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General linear methods: connection to one step methods and invariant curves

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Summary. We generalize a result of Kirchgraber (1986) on multistep methods. We show that every strictly stable general linear method is essentially conjugate to a one step method of the same order. This result may be used to show that general properties of one step methods carry over to general linear methods. As examples we treat the existence of invariant curves and the construction of attracting sets.

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In this paper we deal with general properties of integration methods. We show how such general properties of one step methods carry over to multistep methods and, even more general, to general linear methods. We discuss the following two examples of such properties.

Property A. Assume that the differential equation $y'=f(y)$ admits a hyperbolic periodic solution. Beyn (1987) and Eirola (1988) showed that for every one step method there exists a hyperbolic invariant closed curve near the orbit of the periodic solution [see also Brown and Hershenov (1977) and others].

Property B. Assume that the differential equation $y'=f(y)$ admits a compact, uniformly asymptotically stable set Λ (an attractor). Apply a one step method with step size h to the differential equation. Kloeden and Lorenz (1986) showed that for every small h one can find a compact, uniformly asymptotically stable set $\Lambda(h)$ containing Λ such that $\Lambda(h)\to\Lambda$ with respect to the Hausdorff metric as $h\to 0$.

Later, these two properties of one step methods were also shown to hold for multistep methods, see Eirola and Nevanlinna (1988) for Property A and Kloeden and Lorenz (1990) for Property B.

Our approach is different from the construction of Eirola and Nevanlinna. We use a result of Kirchgraber (1986) stating that every strictly stable multistep method is essentially equivalent to a one step method. This equivalence allows to show that Property A and Property B carry over to multistep methods.

First let us generalize the result of Kirchgraber to general linear methods. These methods include linear multistep methods, predictor-corrector methods in the $P(EC)^m E$ and $P(EC)^m$ mode, split linear multistep methods as discussed in Cash (1983) and Voss and Casper (1989), cyclic multistep methods and many more. The generalization to general linear methods is not trivial because of the difficulties encountered when defining the order of general linear methods. Using the fact that any strictly stable general linear method is essentially conjugate to some one step method of the same order, we give a new and more general proof that Properties A and B hold for these methods.

1. Basic assumptions and definitions

Consider the differential equation

$$\frac{dy}{dt}=f(y), \qquad y\in\mathbb{R}^d \tag{1.1}$$

where f is of class C^{p+k}, $p\geqq 1$, $k\geqq 0$, and f and all its derivatives up to order $p+k$ are continuous and bounded. We denote the solution of Eq.(1.1) by $\varphi(t, y)$. A *one step method of order* p is a map Φ: $\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ such that the *local error*

$$l(h, y)=\Phi(h, y)-\varphi(h, y)$$

may be estimated by $|l(h, y)|\leqq c h^{p+1}$ for some constant c and sufficiently small h. We consider a *general linear method* C in its partitioned form, i.e. a map

$$C:\mathbb{R}\times\mathbb{R}^{rd}\to\mathbb{R}^{rd}$$

defined as follows. Let $y^{(n)}\in\mathbb{R}^{rd}$ be the vector consisting of the vectors $y_1^{(n)}, y_2^{(n)}, \ldots, y_r^{(n)}\in\mathbb{R}^d$. The vector $y^{(n)}$ is the information passed from step n to step $n+1$. In order to compute $y^{(n+1)}$, s internal stages are performed and some quantities $Y_1^{(n)}, Y_2^{(n)}, \ldots Y_s^{(n)}$ are computed. These vectors are not needed again after the step is completed. The method C is defined by

$$\begin{aligned} Y^{(n)}&=(C_{11}\otimes I)\,hF(Y^{(n)})+(C_{12}\otimes I)\,y^{(n)}\\ y^{(n+1)}&=(C_{21}\otimes I)\,hF(Y^{(n)})+(C_{22}\otimes I)\,y^{(n)} \end{aligned} \tag{1.2}$$

where $F(Y^{(n)})$ denotes the vector in $\mathbb{R}^{sd}$ consisting of $f(Y_1^{(n)}), \ldots, f(Y_s^{(n)})\in\mathbb{R}^d$. For given $y^{(n)}$ and sufficiently small h the first equation uniquely defines $Y^{(n)}$. The second equation then determines $y^{(n+1)}=C(h, y^{(n)})$. The method is characterized by the $(s+r)\times(s+r)$ matrix

$$C=\begin{bmatrix} C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}.$$

As usual we denote the map as well as the matrix characterizing the method by the same symbol C.

