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Inventiones mathematicae

The e-invariant and the spectrum of the Laplacian for compact nilmanifolds covered by Heisenberg groups

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§ 1. Introduction

The main purpose of this paper is to present a natural and explicit family of framed manifolds which represent, via the Thom-Pontrjagin construction, elements of arbitrarily large order in the stable homotopy groups of spheres. In the course of the proof we also give a complete spectral analysis of the classical Laplace operator on these manifolds.

Atiyah and Smith [6] have shown that for compact connected Lie groups of dimension m the e-invariant vanishes for $m \equiv 3 \bmod 4$ and $m > 3$. From this and subsequent work of Knapp, Ossa and others it became increasingly clear that compact Lie groups are far from sufficient to describe the stable homotopy of spheres.

Therefore we consider a more general situation:

Let G be a not necessarily compact oriented Lie group of dimension m with a discrete subgroup Γ such that G/Γ is compact, and let G/Γ be endowed with the parallelization induced by the choice of a Lie algebra basis belonging to the orientation. We denote by $[G/\Gamma]$ the corresponding element in π_m^S. For G we select the nilpotent Lie groups with the simplest representation theory, the so called Heisenberg groups $H(n)$. They can be defined as follows:

$$H(n)=\left\{[x,y,z]=\begin{pmatrix} 1 & x_1 & \cdots & x_n & z \\ & \ddots & & & y_1 \\ & & \ddots & 0 & \vdots \\ & 0 & & \ddots & y_n \\ & & & & 1 \end{pmatrix} \,\middle|\, x_i, y_i, z \in \mathbb{R}\right\}.$$

In $H(n)$ we consider the arithmetic subgroups $\Gamma_k(n)$ defined by $x_i, y_i \in \mathbb{Z}$, $z \in \frac{1}{k}\mathbb{Z}$ for k in $\mathbb{N}$.

Theorem. *Let n be an odd integer. Then:*

$$e[H(n)/\Gamma_k(n)] = \varepsilon(n)\, k^n\, \zeta(-n) + \delta(n)$$

in $\mathbb{Q}/\mathbb{Z}$, *where* ζ *is Riemann's zeta function and*

$$\varepsilon(n) = \begin{cases} -1 & \text{for } n \equiv 3 \bmod 4 \\ \frac{1}{2} & \text{for } n \equiv 1 \bmod 4 \end{cases}, \qquad \delta(n) = \begin{cases} \frac{1}{2} & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases}.$$

Thus $e[H(n)/\Gamma_1(n)]$ is a generator of the image of $e\colon \pi^S_{2n+1} \to \mathbb{Q}/\mathbb{Z}$ for $n \equiv 1 \bmod 4$ and twice a generator for $n \equiv 3 \bmod 4$. This follows from [1] and the formula $\zeta(1-2\nu) = (-1)^\nu \frac{B_\nu}{2\nu}$ for the values of the zeta function at the negative odd integers. By way of contrast, for every arithmetic subgroup Γ of a simply connected nilpotent non-abelian Lie group G the d-invariant of $[G/\Gamma]$ vanishes [15].

The proof of our result is based on the relation between the e-invariant and the η-function of the Dirac-operator established by Atiyah, Patodi and Singer in [4]. In general it is hopeless to determine the spectrum of an elliptic operator explicitly and to calculate the η-invariant directly. Therefore we have to introduce a modified version of the Dirac operator for which the eigenvalue problem can be solved completely. This approach is possible because of the local nature of $\dot{\eta}$ and since our manifold is a covering space of itself with an arbitrarily large number of sheets.

In more detail, the plan of proof is as follows:

We concentrate on the case $k=1$ and set $\Gamma(n) = \Gamma_1(n)$, as the general case requires only trivial modifications. We start out by finding the eigenvalues and eigenfunctions of the Laplace operator on $H(n)/\Gamma(n)$ thus obtaining a basis of $L^2(H(n)/\Gamma(n))$. Unfortunately the Dirac operator D is complicated with respect to the corresponding basis of the Hilbert space of spinors. We simplify D by neglecting the constant terms of this first order elliptic differential operator to obtain an operator $\tilde{D}$. The square of $\tilde{D}$ is closely related to the Laplacian on functions studied in §2. Therefore it becomes possible to solve the eigenvalue problem for $\tilde{D}^2$ explicitly. By considering $\tilde{D}$ on the different eigenspaces of $\tilde{D}^2$ we are able to determine the spectrum of $\tilde{D}$ and finally to derive the value of the e-invariant.

Our interest in this problem was aroused by the work of Seade and Steer [14] which, incidentally, is completed by the special case $n=1$ of our result.

