V/W_k),$$

and by (4.11) and (4.12)

$$^*E_1^{k,1-k}\simeq H^1(\bar{S},j_*\mathrm{Gr}_{-k}^W\mathbb{V}_{\mathbb{C}}^*)\oplus\mathrm{Gr}_{-k-1}^M(W_{-k-1}V^*/N\,W_{-k-1}V^*)(-1).$$

Because $\mathrm{Gr}_k^W\mathbb{V}$ is polarizable,

$$\mathrm{Gr}_{-k}^W\mathbb{V}^*\simeq(\mathrm{Gr}_k^W\mathbb{V})^*\simeq\mathrm{Gr}_k^W\mathbb{V}(-k),$$

and it follows from [11, §7] that the pairing

$$H^1(\bar{S},j_*\mathrm{Gr}_k^W\mathbb{V})\times H^1(\bar{S},j_*\mathrm{Gr}_{-k}^W\mathbb{V}^*)\to\mathbb{F}(-1)$$

is nonsingular. Also it is obvious that

$$\mathrm{Gr}_{k+1}^M(\mathrm{Ker}\,N;\,V/W_k)=\mathrm{Hom}(\mathrm{Gr}_{-k-1}^M(\mathrm{Coker}\,N;\,W_{-k-1}V^*),\mathbb{F}(-1)).$$

It is easy to see that the "cross-pairings" are trivial, so

$$E_1^{-k,1+k} \times {}^*E_1^{k,1-k} \to \mathbb{C}$$

is a nonsingular pairing. To conclude the same about the E_2-terms, we must understand the d_1-maps in the spectral sequences (4.34) and (4.35). The map

$$d_1^{-1-k,1+k}\colon E_1^{-1-k,1+k} \to E_1^{-k,1+k}$$

decomposes into

$$u_1\colon H^0(\bar{S}, j_* \operatorname{Gr}_{k+1}^W \mathbb{V}_{\mathbb{C}}) \to H^1(\bar{S}, j_* \operatorname{Gr}_k^W \mathbb{V}_{\mathbb{C}}),$$
$$u_2\colon H^0(\bar{S}, j_* \operatorname{Gr}_{k+1}^W \mathbb{V}_{\mathbb{C}}) \to \operatorname{Gr}_{k+1}^M(\operatorname{Ker} N;\, V/W_k).$$

We will describe u_1 and u_2 explicitly. The other non-trivial d_1-maps and $*d_1$-maps have a similar description, and it will follow from this that the $*d_1$'s are the dual maps of the d_1's.

Because of the admissibility of $\mathbb{V}$ all maps N_σ are quasi-strict (see (2.18)). It is not hard to see that this implies that the following sequence is exact for all k:

(4.36) $$0 \to j_* \operatorname{Gr}_k^W \mathbb{V} \to j_*(W_{k+1}\mathbb{V}/W_{k-1}\mathbb{V}) \to j_* \operatorname{Gr}_{k+1}^W \mathbb{V} \to 0.$$

The mapping u_1 is nothing but a connecting homomorphism in the long exact cohomology sequence of (4.36). To compute u_2, we observe that the decomposition of d_1 into u_1 and u_2 is functorial in $\mathbb{V}$, hence u_2 factors as

$$H^0(\bar{S}, j_* \operatorname{Gr}_{k+1}^W \mathbb{V}_{\mathbb{C}}) \xrightarrow{u_2'} \operatorname{Gr}_{k+1}^M(\operatorname{Ker} N; \operatorname{Gr}_{k+1}^W V) \xrightarrow{u_2''} \operatorname{Gr}_{k+1}^M(\operatorname{Ker} N;\, V/W_k);$$

here u_2'' is obtained from the inclusion $\operatorname{Gr}_{k+1}^W V \subset V/W_k$, and we can see that u_2' is the natural mapping

$$\bigcap_\sigma \operatorname{Ker}(\operatorname{Gr}_{k+1} N_\sigma) \to \bigoplus_\sigma \operatorname{Gr}_{k+1}^M \operatorname{Ker}(\operatorname{Gr}_{k+1} N_\sigma).$$

(Because $\operatorname{Gr}_{k+1}^W \mathbb{V}$ is pure of weight $k+1$, each $\operatorname{Ker}(\operatorname{Gr}_{k+1} N_\sigma)$ is contained in M_{k+1}.) By the complete reducibility of polarizable variations of Hodge structure ([3, (4.2.6)]), it suffices to check this for a constant local system, in which case it is obvious.

§ 5. The geometric case

In this chapter, we show that a family of algebraic varieties $f'\colon X^* \to \Delta^*$ gives rise to a variation of mixed Hodge structure over Δ^* that is graded-polarizable (see also [19]) and satisfies the conditions of (3.13). We restrict ourselves here to the case of smooth varieties, for which we had worked out most of the story from a slightly different angle from the one that El Zein takes in [6]. For a construction in the general case, we refer the reader to [6], in which the case of a family of varieties with normal crossings is taken as the point of departure.

(5.1) *Notation.* Let X be a complex Kähler manifold, Δ the unit disc in the complex plane, $f\colon X\to\Delta$ a proper mapping, $D\subset X$ a divisor (we write D as $\bigcup D_i$, the union of its components), $Y=f^{-1}(0)$. We suppose that $D\cup Y$ is a divisor with normal crossings, and that f and its restriction to each intersection $D_{i_1}\cap\ldots\cap D_{i_p}$ ($p\geqq 1$) are flat over Δ and smooth over the punctured disc Δ^*. Moreover, we assume here that Y is reduced (the non-reduced case can then be handled as in [10]), and thus the monodromy will be unipotent.

(5.2) We put $U=X-(D\cup Y)$. We are interested in the cohomology of the fibers of the restriction $f'\colon U\to\Delta^*$ of f, which we study in a way similar to the method of [10]. (We remark that for any smooth quasi-projective morphism $Z\to\Delta$, one sees that there exists, perhaps after shrinking Δ, a manifold X as in (5.1), with $Z|_{\Delta^*}\simeq X-(D\cup Y)$, by using Hironaka's resolution theorem. By semi-stable reduction, we can arrange that Y is reduced, after taking a finite covering of Δ.)

Let $m\geqq 0$ be an integer, and

$$\mathbb{V}_{\mathbb{Q}}=R^m(f')_*\mathbb{Q}_U.$$

We obtain an increasing filtration W on $\mathbb{V}_{\mathbb{Q}}$ by consideration of the spectral sequence

$$E_2^{p,q}=R^p f''_*(R^q g_*\mathbb{Q}_U)\Rightarrow R^{p+q}f'_*\mathbb{Q}_U,$$

where $g\colon U\hookrightarrow X-Y$ and $f''\colon X-Y\to\Delta^*$; we have

$$\mathrm{Gr}_l^W\mathbb{V}_{\mathbb{Q}}\simeq E_\infty^{m-l,l}.$$

By the results of [3, § 3] applied to the fibers of f', we conclude that $E_\infty\simeq E_3$ and W indeed gives a filtration by local subsystems; in other words, the mixed Hodge theoretic weight filtration of the fibers is locally constant.

Again imitating [3], we see that

$$\mathscr{V}=\mathcal{O}_{\Delta^*}\otimes_{\mathbb{C}}\mathbb{V}\simeq\mathbf{R}^m f''_*\,\Omega^\cdot_{(X-Y)/\Delta^*}(\log D)$$

is filtered by its locally free subsheaves

$$\mathscr{F}^p=\mathbf{R}^m f''_*\,F^p\Omega^\cdot_{(X-Y)/\Delta^*}(\log D),$$

and

$$\mathscr{F}^p/\mathscr{F}^{p+1}\simeq R^{m-p}f''_*\,\Omega^p_{(X-Y)/\Delta^*}(\log D)$$

is also locally free.

As in [7], one sees that the connection ∇ on $\mathscr{V}$ is the connecting homomorphism in the long exact sequence of relative hypercohomology associated to

$$\begin{aligned}0\to(f'')^*\Omega^1_{\Delta^*}\otimes\Omega^\cdot_{(X-Y)/\Delta^*}(\log D)[-1]\\ \to\Omega^\cdot_{X-Y}(\log D)\to\Omega^\cdot_{(X-Y)/\Delta^*}(\log D)\to 0,\end{aligned}$$

and hence maps $\mathscr{F}^p$ to $\Omega^1_{\Delta^*}\otimes\mathscr{F}^{p-1}$.

(5.3) Next, we assert that the canonical extension $\tilde{\mathscr{V}}$ of $\mathscr{V}$ is given by the formula

$$\tilde{\mathscr{V}} \simeq \mathbf{R}^m f_* \Omega^\cdot_{X/\Delta}(\log(D+Y)).$$

To prove this, it is enough to check the following:

i) $\mathbf{R}^m f_* \Omega^\cdot_{X/\Delta}(\log(D+Y))$ is locally free on Δ,

ii) ∇ extends to a connection with a logarithmic pole at 0 on $\mathbf{R}^m f_* \Omega^\cdot_{X/\Delta}(\log(D+Y))$,

iii) $\mathrm{Res}_0(\nabla)$ is nilpotent.