The method is called *consistent with preconsistency vector u and consistency vector v* if

$$C_{12}\,u=e, \qquad e=(1,1,\ldots 1)^{\mathrm{T}}$$
$$C_{22}\,u=u$$
$$C_{21}\,e+C_{22}\,v=u+v.$$

The method C is called *strictly stable* if C_{22} has 1 as a simple eigenvalue and all other eigenvalues of C_{22} lie inside the unit circle. To start a general linear method, one needs a starting method S, i.e. a map S: $\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^{rd}$ providing starting values $y_1^{(0)},\ldots,y_r^{(0)}$ for any given initial value η of Eq. (1.1)

$$S(h,\eta)=y^{(0)}=\begin{pmatrix} y_1^{(0)} \\ \vdots \\ y_r^{(0)} \end{pmatrix}\in\mathbb{R}^{rd}.$$

A linear starting method S is defined similarly as in Eq. (1.2) and may be described by an $(\bar{s}+r)\times(\bar{s}+1)$ matrix

$$S=\begin{pmatrix} S_{11} & e \\ S_{21} & u \end{pmatrix} \tag{1.3}$$

where $e=(1,\ldots,1)^{\mathrm{T}}$ and u is the preconsistency vector of C. The *local error $L(h,\eta)$ of the method C relative to S* is defined as

$$L(h,\eta)=C(h,S(h,\eta))-S(h,\varphi(h,\eta))$$

The *order of C relative to S* is the greatest integer p such that

$$|L(h,\eta)|\leqq c\,h^{p+1}$$

The *order of* C is the greatest p such that there is a starting method S such that C is of order p relative to S. For more details on general linear methods see e.g. Butcher (1987).

Lemma 1.1. *Let f be bounded, of class C^{p+k}, $p\geqq 1$, $k\geqq 0$, with bounded derivatives and let C be a general linear method of order p.*
Then for sufficiently small h the local error $L(h,\eta)$ of C satisfies

$$L(h,\eta)=h^{p+1}\,R(h,\eta)$$

where $R(h,\cdot)$ is of class C^k.

Proof. The general linear method C is defined by Eq. (1.2). Put $n=0$. By the Implicit Function Theorem $Y^{(0)}$ is of class C^{p+k} with respect to h and $y^{(0)}$. Hence by the second equation of (1.2)

$$\frac{y^{(1)}-(C_{22}\otimes I)\,y^{(0)}}{h}=(C_{21}\otimes I)\,F(Y^{(0)})$$

is of class C^{p+k}. Puting $y^{(0)}=S(h,\eta)$ we get

$$\frac{C(h,S(h,\eta))-(C_{22}\otimes I)\,S(h,\eta)}{h}=u(h,\eta)\in C^{p+k} \tag{1.4}$$

for some function $u(h,\eta)$ of class C^{p+k}. According to Eq. (1.3) the starting method S is defined by

$$Y=(S_{11}\otimes I)\,hF(Y)+(e\otimes I)\,\eta$$
$$y^{(0)}=S(h,\eta)=(S_{21}\otimes I)\,hF(Y)+(u\otimes I)\,\eta.$$

By the Implicit Function Theorem Y is of class C^{p+k} with respect to h and η. Next we want to eliminate the factor $(C_{22}\otimes I)$ in Eq. (1.4). By definition of the starting method we get

$$\begin{aligned}\frac{(C_{22}\otimes I)\,S(h,\eta)-S(h,\eta)}{h}&=\frac{((C_{22}\,S_{21}-S_{21})\otimes I)\,hF(Y)+((C_{22}\,u-u)\otimes I)\,\eta}{h}\\&=((C_{22}\,S_{21}-S_{21})\otimes I)\,F(Y)=v(h,\eta)\in C^{p+k}\end{aligned} \tag{1.5}$$

where we have used that the preconsistency vector u satisfies $C_{22}\,u=u$. Finally let us relate $S(h,\eta)$ to $S(h,\varphi(h,\eta))$. To save writing we set $\bar{\eta}=\varphi(h,\eta)$ and denote the internal stages and the starting value corresponding to $\bar{\eta}$ by $\bar{Y}$ and $\bar{y}^{(0)}$. One gets

$$\begin{aligned}\frac{S(h,\eta)-S(h,\bar{\eta}))}{h}&=\frac{y^{(0)}-\bar{y}^{(0)}}{h}\\&=(S_{21}\otimes I)(F(Y)-F(\bar{Y}))+(u\otimes I)\,\frac{\bar{\eta}-\eta}{h}\\&=w(h,\eta)\in C^{p+k}.\end{aligned} \tag{1.6}$$

Adding the three equations (1.4), (1.5), (1.6) one gets

$$\frac{L(h,\eta)}{h}=\frac{C(h,S(h,\eta))-S(h,\varphi(h,\eta))}{h}=u(h,\eta)+v(h,\eta)+w(h,\eta)\in C^{p+k}.$$