Let $M(p, q, r)$ be the Brieskorn manifold

$$\{(z_0, z_1, z_2) \in \mathbb{C}^3 \mid z_0^p + z_1^q + z_2^r = 0,\ |z_0|^2 + |z_1|^2 + |z_2|^2 = \varepsilon\},$$

$\varepsilon > 0$ small. It is known [10] that there are diffeomorphisms between $H(1)/\Gamma_1(1)$ and $M(6, 3, 2)$, between $H(1)/\Gamma_2(1)$ and $M(4, 4, 2)$ and between $H(1)/\Gamma_3(1)$ and $M(3, 3, 3)$. Brieskorn manifolds admit a canonical framing and with respect to this framing $e[M(p, q, r)] = \frac{\mu+1}{24}$ where $\mu = (p-1)(q-1)(r-1)$ is the Milnor number [13]. The agreement of the e-invariants of the respective homogeneous spaces and Brieskorn manifolds makes it plausible that there exist diffeomorphisms which respect the natural framings.

§ 2. Spectrum and eigenfunctions of the Laplace operator

We begin by describing the irreducible unitary representations of the Heisenberg group $H(n)$. Their knowledge goes back to v. Neumann [11]. Of course they can also be read off from the general representation theory of nilpotent Lie groups [9]:

For $\lambda, \mu \in \mathbb{R}^n$ define the one-dimensional representation $U_{\lambda,\mu}$ of $H(n)$ by

$$U_{\lambda,\mu}([x, y, z]) = \exp(2\pi i(\langle\lambda|x\rangle + \langle\mu|y\rangle))$$

where $\langle\ |\ \rangle$ is the usual scalar product in $\mathbb{R}^n$.
For $\tau \in \mathbb{R}\setminus 0$ define a representation of $H(n)$ in $L^2(\mathbb{R}^n)$ by

$$(U_\tau([x, y, z]) \cdot f)(v) = \exp(2\pi i\tau(z + \langle v|y\rangle)) \cdot f(v + x).$$

In this way we obtain all classes of irreducible unitary representations of $H(n)$. Next we consider the "regular" representation R of $H(n)$ on $L^2(H(n)/\Gamma(n))$ given by $((Rg) \cdot f)(\Gamma(n)g') = f(\Gamma(n)g'g)$. (We denote by G/Γ the cosets Γg.) By results of [7] or [12],

$$R = \sum_{\lambda,\mu \in \mathbb{Z}^n} U_{\lambda,\mu} + \sum_{m \in \mathbb{Z}\setminus 0} |m|^n U_m. \tag{2.1}$$

The Hermite functions are basic for our construction of the eigenfunctions of the Laplace operator. We adopt the following convention:

$$F_\nu(t) = (-1)^\nu e^{t^2/2} \frac{d^\nu}{dt^\nu}(e^{-t^2}) \qquad \text{for } t \in \mathbb{R},\ \nu \geqq 0$$

and

$$F_\nu(t) = 0 \qquad \text{for } \nu < 0.$$

The following identities valid for $\nu \geqq 0$ are classical:

$$\begin{aligned} F_\nu'(t) &= t F_\nu(t) - F_{\nu+1}(t) \\ F_\nu'(t) &= 2\nu F_{\nu-1}(t) - t F_\nu(t) \\ F_\nu''(t) &= t^2 F_\nu(t) - (2\nu+1) F_\nu(t). \end{aligned} \tag{2.2}$$

We use the F_ν to construct functions on $H(n)$ which are left invariant under $\Gamma(n)$ and will also be considered as functions on $H(n)/\Gamma(n)$:

For $k, h \in \mathbb{Z}^n$ let

$$f_{k,h}([x, y, z]) = \exp(2\pi i(\langle k|x\rangle + \langle h|y\rangle))$$

and for $m\in\mathbb{Z}\backslash 0$, $q\in\mathbb{Z}^n$ and $h=(h_1,\dots,h_n)\in\mathbb{N}_0^n$ let

$$g_{q,m,h}([x,y,z])=\exp(2\pi i(mz+\langle q|y\rangle))\cdot\prod_{j=1}^{n}\left[\sum_{k\in\mathbb{Z}}F_{h_j}\left(\sqrt{2\pi|m|}\left(x_j+\frac{q_j}{m}+k\right)\right)\exp(2\pi ikmy_j)\right].$$

For $q\equiv q' \bmod (m\mathbb{Z})^n$ we have $g_{q,m,h}=g_{q',m,h}$.