We proceed along the lines of [10, §2]. Let $X_\infty = X \times_\Delta \tilde{\Delta}^*$, where $\tilde{\Delta}^*$ is the universal covering of Δ^*, and let $k: X_\infty \to X$ and $i: Y \to X$ be the obvious maps. Put $D_\infty = k^{-1}(D)$, $U_\infty = X_\infty - D_\infty$. Then U_∞ is homotopy equivalent to any fiber of f', and

$$H^m(U_\infty, \mathbb{C}) \simeq \mathbf{H}^m(X_\infty, \Omega^\cdot_{X_\infty}(\log D_\infty)).$$

As in [10, (2.5)], this is isomorphic to

$$\mathbf{H}^m(Y, i^{-1} k_* \Omega^\cdot_{X_\infty}(\log D_\infty)).$$

We have quasi-isomorphisms of complexes of sheaves on Y:

$$i^{-1}\Omega^\cdot_X(\log(D+Y))[\log t] \to i^{-1} k_* \Omega^\cdot_{X_\infty}(\log D_\infty)$$

and

$$i^{-1}\Omega^\cdot_X(\log(D+Y))[\log t] \to \Omega^\cdot_{X/\Delta}(\log(D+Y)) \otimes_{\mathcal{O}_X} \mathcal{O}_Y,$$

where t is a coordinate on Δ. Therefore

$$\mathbf{H}^m(Y, \Omega^\cdot_{X/\Delta}(\log(D+Y)) \otimes \mathcal{O}_Y) \simeq \mathbf{H}^m(Y, i^{-1} k_* \Omega^\cdot_{X_\infty}(\log D_\infty)) \simeq H^m(U_\infty, \mathbb{C}).$$

As a similar formula holds for the other fibers of f, one concludes that $\mathbf{R}^m f_* \Omega^\cdot_{X/\Delta}(\log(D+Y))$ is (locally) free on Δ and commutes with base-change.

The extension of ∇ is obtained as the connecting homomorphism in the long exact sequence of relative hypercohomology associated to

$$0 \to f^*\Omega^1_\Delta(\log 0) \otimes \Omega^\cdot_{X/\Delta}(\log(D+Y))[-1]$$
$$\to \Omega^\cdot_X(\log(D+Y)) \to \Omega^\cdot_{X/\Delta}(\log(D+Y)) \to 0$$

so

$$\nabla: \mathbf{R}^m f_* \Omega^\cdot_{X/\Delta}(\log(D+Y)) \to \Omega^1_\Delta(\log 0) \otimes \mathbf{R}^m f_* \Omega^\cdot_{X/\Delta}(\log(D+Y)).$$

The fact that $\mathrm{Res}_0(\nabla)$ is nilpotent follows directly as in the proof of [10, (2.20)], because Y is reduced.

(5.4) We define a filtration $W(D)$ on $\Omega^\cdot_X(\log(D+Y))$ (the weight filtration with respect to D) by:

$$W(D)_l \Omega^p_X(\log(D+Y)) = \Omega^l_X(\log(D+Y)) \wedge \Omega^{p-l}_X(\log Y)$$

We also denote by $W(D)$ the induced filtrations on $\Omega^\cdot_{X/\Delta}(\log(D+Y))$ and $\Omega^\cdot_{X/\Delta}(\log(D+Y)) \otimes \mathcal{O}_Y$, or on $\mathbf{H}^m(Y, \Omega^\cdot_{X/\Delta}(\log(D+Y)) \otimes \mathcal{O}_Y)$. From (5.3) we obtain the isomorphism (which depends on the choice of t)

$$\psi\colon H^m(U_\infty, \mathbb{C}) \xrightarrow{\sim} \mathbf{H}^m(Y, \Omega^{\cdot}_{X/\Delta}(\log(D+Y)) \otimes \mathcal{O}_Y).$$

On the left-hand side we have the monodromy transformation T and the weight filtration W from mixed Hodge theory, and on the right-hand side we have the endomorphism $\mathrm{Res}_0(\nabla)$ and the filtration $W(D)$. Under the identification ψ we have (cf. [10, (2.21)]) $T = \exp(-2\pi i \,\mathrm{Res}_0(\nabla))$ and $W = W(D)$. Hence T is unipotent, and the admissibility of $N = \log T$ relative to W is equivalent to the admissibility of $\mathrm{Res}_0(\nabla)$ relative to $W(D)$.

(5.5) The complex $\Omega^{\cdot}_X(\log(D+Y))$ also carries the filtration $W(Y)$ (weight filtration with respect to Y), defined analogously to $W(D)$. We use this to replace $\Omega^{\cdot}_{X/\Delta}(\log(D+Y)) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ by the following double complex. Let (compare [10, (4.14)])

$$A^{p,q} = \Omega^{p+q+1}_X(\log(D+Y))/W(Y)_q \qquad (p, q \geqq 0),$$

$d'\colon A^{p,q} \to A^{p+1,q}$ be ordinary differentiation, $d''\colon A^{p,q} \to A^{p,q+1}$ be cup product with the 1-form $\theta = f^*(dt/t)$. Cup product of forms with θ defines a quasi-isomorphism

$$\phi\colon \Omega^{\cdot}_{X/\Delta}(\log(D+Y)) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathbf{s}A^{\cdot\cdot}$$

(cf. [10, (4.16)]), where $\mathbf{s}$ denotes the associated single complex of a double complex.

We put

$$W(D)_l A^{p,q} = \text{image of } W(D)_l \Omega^{p+q+1}_X(\log(D+Y)) \quad \text{in } A^{p,q};$$
$$F^p A^{\cdot\cdot} = \bigoplus_{r \geqq p} A^{r,\cdot};$$
$$M_k A^{p,q} = W_{2q+k+1} \Omega^{p+q+1}_X(\log(D+Y))/W(Y)_q$$

(this can be shown to coincide with the definition in [6, II, (3.1)]). Then ϕ is even a bifiltered quasi-isomorphism with respect to the filtrations $W(D)$ and F; if one grades for $W(D)$, one obtains the situation of [10].

Let $\tilde{\nu}$ be the endomorphism of $\mathbf{s}A^{\cdot\cdot}$ given by the canonical projections $\nu\colon A^{p,q} \to A^{p-1,q+1}$. Then $\tilde{\nu}$ is a lifting of $\mathrm{Res}_0(\nabla)$ to the level of complexes.

(5.6) **Theorem.** *With the same notations as above, $(\mathbf{s}A^{\cdot\cdot}, M, F)$ is part of a cohomological mixed Hodge complex, filtered by $W(D)$.*

The proof of this theorem is rather involved, and we postpone it till the end of this chapter.

(5.7) We show how the Properties (3.13) follow from the theorem.

As to the existence of the relative weight filtration $M(N, W(D))$, we observe that

$$\tilde{\mathcal{V}}(0) = \mathbf{H}^m(Y, \Omega^{\cdot}_{X/\Delta}(\log(D+Y)) \otimes \mathcal{O}_Y) = \mathbf{H}^m(Y, A^{\cdot\cdot})$$

and put (introducing the customary shift)

$$M_k \tilde{\mathcal{V}}(0) = \text{Image of } \mathbf{H}^m(Y, M_{k-m} A^{\cdot\cdot}) \text{ in } \tilde{\mathcal{V}}(0).$$

To show that this defines the relative weight filtration, first observe that $\tilde{\nu}$ maps $M_k A^{\cdot\cdot}$ to $M_{k-2}A^{\cdot\cdot}$, hence $\mathrm{Res}_0 \nabla$ shifts M by 2. It remains to be checked that M induces the monodromy weight filtration on $\mathrm{Gr}_l^{W(D)}\tilde{\mathscr{V}}(0)$ for each l. From [10, § 5], it follows that M induces the monodromy weight filtration on $\mathbf{H}^m(Y, \mathrm{Gr}_l^{W(D)} A^{\cdot\cdot})$ for each l, hence we are led to consider the spectral sequence

$$(*) \qquad E_1^{-l, l+m} = \mathbf{H}^m(Y, \mathrm{Gr}_l^{W(D)} A^{\cdot\cdot}) \Rightarrow \mathbf{H}^m(Y, A^{\cdot\cdot}).$$

This has an analogue for each $t \in \Delta^*$:

$$E_1^{-l, l+m} = \mathbf{H}^m(X_t, \mathrm{Gr}_l^W \Omega_{X_t}^{\cdot}(\log D_t)) \Rightarrow H^m(X_t - D_t, \mathbb{C}),$$

which degenerates at E_2 for each $t \in \Delta^*$. As its d_1-mappings are horizontal over Δ^*, the sequence $(*)$ also degenerates at E_2.

The next step in the argument is due to El Zein [6]. The d_1-mappings in $(*)$ are morphisms of mixed Hodge structures (because $W(D)$ extends to a filtration of cohomological mixed Hodge complexes), hence are strictly compatible with the filtrations induced by M [7] on the E_1 terms. Because, for fixed q, M_k induces a single L-weight filtration level on $E_1^{\cdot, q}$, it follows that M induces also the monodromy weight filtration on the terms

$$E_\infty^{-l, l+m} = E_2^{-l, l+m} \cong \mathrm{Gr}_l^{W(D)} \mathbf{H}^m(Y, A^{\cdot\cdot}).$$

From this, we also obtain (3.13, iii) and the graded-polarizability at once.