The function $u+v+w$ may be expanded with respect to h. This leads to

$$L(h,\eta)=h\,[r^0(\eta)+h^1\,r^1(\eta)+\ldots+h^{p-1}\,r^{p-1}(\eta)+h^p\,R(h,\eta)]$$

where $R(h,\eta)\in C^k$. From the assumption that the general linear method C is of order p it follows that

$$r^0(\eta)=r^1(\eta)=\ldots=r^{p-1}(\eta)=0.$$

This completes the proof of Lemma 1.1. □

Now assume that the general linear method C is strictly stable. The $r\times r$ matrix C_{22} has 1 as simple eigenvalue and all remaining eigenvalues λ_i have modulus

$|\lambda_i| < \vartheta < 1$, $i = 2, \ldots, r$. Thus there is a transformation matrix $T = (u, \ldots)$ such that

$$T^{-1} C_{22} T = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{C}_{22} \end{array}\right] \qquad \text{with } \|\bar{C}_{22}\| < \vartheta < 1.$$

Setting

$$y^{(n)} = (T \otimes I) \begin{pmatrix} u^{(n)} \\ v^{(n)} \end{pmatrix}, \qquad u^{(n)} \in \mathbb{R}^d, v^{(n)} \in \mathbb{R}^{(r-1)d}$$

$y^{(n)}$ is transformed to new coordinates $u^{(n)}, v^{(n)}$. The transformed general linear method is determined by the matrix

$$\bar{C} = \begin{bmatrix} C_{11} & C_{12} T \\ T^{-1} C_{21} & T^{-1} C_{22} T \end{bmatrix}.$$

From now on we assume that the method C has already this special form, i.e. we assume without loss of generality that

$$C_{22} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{C}_{22} \end{array}\right] \qquad \text{with } \|\bar{C}_{22}\| < \vartheta < 1, \tag{1.7}$$

that $y^{(n)}$ is split into $u^{(n)} \in \mathbb{R}^d$, $v^{(n)} \in \mathbb{R}^{(r-1)d}$ and that the preconsistency vector is $u = e_1 = (1, 0, \ldots, 0)^T$.

2. General linear methods are essentially conjugate to one step methods

Let C be a stricly stable general linear method and assume without loss of generality that C_{22} has the special form (1.7). For $h = 0$ the method reduces to

$$\begin{aligned} u^{(n+1)} &= u^{(n)} \\ v^{(n+1)} &= \bar{C}_{22} v^{(n)}. \end{aligned} \tag{2.1}$$

Hence the set

$$M_0 := \left\{ y = \begin{pmatrix} u \\ v \end{pmatrix} \middle| \, u \in \mathbb{R}^d, \quad v = 0 \in \mathbb{R}^{(r-d)d} \right\}$$

is invariant under $C(0, \cdot)$ and is exponentially attractive. Note that if $f \in C^{p+k}$ then the method C is also of class C^{p+k} (see the proof of Lemma 1.1).

One may show that the manifold M_0 persists under perturbations. For our situation we may use the following

Proposition 2.1. *Let the map*

$$P: (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto (\bar{x}, \bar{y}) \in \mathbb{R}^m \times \mathbb{R}^n$$

be of the form

$$\bar{x} = Ax + \hat{f}(x, y)$$
$$\bar{y} = g(x, y).$$

Let the following assumptions be satisfied:

a) The functions $\hat{f}$, g are of class C^k with bounded derivatives.
b) The matrix A is regular and $\|A^{-1}\| \leqq \alpha$.
c) $\hat{f}$ and g satisfy the following Lipschitz conditions

$$|\hat{f}(x_1, y_1) - \hat{f}(x_2, y_2)| \leqq L_{11}|x_1 - x_2| + L_{12}|y_1 - y_2|$$
$$|g(x_1, y_1) - g(x_2, y_2)| \leqq L_{21}|x_1 - x_2| + L_{22}|y_1 - y_2|.$$

d) The Lipschitz constants α and L_{ij} satisfy the conditions

$$L_{11} + L_{22} + 2\sqrt{L_{12} L_{21}} < \frac{1}{\alpha}$$

and

$$L_{22} + \frac{2 L_{12} L_{21}}{\frac{1}{\alpha} - L_{11} - L_{22}} < \min\left\{1, \left(\frac{1}{\alpha} - L_{11} - \frac{2 L_{12} L_{21}}{\frac{1}{\alpha} - L_{11} - L_{22}}\right)^k\right\}.$$