The Lie algebra $\mathfrak{h}(n)$ of $H(n)$ consists of the matrices

$$[\![u,v,w]\!]=\begin{pmatrix}0 & u_1 & \dots & u_n & w\\ & \ddots & & & v^1\\ & & \ddots & 0 & \vdots\\ & 0 & & \ddots & v_n\\ & & & & 0\end{pmatrix};\qquad u_i,v_i,w\in\mathbb{R}.$$

Our left invariant Riemannian metric on $H(n)$ is defined by considering $[\![u,v,w]\!]$ as element of $\mathbb{R}^{2n+1}$ with the standard scalar product. We endow $H(n)/\Gamma(n)$ with the induced Riemannian structure. The volume element on $H(n)$ is left invariant and thus equals Haar measure which can be chosen to be the standard Lebesgue measure $dx\,dy\,dz$ on $H(n)\approx\mathbb{R}^{2n+1}$. For purposes of integration we note that a fundamental domain for the operation of $\Gamma(n)$ on $H(n)$ is the unit cube in $\mathbb{R}^{2n+1}$. The Laplace operator on functions corresponding to the Riemannian structure is given by:

$$\Delta=\sum_{j=1}^{n}(D_j^2+D_j'^2)+\partial_z^2$$

where $D_j=\frac{\partial}{\partial x_j}$, $D_j'=\frac{\partial}{\partial y_j}+x_j\frac{\partial}{\partial z}$ and $\partial_z=\frac{\partial}{\partial z}$.

A straightforward calculation based on (2.2) yields:

(2.4)
$$\Delta f_{k,h}=-4\pi^2(\|k\|^2+\|h\|^2)f_{k,h}$$
$$\Delta g_{q,m,h}=-2\pi|m|(2h_1+\dots+2h_n+n+2\pi|m|)\,g_{q,m,h}.$$

(2.5) **Theorem.** *The functions $f_{k,h}$ $(k,h\in\mathbb{Z}^n)$ and $g_{q,m,h}$ $(m\in\mathbb{Z}\backslash 0;\ q\in(\mathbb{Z}/m)^n$, $h\in\mathbb{N}_0^n)$ form a complete orthogonal basis of $L^2(H(n)/\Gamma(n))$ consisting of eigenfunctions of Δ and ∂_z.*

Remark. This basis is the union of bases of the representation spaces of the isotypical components $U_{\lambda,\mu}$ and $|m|^n U_m$ of R.

Proof. The orthogonality assertion can be verified using the orthogonality of the Hermite functions F_ν in $L^2(\mathbb{R})$. The proof of completeness will be given in three steps:

First we prove the theorem for $n=1$ by an explicit spectral analysis of Δ. In step two we establish, for $n=1$, the connection with the representation theory. Finally in step three this together with (2.1) is used to derive the theorem in the general case.

Step 1. The functions f on $H(1)/\Gamma(1)$ can be identified with those f on $H(n)$ which satisfy:

$$f(x-1, y, z)=f(x, y, y+z),$$
$$f(x, y, z)=f(x, y+1, z),$$
$$f(x, y, z)=f(x, y, z+1).$$

By the regularity theory of elliptic differential operators the eigenfunctions of Δ have the following form:

(2.6) $$f(x, y, z)=\sum_{\nu_1, \nu_2 \in \mathbb{Z}} A_{\binom{\nu_1}{\nu_2}}(x)\, e^{2\pi i(\nu_1 z+\nu_2 y)}$$

where A_ν: $\mathbb{R}\to\mathbb{C}$ is a C^∞-function with:

(2.7) $$A_{\binom{\nu_1}{\nu_2}}(x-1)=A_{\binom{\nu_1}{\nu_2-\nu_1}}(x).$$

Partial integration shows that for some constant b and all $\nu_1, \nu_2 \neq 0$ the estimate $|A_\nu(x)| \leqq b(\nu_1 \nu_2)^{-1}$ holds. Together with (2.7) this implies the boundary condition:

$$A_\nu(x)\to 0 \quad \text{for } x\to\pm\infty \text{ if } \nu_1 \neq 0.$$

The function f is a solution of $\Delta f=\alpha f$ iff:

$$A_\nu''(x)=(4\pi^2 \nu_1^2 x^2+8\pi^2 \nu_1 \nu_2 x+4\pi^2(\nu_1^2+\nu_2^2)+\alpha)A_\nu(x).$$

For $\nu_1=0$ this yields the eigenfunctions $f_{k,h}$. For $\nu_1 \neq 0$ the substitutions $t=\left(x+\frac{\nu_2}{\nu_1}\right)\sqrt{2\pi|\nu_1|}$ and $C_\nu(t)=A_\nu(x)$ lead to the differential equation

$$C_\nu''(t)-\left(2\pi|\nu_1|+\frac{\alpha}{2\pi|\nu_1|}+t^2\right) C_\nu(t)=0$$

with $C_\nu(t)\to 0$ for $t\to\pm\infty$. This equation is classically studied and appears for example as the radial component of the Schrödinger equation of the harmonic oscillator. Its solutions are the Hermite functions (e.g. [8]). This completes the proof of (2.5) for $n=1$ since the eigenfunctions of elliptic operators are complete.