As for the extendability of the Hodge filtration, we conclude from the theorem again (compare [12, § 2]) that the spectral sequence

$$E_1^{p,q} = H^q(Y, \Omega_{X/\Delta}^p(\log(D+Y)) \otimes \mathcal{O}_Y) \Rightarrow \mathbf{H}^{p+q}(Y, \Omega_{X/\Delta}^{\cdot}(\log(D+Y)) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

degenerates at E_1. This implies that the sheaves

$$R^q f_* \Omega_{X/\Delta}^p(\log(D+Y))$$

are locally free on Δ. Moreover, strictness of the d_1-mappings in the spectral sequence* with respect to the Hodge filtration implies that the various filtrations induced by F on E_2, directly and from E_1, coincide (see [3, (1.3.16)]). Thus, we have checked (3.13, ii) also.

(5.8) We now prove Theorem (5.6). To do this, we have to imitate the construction of the double complex $A^{\cdot\cdot}$, with its filtrations M and $W(D)$, on the level of complexes of $\mathbb{Q}$-vectorspaces. As [10] is a bit obscure (though essentially correct) about this, we use the formalism of [8, § 1].

For any space Z, we let $C^{\cdot}(Z)$ denote the complex of sheaves of germs of rational-valued singular cochains on Z; it is a fine resolution of the constant sheaf $\mathbb{Q}$ on Z, hence $\mathbf{H}^m(Z, C^{\cdot}(Z)) \cong H^m(Z, \mathbb{Q})$. Let $\bar{j}: X^* = X - Y \hookrightarrow X$, and put

$$K^{\cdot}(X^*) = i^{-1}\bar{j}_* C^{\cdot}(X^*), \qquad K^{\cdot}(X_\infty) = i^{-1} k_* C^{\cdot}(X_\infty).$$

---

[7] More accurately, Dec M (see (6.2))

Then one shows as in [10, (2.4), (2.5)] that

$$\mathbf{H}^m(Y, K^\bullet(X^*)) \cong H^m(X^*, \mathbb{Q}),$$
$$\mathbf{H}^m(Y, K^\bullet(X_\infty)) \cong H^m(X_\infty, \mathbb{Q}).$$

The monodromy transformation T on $H^m(X_\infty, \mathbb{Q})$ is induced by the automorphism $(x,u) \mapsto (x, u-1)$ of $X_\infty = X^* \times_{\Delta^*} H$, where $H = \{u \in \mathbb{C} \mid \operatorname{Im} u > 0\} \simeq \tilde{\Delta}^*$ is mapped to Δ^* by $u \mapsto \exp 2\pi i u$. Hence T lifts to an automorphism of $K^\bullet(X_\infty)$ such that

$$K^\bullet(X^*) = \{\sigma \in K^\bullet(X_\infty) \mid T\sigma = \sigma\}.$$

Let $B^\bullet_m = \operatorname{Ker}(T-I)^{m+1} \subset K^\bullet(X_\infty)$ and $B^\bullet = \bigcup_{m \geqq 0} B^\bullet_m$.

(5.9) **Lemma.** *The inclusion $B^\bullet \to K^\bullet(X_\infty)$ is a quasi-isomorphism.*

Proof. Let $B^\bullet_{\mathbb{C}}$, $K^\bullet(X_\infty)_{\mathbb{C}}$ etc. be the sheaf complexes obtained by replacing $\mathbb{Q}$ by $\mathbb{C}$ in the above definitions. One obtains a morphism

$$s\colon i^{-1}\Omega^\bullet_X(\log Y)[\log t] \to B^\bullet_{\mathbb{C}}.$$

Let λ denote the image of $\log t$ under s, so $B^\bullet_{m,\mathbb{C}} = \sum_{k=0}^{m} \lambda^k K^\bullet(X^*)_{\mathbb{C}}$. As $d(\lambda^m) \in B^\bullet_{m-1,\mathbb{C}}$, one obtains a quasi-isomorphism

$$\lambda^m\colon K^\bullet(X^*)_{\mathbb{C}} \xrightarrow{\sim} B^\bullet_{m,\mathbb{C}}/B^\bullet_{m-1,\mathbb{C}}.$$

This shows that s is also a quasi-isomorphism. Because the complex $i^{-1}\Omega^\bullet_X(\log Y)[\log t]$ is quasi-isomorphic to $K^\bullet(X_\infty)_{\mathbb{C}}$ (see the discussion in (5.3) or [10], §2) the inclusion $B^\bullet_{\mathbb{C}} \subset K^\bullet(X_\infty)_{\mathbb{C}}$ is a quasi-isomorphism. This suffices to conclude that $B^\bullet \subset K^\bullet(X_\infty)$ is also a quasi-isomorphism. □

We thank Navarro Aznar for pointing out an error in a previous version of the proof of Lemma (5.9).

(5.10) *Remark.* The advantage of working with $B^\bullet$ rather than with $K^\bullet(X_\infty)$ is that every local section of $B^\bullet$ is killed by some power of $T-I$.

(5.11) *Definition.* If $K^\bullet$ is a complex of $\mathbb{Q}$-vector spaces and $r \in \mathbb{Z}$, we put $K^\bullet(r) = (2\pi i)^r K^\bullet \subset K^\bullet \otimes_{\mathbb{Q}} \mathbb{C}$. We define $\delta\colon B^\bullet \to B^\bullet(-1)$ by

$$\delta = -\frac{1}{2\pi i} \log T.$$

(Note that $\log T$ makes sense on $B^\bullet$, by (5.10).)

(5.12) We let $\rho(B)^\bullet$ denote the mapping cone of the morphism δ, i.e.

$$\rho(B)^p = B^p \oplus B^{p-1}(-1),$$
$$d(x,y) = (dx, -dy + \delta(x)).$$

One defines $\theta\colon \rho(B)^\cdot \to \rho(B)^\cdot(1)[1]$ by $\theta(x,y)=(0,x)$. We finally let

$$C^{p,q}=\begin{cases}\rho(B)^{p+q+1}(q+1) & \text{if } p\geqq 0;\\ \rho(B)^q(q+1)/\operatorname{Ker}(d) & \text{if } p=-1;\\ 0 & \text{if } p<-1.\end{cases}$$

We provide $\oplus C^{p,q}$ with the structure of a double complex by defining $d'\colon C^{p,q}\to C^{p+1,q}$ as the differentiation in $\rho(B)^\cdot$, and $d''\colon C^{p,q}\to C^{p,q+1}$ by $d''(z)=\theta(z)$.

(5.13) **Lemma.** *The inclusion $x\mapsto(0,(-1)^p x)$ from B^p into $C^{p,0}$ induces a quasi-isomorphism*

$$B^\cdot\to \mathbf{s}(C^{\cdot\cdot}).$$

Proof. It is easy to see that the sequence $0\to B^p\to C^{p,0}\xrightarrow{\theta} C^{p,1}\to\dots$ is exact for every $p\geqq 0$. It remains to be shown that the complex $C^{-1,\cdot}$ is acyclic: let $(x,y)\in B^q(q+1)\oplus B^{q-1}(q)$ such that $d\theta(x,y)=0$. Then $dx=0$. Because δ acts as the zero map on $\mathscr{H}^q(B^\cdot)$, there exists $\eta\in B^{q-1}(q)$ such that $\delta(x)=d\eta$. Then $d(x,\eta)=0$, so $(x,y)=(x,\eta)+\theta(y-\eta,0)\in\operatorname{Im}\theta+\operatorname{Ker} d$, i.e., $(x.y)$ represents a coboundary in $C^{-1,q}$. $\square$

(5.14) **Lemma.** *Let $\tilde\delta\colon \mathbf{s}(C^{\cdot\cdot})\to\mathbf{s}(C^{\cdot\cdot})(-1)$ be given by $\tilde\delta(x,y)=(\delta x,\delta y)$, and $v\colon \mathbf{s}(C^{\cdot\cdot})\to\mathbf{s}(C^{\cdot\cdot})(-1)$ by $v(x,y)=(-1)^{p+q+1}(x',y')$ for $(x,y)\in C^{p,q}$, (x',y') its canonical image in $C^{p-1,q+1}$. Then $\tilde\delta$ and v are homotopic.*

Proof. See [8, (1.7)]. $\square$

(5.15) Because $\tilde\delta$ is a lifting of $-\frac{1}{2\pi i}\log T$ to the level of complexes, v induces $\operatorname{Res}_0(\nabla)$ on hypercohomology. We now first finish the proof of Theorem (5.6) for the case $D=\emptyset$, and afterwards indicate the modifications that have to be made in the general case. Assuming $D=\emptyset$, we let for $k\in\mathbf{Z}$:

$$M_k C^{p,q}=\text{the image of } \tau_{k+2q+1}\rho(B)^{p+q+1}(q+1) \quad \text{in } C^{p,q}.$$

Here, for a complex $K^\cdot$,

$$\tau_r K^p=\begin{cases}K^p & \text{if } p<r,\\ (\ker d)\cap K^p & \text{if } p=r,\\ 0 & \text{if } p>r;\end{cases}$$

$\operatorname{Gr}_r^\tau K^\cdot\simeq\mathscr{H}^r(K^\cdot)[-r]$.

