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Inventiones mathematicae

Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville

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Introduction

Our purpose is to apply deformation theory to prove (strengthenings of) some conjectures of Beauville and Catanese concerning the vanishing of cohomology groups of generic topologically trivial line bundles on a Kähler manifold. As a corollary, we obtain a bound – suggested by Enriques – on the dimension of the paracanonical system of curves on a surface without irrational pencils.

Let X be a compact, connected Kähler manifold of dimension n, and let $\mathrm{Pic}^0(X)$ denote the identity component of the Picard group of X, which parametrizes topologically trivial holomorphic line bundles on X. We wish to study the cohomology groups $H^i(X,L)$ for a general line bundle $L\in\mathrm{Pic}^0(X)$. To this end, for a given integer $i\geqq 0$ let $S^i(X)\subseteq\mathrm{Pic}^0(X)$ be the analytic subvariety defined by

$$S^i(X)=\{L\in\mathrm{Pic}^0(X)\,|\,H^i(X,L)\neq 0\},$$

and let

$$a: X\to\mathrm{Alb}(X)$$

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denote the Albanese mapping of X. Our first result is

Theorem 1. *One has the inequality*

$$\operatorname{codim}(S^i(X), \operatorname{Pic}^0(X)) \geqq \dim a(X) - i.$$

In particular, if $L \in \operatorname{Pic}^0(X)$ *is a generic line bundle, then* $H^i(X, L) = 0$ *for* $i < \dim a(X)$.

The second statement of the theorem was conjectured by Beauville in [U, p. 620] on the basis of some questions of Catanese [C]. By way of application, recall that $\chi(X, \mathcal{O}_X) = \chi(X, L)$ for every $L \in \operatorname{Pic}^0(X)$ thanks to the fact that the Euler characteristic of a coherent sheaf is a deformation invariant. Hence:

Corollary. *If* X *has maximal Albanese dimension, i.e. if* $\dim a(X) = n$, *then* $(-1)^n \cdot \chi(X, \mathcal{O}_X) \geqq 0$.

We remark that when $i \geqq \dim a(X)$, it can happen that $H^i(X, L) \neq 0$ for every $L \in \operatorname{Pic}^0(X)$.

One may view Theorem 1 as a Kodaira-type vanishing theorem for generic topologically trivial line bundles. It is then natural to ask for analogous results à la Nakano. There are examples where $i + j < \dim a(X)$ but $H^i(X, \Omega^j_X \otimes L) \neq 0$ for every $L \in \operatorname{Pic}^0(X)$, so we consider instead the invariant

$$w(X) \underset{\text{def}}{=} \max\{\operatorname{codim}_X Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega^1_X)\},$$

where $Z(\omega)$ denotes the zero-locus of a holomorphic one-form ω. For instance if X carries a one-form ω with only isolated zeroes – which is automatic e.g. if the Albanese mapping is an immersion – then $w(X) = \dim(X)$.

Theorem 2. *For a generic line bundle* $L \in \operatorname{Pic}^0(X)$,

$$H^i(X, \Omega^j_X \otimes L) = 0 \quad \text{whenever} \quad i + j < w(X).$$

If there is a nowhere vanishing one-form on X, then the Theorem holds with $w(X) = \infty$.

Suppose now that $\dim X = 2$. We prove a variant of Theorem 1 for surfaces which leads to an upper bound on the dimensions of algebraic deformations of a canonical divisor on X, as sought by Enriques [E].

Theorem 3. *Let* X *be an irregular surface without irrational pencils. Then the trivial bundle* $\mathcal{O}_X$ *is an isolated point of* $S^1(X)$, *and consequently any (effectively parametrized) irreducible family of curves on* X *containing at least one canonical divisor has dimension* $\leqq p_g(X)$.

It was this application to surfaces that led Catanese and Beauville to ask about the sort of generic vanishing theorems we prove here (cf. [C]).

These results are proved by studying the deformation theory of the groups $H^i(X, L)$ as L varies, and we are led to consider the following general situation. Let M be a complex manifold and suppose given a bounded complex

$$E^{\cdot}: \ldots \to E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \to \ldots.$$

of holomorphic vector bundles on M. For $y \in M$, denote by $E^\cdot(y)$ the fibrewise complex of vector spaces at y determined by $E^\cdot$: i.e.

$$E^\cdot(y) = E^\cdot \otimes \mathbb{C}(y),$$

where $\mathbb{C}(y)$ is the residue field of M at y. We are interested in the cohomological support loci

$$S^i(E^\cdot) = \{y \in M \mid H^i(E^\cdot(y)) \neq 0\}$$

and

$$S^i_m(E^\cdot) = \{y \in M \mid \dim H^i(E^\cdot(y)) \geqq m\};$$

in particular, we wish to understand these loci infinitesimally. To this end, fix a point $y \in M$, and a tangent vector $v \in T_y M$. Then $E^\cdot$ induces a *derivative complex*

$$D_v(E^\cdot, y) \ldots \to H^{i-1}(E^\cdot(y)) \xrightarrow{D_v(d^{i-1})} H^i(E^\cdot(y)) \xrightarrow{D_v(d^i)} H^{i+1}(E^\cdot(y)) \to \ldots$$

of vector spaces in which the differentials are obtained by differentiating the maps d^i at y in the direction v. We prove

Theorem 4. *Given $y \in S^i_m(E^\cdot)$, let $TC_y(S^i_m(E^\cdot)) \subseteqq T_y M$ denote the tangent cone to $S^i_m(E^\cdot)$ at y. Then*

$$TC_y(S^i_m(E^\cdot)) \subseteqq \{v \in T_y M \mid \dim H^i(D_v(E^\cdot, y)) \geqq m\}.$$

The theorem immediately yields various criteria which enable one to get a handle on the cohomological support loci associated to $E^\cdot$:

(a) Let $m = \dim H^i(E^\cdot(y))$. If there exists a tangent vector $v \in T_y M$ for which either $D_v(d^{i-1}) \neq 0$ or $D_v(d^i) \neq 0$, then $S^i_m(E^\cdot)$ is a proper subvariety of M.

(b) Suppose there is a point $y \in M$ and a tangent vector $v \in T_y M$ such that $H^i(D_v(E^\cdot, y)) = 0$. Then $S^i(E^\cdot)$ is a proper subvariety of M.

(c) If $y \in S^i(E^\cdot)$, and if $H^i(D_v(E^\cdot, y)) = 0$ for *every* non-zero tangent vector $v \in T_y M$, then y is an isolated point of $S^i(E^\cdot)$.

These generalize well-known results for two term complexes $E^0 \to E^1$, which arise for example in Brill-Noether theory (cf. [ACGH]).

Returning now to the situation of Theorems 1–3, recall that one can construct locally on $\mathrm{Pic}^0(X)$ a finite complex $E^\cdot$ of vector bundles with the property that for $y \in \mathrm{Pic}^0(X)$:

$$H^i(E^\cdot(y)) = H^i(X, L_y),$$

where L_y is the line bundle on X corresponding to the point y. Then the derivative complex $D_v(E^\cdot, y)$ associated to a tangent vector

$$v \in T_y \mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)$$

takes the form

$$(*) \qquad \ldots \to H^{i-1}(X, L_y) \to H^i(X, L_y) \to H^{i+1}(X, L_y) \to \ldots,$$

the differentials being cup product with the element $v \in H^1(X, \mathcal{O}_X)$. On the other hand, since L_y is the flat line bundle associated to a one-dimensional unitary representation of $\pi_1(X)$, one can apply Hodge theory with twisted coefficients to interpret this complex. Specifically, v is conjugate to a holomorphic 1-form ω, and (*) is conjugate to

$$(**) \quad \ldots \to H^0(X, \Omega_X^{i-1} \otimes L_y^*) \to H^0(X, \Omega_X^i \otimes L_y^*) \to H^0(X, \Omega_X^{i+1} \otimes L_y^*) \to \ldots,$$

where here the differentials are given by wedge product with ω. Theorem 1 – or at least the second statement – then follows easily from criterion (a) above. For Theorem 2, we take $L_y = \mathcal{O}_X$ and use (b): we show in the spirit of Carrell and Lieberman [CL] that the spectral sequence $E_1^{i,j} = H^j(X, \Omega_X^i)$ associated to the Koszul complex constructed from $\omega: \mathcal{O}_X \to \Omega_X^1$ degenerates at E_2, which implies exactness in (**) and the analogous complexes for H^j. Finally, for Theorem 3 we simply invoke the classical lemma of Castelnuovo concerning holomorphic 1-forms on a surface which wedge to zero, and apply (c) at $y = 0 \in \mathrm{Pic}^0(X)$. We hope that some of these techniques may find other applications in the future.