Step 2. We continue to consider the case $n=1$. Let E_1 (resp. E_2) be the closed subspace of $L^2(H(1)/\Gamma(1))$ generated by the $f_{k,h}$ (resp. $g_{q,m,h}$). Since $\mathbb{C}\cdot f_{k,h}$ is R-invariant, E_1 is also R-invariant. According to step 1, $L^2(H(1)/\Gamma(1))=E_1 \perp E_2$ and thus E_2 is R-invariant as well. In E_2 we consider the closed subspace V_m spanned by the $g_{q,m,h}$ with fixed m. For $m \neq m'$,

$$\langle R(a)\, g_{q,m,h} | g_{q',m',h'}\rangle=0 \quad \text{for all } a \in H(1).$$

Thus every V_m is R-invariant. By R_m we denote the restriction of R to V_m. Since the U_τ are determined by their restriction to the center of $H(1)$ it is obvious that R_m is isotypical of type U_m. According to step 1 and (2.1), $R_m=|m|\cdot U_m$. Let $V_m=V_{m,1}\perp\ldots\perp V_{m,|m|}$ be a complete reduction of R_m. By W_m we denote the vector space of all complex valued functions φ on $\mathbb{R}^2$ such that $e^{2\pi imz}\varphi(x,y)$ lies in V_m. Then we have a corresponding decomposition $W_m = W_{m,1} \oplus \ldots \oplus W_{m,|m|}$.

Step 3. Now let n be arbitrary and let $V_m(n)$ be the closed subspace of $L^2(H(n)/\Gamma(n))$ generated by $g_{q,m,h}$ for fixed m. For $\varphi_1, \ldots, \varphi_n$ in W_m the function

$$\Phi(\varphi_1, \ldots, \varphi_n)\colon [x, y, z] \mapsto e^{2\pi imz}\varphi_1(x_1, y_1)\ldots\varphi_n(x_n, y_n)$$

is in $V_m(n)$. For any map $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, |m|\}$ we denote by Y_σ the closed subspace of $V_m(n)$ generated by all functions $\Phi(\varphi_1, \ldots, \varphi_n)$ with $\varphi_j \in W_{m,\sigma(j)}$. The $|m|^n$ spaces Y_σ are pairwise orthogonal and invariant under R. To see this observe that $H(n)$ is generated by the images of the n canonical embeddings of $H(1)$ in $H(n)$ given by $[x, y, z] \mapsto [xa_i, ya_i, z]$ where $a_1, \ldots, a_n$ is the canonical base of $\mathbb{R}^n$. Now the theorem follows from (2.1).

§ 3. The Dirac operator

Let M be an m-dimensional manifold without boundary and with a trivialization of the tangent bundle given by vector fields $X_1, \ldots, X_m$. They define on M a Riemannian metric and a spin structure, that is a principal $\mathrm{Spin}(m)$-bundle P on M. Writing $m=2n+2$ if m is even and $m=2n+1$ for odd m we have an inclusion of $\mathrm{Spin}(m)$ into the Clifford algebra C_{2n+2} (see [3]). The complexification $C_{2n+2}\otimes\mathbb{C}$ of C_{2n+2} is isomorphic to $\mathrm{End}(W)$, W a complex 2^{n+1}-dimensional vector space. We make the following explicit choice [2]:

We will henceforth assume n odd. For $1 \leq j \leq n+1$ let $Q_j = ie_{2j-1}e_{2j}$. Put $W = \{a \in C_{2n+2}\otimes\mathbb{C} \mid aQ_j = a \text{ for } 1 \leq j \leq n+1\}$. Then $C_{2n+2}\otimes\mathbb{C}$ operates on W from the left. In W we consider the 2^n-dimensional subspaces $S^+ = \{a \in W \mid \omega a = (-1)^{\frac{n+1}{2}} a\}$, where $\omega = e_1 \ldots e_{2n+2}$.

The spinor bundle is defined as $E = P \times_{\mathrm{Spin}(m)} W$. This is a trivial vector bundle. The Levi-Civita connection on M lifts to a principal connection on P which in turn gives a covariant derivative ∇ on E. We consider the tangent bundle of M as a subbundle of the Clifford-algebra bundle C by identifying the vector field X_j with e_j considered as a constant section of C. Then the Dirac operator on M is given by $D(s) = \sum_{j=1}^{m} e_j \nabla_{e_j}(s)$ for sections s of E. For odd m we need another operator D^+ on $E^+ = P \times_{\mathrm{Spin}(m)} S^+ \subset E$ defined as the restriction of $e_{2n+2}D$ to E^+. Then D^+ is elliptic and formally self-adjoint.