(5.16) **Lemma.** *The map $x\mapsto(x,0)$ defines a quasi-isomorphism $K^\cdot(X^*)\to\rho(B)^\cdot$.*

Proof. This is a general fact about cones: we use the fact that $\delta\colon B^p\to B^p(-1)$ is surjective, with $\operatorname{Ker}(\delta)\simeq K^p(X^*)$. $\square$

(5.17) Observe that $\theta\colon\rho(B)\to\rho(B)(1)[1]$ maps $\tau_s\rho(B)^l$ to $\tau_{s+1}\rho(B)^{l+1}(1)$, and hence $M_k C^{p,q}$ to $M_{k-1}C^{p,q+1}$. As a consequence, in the complex $\operatorname{Gr}_k^M C^{\cdot\cdot}$, θ

induces the zero map. We obtain a splitting of the double complex

$$\begin{aligned}\mathrm{Gr}_k^M C^{\cdot\cdot} &\cong \bigoplus_{\substack{q\geqq 0\\ q\geqq -k}} \mathrm{Gr}_k^M C^{\cdot,q}[-q] \cong \bigoplus_{\substack{q\geqq 0\\ q\geqq -k}} (\mathrm{Gr}^\tau_{k+2q+1}\rho(B)^\cdot(q+1))[1]\\ &\cong \bigoplus_{\substack{q\geqq 0\\ q\geqq -k}} (\mathrm{Gr}^\tau_{k+2q+1} K^\cdot(X^*)(q+1))[1].\end{aligned}$$

Moreover, from [3, (3.1.4)] we know what the cohomology sheaves of $K^\cdot(X^*)$ are: if $\tilde{Y}^{(q)}$ is the disjoint union of all q-fold intersections of the components of Y, and $a_q\colon \tilde{Y}^{(q)} \to Y$ is the natural map, then we have a canonical isomorphism

$$\mathscr{H}^q(K^\cdot(X^*)) \xrightarrow{\sim} (a_q)_* \mathbb{Q}_{\tilde{Y}^{(q)}}(-q) \qquad (q \geqq 1).$$

Hence we obtain [10, Lemma (4.13)]:

$$\mathrm{Gr}_k^M C^{\cdot\cdot} \cong \bigoplus_{\substack{q\geqq 0\\ q\geqq -k}} (a_{k+2q+1})_* \mathbb{Q}_{\tilde{Y}^{(k+2q+1)}}(-k-q)[-k-2q].$$

(5.18) To connect the filtered complex $(C^{\cdot\cdot}, M)\otimes_{\mathbb{Q}}\mathbb{C}$ with the complex $(A^{\cdot\cdot}, M)$ as defined in (5.5), we use an intermediate complex $(\tilde{C}^{\cdot\cdot}, M)$, which has filtered quasi-isomorphisms:

$$(C^{\cdot\cdot}, M)\otimes\mathbb{C} \xleftarrow{\psi} (\tilde{C}^{\cdot\cdot}, M) \xrightarrow{\phi} (A^{\cdot\cdot}, M).$$

One obtains $\tilde{C}^{\cdot\cdot}$ as a subcomplex of $C^{\cdot\cdot}\otimes\mathbb{C}$ by performing the same construction which transforms $B^\cdot$ into $C^{\cdot\cdot}$, but now starting out with the complex $\tilde{B}^\cdot = i^{-1}\Omega_X^\cdot(\log Y)[\log t]$, with its endomorphism $T\colon \log t \mapsto \log t + 2\pi i$. If we put $u = \log t/2\pi i$, the endomorphism δ is given by

$$-2\pi i\delta\left(\sum_{j=0}^{r} \omega_j u^j/j!\right) = \sum_{j=0}^{r-1} \omega_{j+1} u^j/j!.$$

It is easily proved that $\tilde{B}^\cdot \hookrightarrow B^\cdot\otimes\mathbb{C}$ is a quasi-isomorphism which is δ-equivariant, hence the natural inclusion $\tilde{C}^{\cdot\cdot} \hookrightarrow C^{\cdot\cdot}\otimes\mathbb{C}$ is a filtered quasi-isomorphism with respect to the M-filtrations.

The map $\phi\colon \tilde{C}^{\cdot\cdot} \to A^{\cdot\cdot}$ is induced by

$$(x, y) \mapsto x_0 - 2\pi i\, du \wedge y_0$$

for

$$x = \sum_{j=0}^{r} x_j u^j/j!, \qquad \text{a section of } i^{-1}\Omega_X^{p+q+1}(\log Y)[u],$$

$$y = \sum_{j=0}^{s} y_j u^j/j!, \qquad \text{a section of } i^{-1}\Omega_X^{p+q}(\log Y)[u].$$

One checks easily that ϕ is a morphism of double complexes, compatible with M, such that $\mathrm{Gr}_k^M(\phi)$ is a direct sum of mappings

$$\mathrm{Gr}^\tau_j \rho(\tilde{B}) \xrightarrow{\mu_j} \mathrm{Gr}^W_j \Omega_X^\cdot(\log Y).$$

By an argument as in Lemma (5.16) the inclusion

$$i^{-1}\Omega^{\cdot}_X(\log Y) \hookrightarrow \rho(\tilde{B}^{\cdot})$$

is a quasi-isomorphism, hence a filtered quasi-isomorphism if both are equipped with the canonical filtration τ. Because $(\Omega^{\cdot}_X(\log Y), \tau) \xrightarrow{\text{id}} (\Omega^{\cdot}_X(\log Y), W)$ is a filtered quasi-isomorphism [3, (3.18)] and $\phi(x_0, 0) = x_0$, we may conclude that ϕ is a filtered quasi-isomorphism. Thus $\mathrm{Gr}^M_k C^{\cdot\cdot}$ becomes a direct sum of Hodge complexes of weight k.

(5.19) In the general case $D \neq \emptyset$, we modify the previous construction a bit. From now on we will write $C^{\cdot\cdot}(X)$, $B^{\cdot\cdot}(X)$ etc. instead of $C^{\cdot\cdot}, B^{\cdot}$, and use the fact that we can apply the same formalism to the composition of f with the natural mappings

$$b_q\colon \tilde{D}^{(q)} \to X$$

where $\tilde{D}^{(q)}$ is defined analogous to $\tilde{Y}^{(q)}$ (see (5.17)). Hence it makes sense to speak of $C^{\cdot\cdot}(\tilde{D}^{(q)})$ or $B^{\cdot}(\tilde{D}^{(q)})$ as well. We put $X' = X - D$; $j'\colon X' \hookrightarrow X$ is the inclusion map. We define

$$\begin{aligned} &K^{\cdot}(X') = i^{-1} j'_* C^{\cdot}(X') \quad \text{and} \\ &K^{\cdot}(U) = K^{\cdot}(X^*) \otimes_{\mathbb{Q}} K^{\cdot}(X'); \qquad K^{\cdot}(U_\infty) = K^{\cdot}(X_\infty) \otimes_{\mathbb{Q}} K^{\cdot}(X') \\ &B^{\cdot}(U) = B^{\cdot}(X) \otimes_{\mathbb{Q}} K^{\cdot}(X') \cong K^{\cdot}(U)[\lambda] \subset K^{\cdot}(U_\infty). \end{aligned}$$

For a tensor product $K^{\cdot} \otimes_{\mathbb{Q}} L^{\cdot}$ of complexes we have partial canonical filtrations

$$\begin{aligned} &\tau'_q(K^{\cdot} \otimes L^{\cdot}) = (\tau_q K^{\cdot}) \otimes L^{\cdot}; \\ &\tau''_q(K^{\cdot} \otimes L^{\cdot}) = K^{\cdot} \otimes (\tau_q L^{\cdot}). \end{aligned}$$

This applies in particular to $K^{\cdot}(U)$, $B^{\cdot}(U)$ and $K^{\cdot}(U_\infty)$. Observe that we have a filtered quasi-isomorphism

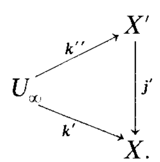
$$(B^{\cdot}(U), \tau'') \to (K^{\cdot}(U_\infty), \tau'')$$

because $\mathrm{Gr}^{\tau''}_q B^{\cdot}(U) \cong B^{\cdot}(X) \otimes_{\mathbb{Q}} \mathrm{Gr}^{\tau}_q K^{\cdot}(X')$ and similarly for $K^{\cdot}(U_\infty)$.