Then there is a function $s\colon \mathbb{R}^m \to \mathbb{R}^n$ such that the following holds

i) The set $M := \{(u, v) \mid u \in \mathbb{R}^m, v = s(x)\}$ is invariant under P.
ii) The function s is of class C^k with bounded derivatives.
iii) The invariant manifold M is exponentially attractive with constant

$$\chi = L_{22} + \frac{2 L_{12} L_{21}}{\frac{1}{\alpha} - L_{11} - L_{22}} < 1.$$

iv) The property of "asymptotic phase" holds, i.e. there is a constant c such that for every (x_0, y_0) there is a $(\tilde{x}_0, \tilde{y}_0) \in M$ such that for $i \in \mathbb{N}_0$

$$|x_i - \tilde{x}_i| \leqq c\chi^i |y_0 - s(x_0)|$$
$$|y_i - \tilde{y}_i| \leqq \chi^i |y_0 - s(x_0)|$$

where $(x_i, y_i) := P^i(x_0, y_0)$ and $(\tilde{x}_i, \tilde{y}_i) := P^i(\tilde{x}_0, \tilde{y}_0) \in M$.

A detailed proof of this Theorem may be found in Nipp and Stoffer (1992). For invariant manifold theory in a more general set up see e.g. Hirsch et al. (1977) or Shub (1987).

In the present case we take h as an additional variable in order to get differentiability of the invariant manifold with respect to h. We put $x := (h, u)$, $y := v$. The general linear method then has the form

$$\bar{x} = \quad x + O(h)$$
$$\bar{y} = \bar{C}_{22}\, y + O(h).$$

Hence we have $\alpha=1$, $L_{11}=L_{12}=L_{21}=O(h)$, $L_{22}=\|C_{22}\|+O(h)$ and thus the assumptions of Proposition 2.1 are satisfied for sufficiently small h. As a consequence we get the following

Proposition 2.2. *Let f in (1.1) be bounded, of class C^{p+k}, $p\geqq 1$, $k\geqq 0$, with bounded derivatives and let C be a strictly stable general linear method satisfying (1.7).*

Then there are constants h_0 and K and a function $\sigma^*(h,u)$ of class C^{p+k} with bounded derivatives such that for all $h\in(0,h_0)$ the following holds

i) The set

$$M_h:=\left\{y=\begin{pmatrix}u\\ v\end{pmatrix}\middle|\, u\in\mathbb{R}^d, \quad v=\sigma^*(h,u)\right\}$$

is invariant under C, i.e. $y^{(0)}\in M_h$ implies $C(h, y^{(0)})\in M_h$.

ii) M_h is exponentially attractive, more precisely

$$\operatorname{dist}(C^n(h, y^{(0)}), M_h)\leqq \vartheta^n \operatorname{dist}(y^{(0)}, M_h)$$

where $\operatorname{dist}(y, M_h)$ *is defined as* $\operatorname{dist}((u,v), M_h):=|v-\sigma^*(h,u)|$.

iii) The property of asymptotic phase holds, i.e. for all $y^{(0)}\in\mathbb{R}^{rd}$ there is a unique $y^{*(0)}\in M_h$ such that $|y^{(n)}-y^{*(n)}|\leqq K\vartheta^n \operatorname{dist}(y^{(0)}, M_h)$ holds for all $n\in\mathbb{N}$.

Remark. This proposition states that the behaviour of the general linear method C is essentially determined by the dynamics of C restricted to M_h.

Theorem 2.3. *Let f satisfy the assumptions made in Proposition 2.2 and let C be a strictly stable general linear method of order p.*

Then there is a constant h_0, a (nonlinear) starting method S^* and a one step method Φ both of class C^{p+k} such that for all $h\in(0,h_0)$ the following holds

i) $S^*(h,\eta)\in M_h$ for all initial conditions $\eta\in\mathbb{R}^d$.

ii) C is of order p relative to S^*.

iii) The one step method Φ is conjugate to C restricted to M_h, i.e.

$$C(h, S^*(h,\eta))=S^*(h, \Phi(h,\eta))$$

or equivalently: the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{\ \Phi\ } & \mathbb{R}^d \\ {\scriptstyle S^*}\downarrow & & \downarrow{\scriptstyle S^*}. \\ M_h & \xrightarrow{\ C\ } & M_h \end{array}$$

iv) The one step method Φ is of order p. More precisely: the local error satisfies

$$l(h,\eta):=\Phi(h,\eta)-\varphi(h,\eta)=h^{p+1}r(h,\eta)$$

where r is of class C^k.