While the results here are of a somewhat different nature, we remark that the classical vanishing theorems and their generalizations have recently received a considerable amount of attention. The reader may consult the survey [K] for a general overview, and [EV] for a new approach to these theorems.

The paper is organized as follows. In §1 we study the deformation theory associated to a complex of vector bundles on a complex manifold, and prove Theorem 4 above. This is applied in §2 to prove Theorem 1. The Nakano-type Theorem 2 occupies §3, and finally in §4 we give the application to surfaces.

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§0. Notation and conventions

(0.1) Except when mention is made to the contrary, X will denote a compact connected complex manifold on which a Kähler metric has been fixed. In particular, via the Dolbeaut isomorphisms and the Hodge theorem, we may view elements in the cohomology groups $H^j(X, \Omega_X^i)$ as being represented by harmonic (i,j) forms.

(0.2) If V is a complex vector space, and if M is a complex manifold, we denote by V_M the trivial vector bundle on M with fibre V.

(0.3) Let $u: E \to F$ be a map of holomorphic vector bundles on ranks e and f respectively on a complex manifold M. For any integer $a \geqq 0$, we denote by $\mathcal{J}_a(u)$ the ideal sheaf on M locally generated by the determinants of the $a \times a$ minors of u, with the conventions that $\mathcal{J}_0(u) = \mathcal{O}_M$ and $\mathcal{J}_a(u) = 0$ if $a > \min\{e, f\}$.

§1. Deformations of cohomology groups

Our purpose in this section is to prove an elementary but useful result concerning the deformation theory associated to a complex of vector bundles on a complex manifold. The reader will note that everything we do here works just as well for a complex of bundles on a smooth algebraic variety over an algebraically closed ground field of arbitrary characteristic $\neq 2$.

Let M be a complex manifold, and let $E^{\cdot} = \{E^i, d^i\}$ $(0 \leqq i \leqq N)$ be a bounded complex of locally free sheaves on M, with $\operatorname{rk} E^i = e_i$:

$$E^{\cdot}: \ldots \to E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \to \ldots .$$

Given a point $y \in M$, we denote by $E^{\cdot}(y)$ the complex of vector spaces at y determined by the fibres of $E^{\cdot}$; in other words,

$$E^{\cdot}(y) = E^{\cdot}/\underline{m}_y E^{\cdot} = E^{\cdot} \otimes \mathbb{C}(y),$$

where $\underline{m}_y \subset \mathcal{O}_M$ is the maximal ideal of y, and $\mathbb{C}(y) = \mathcal{O}_M/\underline{m}_y$.

Associated to $E^{\cdot}$ are the *cohomological support loci*

$$S^i(E^{\cdot}) = \{y \in M \mid H^i(E^{\cdot}(y)) \neq 0\},$$

and

$$S^i_m(E^{\cdot}) = \{y \in M \mid \dim H^i(E^{\cdot}(y)) \geqq m\}.$$

These are analytic subvarieties of M. We wish to give $S^i_m(E^{\cdot})$ (and hence also $S^i(E^{\cdot}) = S^i_1(E^{\cdot})$) the structures of complex subspaces of M. To this end, note first that

$$\dim H^i(E^{\cdot}(y)) \geqq m \Leftrightarrow \operatorname{rk} d^{i-1}(y) + \operatorname{rk} d^i(y) \leqq e_i - m, \tag{1.1}$$

where we set $d^{-1} = d^N = 0$ by convention. It follows that, as sets:

$$S^i_m(E^{\cdot}) = \bigcap_{\substack{a+b=e_i-m+1 \\ a,b \geqq 0}} \{y \in M \mid \operatorname{rk} d^{i-1}(y) \leqq a-1 \quad \text{or} \quad \operatorname{rk} d^i(y) \leqq b-1\}.$$

Hence we may take the ideal sheaf of $S^i_m(E^{\cdot})$ to be

$$\mathscr{I}(S^i_m(E^{\cdot})) = \sum_{\substack{a+b=e_i-m+1 \\ a,b \geqq 0}} \mathscr{I}_a(d^{i-1}) \cdot \mathscr{I}_b(d^i), \tag{1.2}$$

where as in (0.3) $\mathscr{I}_a(d^{i-1})$ denotes the ideal sheaf locally generated by the $a \times a$ minors of d^{i-1}, and similarly for $\mathscr{I}_b(d^i)$.

We now turn to the study of the infinitesimal behavior of $E^{\cdot}$. Fix a point $y \in M$, and denote by T the holomorphic tangent space to M at y. Let $\mathcal{O} = \mathcal{O}_y M$ be the local ring of M at y, and $\underline{m} = \underline{m}_y \subset \mathcal{O}$ its maximal ideal. The exact sequence of complexes

$$\begin{array}{ccccccccc}
0 \to & \underline{m}E^{\cdot}/\underline{m}^2 E^{\cdot} & \to & E^{\cdot}/\underline{m}^2 E^{\cdot} & \to & E^{\cdot}/\underline{m}E^{\cdot} & \to 0 \\
& \| & & & & \| & \\
& E^{\cdot}(y) \otimes T^* & & & & E^{\cdot}(y) &
\end{array}$$

gives rise to a map

$$D(d^i(y)) : H^i(E^{\cdot}(y)) \to H^{i+1}(E^{\cdot}(y)) \otimes T^*. \tag{1.3}$$

Thus for each tangent vector $v \in T = T_y M$, $D(d^i(y))$ determines a homomorphism

(1.4) $$D_v(d^i(y)) : H^i(E^{\cdot}(y)) \to H^{i+1}(E^{\cdot}(y)),$$

and these vary linearly with v. More concretely, $D_v(d^i(y))$ may be described as follows: after choosing local trivializations of the E^j near y, d^i will be given by a matrix of functions. Differentiating this matrix at y in the direction v yields a linear map $E^i(y) \to E^{i+1}(y)$ which passes to cohomology, inducing the homomorphism in (1.4).

By breaking up the filtered complex

$$E^{\cdot}/\underline{m}^3 E^{\cdot} \supseteq \underline{m}E^{\cdot}/\underline{m}^3 E^{\cdot} \supseteqq \underline{m}^2 E^{\cdot}/\underline{m}^3 E^{\cdot}$$

into short exact sequences, one verifies that $D_v(d^i(y)) \circ D_v(d^{i-1}(y)) = 0$ for every tangent vector $v \in T$. Hence for a given $v \in T$, the homomorphisms $D_v(d^i(y))$ define a complex of vector spaces

$$D_v(E^{\cdot}, y) \colon \ldots \to H^{i-1}(E^{\cdot}(y)) \to H^i(E^{\cdot}(y)) \to H^{i+1}(E^{\cdot}(y)) \to \ldots$$

which we call the *derivative complex of* $E^{\cdot}$ *at* y *in the direction* v. Thinking of T itself as a complex manifold, these directional derivative complexes fit together into a complex of trivial vector bundles on T:

$$D(E^{\cdot}, y) \colon \ldots \to H^{i-1}(E^{\cdot}(y))_T \to H^i(E^{\cdot}(y))_T \to H^{i+1}(E^{\cdot}(y))_T \to \ldots$$

(cf. (0.2)); we call this the *derivative complex of* $E^{\cdot}$ *at* y. In particular, the complex subspaces $S^i_m(D(E^{\cdot}, y)) \subset T$ are defined. Note that

(1.5) $$v \in S^i_m(D(E^{\cdot}, y)) \Leftrightarrow \dim H^i(D_v(E^{\cdot}, y)) \geqq m.$$

Observe also that since the differentials in $D(E^{\cdot}, y)$ are linear with respect to the vector space structure on T, the loci $S^i_m(D(E^{\cdot}, y))$ are cones in T, i.e. algebraic sets defined by homogeneous ideals in $\mathrm{Sym}(T^*)$.