The spinors in $\mathrm{Ker}\, D$ are called harmonic spinors. We set:

$$h(D^+) = \tfrac{1}{2}\dim(\mathrm{Ker}\, D) = \dim(\mathrm{Ker}\, D^+).$$

Then the theorem of Atiyah, Patodi and Singer [4], (4.14) which is fundamental for our paper asserts that

$$e[M] = \kappa(m)(\tfrac{1}{2}(h(D^+) + \eta(D^+, 0)) + I(M)) \bmod \mathbb{Z}$$

in $\mathbb{Q}/\mathbb{Z}$. Here $[M] \in \pi_m^S$ is the element in the stable homotopy obtained from the framed manifold M by the Thom-Pontrjagin construction. The η-invariant $\eta(D^+,0)$ is the finite value at $s=0$ of the analytic continuation of

$$\eta(D^+,s)=\sum_{\substack{\lambda\in\sigma(D^+)\\ \lambda\neq 0}}\frac{\operatorname{sgn}\lambda}{|\lambda|^s},\qquad \operatorname{Re}s>\dim M.$$

For our purposes the precise definition of the Chern-Simons invariant $I(M)$ is not needed; all we have to know is that it is local and thus behaves multiplicatively for finite coverings. Finally $\kappa(m)=\frac{1}{2}$ for $m\equiv 3 \bmod 8$ and $\kappa(m)=1$ for $m\equiv 7 \bmod 8$.

In addition to our main object of study $H(n)/\Gamma(n)$ we have to introduce a family of λ^{2n+2} sheeted covering spaces $H(n)/\Gamma(n,\lambda)$ for $\lambda\in\mathbb{N}$. Here $\Gamma(n,\lambda)$ is the subgroup of matrices $[x,y,z]\in H(n)$ where $x\in\lambda\mathbb{Z}^n$, $y\in\lambda\mathbb{Z}^n$, $z\in\lambda^2\mathbb{Z}$ and the covering map onto $H(n)/\Gamma(n)$ is the natural projection. For $M=H(n)/\Gamma(n,\lambda)$ the vector fields X_j are constructed from the (natural) basis

$$[\![a_1,0,0]\!],\ldots,[\![a_n,0,0]\!],[\![0,a_1,0]\!],\ldots,[\![0,a_n,0]\!],[\![0,0,1]\!]\qquad \text{of } \mathfrak{h}(n)$$

by left translation on $H(n)$ and projection to $H(n)/\Gamma(n,\lambda)$. Here $a_1,\ldots,a_n$ is the standard basis of $\mathbb{R}^n$.

A vector field X on $H(n)/\Gamma(n,\lambda)$ is called constant if it is induced by a left invariant vector field on $H(n)$. Correspondingly for spinors; hence the constant spinors can be identified with the elements of S^+. It is clear that $D^{+,\lambda}$, the Dirac operator on $H(n)/\Gamma(n,\lambda)$, maps the space of constant spinors into itself and that:

$$(3.1)\quad D^{+,\lambda}(fc)=fD^{+,\lambda}(c)+\left[\sum_{j=1}^{n}(D_jfe_{2n+2}e_j+D'_jfe_{2n+2}e_{n+j})+\partial_zfe_{2n+2}e_{2n+1}\right]\cdot c$$

for $f\in C^\infty(H(n)/\Gamma(n,\lambda))$ and c a constant spinor.

By $D^{+,\lambda}_{\text{const}}$ we denote the operator of order zero defined by $D^{+,\lambda}_{\text{const}}(fc)=fD^{+,\lambda}(c)$ and by $D^{\sim,\lambda}$ the modified first order differential operator $D^{\sim,\lambda}=D^{+,\lambda}-D^{+,\lambda}_{\text{const}}$.

§ 4. The spectrum of $D^{\sim,\lambda}$

The aim of this section is the proof of the following proposition:

(4.1) **Proposition.** *For n odd (as always) and $\lambda\in\mathbb{N}$ we have:*

$$\eta(D^{\sim,\lambda},0)+\dim(\operatorname{Ker}D^{\sim,\lambda})=(-1)^{\frac{n-1}{2}}2\zeta(-n)+2^n.$$

We fix $\lambda\in\mathbb{N}$ and consider the diffeomorphism $\varphi\colon H(n)/\Gamma(n,\lambda)\to H(n)/\Gamma(n)$ defined by

$$\varphi(\Gamma(n,\lambda)[x,y,z])=\Gamma(n)\left[\frac{x}{\lambda},\frac{y}{\lambda},\frac{z}{\lambda^2}\right].$$