Proof. We write the general linear method C as

$$\text{(2.2)}\qquad \begin{aligned} u^{(n+1)}&=G(h,u^{(n)},v^{(n)})\in C^{p+k}\\ v^{(n+1)}&=H(h,u^{(n)},v^{(n)})\in C^{p+k}. \end{aligned}$$

Note that for $h=0$ the map reduces to Eq.(2.1). Let S be a linear starting method such that C is of order p relative to S. S is as regular as f and hence of class C^{p+k}. One may split S into a u-part and a v-part and we may write

$$S(h,\eta)=\begin{pmatrix}u^{(0)}\\ v^{(0)}\end{pmatrix}=\begin{pmatrix}S_u(h,\eta)\\ S_v(h,\eta)\end{pmatrix}.$$

By Eq.(1.3) and our special choice of the preconsistency vector $u=(1,0,\dots,0)^T$ we have for $h=0$

$$S_u(0,\eta)=\eta$$
$$S_v(0,\eta)=0.$$

Hence the near identity map $S_u(h,\eta)$ is invertible for small h. Now we define the starting method S^* as

$$S^*(h,\eta)=\begin{pmatrix}S_u^*(h,\eta)\\ S_v^*(h,\eta)\end{pmatrix}:=\begin{pmatrix}S_u(h,\eta)\\ \sigma^*(h,S_u(h,\eta))\end{pmatrix}$$

where σ^* is the function given in Proposition 2.2. Note that $S^*(h,\cdot)$ is a diffeomorphism from $\mathbb{R}^d$ to M_h. We therefore may define

$$\Phi:=S^{*-1}\circ C\circ S^*.$$

By our definitions of S^* and Φ clearly assertions i) and iii) hold.

Next we show that S^*-S is of order h^{p+1}. Set

$$\sigma(h,u):=S_v(h,S_u^{-1}(h,u))$$

and similarly to S^* the starting method S may be written as

$$S(h,\eta)=\begin{pmatrix}S_u(h,\eta)\\ \sigma(h,S_u(h,\eta))\end{pmatrix}.$$

Now we transform the coordinates $\begin{pmatrix}u\\ v\end{pmatrix}$ in Eq.(2.2) to new coordinates $\begin{pmatrix}u\\ w\end{pmatrix}$ by setting

$$v=\sigma(h,u)+h^{p+1}w.$$

The general linear method C is transformed to

$$\begin{aligned}u^{(n+1)}&=G(h,u^{(n)},\sigma(h,u^{(n)})+h^{p+1}w^{(n)})\\ h^{p+1}w^{(n+1)}&=H(h,u^{(n)},\sigma(h,u^{(n)})+h^{p+1}w^{(n)})\\ &\quad-\sigma(h,G(h,u^{(n)},\sigma(h,u^{(n)})+h^{p+1}w^{(n)})).\end{aligned}\tag{2.3}$$

Since C is order p relative to S we know by Lemma 1.1 that for some $R(h,\eta)\in C^k$

$$C(h,S(h,\eta))-S(h,\varphi(h,\eta))=L(h,\eta)=h^{p+1}R(h,\eta)\tag{2.4}$$

or equivalently

$$G(h, S_u(h,\eta), S_v(h,\eta)) - S_u(h, \varphi(h,\eta)) = h^{p+1} U(h,\eta) \tag{2.5}$$
$$H(h, S_u(h,\eta), S_v(h,\eta)) - S_v(h, \varphi(h,\eta)) = h^{p+1} V(h,\eta)$$

for some functions $U(h,\eta)$, $V(h,\eta) \in C^k$. S_u being invertible one may choose η such that $S_u(h,\eta) = u^{(n)}$. Hence

$$S_u(h, \varphi(h,\eta)) = G(h, u^{(n)}, \sigma(h, u^{(n)})) - h^{p+1} \tilde{U}(h, u^{(n)})$$
$$S_v(h, \varphi(h,\eta)) = H(h, u^{(n)}, \sigma(h, u^{(n)})) - h^{p+1} \tilde{V}(h, u^{(n)})$$

where we put $\tilde{U}(h, u^{(n)}) := U(h, S_u^{-1}(h, u^{(n)}))$, $\tilde{V}(h, u^{(n)}) := V(h, S_u^{-1}(h, u^{(n)}))$. Combining these two equtions we have

$$H(h, u^{(n)}, \sigma) - h^{p+1} \tilde{V} - \sigma(h, G(h, u^{(n)}, \sigma) - h^{p+1} \tilde{U}) = 0$$

where we have omitted the arguments $(h, u^{(n)})$ in the functions σ, $\tilde{U}$ and $\tilde{V}$. Subtracting this last equation from the w equation in (2.3) one gets

$$\begin{aligned} h^{p+1} w^{(n+1)} = {} & H(h, u^{(n)}, \sigma + h^{p+1} w^{(n)}) - H(h, u^{(n)}, \sigma) + h^{p+1} \tilde{V} \\ & + \sigma(h, G(h, u^{(n)}, \sigma) - h^{p+1} \tilde{U}) - \sigma(h, G(h, u^{(n)}, \sigma + h^{p+1} w^{(n)})). \end{aligned}$$