The deformation-theoretic significance of the derivative complex is illustrated by

(1.6) **Theorem.** *Fix a positive integer* m, *and assume that* $y \in S^i_m(E^{\cdot})$. *Then*

$$TC_y(S^i_m(E^{\cdot})) \subseteqq S^i_m(D(E^{\cdot}, y)),$$

where $TC_y(S^i_m(E^{\cdot}))$ *denotes the tangent cone to* $S^i_m(E^{\cdot})$ *at* y.

Since $\dim_y TC_y(S^i_m(E^{\cdot})) = \dim_y S^i_m(E^{\cdot})$, one immediately obtains the following corollaries.

(1.7) **Corollary.** *Set* $m = \dim H^i(E^{\cdot}(y))$. *Then*

$$\operatorname{codim}_y(S^i_m(E^{\cdot}), M) \geqq \operatorname{codim}_T \{v \in T \mid D_v(d^i(y)) = D_v(d^{i-1}(y)) = 0\}.$$

In particular, if either $D_v(d^i(y)) \neq 0$ *or* $D_v(d^{i-1}(y)) \neq 0$ *for some tangent vector* $v \in T$, *then* $S^i_m(E^{\cdot})$ *is a proper subvariety of* M. □

(1.8) **Corollary.** *If* $H^i(D_v(E^{\cdot}, y)) = 0$ *for some point* $y \in M$ *and some tangent vector* $v \in T_y M$, *then* $S^i(E^{\cdot})$ *is a proper subvariety of* M. □

(1.9) **Corollary.** *If* $y \in S^i(E^{\cdot})$, *and if* $H^i(D_v(E^{\cdot}, y)) = 0$ *for every non-zero tangent vector* $v \in T_y M$, *then* y *is an isolated point of* $S^i(E^{\cdot})$. $\square$

Remark. Note that the Corollaries do not involve the structure of $S^i_m(E^{\cdot})$ as a complex space. Along the same lines, observe that the Theorem holds *a fortiori* if one were to consider $S^i_m(E^{\cdot})$ as a reduced analytic space.

Proof of Theorem (1.6). Keeping notation as above, we start by establishing some conventions. First, given vector spaces V and W, we may view a linear map $\delta: V \to W \otimes T^*$ as being given by a matrix with entries in T^*. The (determinants of) the $a \times a$ minors of this matrix are then elements in $\operatorname{Sym}^a T^* = \underline{m}^a/\underline{m}^{a+1}$. We denote by $J_a(\delta)$ the homogeneous ideal of $\operatorname{Sym} T^*$ spanned by these minors, and by $J_a(\delta)_k \subseteq \underline{m}^k/\underline{m}^{k+1}$ the degree k piece of this ideal. Secondly, for any ideal sheaf $\mathscr{J}$ of $\mathcal{O}_M$, set

$$Gr_k(\mathscr{J}) = \mathscr{J} \cap \underline{m}^k / \mathscr{J} \cap \underline{m}^{k+1},$$

and put $Gr(\mathscr{J}) = \oplus Gr_k(\mathscr{J})$. We view $Gr(\mathscr{J})$ as a homogeneous ideal in $Gr(\mathcal{O}) = \oplus \underline{m}^k/\underline{m}^{k+1} = \operatorname{Sym} T^*$.

Note now the following

(1.10) **Lemma.** *Let* $d: E \to F$ *be a map of vector bundles on* M, *giving rise as in* (1.3) *to a "derivative" homomorphism* $\delta: (\ker d(y)) \to (\operatorname{coker} d(y)) \otimes T^*$. *Let* $r = \operatorname{rk} d(y)$. *Then for* $k \geqq r$:

(i) $\mathscr{J}_k(d) \subseteq \underline{m}^{k-r}$

and

(ii) $Gr_{k-r}(\mathscr{J}_k(d)) = J_{k-r}(\delta)_{k-r}$.

Proof. The question being local near y, we assume E and F are trivial. We may choose adapted bases $v_1, \ldots, v_n$ for E and $w_1, \ldots, w_m$ for F such that

$$d(v_i) \equiv w_i \pmod{\underline{m}} \quad \text{for} \quad 1 \leqq i \leqq r,$$

and

$$d(v_i) \equiv \delta(v_i) \in \underline{m}F \pmod{(w_1, \ldots, w_r) + \underline{m}^2 F} \quad \text{for} \quad i > r.$$

For multi-indices L, I of length k, denote by Δ^L_I the determinant of the $k \times k$ minor corresponding to L and I of a given matrix. Then for $i_1 < i_2 < \ldots < i_k$ and $\ell_1 < \ell_2 < \ldots < \ell_k$, one has

$$\Delta^{\ell_1, \ldots, \ell_k}_{i_1, \ldots, i_k}(d) \equiv \Delta^{\ell_{r+1}, \ldots, \ell_k}_{i_{r+1}, \ldots, i_k}(\delta) \pmod{\underline{m}^{k-r+1}}$$

if $i_1 = \ell_1 = 1, \ldots, i_r = \ell_r = r$, and

$$\Delta^{\ell_1, \ldots, \ell_k}_{i_1, \ldots, i_k}(d) \equiv 0 \pmod{\underline{m}^{k-r+1}}$$

otherwise. The lemma follows.

Returning to the proof of the theorem, denote simply by δ^i the map $D(d^i(y))$ defined in (1.3). We assert next:

(1.11) One has

$$Gr_{k-r_i}(\mathscr{J}_k(d^i)) = J_{k-r_i}(\delta^i)_{k-r_i} \quad \text{for} \quad k \geqq r_i \underset{\text{def}}{=} \operatorname{rk} d^i(y).$$

In fact, the differential $d^i: E^i \to E^{i+1}$ gives rise as above to a homomorphism $\underline{\delta}^i: (\ker d^i(y)) \to (\operatorname{coker} d^i(y)) \otimes T^*$ fitting into a commutative diagram

$$\begin{tikzcd} \ker d^i(y) \arrow[r, "\underline{\delta}^i"] \arrow[d] & (\operatorname{coker} d^i(y)) \otimes T^* \\ H^i(E^{\cdot}(y)) \arrow[r, "\delta^i"'] & H^{i+1}(E^{\cdot}(y)) \otimes T^* \arrow[u] \end{tikzcd},$$

where the vertical map on the left is surjective, and that on the right is injective. One verifies that $J_k(\delta^i) = J_k(\underline{\delta}^i)$, so the assertion follows from the Lemma.