(5.20) **Proposition.** *We have a canonical isomorphism*

$$H^m(U_\infty, \mathbb{Q}) \to \mathbf{H}^m(Y, K^{\cdot}(U_\infty)).$$

Proof. We have a commutative diagram



Because $C^{\cdot}(X')$ is a resolution of $\mathbb{Q}_{X'}$, we have a quasi-isomorphism $k''_* C^{\cdot}(U_\infty) \to k''_* C^{\cdot}(U_\infty) \otimes_{\mathbb{Q}} C^{\cdot}(X')$. Moreover we have a natural morphism of double complexes

$$\alpha\colon K^{\cdot}(X_\infty)\otimes_{\mathbb{Q}} K^{\cdot}(X')\to i^{-1}j'_*(k''_*\,C^{\cdot}(U_\infty)\otimes_{\mathbb{Q}} C^{\cdot}(X')).$$

We claim that α is also a quasi-isomorphism. This is easily checked using local coordinates $(z_0, \ldots, z_n)$ on X such that $f(z_0, \ldots, z_n) = z_0 \ldots z_r$, and D is given by $z_{r+1} \ldots z_{r+s} = 0$ (for suitable r and s). The claim can be reduced to the statement that for such a polydisc coordinate neighborhood V, the map

$$\phi\colon (k')^{-1}(V)\to k^{-1}(V)\times (j')^{-1}(V),$$

defined by $\phi(x) = (x, k''(x))$ is a homotopy equivalence, and application of the Künneth formula.

As a consequence we obtain

$$\begin{aligned} H^m(U_\infty, \mathbb{Q}) &\cong \mathbf{H}^m(Y, i^{-1}k'_*\,C^{\cdot}(U_\infty)) \cong \mathbf{H}^m(Y, i^{-1}j'_*k''_*\,C^{\cdot}(U_\infty)) \\ &\cong \mathbf{H}^m(Y, i^{-1}j'_*(k''_*\,C^{\cdot}(U_\infty))\otimes_{\mathbb{Q}} C^{\cdot}(X')) \cong \mathbf{H}^m(Y, K^{\cdot}(U_\infty)). \quad \square \end{aligned}$$

(5.21) We define the double complex $C^{\cdot\cdot}(U)$ by

$$C^{p,q}(U) = \begin{cases} \rho(B^{p+q+1}(U))(q+1) & \text{if } p \geqq 0; \\ \rho(B^q(U))(q+1)/\operatorname{Ker}(d'\otimes 1) & \text{if } p = -1; \\ 0 & \text{if } p < -1. \end{cases}$$

The filtration $W(D)$ on $C^{\cdot\cdot}(U)$ is induced by the filtration τ'' of $B^{\cdot}(U)$, and we define $M_k C^{p,q}(U)$ as the image of $\tau_{k+2q+1}\,\rho(B^{p+q+1}(U))(q+1)$ in $C^{p,q}(U)$.

(5.22) **Lemma.** i) $\mathrm{Gr}_l^{W(D)} C^{\cdot\cdot}(U) \cong (b_l)_*\, C^{\cdot\cdot}(\tilde{D}^{(l)})(-l)[-l]$,

ii) $\mathrm{Gr}_k^M C^{\cdot\cdot}(U) \cong \bigoplus_{\substack{q\geqq 0\\ q\geqq -k}} \bigoplus_{s>q} (c_{2q+k+1})_*\, \mathbb{Q}_{\tilde{Y}^{(s)}\cap \tilde{D}^{(2q+k+1-s)}}(-k-q)[-k-2q]$.

Here $c_t\colon \widetilde{(Y\cup D)}^{(t)} \to X$.

Proof. From (5.19) we have

$$\begin{aligned} \mathrm{Gr}_l^{\tau''} B^{\cdot}(U) &\cong B^{\cdot}(X)\otimes_{\mathbb{Q}} \mathrm{Gr}_l^{\tau} K^{\cdot}(X') \\ &\cong B^{\cdot}(X)\otimes (b_l)_*\,\mathbb{Q}_{\tilde{D}^{(l)}}(-l)[-l] \\ &\cong B^{\cdot}(\tilde{D}^{(l)})(-l)[-l], \end{aligned}$$

because for any $A\subset X$ one has $C^{\cdot}(X)|_A \xrightarrow{\sim} C^{\cdot}(A)$. Applying the construction of $C^{\cdot\cdot}$ to both sides we obtain the first equality. The second one is a consequence of the splitting

$$\mathrm{Gr}_k^M C^{\cdot\cdot}(U) \cong \bigoplus_{\substack{q\geqq 0\\ q\geqq -k}} \mathrm{Gr}_k^M C^{\cdot q}(U)[-q]$$

and the isomorphisms

$$\mathrm{Gr}_k^M C^{\cdot q}(U)[-q] \cong \bigoplus_{s>q} \mathrm{Gr}_s^{\tau'} K^{\cdot}(X^*)\otimes \mathrm{Gr}_{2q+k+1-s}^{\tau''} K^{\cdot}(X'),$$

$$\mathbb{Q}_{Y_1\cap\ldots\cap Y_s}\otimes_{\mathbb{Q}_X} \mathbb{Q}_{D_1\cap\ldots\cap D_t} = \mathbb{Q}_{Y_1\cap\ldots\cap Y_s\cap D_1\cap\ldots\cap D_t}. \quad \square$$

(5.23) The isomorphism between $(C^{\cdot\cdot}(U)\otimes\mathbb{C}, M)$ and the complex $(A^{\cdot\cdot}, M)$, as defined in (5.5), is defined in a similar way as in the case $D=\emptyset$. One uses the intermediate filtered double complex $\tilde{C}^{\cdot\cdot}(U)\subset C^{\cdot\cdot}(U)\otimes\mathbb{C}$, starting with

$$\tilde{B}^{\cdot}(U)=\mathbf{s}(i^{-1}(\Omega^{\cdot}_X(\log Y)[\log t]\otimes_{\mathbb{C}}\Omega^{\cdot}_X(\log D)).$$

which is in a natural way a subcomplex of $B^{\cdot}(U)\otimes\mathbb{C}$. Then $\tilde{C}^{\cdot\cdot}(U)$ carries natural filtrations M and $W(D)$ such that its inclusion in $C^{\cdot\cdot}(U)\otimes\mathbb{C}$ is a bifiltered quasi-isomorphism. To end the story, we define the mapping $\tilde{C}^{p,q}(U)\xrightarrow{\phi}A^{p,q}$. Recall that by definition, $\tilde{C}^{p,q}(U)$ is a quotient of

$$\bigoplus_{r+s=p+1} i^{-1}\Omega^r_X(\log Y)[u]\otimes_{\mathbb{C}}\Omega^s_X(\log D)$$

$$\oplus\bigoplus_{r+s=p} i^{-1}\Omega^r_X(\log Y)[u]\otimes_{\mathbb{C}}\Omega^s_X(\log D).$$

We put

$$\phi(\sum x'_{ri}u^i\otimes x''_s, \sum y'_{ri}u^i\otimes y''_s)$$

to be the image in $A^{p,q}$ of

$$\sum x'_{r0}\wedge x''_s-2\pi i\, du\wedge\sum y'_{r0}\wedge y''_s.$$

The remaining details are left to the reader.

(5.24) *Remark.* The change of index from s to $l=2q+k+1-s$ in (5.22, ii) gives

$$\mathrm{Gr}^M_k C^{\cdot\cdot}\cong\bigoplus_{q\geq\max\{-k,0\}}\bigoplus_{l\leq q+k}(c_{2q+k-1})_*\mathbb{Q}_{\tilde{Y}^{(2q+k+1-l)}\cap\tilde{D}^{(l)}}.$$

Analogously, for $A^{\cdot\cdot}$ we get

$$\mathrm{Gr}^M_k A^{\cdot\cdot}\cong\bigoplus_l \mathrm{Gr}^M_k\,\mathrm{Gr}^{W(D)}_l A^{\cdot\cdot}.$$

(5.25) **Proposition.** *The natural mapping*

$$H^m(Y-D,\mathbb{C})\to\mathbf{H}^m(C^{\cdot}\otimes\mathbb{C})\cong\tilde{\mathscr{V}}(0)$$

is a morphism of mixed Hodge structures.

Proof. As in [10, (4.27)], we show that

$$D^{\cdot\cdot}=\ker\{\tilde{\nu}\colon A^{\cdot\cdot}\to A^{\cdot\cdot}\}$$

(see (5.5)) is naturally isomorphic, as a bifiltered complex, to the one which is used to define the mixed Hodge structure on $Y-D$. One has

$$D^{p,q}\cong\mathrm{Gr}^{q+1}_{W(Y)}\Omega^{p+q+1}_X(\log(D+Y))\cong\Omega^p_{\tilde{Y}^{(q+1)}}(\log(\tilde{D}\cap\tilde{Y}^{(q+1)})),$$

with d' inducing differentiation, and d'' inducing restriction. For the induced filtrations,

$$F^rD^{p,q}\cong F^r\Omega^p_{\tilde{Y}^{(q+1)}}(\log(\tilde{D}\cap\tilde{Y}^{(q+1)})),$$
$$M_kD^{p,q}\cong W(D)_{k+q}\Omega^p_{\tilde{Y}^{(q+1)}}(\log(\tilde{D}\cap\tilde{Y}^{(q+1)})).$$

From this, we see that we have recovered the mixed Hodge complex for $Y-D$ (one checks that the above is also compatible with the structure over $\mathbb{Q}$). □

As an application of the fact that Conditions (3.13) are satisfied for variations of mixed Hodge structures coming from geometry, we give a neat proof of [15, Lemma (2.6)], which is related to the extension of Abel-Jacobi mappings. Our assumptions will be less restrictive than those in [15]. The same kind of reasoning was independently known to Donagi and Griffiths.