By means of φ we transport the operator $D^{\sim,\lambda}$ defined on $H(n)/\Gamma(n,\lambda)$ to $H(n)/\Gamma(n)$ by setting

$$(P(s))(\bar{g})=(D^{\sim,\lambda}(s\circ\varphi))(\varphi^{-1}(\bar{g})).$$

Then $\sigma(P)=\sigma(D^{\sim,\lambda})$ and for $f\in C^\infty(H(n)/\Gamma(n))$ and any constant spinor on $H(n)/\Gamma(n)$ we have by (3.1):

$$P(fc)=\left[\frac{1}{\lambda}\sum_{j=1}^{n}(D_j f e_{2n+2}e_j+D'_j f e_{2n+2}e_{n+j}) + \frac{1}{\lambda^2}\partial_z f e_{2n+2}e_{2n+1}\right]\cdot c. \tag{4.2}$$

A straightforward calculation gives:

$$P^2(fc)=-\Delta_\lambda(f)\cdot c-\frac{i}{\lambda^2}\partial_z f\cdot\Omega\cdot c \tag{4.3}$$

with

$$\Delta_\lambda=\frac{1}{\lambda^2}\sum_{j=1}^{n}(D_j^2+D_j'^2)+\frac{1}{\lambda^4}\partial_z^2=\frac{1}{\lambda^2}\left(\Delta+\left(\frac{1}{\lambda^2}-1\right)\partial_z^2\right)$$

and

$$\Omega=\sum_{j=1}^{n} i e_j e_{n+j}.$$

In order to find the eigenvectors of P^2 it is, in the light of (2.5), sufficient to diagonalize $\Omega\cdot: S^+\to S^+$.

Left multiplication in S^+ with ie_je_{n+j} will be denoted by α_j for $1\leqq j\leqq n$. The α_j's are commuting involutions, hence there is a decomposition of S^+ in the form: $S^+=\bigoplus_{\underline{\varepsilon}} V_{\underline{\varepsilon}}$, for $\underline{\varepsilon}\in\{+1,-1\}^n$ and

$$V_{\underline{\varepsilon}}=\{v\in S^+\mid \alpha_j v=\varepsilon_j v \text{ for } 1\leqq j\leqq n\}.$$

Setting

$$Y_{\underline{\varepsilon}}=\prod_{\substack{j=1,\dots,n\\ \varepsilon_j=-1}} e_j \qquad \text{if } \#\{j\mid\varepsilon_j=-1\} \text{ is even}$$

and

$$Y_{\underline{\varepsilon}}=\Big(\prod_{\substack{j=1,\dots,n\\ \varepsilon_j=-1}} e_j\Big)\cdot e_{2n+2} \qquad \text{if } \#\{j\mid\varepsilon_j=-1\} \text{ is odd,}$$

we obtain isomorphisms from $V_{\underline{\varepsilon}}$ to $V_{\underline{\delta}}$ by $v\mapsto Y_{\underline{\varepsilon}\cdot\underline{\delta}}\cdot v$. Hence all $V_{\underline{\varepsilon}}$ are one-dimensional and choosing nonzero vectors $v_{\underline{\varepsilon}}$ in $V_{\underline{\varepsilon}}$ we get a basis $\{v_{\underline{\varepsilon}}\}$ of S^+ with $\Omega v_{\underline{\varepsilon}}=\left(\sum_{j=1}^{n}\varepsilon_j\right)v_{\underline{\varepsilon}}$. Applying (2.4) and (4.3) yields:

$$P^2(f_{k,h}v_{\underline{\varepsilon}})=\frac{4\pi^2}{\lambda^2}(\|k\|^2+\|h\|^2)f_{k,h}v_{\underline{\varepsilon}} \tag{4.4}$$

$$P^2(g_{q,m,h}v_{\underline{\varepsilon}})=\left(\frac{2\pi|m|}{\lambda^2}\left(2\sum_{j=1}^{n}h_j+n\right)+\frac{4\pi^2m^2}{\lambda^4} + \frac{2\pi m}{\lambda^2}\left(\sum_{j=1}^{n}\varepsilon_j\right)\right)g_{q,m,h}v_{\underline{\varepsilon}}.$$