Let $h = \dim H^i(E^{\cdot}(y))$. Then, according to (1.2), $S^i_m(D(E^{\cdot}, y))$ is defined in $T = T_y M$ by the homogeneous ideal

$$J = \sum_{\substack{a+b=h-m+1 \\ a,b \geqq 0}} J_a(\delta^{i-1}) \cdot J_b(\delta^i)$$

in $\operatorname{Sym} T^*$. On the other hand, if

$$\mathscr{J} = \sum_{\substack{a+b=e_i-m+1 \\ a,b \geqq 0}} \mathscr{J}_a(d^{i-1}) \cdot \mathscr{J}_b(d^i)$$

denotes the ideal sheaf defining $S^i_m(E^{\cdot})$ in M, then the tangent cone $TC_y(S^i_m(E^{\cdot}))$ is defined in T by the homogeneous ideal $Gr(\mathscr{J}) \subseteq \operatorname{Sym} T^*$ (cf. [ACGH, p. 62]). Since J is generated by homogeneous forms of degree $h-m+1$, the theorem will follow if we prove

$$J_{h-m+1} \subseteqq Gr_{h-m+1}(\mathscr{J}). \tag{1.12}$$

To this end, set $r_k = \operatorname{rk} d^k(y)$, and fix integers $a, b \geqq 0$ such that $a+b=h-m+1$. Then by (1.11):

$$J_a(\delta^{i-1})_a = Gr_a(\mathscr{J}_{a+r_{i-1}}(d^{i-1}))$$

and

$$J_b(\delta^i)_b = Gr_b(\mathscr{J}_{b+r_i}(d^i)).$$

Hence

$$\{J_a(\delta^{i-1}) \cdot J_b(\delta^i)\}_{a+b} \subseteqq Gr_{a+b}(\mathscr{J}_{a+r_{i-1}}(d^{i-1}) \cdot \mathscr{J}_{b+r_i}(d^i)). \tag{$*$}$$

But $r_i + r_{i-1} = e_i - h$ thanks to (1.1), and consequently

$$(a + r_{i-1}) + (b + r_i) = e_i - m + 1.$$

Therefore the product of ideals on the right in $(*)$ is contained in $\mathscr{J}$, and (1.12) follows. This completes the proof of the Theorem. □

Remark. In fact, one has equality in $(*)$ and in (1.12). It follows e.g. that if $m = h^i(E^{\cdot}(y))$, then $S^i_m(D(E^{\cdot}, y))$ is the Zariski tangent space to $S^i_m(E^{\cdot})$ at y.

§2. Generic vanishing criteria for topologically trivial line bundles

In this section we apply the results of §1 to study the cohomology of a generic topologically trivial line bundle on a Kähler manifold, and we prove Theorem 1 stated in the Introduction.

We start by fixing notation. Let X be a compact connected Kähler manifold of dimension n, and let $\mathrm{Pic}^0(X)=H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ be the identity component of the Picard group of X. Given a point $y\in\mathrm{Pic}^0(X)$, we denote by L_y the corresponding topologically trivial line bundle on X. Recall that there exists a Poincaré line bundle $\mathscr{L}$ on $X\times\mathrm{Pic}^0(X)$, i.e. a bundle with the property that for any point $y\in\mathrm{Pic}^0(X)$,

$$\mathscr{L}\,|\,X\times\{y\}=L_y.$$

When X is a complex torus this is proved e.g. in [GH, p. 328], and the existence of a Poincaré line bundle for an arbitrary Kähler manifold then follows from the isomorphism $\mathrm{Pic}^0(X)=\mathrm{Pic}^0(\mathrm{Alb}(X))$ (cf. [GH, p. 332]). Finally, for given integers $i,m\geqq 0$, we denote by $S^i(X)$ and $S^i_m(X)$ the subsets of $\mathrm{Pic}^0(X)$ defined by:

$$S^i(X)=\{y\in\mathrm{Pic}^0(X)\,|\,H^i(X,L_y)\neq 0\}$$

and

$$S^i_m(X)=\{y\in\mathrm{Pic}^0(X)\,|\,\dim H^i(X,L_y)\geqq m\}.$$

It follows from the semicontinuity theorem applied to $\mathscr{L}$ that these are analytic subsets of $\mathrm{Pic}^0(X)$. It is not necessary for our purposes to give these varieties the structures of analytic spaces.

We wish to apply the results of the previous section to study these loci. To this end, the basic fact is that at least locally on $\mathrm{Pic}^0(X)$, the groups $H^i(X,L_y)$ are computed as the pointwise cohomology of a complex of vector bundles. In fact, one has:

(2.1) Let $\mathscr{P}$ be a coherent sheaf on $X\times\mathrm{Pic}^0(X)$, flat over $\mathrm{Pic}^0(X)$, and let $y\in\mathrm{Pic}^0(X)$ be a fixed point. Then there exists a neighborhood $V=V(y)\subseteq\mathrm{Pic}^0(X)$ of y, and a finite complex $E^\cdot$ of vector bundles on V, with the property that for every coherent sheaf $\mathscr{F}$ on V, there is an isomorphism

$$(*)\qquad R^ip_*(\mathscr{P}\otimes p^*\mathscr{F})=H^i(E^\cdot\otimes\mathscr{F}),$$

where $p:X\times V\to V$ is projection onto the second factor. Furthermore, the isomorphism $(*)$ is compatible with short exact sequences of $\mathcal{O}_V$-modules, i.e. it is an isomorphism of δ-functors in the variable $\mathscr{F}$.

This is a special case of [BS, Chap. 3, Theorem 4.1]. (For the analogous result in the algebraic setting, cf. [M, p. 46].)

Given $y\in\mathrm{Pic}^0(X)$, fix once and for all a neighborhood $V=V(y)$ and a complex $E^\cdot$ for which (2.1) applies to the Poincaré bundle $\mathscr{L}$. Taking $\mathscr{F}=\mathcal{O}_V/\underline{m}_y$, it follows in the first place that $H^i(E^\cdot(y))=H^i(X,L_y)$. Therefore one has set-theoretically

$$(2.2)\qquad S^i(X)\cap V=S^i(E^\cdot)\quad\text{and}\quad S^i_m(X)\cap V=S^i_m(E^\cdot).$$

Furthermore, the derivative complexes defined in §1 take in the present situation a very simple form:

(2.3) **Lemma.** *Let $v \in H^1(X, \mathcal{O}_X)$ be a cohomology class corresponding to a tangent vector to* $\mathrm{Pic}^0(X)$ *at* y *under the canonical identification*

$$T_y \operatorname{Pic}^0(X) = H^1(X, \mathcal{O}_X).$$

Then the derivative complex $D_v(E^{\cdot}, y)$ of $E^{\cdot}$ at y in the direction v is identified with the complex

$$\text{(2.4)} \qquad \ldots \to H^{i-1}(X, L_y) \to H^i(X, L_y) \to H^{i+1}(X, L_y) \to \ldots,$$

where the differentials are given by cup product with v.

In particular, Corollaries (1.7), (1.8), and (1.9) apply to this complex.

Proof of Lemma (2.3). Let D be the complex space corresponding to the dual numbers $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, and let $\underline{n} = (\varepsilon)$ be the maximal ideal of $\mathcal{O}_D$. A tangent vector $v \in T_y \operatorname{Pic}^0(X)$ gives rise to a homomorphism $\mathcal{O}_V \to \mathcal{O}_D$ centered at y, and one has the exact sequence

$$\mathcal{F}^{\cdot}: 0 \to \underline{n} \to \mathcal{O}_D \to \mathcal{O}_D/\underline{n} \to 0$$

of $\mathcal{O}_V$-modules. Then, as is well-known, $\mathcal{L} \otimes p^* \mathcal{F}^{\cdot}$ is identified with the sequence

$$0 \to L_y \to \mathcal{L} \otimes p^* \mathcal{O}_D \to L_y \to 0$$

of $\mathcal{O}_X$-modules given by the extension class $v \in H^1(X, \mathcal{O}_X)$. In particular, the corresponding coboundary homomorphism

$$\begin{array}{ccc} \delta: H^i(X, L_y) & \longrightarrow & H^{i+1}(X, L_y) \\ \| & & \| \\ R^i p_*(\mathcal{L} \otimes p^* \mathcal{O}_D/\underline{n}) & & R^{i+1} p_*(\mathcal{L} \otimes p^* \underline{n}) \end{array}$$

is cup product with v. On the other hand, the map $D_v(d^i(y))$ defined in (1.4) is just the connecting homomorphism $H^i(E^{\cdot} \otimes \mathcal{O}_D/\underline{n}) \to H^{i+1}(E^{\cdot} \otimes \underline{n})$ determined by the exact sequence

$$0 \to E^{\cdot} \otimes \underline{n} \to E^{\cdot} \otimes \mathcal{O}_D \to E^{\cdot} \otimes \mathcal{O}_D/\underline{n} \to 0.$$