We first state an easy consequence of the existence and functoriality of the limit mixed Hodge structures:

(5.26) **Lemma.** *Let $0 \to \mathbb{V}' \to \mathbb{V} \to \mathbb{V}'' \to 0$ be an exact sequence of variations of mixed Hodge structure over the punctured disc Δ^*, all of which satisfy the Conditions (3.13). Let the canonical extensions of their sheaves of holomorphic sections be denoted by $\tilde{\mathscr{V}}', \tilde{\mathscr{V}}$ and $\tilde{\mathscr{V}}''$ respectively. Then for each p the sequence*

$$0 \to F^p\tilde{\mathscr{V}}' \to F^p\tilde{\mathscr{V}} \to F^p\tilde{\mathscr{V}}'' \to 0$$

is still exact.

Proof. Because $\tilde{\mathscr{V}}'$, $\tilde{\mathscr{V}}$ and $\tilde{\mathscr{V}}''$ are locally free on the disc Δ and each F^p is a subbundle it is enough to show that the sequence

$$0 \to F^p\tilde{\mathscr{V}}'(0) \to F^p\tilde{\mathscr{V}}(0) \to F^p\tilde{\mathscr{V}}''(0) \to 0$$

is exact for each p. But this follows immediately from the fact that we have an exact sequence of the limit mixed Hodge structures, as morphisms of mixed Hodge structures are strictly compatible with F. □

(5.27) *Remark.* A priori, i.e. without (3.13), one knows only that elements of $F^p\tilde{\mathscr{V}}''$ can be lifted to $j_* F^p\mathscr{V}$ and to $\tilde{\mathscr{V}}$ separately.

(5.28) **Proposition.** *Let X be a complex manifold, $f\colon X \to \Delta$ a projective mapping which is smooth outside the fiber over 0. Let $(Z_t)_{t\in\Delta}$ be a family of algebraic $(m-1)$-cycles on X with $|Z_t| \subset X_t = f^{-1}(t)$. Let $\mathbb{V}_t'' = H^{2m-1}(X_t, \mathbb{C})$ and let ω be a section of $F^m\tilde{\mathscr{V}}''$. Let $(\gamma(t))_{t\in\Delta^*}$ be a continuously varying family of relative cycles, $\gamma(t) \in H_{2m-1}(X_t, |Z_t|, \mathbb{Z})$. Then the Abel-Jacobi integral $\int_{\gamma(t)} \omega(t)$ is a multivalued function on Δ^* of the form*

$$\sum_{a,q} f_{a,q}(t)\, t^a (\log t)^q \tag{5.29}$$

with $f_{a,q}$ holomorphic on Δ, and $f_{a,q} \neq 0 \Rightarrow 0 \leqq q \leqq 2m-1$, $a \in \mathbb{Q}$, $0 \leqq a < 1$.

Proof. For $t \in \Delta^*$ one has the exact sequence of mixed Hodge structures

$$H^{2m-2}(|Z_t|) \xrightarrow{u_t} H^{2m-1}(X_t, |Z_t|) \to H^{2m-1}(X_t) \to H^{2m-1}(|Z_t|).$$

Because $|Z_t|$ is analytic of dimension $m-1$, $H^{2m-2}(|Z_t|)$ is purely of type $(m-1, m-1)$ and $H^{2m-1}(|Z_t|) = 0$. Hence if $\mathbb{V}_t' = $ image of u_t, and $\mathbb{V}_t = H^{2m-1}(X_t, |Z_t|)$, we obtain an exact sequence of variations of mixed Hodge structures on Δ^*:

$$0 \to \mathbb{V}' \to \mathbb{V} \to \mathbb{V}'' \to 0,$$

all of which come from geometry, so they satisfy the Conditions (3.13). Because $F^m\tilde{\mathscr{V}}'=0$ we get $F^m\tilde{\mathscr{V}}\cong F^m\tilde{\mathscr{V}}''$ by Lemma (5.26). Hence ω lifts uniquely to a section of $F^m\tilde{\mathscr{V}}$. The proposition now follows from the observation that sections of $\tilde{\mathscr{V}}$ are characterized by the fact that their values on multivalued horizontal sections of $\mathbb{V}^*$ are of the form (5.29). To see this, observe that $\tilde{\mathscr{V}}_0$ has an $\mathcal{O}_{\Delta,0}$-basis $e_1,\dots,e_n$ on which the operator $\nabla_{t\frac{d}{dt}}$ has a *constant* matrix U ([2], proofs of Prop. II.5.2 and II.5.4). A basis of multivalued horizontal sections of $\mathbb{V}$ is given by $f_i=\exp(-\log t\cdot U)e_i$, $i=1,\dots,n$ ([2], Lemma II.1.17.1). Thus $v\in\tilde{\mathscr{V}}_0$ if and only if $v=\sum_{i=1}^{n} g_i\exp(\log t\cdot U)f_i$, with g_i holomorphic. The claim is easily deduced from this. □

§6. On filtered mixed Hodge complexes

Let M be an increasing filtration of a complex $K^\bullet$. We first recall the following definition, adapted from [3, (1.3.3)], of the increasing filtration $\operatorname{Dec} M$:

(6.1) $$(\operatorname{Dec} M)_k K^i=\{x\in M_{k-i}K^i\colon dx\in M_{k-i-1}K^{i+1}\}.$$

Except for a shift depending on i, M and $\operatorname{Dec} M$ define the same filtration on the cocycles of $K^\bullet$, hence on cohomology; precisely,

(6.2) $$(\operatorname{Dec} M)_k H^i(K^\bullet)=\text{image of } H^i(M_{k-i}K^\bullet).$$

(In other words, the use of $\operatorname{Dec} M$ instead of M effects the shift that one sees in the definition, in mixed Hodge theory, of the weight filtration on cohomology: convolution with degree.)

The spectral sequences of $\operatorname{Dec} M$ and M are related by

(6.3) $$\text{i) } {}_{\operatorname{Dec} M}E_0 \quad \text{is quasi-isomorphic to } {}_M E_1,$$
$$\text{ii) } {}_{\operatorname{Dec} M}E_r\cong {}_M E_{r+1} \quad (r\geqq 1)$$

[3, (1.3.4)]. In particular, if the spectral sequence of M degenerates at E_2, that of $\operatorname{Dec} M$ degenerates at E_1.

In [13], the notion of a filtered mixed Hodge complex is introduced. We recall the definition (including the "optional" axiom AIV of El Zein), adopting the abuse of language of [12, p. 126]:

(6.4) *Definition.* A *filtered cohomological mixed Hodge complex* is a cohomological mixed Hodge complex $(K^\bullet, M, F)$ defined over $\mathbb{F}$, together with a third (increasing) filtration W of $K^\bullet$, also defined over $\mathbb{F}$, such that

i) For each l, $\operatorname{Gr}_l^W K^\bullet$, with the filtrations induced by M and F, is a cohomological mixed Hodge complex,

ii) For each l, $\operatorname{Dec}(M\operatorname{Gr}_l^W)$ and $(\operatorname{Dec} M)\operatorname{Gr}_l^W$ coincide on $\tilde{K}^\bullet=R\Gamma(K^\bullet)$,

iii) The spectral sequence of W on $\tilde{K}^\bullet$ degenerates at E_2.

The notion of a *cohomological limit mixed Hodge complex* is obtained by replacing (ii) in the above definition by

ii'). For each l, $W_l K^\bullet$, with the filtrations induced by M and F, is a cohomological mixed Hodge complex.

(6.5) **Proposition** [13, I(2.5)]: *Let* $K^\cdot$ *underlie a filtered cohomological mixed Hodge complex (or a cohomological limit mixed Hodge complex). Then*

i) *The filtration induced by* W *on* $H^i(\tilde{K}^\cdot)$ *is a filtration of mixed Hodge structures.*

ii) *The induced mixed Hodge structure on* $\mathrm{Gr}_k^W H^i(\tilde{K}^\cdot)$ *coincides with the one coming via the isomorphism*

$$(6.6) \qquad {}_W E_\infty(\tilde{K}^\cdot) \simeq {}_W E_2(\tilde{K}^\cdot) \simeq H^\cdot(H^\cdot(\mathrm{Gr}^W \tilde{K}^\cdot), d_1).$$

Proof. According to [4, (8.1.18)] (cf. our (3.17)), (i) is equivalent to the assertion that $\mathrm{Gr}_k^W H^i(\tilde{K}^\cdot)$ becomes a mixed Hodge structure under the filtrations induced by M and F. In other words, we need only check, to prove the proposition, that the "recursive" filtrations induced on ${}_W E_2(\tilde{K}^\cdot)$ by M and F, which provide a mixed Hodge structure by (6.4, i) coincide with the induced filtrations under the above isomorphism (6.6). Now, with the spectral sequences associated to F degenerating at E_1, the filtration F satisfies the conditions of the lemma on two filtrations [3, (1.3.17)]; whereas (6.4, ii) implies that $\mathrm{Dec}\, M$ does likewise, given (6.4, iii). Thus, we are done. (In the "limit" case, the hypothesis (ii′) allows one to bypass the lemma on two filtrations and use a more direct argument.) □

Concerning the somewhat cumbersome condition (6.4, ii), we note that $(\mathrm{Dec}\, M)_k \mathrm{Gr}_l^W \tilde{K}^\cdot$ is represented by elements of W_l that represent cycles in $W_l \mathrm{Gr}_k^M \tilde{K}^\cdot$, and $\mathrm{Dec}(M\, \mathrm{Gr}_l^W)_k \tilde{K}^\cdot$ is represented by elements providing cycles in $\mathrm{Gr}_k^M \mathrm{Gr}_l^W \tilde{K}^\cdot$. It follows that one always has

$$(6.7) \qquad (\mathrm{Dec}\, M)_k\, \mathrm{Gr}_l^W \tilde{K}^\cdot \subseteq \mathrm{Dec}(M\, \mathrm{Gr}_l^W)_k \tilde{K}^\cdot,$$

as is remarked in [13]. One obtains rather easily the following criterion:

(6.8) **Proposition.** *If* $\mathrm{Gr}_k^M W_l K^\cdot \simeq \bigoplus_{j \leq l} \mathrm{Gr}_k^M \mathrm{Gr}_j^W K^\cdot$, *then* (6.7) *is an equality,* i.e. (6.4, ii) *is satisfied. Also,* (6.4, i) *and* (6.4, ii′) *are equivalent if the isomorphism is of F-filtered complexes* (cf. (3.14, ii)).