Defining the functions δG, δH, $\delta \sigma$ as

$$\delta G(h, u, v; z) := \int_0^1 \frac{\partial G}{\partial v}(h, u, v + tz)\, dt$$

$$\delta H(h, u, v; z) := \int_0^1 \frac{\partial H}{\partial v}(h, u, v + tz)\, dt$$

$$\delta \sigma(h, u; z) := \int_0^1 \frac{\partial \sigma}{\partial u}(h, u + tz)\, dt$$

we readily obtain the general relations

$$G(h, u, v+z) - G(h, u, v) = \delta G(h, u, v; z)\, z$$
$$H(h, u, v+z) - H(h, u, v) = \delta H(h, u, v; z)\, z$$
$$\sigma(h, u+z) - \sigma(h, u) = \delta \sigma(h, u; z)\, z.$$

Note that $\delta G, \delta H, \delta \sigma \in C^{p+k-1}$. Now the general linear method C in u, w coordinates may be written as

$$\begin{aligned} u^{(n+1)} &= G(h, u^{(n)}, \sigma + h^{p+1} w^{(n)}) \\ w^{(n+1)} &= \delta H(h, u^{(n)}, \sigma; h^{p+1} w^{(n)})\, w^{(n)} + \tilde{V} \\ &\quad - \delta \sigma(h, G(h, u^{(n)}, \sigma); -h^{p+1} \tilde{U})\, \tilde{U} - \delta \sigma(h, G(h, u^{(n)}, \sigma); h^{p+1} \Delta)\, \Delta \end{aligned}$$

where $\Delta = \delta G(h, u^{(n)}, \sigma; h^{p+1} w^{(n)}) w^{(n)}$. This map is of class C^k. It is easy to verify that for $h=0$ the map reduces to

$$u^{(n+1)} = u^{(n)}$$
$$w^{(n+1)} = \bar{C}_{22} w^{(n)} + \tilde{V}(0, u^{(n)})$$

where again $\bar{C}_{22}$ denotes the matrix in Eq. (1.7). Hence for $h=0$ the manifold

$$I_0 := \left\{ \begin{pmatrix} u \\ w \end{pmatrix} \middle| u \in \mathbb{R}^d, \quad w = (I - \bar{C}_{22})^{-1} \tilde{V}(0, u) \right\}$$

is invariant and exponentially attractive. It follows from Proposition 2.1 that for small h there is an invariant manifold I_h near I_0 which may be described by some function $\hat{\sigma}$ of class C^k as

$$I_h := \left\{ \begin{pmatrix} u \\ w \end{pmatrix} \middle| u \in \mathbb{R}^d, \quad w = \hat{\sigma}(h, u) \right\}.$$

By uniqueness the two manifolds I_h and M_h describe the same invariant manifold of the method C, but in different coordinates. It follows that

$$\sigma^*(h, u) = \sigma(h, u) + h^{p+1} \hat{\sigma}(h, u)$$

or that $S^*(h, \eta) - S(h, \eta) = h^{p+1} R_0(h, \eta)$ is of order $O(h^{p+1})$. This means that S^* is $O(h^{p+1})$ C^k-close to S. Hence Eq. (2.4) implies that $C(h, S^*(h, \eta)) - S^*(h, \varphi(h, \eta)) = L^*(h, \eta) = O(h^{p+1})$ holds. This proves assertion ii) of the Theorem.

We now show that $\Phi(h, \eta)$ is $O(h^{p+1})$ C^k-close to $\varphi(h, \eta)$. From the definition of the integration method Φ and from the definition of the local error $L(h, \eta)$ we get

$$\begin{aligned} S^*(h, \Phi(h, \eta)) - S(h, \varphi(h, \eta)) &= C(h, S^*(h, \eta)) - C(h, S(h, \eta)) + L(h, \eta) \\ &= h(C_{21} \otimes I)(F(Y^*) - F(Y)) + (C_{22} \otimes I)(S^*(h, \eta) - S(h, \eta)) + h^{p+1} R(h, \eta). \end{aligned}$$

We already showed that S^* is C^k $O(h^{p+1})$-close to S. Hence Y^* is also C^k $O(h^{p+1})$-close to Y. We restrict ourself to the u-part of the equation above. Since $S_u^* = S_u$ holds we obtain

$$S_u(h, \Phi(h, \eta)) - S_u(h, \varphi(h, \eta)) = h^{p+1} T(h, \eta) \tag{2.6}$$

for some function $T(h, \eta)$ of class C^k. Note that $S_u(h, \eta)$, $\Phi(h, \eta)$ and $\varphi(h, \eta)$ are of class C^{p+k} and that $S_u(0, y) = y$. It follows that $S_u^{-1}(h, y)$ (satisfying $S_u^{-1}(h, S_u(h, y)) = y$) is of class C^{p+k}. From Eq. (2.6) we get

$$\begin{aligned} \Phi(h, \eta) &= S_u^{-1}(h, S_u(h, \varphi(h, \eta)) + h^{p+1} T(h, \eta)) = \varphi(h, \eta) \\ &\quad + h^{p+1} \int_0^1 \frac{\partial}{\partial y} S_u^{-1}(h, S_u(h, \varphi(h, \eta)) + t h^{p+1} T(h, \eta)) \, T(h, \eta) \, dt \end{aligned}$$