The Lemma is thus a consequence of the functoriality of the isomorphism (*) in (2.1). □

Remark. The discussion so far is not specific to topologically trivial line bundles on a Kähler manifold. In fact, suppose quite generally given a compact complex space X, a complex manifold M, and a coherent sheaf $\mathcal{P}$ on $X \times M$, flat over M. Fix a point $y \in M$, and let P denote the restrictuon of $\mathcal{P}$ to $X = X \times \{y\}$. Then locally on M, there is a finite complex $E^{\cdot}$ of vector bundles which computes the cohomology of $\mathcal{P}$ as in (2.1). On the other hand, a tangent vector $v \in T_y M$ determines a class

$e_v \in \mathrm{Ext}^1(P,P)$, which in turn gives rise in a natural way to homomorphisms

$$\delta_v : H^i(X,P) \to H^{i+1}(X,P).$$

Then just as in Lemma 2.3, the derivative complex $D_v(E^{\cdot}, y)$ of $E^{\cdot}$ at y in the direction v is identified with the complex

$$\ldots \to H^{i-1}(X,P) \to H^i(X,P) \to H^{i+1}(X,P) \to \ldots,$$

the differentials being the maps δ_v.

The complex (2.4) becomes particularly tractable if it is reinterpreted Hodge-theoretically. To this end, we draw upon the following version of the Hodge theorem:

(2.5) A Kähler metric being fixed on X, let F be the flat unitary vector bundle on X obtained from a unitary representation of $\pi_1(X)$, so that F carries in a natural way a flat Hermitian metric. Then conjugation of harmonic forms determines via the Dolbeaut isomorphisms a complex anti-linear isomorphism

$$(*) \qquad H^i(X, \Omega_X^j \otimes F) \to H^j(X, \Omega_X^i \otimes F^*).$$

Furthermore, if $v \in H^1(X, \mathcal{O}_X)$ is conjugate to a holomorphic 1-form $\omega \in H^0(X, \Omega_X^1)$, then one has a commutative diagram

$$\begin{array}{ccc} H^i(X, \Omega_X^j \otimes F) & \xrightarrow{\cup v} & H^{i+1}(X, \Omega_X^j \otimes F) \\ \downarrow & & \downarrow \\ H^j(X, \Omega_X^i \otimes F^*) & \xrightarrow[\wedge \omega]{} & H^j(X, \Omega_X^{i+1} \otimes F^*) \end{array}.$$

This is proved by doing Kodaira harmonic theory using the given Kähler metric on X, and the flat metric on F. (Cf. Appendix D of [EV] for a more general result.)

Recall now that any line bundle L on X corresponding to a point in $\mathrm{Pic}^0(X)$ is in fact obtained from a unitary character of $\pi_1(X)$: this is well-known when X is a torus (cf. [M, p. 20]), and as above the general case follows. Hence:

(2.6) **Lemma.** *Fix a point $y \in \mathrm{Pic}^0(X)$. Then under the anti-linear isomorphism* $(*)$ *in* (2.5), *the derivative complex* (2.4) *is identified with the complex*

$$(2.7) \quad \ldots \to H^0(X, \Omega_X^{i-1} \otimes L_y^*) \to H^0(X, \Omega_X^i \otimes L_y^*) \to H^0(X, \Omega_X^{i+1} \otimes L_y^*) \to \ldots,$$

where the differentials are given by wedge product with the holomorphic 1-form $\omega \in H^0(X, \Omega_X^1)$ conjugate to $v \in H^1(X, \mathcal{O}_X)$. □

Note that an anti-linear isomorphism between two complexes of vector spaces induces an anti-linear isomorphism of their cohomology groups. In view of (2.2), Corollaries (1.7)–(1.9) then yield:

(2.8) **Proposition.** *Fix a point $y \in \mathrm{Pic}^0(X)$, and set $m = \dim H^i(X, L_y)$.*

(a) *Let $N \subseteq H^0(X, \Omega_X^1)$ be the subspace consisting of all 1-forms ω such that*

$$\omega \wedge \alpha = \omega \wedge \beta = 0 \quad \textit{for every} \quad \alpha \in H^0(X, \Omega_X^{i-1} \otimes L_y^*) \quad \textit{and} \quad \beta \in H^0(X, \Omega_X^i \otimes L_y^*).$$

Then $\mathrm{codim}_y(S_m^i(X), \mathrm{Pic}^0(X)) \geqq \mathrm{codim}(N, H^0(X, \Omega_X^1))$.

(b) *If there exists a point* $y \in \mathrm{Pic}^0(X)$, *and a holomorphic* 1*-form* ω *on* X *for which the sequence*

$$(2.9)\qquad H^0(X, \Omega_X^{i-1} \otimes L_y^*) \xrightarrow{\wedge\omega} H^0(X, \Omega_X^i \otimes L_y^*) \xrightarrow{\wedge\omega} H^0(X, \Omega_X^{i+1} \otimes L_y^*)$$

is exact, then $S^i(X)$ *is a proper subvariety of* $\mathrm{Pic}^0(X)$.

(c) *If* $m > 0$, *and if the sequence* (2.9) *is exact for every non-zero holomorphic* 1*-form* $\omega \in H^0(X, \Omega_X^1)$, *then* y *is an isolated point of* $S^i(X)$. □

We are now set to prove

(2.10) **Theorem.** *Let* X *be a compact Kähler manifold, and let* $a: X \to \mathrm{Alb}(X)$ *be the Albanese mapping of* X. *Then*

$$\operatorname{codim}(S^i(X), \mathrm{Pic}^0(X)) \geqq \dim a(X) - i.$$

Proof. Let $k = \dim a(X)$, and let Y be an irreducible component of $S^i(X)$. Fix a point $y_0 \in Y$ at which $\dim H^i(X, L_y)$ achieves its minimum on Y. Setting $m = \dim H^i(X, L_{y_0})$, it is enough to show that

$$\operatorname{codim}_{y_0}(S^i_m(X), \mathrm{Pic}^0(X)) \geqq k - i.$$

Fix next a non-zero section $\beta \in H^0(X, \Omega_X^i \otimes L_{y_0}^*)$; this is possible since $m = \dim H^0(X, \Omega_X^i \otimes L_{y_0}^*) \geqq 1$. By (2.8)(a), it suffices in turn to prove:

(2.11) The space $W \underset{\text{def}}{=} \{\omega \in H^0(X, \Omega_X^1) \mid \omega \wedge \beta = 0\}$ has codimension $\geqq k - i$ in $H^0(X, \Omega_X^1)$.

To this end, for any $x \in X$ let $W(x) \subset T_x^* X$ be the subspace defined by

$$W(x) = \{v \in T_x^* X \mid v \wedge \beta(x) = 0\},$$

and consider the evaluation homomorphism

$$e: H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega_X^1.$$

If $\omega \in W$ then evidently $\omega(x) \in W(x)$, and hence

$$\dim W - \dim \ker(e(x)) \leqq \dim W(x).$$

On the other hand, e is just the codifferential of the Albanese map, and consequently at a *general* point $x \in X$, $\ker(e(x))$ has codimension k in $H^0(X, \Omega_X^1)$. Thus

$$k - \operatorname{codim}(W, H^0(X, \Omega_X^1)) \leqq \dim W(x)$$

for general $x \in X$, so for (2.11) we are reduced to proving

(2.12) If $x \in X$ is a sufficiently general point, then $\dim W(x) \leqq i$.