(4.5) **Lemma.** *Let $\mathcal{S}_f$ and $\mathcal{S}_g$ denote the closed subspaces of the L^2-spinors generated by the $f_{k,h}\cdot c$ and the $g_{q,m,h}\cdot c$ respectively with constant spinors c. Then $\mathcal{S}_f$ and $\mathcal{S}_g$ are P-invariant and the following holds:*

a) *If $\mathcal{E}$ is any eigenspace of $P^2|\mathcal{S}_f$ then*

$$\mathrm{Trace}(P|\mathcal{E})=0.$$

b) *If $\mathcal{E}$ is any eigenspace of $P^2|\mathcal{S}_g$ then*

$$\mathrm{Trace}(P|\mathcal{E})\in\frac{2\pi}{\lambda^2}\mathbb{Z}.$$

Proof. For $1\leqq j\leqq n$, $e_{2n+2}e_j v_{\underline{\varepsilon}}\in V_{\underline{\delta}}$ and $e_{2n+2}e_{n+j}v_{\underline{\varepsilon}}\in V_{\underline{\delta}}$ where $\underline{\delta}$ differs from $\underline{\varepsilon}$ exactly in the j-th place. Assertion a) thus follows from $\partial_z f_{k,h}=0$. Similarly b) follows from $\partial_z(g_{q,m,h})=2\pi i m\, g_{q,m,h}$ and $e_{2n+2}e_{2n+1}v_{\underline{\varepsilon}}=\pm i v_{\underline{\varepsilon}}$, since $i e_{2n+2}e_{2n+1}$ is an involution on $V_{\underline{\varepsilon}}$.

Proof of (4.1). From (4.4) it is clear that the kernel of P^2 (and hence of P) consists of the constant spinors and hence has dimension 2^n.

If $\mathcal{E}$ is as in (4.5) and if $\mathrm{Trace}(P|\mathcal{E})$ happens to be zero then $\eta(P|\mathcal{E},s)=0$ for all s. Hence:

$$\eta(P,s)=\sum_{\substack{\mathcal{E}\\ \mathrm{Trace}(P|\mathcal{E})\neq 0}}\eta(P|\mathcal{E},s)\qquad\text{for large } \mathrm{Re}\, s.$$

According to (4.5)a) it suffices then to consider an eigenspace $\mathcal{E}$ of $P^2|\mathcal{S}_g$. For $\mathrm{Trace}(P|\mathcal{E})\neq 0$ we have by (4.4):

$$\mathrm{Trace}(P|\mathcal{E})=a\left(\frac{2\pi|m|}{\lambda^2}\left(2\sum_{j=1}^n h_j+n\right)+\frac{4\pi^2m^2}{\lambda^4}+\frac{2\pi m}{\lambda^2}\left(\sum_{j=1}^n\varepsilon_j\right)\right)^{1/2}$$

with $a\in\mathbb{Z}\setminus 0$. On the other hand by (4.5)b)

$$\mathrm{Trace}(P|\mathcal{E})=\frac{2\pi b}{\lambda^2}\qquad\text{with } b\in\mathbb{Z}.$$

Equating gives: $a^2m^2=b^2$ and $|m|\left(2\sum_{j=1}^n h_j+n\right)+m\sum_{j=1}^n\varepsilon_j=0$. Since $h_j\geqq 0$ and $|\sum\varepsilon_j|\leqq n$ this is possible only for $m>0$, $h=0$, $\underline{\varepsilon}=(-1,\dots,-1)$ or for $m<0$, $h=0$, $\underline{\varepsilon}=(1,\dots,1)$. Moreover for $m>0$:

$$P^2(g_{q,m,0}v_{(-1,\dots,-1)})=\frac{4\pi^2m^2}{\lambda^4}g_{q,m,0}v_{(-1,\dots,-1)}$$

and for $m<0$:

$$P^2(g_{q,m,0}v_{(1,\dots,1)})=\frac{4\pi^2m^2}{\lambda^4}g_{q,m,0}v_{(1,\dots,1)}.$$

Thus the only eigenspaces $\mathcal{E}$ of $P^2|\mathcal{S}_g$ with $\mathrm{Trace}(P|\mathcal{E})\neq 0$ belong to the eigenvalues $\frac{4\pi^2m^2}{\lambda^4}$ for $m\in\mathbb{N}$ and are given by:

$$\mathcal{E}_m=\langle\{g_{q,m,0}v_{(-1,\dots,-1)},\, g_{q,-m,0}v_{(1,\dots,1)}|q\in(\mathbb{Z}/m)^n\}\rangle\qquad\text{for } m\in\mathbb{N}.$$

Using the definition of S^+ and $V_{\underline{\varepsilon}}$ we find that:

$$e_{2n+2}\, e_{2n+1}\, v_{(-1,\ldots,-1)} = -i^n\, v_{(-1,\ldots,-1)}$$

and

$$e_{2n+2}\, e_{2n+1}\, v_{(1,\ldots,1)} = i^n\, v_{(1,\ldots,1)}.$$

Hence a consideration of (4.2) reveals that for $m \in \mathbb{N}$:

$$\operatorname{Trace}(P|\mathscr{E}_m) = 2m^n \left(\frac{2\pi m}{\lambda^2}\right)(-1)^{\frac{n-1}{2}}.$$

Since $\dim(\mathscr{E}_m) = 2m^n$ and since the eigenvalues of $P|\mathscr{E}_m$ have absolute value $\frac{2\pi m}{\lambda^2}$ it follows from this equation that they are in fact all equal to $(-1)^{\frac{n-1}{2}} \frac{2\pi m}{\lambda^2}$.