The integral is of class C^k since all ocurring functions are at least of class C^k. This completes the proof of Theorem 2.3. □

Remarks. 1) In the proof of Theorem 2.3 we showed that if C is of order p relative to some starting method S, then $\operatorname{dist}(S(h,\eta), M_h)=O(h^{p+1})$ holds. This means that S approximates the invariant manifold M_h up to order p. The converse is not true. It is also not true in general that C is of order p relative to the starting method

$$\begin{pmatrix} \eta \\ \hline \sigma^*(h,\eta) \end{pmatrix} \in M_h.$$

See e.g. the section on the "effective order of Runge-Kutta methods" in Butcher (1987).

2) The starting method S^* of Theorem 2.3 is not at all uniquely determined. Is there an "optimal" choice of S^*? Is there a simple way to find S^* such that the associated one step method Φ has minimal local error in some reasonable sense? We do not know the answers to these questions.

We give a condition under which the general linear method C is essentially *equivalent* to a one step method (not only conjugate). For the precise statement see Corollary 2.4 ii) below.

The information passed from step to step is the vector $y^{(n)} \in \mathbb{R}^{rd}$ consisting of $y_1^{(n)}, y_2^{(n)}, \ldots, y_r^{(n)}$. In many cases one of these vectors, say $y_1^{(n)}$, approximates the solution directly, i.e. $\varphi(h, y_1^{(n)}) = y_1^{(n+1)} + O(h^{p+1})$. (This holds e.g. for multistep methods, for predictor-corrector methods, etc.). Under this assumption one may choose the starting method to satisfy $y_1^{(0)} = \eta$.

Corollary 2.4. *Let f satisfy the assumptions made in Proposition 2.2. Let C be a strictly stable general linear method of order p relative to the starting method S. Assume that S satisfies $y_1^{(0)} = \eta$.*

Then there is a starting method $S^*(h,\eta)$ with $y_1^{*(0)} = \eta$ and a one step method Φ of class C^{p+k} such that for sufficiently small h

i) The assertions of Theorem 2.3 hold.

ii) For the first d components $y_1^{*(n)}$ of the vector $y^{*(n)} := C^n \circ S^*(h,\eta) \in M_h$ the equality

$$y_1^{*(n+1)} = \Phi(h, y_1^{*(n)}) \text{ holds for all } n \geqq 0.$$

Proof. Construct S^* and Φ as in the proof of Theorem 2.3. Hence i) holds. By assumption we have $S_u(h,\eta) = S_u^*(h,\eta) = \eta$ for all h. According to the definition of Φ we therefore have $\Phi(h,\eta) = G(h, \eta, S_v^*(h,\eta))$. Thus ii) follows at once. □

In order to easily formulate the next corollary, we use so-called "finishing methods" as in Butcher (1987). A *finishing method* F: $(h, y^{(n)}) \in \mathbb{R} \times \mathbb{R}^{rd} \mapsto y_n \in \mathbb{R}^d$ is a L-Lipschitz map which undoes the work of a starting method S, at least up to order $O(h^{p+1})$, i.e.

$$F(h, S(h,\eta)) = \eta + O(h^{p+1}). \tag{2.7}$$

For given initial value $\eta \in \mathbb{R}^d$ the general linear method C with starting method S and finishing method F defines a numerical orbit $y_0, y_1, y_2, \ldots$ defined by

$$\begin{aligned} y_0 &= \eta \\ y_n &= F \circ C^n \circ S(h,\eta) \qquad n = 1, 2, \ldots \end{aligned} \tag{2.8}$$

Corollary 2.5. *Let C be a strictly stable general linear method of order p with starting method S and finishing method F.*

Then there exists a constant $\bar{K}$, a one step method Φ, a starting method S^* and a finishing method F^* such that the following holds:

For every numerical orbit $y_0, y_1, y_2, \ldots$ defined by (2.8) there is a y_0^* such that

$$y_n^* := \Phi^n(h, y_0^*) = F^* \circ C^n \circ S^*(h, y_0^*) \qquad n = 0, 1, 2, \ldots$$

and

$$|y_n - y_n^*| \leqq \bar{K} h^{p+1}.$$

Proof. We choose S^* as in Theorem 2.3 and define F^* as

$$F^*(h, y) := S_u^{*-1}(h, u) = S_u^{-1}(h, u). \tag{2.9}$$

For the definition of S_u and S_u^* see the proof of Theorem 2.3. By definition of F^* we have $F^*(h, S(h, \eta)) = \eta$ and a comparison with Eq. (2.7) yields $|F^* - F| \leqq c_1 h^{p+1}$ for some c_1 in a $O(h^{p+1})$-neighbourhood of M_h.