In view of (4.18, ii) and (5.24), we get:

(6.9) **Corollary.** *The complexes* $\{\mathbf{V}, V\}$ *of* §4 *and* $A^{\cdot\cdot}$ *of* §5 *are filtered cohomological mixed Hodge complexes, as well as cohomological limit mixed Hodge complexes.*

(6.10) *Remark.* In the proof of (6.5), one does not make use of condition (6.4, iii). In fact, the second assertion of the theorem is valid in the "filtered" case independent of where the spectral sequence of W degenerates, for the differentials become morphisms of mixed Hodge structures (see [3, (1.3.13)]).

Appendix

In this appendix, we give the proof, due to Deligne, that Condition (3.13, iii) is in fact a consequence of (3.13, i) and (3.13, ii). We will also derive from his method a proof of the fact that the category of variations of mixed Hodge structure on a smooth curve which satisfy these conditions is closed under tensor product and Hom.

Deligne's approach is as follows. Let $\mathfrak{n}$ be the one-dimensional Lie algebra over the ground field $\mathbb{F}$ with generator N, and let $\mathcal{N}$ denote the category of nilpotent $\mathfrak{n}$-modules.

Given $V_1, V_2 \in \mathcal{N}$, we get an action of $\mathfrak{n}$ on $V_1 \otimes V_2$ by

(1) $$N(v_1 \otimes v_2) = N(v_1) \otimes v_2 + v_1 \otimes N(v_2)$$

and on $\mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$ by

(2) $$N(\varphi) = N \circ \varphi - \varphi \circ N.$$

This corresponds to (1) under the isomorphism $\mathrm{Hom}(V_1, V_2) = V_1^* \otimes V_2$. In particular, $N(\varphi) = 0$ if and only if $\varphi \in \mathrm{Hom}_{\mathcal{N}}(V_1, V_2)$.

In N we have the usual notion of extensions of V_1 by V_2, the isomorphism classes of which we denote by $\mathrm{Ext}_{\mathcal{N}}(V_1, V_2)$.

(A.1) **Proposition.** *There is a canonical identification*

(3) $$\mathrm{Ext}_{\mathcal{N}}(V_1, V_2) = \mathrm{Hom}_{\mathbb{F}}(V_1, V_2) / N\, \mathrm{Hom}_{\mathbb{F}}(V_1, V_2).$$

Proof. Let

(4) $$0 \to V_2 \to V \xrightarrow{\ p\ } V_1 \to 0$$

be an extension in $\mathcal{N}$. Let $s\colon V_1 \to V$ be an $\mathbb{F}$-linear section of p. Then

$$c(s) = N \circ s - s \circ N \in \mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$$

is a measure of the failure of s to be a splitting in $\mathcal{N}$. Any other section of p differs from s by an element φ of $\mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$, and the corresponding homomorphism $c(s+\varphi)$ differs by $N \circ \varphi - \varphi \circ N$, so

$$c(s+\varphi) = c(s) + N(\varphi).$$

The remaining details are left to the reader. □

We need to consider extensions when (finite) filtrations are imposed on the vector spaces in question. Suppose that V_1 and V_2 each have an increasing filtration denoted M and that

$$0 \to V_2 \to V \xrightarrow{\ p\ } V_1 \to 0$$

is an exact sequence of $\mathbb{F}$-vector spaces. The choice of a section s of p determines a filtration M on V by

(5) $$M_i V = M_i V_2 + s(M_i V_1),$$

with the property that it induces the given filtrations on V_1 and V_2. This implies that the sequence remains exact after taking Gr_i^M. Every such filtration on V arises in this way (choose s compatible with M).

We also have a natural filtration M on $\mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$ given by

(6) $$M_i \mathrm{Hom}_{\mathbb{F}}(V_1, V_2) = \{\varphi\colon V_1 \to V_2 \mid \forall j\colon \varphi(M_j V_1) \subset M_{i+j} V_2\}.$$

Now suppose that $V_1, V_2 \in \mathcal{N}$ have increasing filtrations M such that $NM_i \subset M_{i-2}$, and that V is an extension of V_1 by V_2 in $\mathcal{N}$.

(A.2) **Proposition.** *On V there exists a filtration M, inducing the given ones on V_1 and V_2, with $NM_i \subset M_{i-2}$, if and only if there exists a section s of $p\colon V \to V_1$ such that $c(s) \in M_{-2}\, \mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$.*

Proof. Suppose s is such a section; then we can define M as in (5) on V. With respect to the decomposition $V = V_1 \oplus V_2$ the map N has the form

$$\begin{pmatrix} N_{V_1} & 0 \\ c(s) & N_{V_2} \end{pmatrix}$$

so $N(M_i V) \subset M_{i-2} V_2$.

Conversely, if M is given on V, choose s compatible with M. Then $NM_i \subset M_{i-2}$ if and only if $c(s) M_i V_1 \subset M_{i-2} V_2$. □

In order to remove the shift in indices, we make what we call somewhat prematurely a Tate twist, and view N as a morphism of filtered vector spaces; for $V \in \mathcal{N}$ with filtration M:

$$N\colon V \to V(-1)$$

where $V(-1)$ is V with filtration $M_i V(-1) = M_{i-2} V$. This applies as well to $\mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$ with the filtration induced by M, and we rewrite (3) as

(7) $$\mathrm{Ext}_{\mathcal{N}}(V_1, V_2) = \mathrm{Hom}_{\mathbb{F}}(V_1, V_2)(-1)/N\, \mathrm{Hom}_{\mathbb{F}}(V_1, V_2).$$

We give $\mathrm{Ext}_{\mathcal{N}}(V_1, V_2)$ the induced filtration. Thus, the set of extension classes admitting a lifting as in Proposition (A.2) is exactly $M_0 \mathrm{Ext}_{\mathcal{N}}(V_1, V_2)$.

Now we prove the result about the tensor product alluded to in the beginning of this appendix. We first have:

(A.3) **Lemma.** *Let*

$$0 \to V_2 \to V \xrightarrow{p} V_1 \to 0$$

be an exact sequence in $\mathcal{N}$. Suppose W is a finite increasing filtration on V, inducing filtrations W on V_1 and V_2. Suppose that N acts as an admissible nilpotent endomorphism on (V_1, W) and (V_2, W). Let M denote the corresponding relative weight filtrations of N. Suppose that there exists a section s of p compatible with W such that $c(s) = N \circ s - s \circ N$ satisfies

$$c(s)(W_i V_1) \subset W_{i-1} V_2 \quad \textit{and} \quad c(s)(M_i V_1) \subset M_{i-2} V_2$$

for all i. Then N is an admissible endomorphism of (V, W) and its relative weight filtration is given by (5).

Proof. As s is compatible with W, it provides sections s_k of the sequences

$$0 \to \mathrm{Gr}_k^W V_2 \to \mathrm{Gr}_k^W V \xrightarrow{\ p\ } \mathrm{Gr}_k^W V_1 \to 0, \tag{8}$$

which are obviously exact in $\mathcal{N}$. Because $c(s)$ maps M_i to M_{i-2}, the filtration M on V given by (5) satisfies $N(M_i V) \subset M_{i-2} V$. As $c(s)(W_k V_1) \subset W_{k-1} V_2$, we have $c(s_k)=0$, so (8) splits in $\mathcal{N}$. Therefore, M induces on $\mathrm{Gr}_k^W V$ the filtration $L(\mathrm{Gr}_k N)[-k]$, and M is the relative weight filtration of N on V. □

(A.4) **Theorem.** *Suppose that V_1, $V_2 \in \mathcal{N}$ carry finite increasing filtrations W such that N acts as an admissible endomorphism on (V_i, W), $i=1, 2$. Define*

$$W_k(V_1 \otimes V_2) = \sum_{i+j=k} (W_i V_1 \otimes W_j V_2).$$

Then $N \otimes 1 + 1 \otimes N$ is an admissible endomorphism of $V_1 \otimes V_2$, and its relative weight filtration is the filtration induced by those of V_1 and V_2.