We conclude that for $\operatorname{Re} s$ large:

$$\eta(P,s) = \sum_{m \in \mathbb{N}} \eta(P|\mathscr{E}_m, s) = 2(-1)^{\frac{n-1}{2}} \left(\frac{\lambda^2}{2\pi}\right)^s \sum_{m \in \mathbb{N}} \frac{1}{m^{s-n}}.$$

Hence by analytic continuation:

$$\eta(P,0) = 2(-1)^{\frac{n-1}{2}} \zeta(-n).$$

§ 5. Determination of the *e*-invariant

Returning to § 3 we introduce for every $\lambda \in \mathbb{N}$ a C^∞ one-parameter family of elliptic self-adjoint first order differential operators on $H(n)/\Gamma(n,\lambda)$ by setting

$$D_u^{\sim,\lambda} = D^{\sim,\lambda} + u D_{\text{const}}^{+,\lambda} \quad \text{for } u \in \mathbb{R}. \tag{5.1}$$

By construction these operators for different λ and fixed u come from one and the same operator $D_u^{\sim}$ on the common universal covering $H(n)$. The projection operator pr_u^λ onto $\operatorname{Ker}(D_u^{\sim,\lambda})$ is a self-adjoint pseudo-differential operator with C^∞-kernel. Defining

$$F_u^\lambda = D_u^{\sim,\lambda} + pr_u^\lambda$$

we thus obtain for every $\lambda \in \mathbb{N}$ a C^∞ one-parameter family of invertible elliptic pseudo-differential operators on $H(n)/\Gamma(n,\lambda)$ with the same complete symbol as $D_u^{\sim,\lambda}$. Clearly

$$\eta(F_u^\lambda, 0) = \eta(D_u^{\sim,\lambda}, 0) + \dim(\operatorname{Ker} D_u^{\sim,\lambda}). \tag{5.2}$$

By the remarks preceding (2.12) in Atiyah et al. [5] $\left.\frac{d}{du}\eta(F_u^\lambda, s)\right|_{s=0}$ is a local invariant and hence:

$$\begin{aligned}\eta(F_1^\lambda, 0)-\eta(F_0^\lambda, 0) &= \int_0^1 \left.\frac{d}{du}\eta(F_u^\lambda, s)\right|_{s=0} du \\ &= \lambda^{2n+2}\int_0^1 \left.\frac{d}{du}\eta(F_u^1, s)\right|_{s=0} du \\ &= \lambda^{2n+2}(\eta(F_1^1, 0)-\eta(F_0^1, 0)).\end{aligned}$$

This equality holds in $\mathbb{R}$ and not only in $\mathbb{R}/\mathbb{Z}$.

Applying (5.1) and (5.2) this gives

$$\eta(D^{+,\lambda}, 0)+\dim(\operatorname{Ker}(D^{+,\lambda}))-(\eta(D^{\sim,\lambda}, 0)+\dim(\operatorname{Ker}(D^{\sim,\lambda})))=2c\lambda^{2n+2}$$

where c is a real constant independent of λ.

Invoking (4.1) we get

$$\eta(D^{+,\lambda}, 0)+\dim(\operatorname{Ker}(D^{+,\lambda}))+(-1)^{\frac{n+1}{2}}2\zeta(-n)-2^n=2c\lambda^{2n+2}.$$

Since on the other hand the manifolds $H(n)/\Gamma(n, \lambda)$ are framed diffeomorphic for different λ they have the same e-invariant and hence by the theorem of Atiyah, Patodi and Singer quoted in § 3:

$$\begin{aligned}e[H(n)/\Gamma(n)] &= \kappa(2n+1)(c\lambda^{2n+2}+2^{n-1}+(-1)^{\frac{n-1}{2}}\zeta(-n) \\ &\quad +I(H(n)/\Gamma(n, \lambda))) \bmod \mathbb{Z} \\ &= \kappa(2n+1)((-1)^{\frac{n-1}{2}}\zeta(-n)+2^{n-1})+\lambda^{2n+2}c' \bmod \mathbb{Z}\end{aligned}$$

for all λ and some real constant c' since I is local as well.

A trivial argument implies:

$$e[H(n)/\Gamma(n)]=\kappa(2n+1)((-1)^{\frac{n-1}{2}}\zeta(-n)+2^{n-1}) \bmod \mathbb{Z}.$$

This concludes the proof of the theorem on the e-invariant.

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