But we may certainly assume that $\beta(x) \neq 0$ for general x. Therefore (2.12) – and hence also the Theorem – are consequences of the following elementary point of linear algebra:

Let V be a finite dimensional vector space, let $\beta \in \Lambda^i V$, and let $e_1, \ldots, e_m \in V$ be linearly independent vectors such that $e_\mu \wedge \beta = 0$ for $1 \leqq \mu \leqq m$. If $m > i$, then $\beta = 0$.

In fact, let $n = \dim V$, and complete $e_1, \ldots, e_m$ to a basis $e_1, \ldots, e_n$ of V. If $\beta \neq 0$, then there exists an element $\alpha \in \Lambda^{n-i} V$ such that $\alpha \wedge \beta \neq 0$. Since $n-i > n-m$, every term of α must involve one of the e_μ $(1 \leqq \mu \leqq m)$. Therefore $\alpha \wedge \beta = 0$, a contradiction. □

Remarks. (1) By essentially the same argument, one can prove a "twisted" variant of Theorem 2.10. Specifically, let F be a flat unitary vector bundle on X, and consider the analytic subvariety $S^i(X, F) \subseteqq \mathrm{Pic}^0(X)$ defined by

$$S^i(X, F) = \{ y \in \mathrm{Pic}^0(X) \mid H^i(X, F \otimes L_y) \neq 0 \}.$$

Then $\mathrm{codim}(S^i(X, F), \mathrm{Pic}^0(X)) \geqq \dim a(X) - i$. We leave the details to the interested reader.

(2) An argument similar to the one used to prove (2.10) has an amusing consequence for an *arbitrary* line bundle A on X. In fact, for $y \in \mathrm{Pic}^0(X)$ consider the cup product map

$$\delta_y : H^i(X, A \otimes L_y) \otimes H^1(X, \mathcal{O}_X) \to H^{i+1}(X, A \otimes L_y).$$

Then for any A, *the map* δ_y *is zero for general* $y \in \mathrm{Pic}^0(X)$. Indeed, it follows from Corollary (1.7) that this holds for every $y \in \mathrm{Pic}^0(X)$ at which $h^i(X, A \otimes L_y)$ achieves its minimum on $\mathrm{Pic}^0(X)$.

§3. A Nakano-type generic vanishing theorem

We now apply the results of §1 to obtain criteria for the vanishing of the groups $H^i(X, \Omega_X^j \otimes L_y)$ for generic $y \in \mathrm{Pic}^0(X)$. As before, X denotes a compact connected Kähler manifold, and L_y is the line bundle on X corresponding to a point $y \in \mathrm{Pic}^0(X)$.

We are aiming for the following

(3.1) **Theorem.** *Assume that* X *carries a holomorphic* 1*-form* $\omega \in H^0(X, \Omega_X^1)$ *whose zero locus* $Z = Z(\omega)$ *has codimension* $\geqq k$ *in* X. *Then for generic* $y \in \mathrm{Pic}^0(X)$:

$$H^i(X, \Omega_X^j \otimes L_y) = 0 \quad \textit{whenever} \quad i + j < k.$$

We adopt the convention here and below that if ω is nowhere vanishing, then $k = \infty$. By "generic y" we mean all points in the complement of a proper analytic subvariety $S \subset \mathrm{Pic}^0(X)$.

Remark. One might be tempted to hope that the conclusion of the theorem is valid under the hypothesis $i + j < \dim a(X)$, but this is not the case. In fact, let A be an abelian variety of dimension 4, let $C \subset A$ be a smooth algebraic curve of genus $g \geqq 2$, and let $a : X \to A$ be the blowing up of A along C. Then a is the Albanese mapping of X, and in particular any line bundle in $\mathrm{Pic}^0(X)$ is of the form $a^* L_y$ for some $y \in \mathrm{Pic}^0(A)$. Denote by N the normal bundle to C in A, and by $p : \mathbb{P}(N) \to C$ the canonical projection. Using the exact sequence $0 \to a^* \Omega_A^1 \to \Omega_X^1 \to \Omega_{\mathbb{P}(N)/C}^1 \to 0$, the isomorphism $R^1 p_* \Omega^1_{\mathbf{P}(N)/C} = \mathcal{O}_C$, and the fact that $H^i(A, \Omega^1_A \otimes L_y) = 0$ for $i \geqq 0$ and $0 \neq y \in \mathrm{Pic}^0(A)$, one sees that for all $0 \neq y \in \mathrm{Pic}^0(A)$:

$$H^2(X, \Omega^1_X \otimes a^* L_y) = H^1(C, L_y \otimes \mathcal{O}_C).$$

But $L_y \otimes \mathcal{O}_C$ is a line bundle of degree 0 on C, and hence $H^1(C, L_y \otimes \mathcal{O}_C) \neq 0$ for every $y \in \mathrm{Pic}^0(A)$. We remark that in this example, the zero-locus $Z = Z(\omega)$ of a generic 1-form $\omega \in H^0(X, \Omega^1_X)$ has codimension 3 in X.

To prove the Theorem, we apply the results of §1 to study the analytic subvariety

$$S^i(X, \Omega^j_X) \underset{\mathrm{def}}{=} \{y \in \mathrm{Pic}^0(X) \mid H^i(X, \Omega^j_X \otimes L_y) \neq 0\}$$

locally in a neighborhood of the origin $0 \in \mathrm{Pic}^0(X)$. The set-up is much the same as in §2. Specifically, for a fixed integer j we apply (2.1) to the vector bundle $\mathscr{L} \otimes q^* \Omega^j_X$ on $X \times \mathrm{Pic}^0(X)$, where $q: X \times \mathrm{Pic}^0(X) \to X$ is the projection. Then there exists a neighborhood $V \subset \mathrm{Pic}^0(X)$ of 0 and a bounded complex $E^\cdot$ of vector bundles on V such that

$$S^i(X, \Omega^j_X) \cap V = S^i(E^\cdot).$$

As in Lemma 2.3 one verifies that the directional derivative complex $D_v(E^\cdot, 0)$ corresponding to a tangent vector $v \in H^1(X, \mathcal{O}_X) = T_0 \mathrm{Pic}^0(X)$ takes the form

$$\text{(3.2)} \qquad \ldots \to H^{i-1}(X, \Omega^j_X) \to H^i(X, \Omega^j_X) \to H^{i+1}(X, \Omega^j_X) \to \ldots,$$

where the differentials are given by cup product with v. Using the Hodge conjugate-linear isomorphism $H^i(X, \Omega^j_X) \simeq H^j(X, \Omega^i_X)$, (3.2) is in turn identified with the complex

$$\text{(3.3)} \qquad \ldots \to H^j(X, \Omega^{i-1}_X) \xrightarrow{\wedge \omega} H^j(X, \Omega^i_X) \xrightarrow{\wedge \omega} H^j(X, \Omega^{i+1}_X) \to \ldots,$$

where $\omega \in H^0(X, \Omega^1_X)$ is the holomorphic 1-form conjugate to v. Theorem 3.1 is thus a consequence of Corollary 1.8 and the following

(3.4) **Proposition.** *Let $\omega \in H^0(X, \Omega^1_X)$ be a holomorphic 1-form whose zero-locus $Z = Z(\omega)$ has codimension $\geqq k$ in X. Then the sequence*

$$H^j(X, \Omega^{i-1}_X) \xrightarrow{\wedge \omega} H^j(X, \Omega^i_X) \xrightarrow{\wedge \omega} H^j(X, \Omega^{i+1}_X)$$

is exact whenever $i + j < k$.