Proof. By induction on the length of the filtration W on V_1. If the length of W on V_1 and V_2 is one, then $V_1 \otimes V_2$ is also "pure" and there is nothing to prove.

Suppose that l is such that $0 \subsetneqq W_l V_1 \subsetneqq V_1$. Because M exists on V_1, it exists on $V_1/W_l V_1$ and on $W_l V_1$ and there exists a section s: $V_1/W_l V_1 \to V_1$ which is compatible with M and satisfies $c(s) M_i(V_1/W_l V_1) \subset M_{i-2} V_1$. Observe that automatically

$$c(s)\, W_i \subset W_{i-1},$$

because the weights of $W_l V_1$ are smaller than those of $V_1/W_l V_1$.

Consider the exact sequence in $\mathcal{N}$

$$0 \to (W_l V_1) \otimes V_2 \to V_1 \otimes V_2 \xrightarrow{\ p\ } (V_1/W_l V_1) \otimes V_2 \to 0.$$

Observe that at the ends we may assume by induction that M exists; i.e. N is admissible there. The section $s \otimes 1$ of p has $c(s \otimes 1) = c(s) \otimes 1$ and determines the induced filtration. As it satisfies the requirements of Lemma (A.3), the theorem follows. □

Let us turn to uniqueness of M in (A.2):

(A.5) **Proposition.** *Let V be an extension of V_1 by V_2 in $\mathcal{N}$, where V_1 and V_2 have an increasing filtration M with $NM_i \subset M_{i-2}$. Suppose that the extension class of V lies in $M_0 \mathrm{Ext}_{\mathcal{N}}(V_1, V_2)$. Then the set of liftings of M to V with $NM_i \subset M_{i-2}$ is canonically parametrized by*

$$\frac{\{\varphi \in \mathrm{Hom}_{\mathbb{F}}(V_1, V_2) \colon N(\varphi) \in M_0 \mathrm{Hom}_{\mathbb{F}}(V_1, V_2)(-1)\}}{M_0 \mathrm{Hom}_{\mathbb{F}}(V_1, V_2)}.$$

Proof.
We obtain M on V by choosing a section s with $c(s) \in M_0 \mathrm{Hom}_{\mathbb{F}}(V_1, V_2)(-1)$. When we change s by $\varphi \in \mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$, we will have $c(s+\varphi) \in M_0$ if and only if $N(\varphi) \in M_0$. Moreover the filtrations for s and $s+\varphi$ are the same if and only if $\varphi \in M_0 \mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$. □

(A.6) **Corollary.** *Let $V \in M_0 \operatorname{Ext}_{\mathcal{N}}(V_1, V_2)$. Then the filtration M on V is uniquely determined if and only if the mapping*

$$\frac{\operatorname{Hom}(V_1, V_2)}{M_0 \operatorname{Hom}(V_1, V_2)} \xrightarrow{N} \frac{\operatorname{Hom}(V_1, V_2)(-1)}{M_0 \operatorname{Hom}(V_1, V_2)(-1)}$$

induced by N is injective. □

(A.7) **Corollary.** *M is uniquely determined on V if the mapping*

$$N\colon \operatorname{Hom}(V_1, V_2) \to \operatorname{Hom}(V_1, V_2)(-1)$$

is strictly compatible with M, and $\operatorname{Ker} N \subset M_0 \operatorname{Hom}(V_1, V_2)$. □

From now on the ground field will be $\mathbb{C}$, and the filtrations M will have to be defined over a subfield $\mathbb{F}$ of $\mathbb{R}$.

We suppose that V_1 and V_2 have a second, decreasing, filtration F, such that the mappings $N\colon V_i \to V_i(-1)$ $(i=1,2)$ are compatible with F, i.e. we let $F^p V(-1) = F^{p-1} V$ so $N(F^p V_i) \subset F^{p-1} V_i$. By the arguments used to prove Proposition (A.2) we see that F admits a lifting to the extension V with $N(F^p V) \subset F^{p-1} V$ if and only if the class of V lies in $F^0 \operatorname{Ext}_{\mathcal{N}}(V_1, V_2)$. With that said:

(A.8) **Proposition.** *Suppose that V represents an element of $F^0 \operatorname{Ext}_{\mathcal{N}}(V_1, V_2) \cap M_0 \operatorname{Ext}_{\mathcal{N}}(V_1, V_2)$, and let F be a fixed filtration on V with $NF^p \subset F^{p-1}$, extending those of V_1 and V_2. Further assume*

(a) *the filtration M on V (compatible with N) is uniquely determined;*

(b) *(M, F) define a mixed Hodge structure on V_1 and V_2.*

Then

i) *there exists a splitting $V \simeq V_1 \oplus V_2$ over $\mathbb{C}$ compatible with both F and M;*

ii) *(M, F) define on V a mixed Hodge structure.*

Proof. Let $s\colon V_1 \to V$ be a section compatible with F. Then $c(s) \in F^0 \operatorname{Hom}(V_1, V_2)$. We are given that the extension class also lies in $M_0 \operatorname{Ext}_{\mathcal{N}}(V_1, V_2)$. Now, $\operatorname{Hom}(V_1, V_2)$ inherits from (b) a mixed Hodge structure, and N is a morphism of it to its Tate twist, whence it follows that $\operatorname{Ext}_{\mathcal{N}}(V_1, V_2)$ gets an induced mixed Hodge structure. From this, it follows that

1) the extension V can be represented by an element of

$$(F^0 \cap M_0)[\operatorname{Hom}(V_1, V_2)(-1)];$$

2) the kernel of the mapping

$$F^0 \operatorname{Hom}(V_1, V_2)(-1) \to \operatorname{Ext}_{\mathcal{N}}(V_1, V_2)$$

is $NF^0 \operatorname{Hom}(V_1, V_2)$ by strictness.

Thus, $c(s)$ differs from an element of $(F^0 \cap M_0)$ $[\operatorname{Hom}(V_1, V_2)(-1)]$ by $N(\varphi)$ for some $\varphi \in F^0 \operatorname{Hom}(V_1, V_2)$. On the other hand, we know (cf. Prop. (A.5)) that the splittings of V compatible with F are principal homogeneous under

$F^0 \operatorname{Hom}(V_1, V_2)$. Therefore, the splitting $s+\varphi$ defines both a lifting of M and the given lifting of F on V. With assumption (a), this gives (i).

We wish to conclude that the filtration F induces a Hodge structure of weight k on $\operatorname{Gr}_k^M V$ from the fact that the same holds for V_1 and V_2 by appealing to (3.17). Here, one must proceed carefully. In general, given F on V, it induces two a priori different filtrations (see (3.14, ii)) on $\operatorname{Gr}_k^M V_1$ and $\operatorname{Gr}_k^M V_2$ (we view $V_2 \subset V$ as a little filtration, playing the role of W in (3.14)). For instance, for V_1 they are induced respectively by the subspaces

(10) a) $(F^p + V_2) \cap (M_k + V_2)$, or equivalently

$$(F^p + V_2) \cap M_k \quad \text{or} \quad F^p \cap (M_k + V_2),$$

b) $F^p \cap M_k$.

The first, which gives the filtration induced by F on $\operatorname{Gr}_k^M V_1$ as written, is the one which enters in the mixed Hodge structure; it is the second, however, that is induced by F on $\operatorname{Gr}_k^M V$. Thus, we want to know that the two filtrations coincide. This follows immediately from (i), for we have a bifiltered isomorphism

$$V \simeq V_1 \oplus V_2$$

under which

$$F^p V + V_2 \simeq F^p V_1 \oplus V_2, \quad M_k V + V_2 \simeq M_k V_1 \oplus V_2;$$

then it is clear that (a) and (b) in (10) are equal modulo V_2. The treatment of $\operatorname{Gr}_k^M V_2$ is similar, and we get (ii).

(A.9) **Corollary.** *If a variation of mixed Hodge structure on a smooth curve S satisfies Properties* (3.13, i, ii) *at infinity, then it also possesses Property* (3.13, iii).

Proof. Apply (A.8) to successive quotients of the filtration W. □

(A.10) **Corollary.** *If variations of mixed Hodge structure V_1 and V_2 on a smooth curve both satisfy Properties* (3.13), *then the same holds for $V_1 \otimes V_2$ and* $\operatorname{Hom}(V_1, V_2)$.

Proof. For (3.13, i) see (A.4). For (3.13, ii) there is no problem as the local monodromy is assumed to be unipotent, so taking canonical extensions commutes with tensor product and Hom. □

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Note added in proof

a) Concerning the list of problems at the end of the Introduction:

1. Compare (A.2).

3. It is not hard to see directly, by explicit use of a splitting, that the condition of (A.2) implies that of (2.20).

b) We call the reader's attention to two recent manuscripts:

1. Cattani, E., Kaplan, A., Schmid, W.: Degeneration of Hodge structure

2. Kashiwara, M.: The asymptotic behavior of a variation of a variation of Hodge structure.

In #1, the SL_2-orbit theorem of [9] (see (A) of our Introduction) is generalized to variations of Hodge structure in several variables, and estimates for the Hodge norm are obtained. The latter are derived also in #2, from a somewhat different point of view.