To prove (3.4), we use a degeneration argument in the spirit of Carrell and Lieberman [CL]. Specifically, a holomorphic 1-form $\omega \in H^0(X, \Omega^1_X)$ determines a Koszul complex

$$K^\cdot: 0 \to \mathcal{O}_X \xrightarrow{\wedge \omega} \Omega^1_X \xrightarrow{\wedge \omega} \ldots \xrightarrow{\wedge \omega} \Omega^n_X \to 0$$

of vector bundles on X, where $n = \dim X$. Index $K^\cdot$ by setting $K^i = \Omega^i_X$, and denote by $\mathscr{H}^j(K^\cdot)$ the j^{th} cohomology sheaf of $K^\cdot$. Then one has two spectral sequences

$$\text{(3.5)} \qquad {}'E_1^{i,j} = H^j(X, K^i) \Rightarrow \mathbf{H}^*(X, K^\cdot)$$

and

(3.6) $$''E_2^{i,j}=H^i(X,\mathscr{H}^j(K^\cdot))\Rightarrow \mathbf{H}^*(X,K^\cdot)$$

abutting to the hypercohomology of $K^\cdot$. We prove below

(3.7) **Proposition.** *For any holomorphic* 1*-form* $\omega\in H^0(X,\Omega_X^1)$, *the first spectral sequence* $'E_1^{i,j}=H^j(X,\Omega_X^i)$ *degenerates at* E_2, *i.e.* $'E_2={}'E_\infty$.

Granting this for the time being, the next point is the elementary

(3.8) **Lemma.** *Let F be a vector bundle of rank $n=\dim X$ on X, and let $s\in H^0(F)$ be a section of F whose zero-locus $Z=Z(s)$ has codimension $\geqq k$ in X. Denote by $K^\cdot$ the Koszul complex constructed from $s:\mathcal{O}_X\to F$, indexed so that $K^i=\Lambda^iF$. Then $\mathscr{H}^j(K^\cdot)=0$ for $j<k$.*

Proof. Since the sheaves $\mathscr{H}^j(K^\cdot)$ are supported on Z, it is enough to show that the stalks $\mathscr{H}^j(K^\cdot)_x$ vanish when $j<k$ for any given point $x\in Z$. To this end, let $\mathcal{O}_x$ denote the local ring of X at x, and $\underline{m}$ its maximal ideal. Upon trivializing F near x, the section s will be given by n functions $f_1,\dots,f_n\in\underline{m}$ which generate the ideal of Z in $\mathcal{O}_x$. Since Z has dimension $\leqq n-k$, after possibly transforming the vector $(f_1,\dots,f_n)$ by an element of $GL(n,\mathbb{C})$, we can assume that $f_1,\dots,f_k$ locally cut out near x a subvariety of dimension $n-k$. Since $\mathcal{O}_x$ is a regular local ring, it follows that $f_1,\dots,f_k$ is a regular sequence. On the other hand, the complex of $\mathcal{O}_x$-modules determined stalkwise by $K^\cdot$ is isomorphic to the Koszul complex $K(f_1,\dots,f_n)$ constructed from the sequence of elements $f_1,\dots,f_n\in\underline{m}$. And since k of these elements form a regular sequence, it is a consequence of standard arguments (cf. [Mt, p. 135]) that $H^j(K(f_1,\dots,f_n))=0$ for $j<k$. □

Still granting Proposition 3.7, Proposition 3.4 follows at once. In fact, suppose given a holomorphic 1-form $\omega\in H^0(X,\Omega_X^1)$ whose zero-locus has codimension $\geqq k$ in X. Denoting by $K^\cdot$ the corresponding Koszul complex, one has $\mathscr{H}^j(K^\cdot)=0$ for $j<k$ by the preceding lemma. It follows from the spectral sequence (3.6) that $\mathbf{H}^m(X,K^\cdot)=0$ for $m<k$. On the other hand, the first spectral sequence (3.5) degenerates at E_2 thanks to (3.7), and therefore $'E_2^{i,j}=0$ for $i+j<k$. But this means precisely that the sequence

$$H^j(X,\Omega_X^{i-1})\xrightarrow{\wedge\omega}H^j(X,\Omega_X^i)\xrightarrow{\wedge\omega}H^j(X,\Omega_X^{i+1})$$

is exact whenever $i+j<k$.

It remains to give the

Proof of Proposition (3.7). Denote by $A^{i,j}(X)$ the space of all smooth (i,j)-forms on X. We give a direct argument based on the following version of the principle of two types:

(3.9) Let $b\in A^{i,j}(X)$ be a smooth (i,j)-form on X which is both ∂- and $\bar\partial$-closed. If b is either ∂-exact or $\bar\partial$-exact, then

$$b=\partial\bar\partial c=-\bar\partial\partial c$$

for some $c\in A^{i-1,j-1}(X)$. (Cf. [D, Corollary 5.4]).

We first show that the E_2 differentials of the spectral sequence (3.5) vanish. To this end, let $b \in A^{i,j}(X)$ be a harmonic form representing an element in

$$\ker\{H^j(X,\Omega_X^i) \xrightarrow{\wedge\omega} H^j(X,\Omega_X^{i+1})\}.$$

Then $b \wedge \omega = \bar{\partial} b_2$ for some $b_2 \in A^{i+1,j-1}(X)$, and the E_2 differential $d_2([b])$ of $[b]$ is represented by the class $[b_2 \wedge \omega] \in H^{j-1}(X,\Omega_X^{i+2})$; i.e.

$$(*) \qquad d_2([b]) = [b_2 \wedge \omega] \in \frac{\ker\{H^{j-1}(X,\Omega_X^{i+2}) \xrightarrow{\wedge\omega} H^{j-1}(X,\Omega_X^{i+3})\}}{\operatorname{im}\{H^{j-1}(X,\Omega_X^{i+1}) \xrightarrow[\wedge\omega]{} H^{j-1}(X,\Omega_X^{i+2})\}},$$

this being independent of the choice of b_2. Now since b is harmonic and ω is holomorphic (and hence harmonic), one has

$$\partial(b \wedge \omega) = \bar{\partial}(b \wedge \omega) = 0.$$

And since $b \wedge \omega$ is $\bar{\partial}$-exact by assumption, (3.9) applies to give

$$(**) \qquad b \wedge \omega = \bar{\partial}\partial c_1 \quad \text{for some} \quad c_1 \in A^{i,j-1}(X).$$

Then we can take $b_2 = \partial c_1$ in (*), so it is enough to show that $\partial c_1 \wedge \omega$ is $\bar{\partial}$-exact. But note that

$$(***) \qquad \partial c_1 \wedge \omega = \partial(c_1 \wedge \omega),$$

and hence $\partial c_1 \wedge \omega$ is ∂-closed. It follows from (**) that $\partial c_1 \wedge \omega$ is $\bar{\partial}$-closed, and so (3.9) applies thanks to (***) to yield

$$\partial c_1 \wedge \omega = \bar{\partial}\partial c_2$$

for some $c_2 \in A^{i+1,j-2}(X)$. In particular $\partial c_1 \wedge \omega$ is $\bar{\partial}$-exact, and thus $d_2([b]) = 0$.

Assume now inductively that $d_k([b]) = 0$ for $2 \leqq k \leqq r-1$, and that one can construct forms $c_k \in A^{i-1+k,j-k}(X)$ such that

$$\partial c_{k-1} \wedge \omega = \bar{\partial}\partial c_k$$

for $2 \leqq k \leqq r-1$. Then $d_r([b])$ is the image in $E_r^{i+r,j-r+1} = E_2^{i+r,j-r+1}$ of the cohomology class $[\partial c_{r-1} \wedge \omega] \in H^{j-r+1}(X,\Omega_X^{i+r})$. Now

$$\partial c_{r-1} \wedge \omega = \partial(c_{r-1} \wedge \omega),$$

and it follows as above that $\partial c_{r-1} \wedge \omega$ is both ∂- and $\bar{\partial}$-closed. Thus

$$\partial c_{r-1} \wedge \omega = \bar{\partial}\partial c_r,$$

and in particular $[\partial c_{r-1} \wedge \omega] = 0$. This completes the induction. $\square$

Remarks. (1) Let F be a flat unitary vector bundle on X. Given a holomorphic 1-form $\omega \in H^0(X,\Omega_X^1)$, one can construct a twisted Koszul complex

$$K^\cdot \otimes F\colon 0 \to F \xrightarrow{\wedge\omega} F \otimes \Omega_X^1 \xrightarrow{\wedge\omega} \dots \xrightarrow{\wedge\omega} F \otimes \Omega_X^n \to 0.$$

Then Proposition 3.7 remains valid for this complex, and it follows from (3.8) that $\mathscr{H}^j(K^{\cdot}\otimes F)=0$ for $j<\operatorname{codim}_X(Z(\omega))$. This has two consequences. First, consider the analytic subvariety $S^i(X,\Omega_X^j\otimes F)$ of $\operatorname{Pic}^0(X)$ defined by

$$S^i(X,\Omega_X^j\otimes F)=\{y\in\operatorname{Pic}^0(X)\,|\,H^i(X,\Omega_X^j\otimes F\otimes L_y)\neq 0\}.$$

By virtually the same argument as above, one shows that if X carries a holomorphic form ω whose zero-locus has codimension $\geqq k$ in X, then $S^i(X,\Omega_X^j\otimes F)$ is a proper subset of $\operatorname{Pic}^0(X)$ whenever $i+j<k$. Secondly, suppose that $m(X)$ is is the *least* codimension of the zero-locus of a non-zero holomorphic differential ω on X. Then by applying Corollary 1.9 to the analogue of (3.3) with Ω_X^i replaced by $\Omega_X^i\otimes L_y^*$ one sees that:

$S^i(X,\Omega_X^j)$ consists of finitely many isolated points whenever $i+j<m(X)$.

For example, let A be an abelian variety, and let $X\subset A$ be a smooth subvariety with ample normal bundle. Then every non-zero $\omega\in H^0(X,\Omega_X^1)$ vanishes at only finitely many points, i.e. $m(X)=\dim(X)$. (In fact, on the zero-locus of any non-vanishing one-form, the normal bundle $N_{X/A}$ has a trivial quotient.) Hence $S^i(X,\Omega_X^j)$ is finite for $i+j<\dim X$.

(2) When X is a surface, I. Reider has proven the injectivity of the map

$$H^1(X,\mathcal{O}_X)\xrightarrow{\wedge\omega}H^1(X,\Omega_X^1)$$

under the assumption that ω does not pull back from a curve C of genus $\geqq 2$ via an irrational pencil $f\colon Y\to C$. It was this result that led us to suspect that Proposition 3.4 should be true.

§4. An application to surfaces

In this section we take X to be a compact, connected Kähler manifold of dimension 2. Recall that an *irrational pencil of genus* g on X is a surjective holomorphic mapping $f\colon X\to C$, where C is smooth algebraic curve of genus $g\geqq 1$. We denote by ω_X the canonical line bundle on X, and as usual we set $p_g(X)=\dim H^0(X,\omega_X)$ and $q(X)=\dim H^1(X,\mathcal{O}_X)$.

(4.1) **Proposition.** *Assume that $q(X)>0$, and that X does not carry any irrational pencils of genus $\geqq 2$. Then the trivial bundle $\mathcal{O}_X$ is an isolated point of*

$$S^1(X)=\{y\in\operatorname{Pic}^0(X)\,|\,H^1(X,L_y)\neq 0\}.$$

Proof. By Proposition (2.8)(c), it is enough to show that the sequence

$$H^0(X,\mathcal{O}_X)\xrightarrow{\wedge\omega}H^0(X,\Omega_X^1)\xrightarrow{\wedge\omega}H^0(X,\omega_X)$$

is exact for any non-zero holomorphic 1-form $\omega\in H^0(X,\Omega_X^1)$. But this follows immediately from the hypothesis and a classical lemma of Castelnuovo (cf. [BPV, Proposition IV.4.1]), which asserts that if ω and α are linearly independent holomorphic 1-forms on X such that $\omega \wedge \alpha = 0$, then ω and α pull back from a curve C of genus $\geqq 2$ under an irrational pencil $f: X \to C$. $\square$

Let $\mathrm{Pic}^{[\omega]}(X)$ denote the component of $\mathrm{Pic}(X)$ containing the point $[\omega_X] \in \mathrm{Pic}(X)$, and let $\mathrm{Div}^{[\omega]}(X)$ denote the space of all effective divisors on X associated to a line bundle in $\mathrm{Pic}^{[\omega]}(X)$. Thus one has a natural map

$$p: \mathrm{Div}^{[\omega]}(X) \to \mathrm{Pic}^{[\omega]}(X).$$

One then defines the *paracanonical system* $\{K_X\}$ of X to be the union of those irreducible components of $\mathrm{Div}^{[\omega]}(X)$ which contain the complete linear system $p^{-1}([\omega_X]) = |K_X|$ of all canonical divisors on X (cf. [C] or [E]). Enriques [E, pp. 354ff.] raised the question of finding criteria under which $\dim\{K_X\} \leqq p_g(X)$. Proposition 4.1 immediately leads to the following solution of Enriques' problem, as conjectured by Catanese [C].

(4.2) **Theorem.** *Assume that X does not carry any irrational pencils of genus $g \geqq 2$, and let $Z \subseteq \mathrm{Div}^{[\omega]}(X)$ be any irreducible family of curves on X which contains at least one canonical divisor. Then*

$$\dim Z \leqq p_g(X).$$

In particular, $\dim\{K_X\} \leqq p_g(X)$.

Proof. Given $z \in Z$, denote by $D_z \subseteq X$ the corresponding effective divisor on X, and let $q: Z \to \mathrm{Pic}^{[\omega]}(X)$ be the natural map. The hypothesis on Z means that $[\omega_X] \in q(Z)$. If $Z \subseteq |K_X|$ then the assertion of the Theorem is clear. Otherwise, since $\omega_X(-D_z)$ lies in $\mathrm{Pic}^0(X)$, it follows from Serre duality and Proposition 4.1 that if $z \in Z$ is a general point, then

$$\dim H^1(X, \mathcal{O}_X(D_z)) = \dim H^1(X, \omega_X(-D_z)) = 0.$$

Furthermore, if $q(z) \neq [\omega_X]$ then evidently $h^2(X, \mathcal{O}_X(D_z)) = h^0(X, \omega_X(-D_z)) = 0$. Hence for general $z \in Z$:

$$\dim H^0(X, \mathcal{O}_X(D_z)) = \chi(X, \mathcal{O}_X(D_z)) = \chi(X, \omega_X) = 1 - q(X) + p_g(X).$$

But for any $z \in Z$, one has

$$\dim q^{-1}(q(z)) \leqq \dim |D_z| = \dim H^0(X, \mathcal{O}_X(D_z)) - 1,$$

and since $\dim \mathrm{Pic}^{[\omega]}(X) = q(X)$, it follows that $\dim Z \leqq p_g(X)$, as required. $\square$

Remarks. (1) Beauville has given a construction of surfaces X of fixed irregularity, without irrational pencils, on which there exists a line bundle $\eta \in \mathrm{Pic}^0(X)$ with $H^1(X, \eta)$ arbitrarily large. This leads to examples where the inequality in Theorem 4.2 fails for some irreducible component of $\mathrm{Div}^{[\omega]}(X)$. (Of course the component in question doesn't contain any canonical divisors.) Beauville's examples are discussed in [C, pp. 103ff.].

(2) We suspect that if X doesn't carry any irrational pencils, then $S^1(X)$ is finite. More generally, for an arbitrary Kähler surface X it seems plausible that any component of $S^1(X)$ of positive dimension consists of line bundles $L \in \mathrm{Pic}^0(X)$ pulled back from a curve C of genus $g \geqq 2$ under an irrational pencil $f: X \to C$.

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Note added in proof

Beauville has answered affirmatively the questions raised in Remark 2 at the end of §4.