Now consider the sequence $y^{(n)} := C^n \circ S(h, y_0)$, $n = 0, 1, 2, \ldots$ By the property of asymptotic phase, Proposition 2.2, there exists a $y^{*(0)} \in M_h$ such that

$$|y^{(n)} - y^{*(n)}| \leqq K \vartheta^n \operatorname{dist}(y^{(0)}, M_h).$$

In the proof of Theorem 2.3 we showed that $S - S^* = O(h^{p+1})$, we therefore have

$$\operatorname{dist}(y^{(0)}, M_h) \leqq |S(h, y_0) - S^*(h, y_0)| \leqq c_0 h^{p+1}$$

for some constant c_0 and hence

$$|y^{(n)} - y^{*(n)}| \leqq K c_0 \vartheta^n h^{p+1}.$$

F being Lipschitz with Lipschitz constant L we get

$$\begin{aligned} |y_n - y_n^*| &= |F(h, y^{(n)}) - F^*(h, y^{*(n)})| \\ &\leqq |F(h, y^{(n)}) - F(h, y^{*(n)})| + |F(h, y^{*(n)}) - F^*(h, y^{*(n)})| \\ &\leqq LK c_0 \vartheta^n h^{p+1} + c_1 h^{p+1} \end{aligned}$$

and the corollary follows with $\bar{K} := LK c_0 + c_1$. □

3. Applications

Property A: Invariant curves

In this section we show how the existence of invariant closed curves carries over to general linear methods by means of Theorem 2.3.

Theorem 3.1. *Assume that $f \in C^{p+k+1}$ and that the differential equation (1.1) admits a hyperbolic periodic solution. Let C be a strictly stable general linear method of order p and let S^* and F^* be as in Corollary 2.5.*

Then for sufficiently small h there exists a closed curve $\Gamma_h \in \mathbb{R}^d$, $O(h^p)$ C^k-close to the orbit of the periodic solution such that for every $y_0^* \in \Gamma_h$ the points

$$y_n^* := F^*(h, C^n(h, S^*(h, y_0^*))) = F^* \circ C \circ C \circ \ldots \circ C \circ S^*(h, y_0^*) \in \Gamma_h$$

lie on Γ_h for all $n \geqq 0$.

Proof. By our assumptions Theorem 2.3 applies. Thus there is a starting method S^* and a one step method Φ of class $p+k+1$ such that the assertions of Theorem 2.3 hold. We may apply the main result of Eirola (1988) for the one step method Φ. Thus there exists a closed curve Γ_h invariant under Φ and $O(h^p)$ C^k-close to the orbit of the periodic solution. The claim of Theorem 3.1 follows at once. □

Property B: Attractive sets

Here we show how the result of Kloeden and Lorenz (1986) may be generalized to general linear methods.

Theorem 3.2. *Assume that f and its first p derivatives are uniformly bounded and that the system (1.1) admits a compact, uniformly asymptotically stable set Λ.*

Then for all sufficiently small h there exists a set $\tilde{\Lambda}(h) \supset \Lambda$ with $\tilde{\Lambda}(h) \to \Lambda$ with respect to the Hausdorff metric as $h \to 0$. Moreover, there is a bounded open set $\tilde{U}_0$ independent of h containing $\tilde{\Lambda}(h)$ and there is a time $T_0(h)$ such that the following holds: Every numerical orbit $y_0, y_1, y_2, \ldots$ starting in $\tilde{U}_0$ (i.e. $y_0 \in \tilde{U}_0$) and defined by (2.8) satisfies $y_n \in \tilde{\Lambda}(h)$ for all n with $nh \geqq T_0(h)$.

Proof. By our assumptions we may apply Corollary 2.5 and the result of Kloeden and Lorenz (1986) holds for the one step method Φ. Hence there is an open set U_0 and for small h there is a set $\Lambda(h)$ with $\Lambda \subset \Lambda(h) \subset U_0$ and $\lim_{h \to 0} \Lambda(h) = \Lambda$ such that whenever $y_0^* \in U_0$ and $nh > T_0(h)$ then $y_n^* := \Phi^n(h, y_0^*) \in \Lambda(h)$ holds. Let $B(\bar{K} h^{p+1})$ be the closed Ball with radius $\bar{K} h^{p+1}$, $\bar{K}$ as in Corollary 2.5. Now define $\tilde{\Lambda}(h) := \Lambda(h) + B(\bar{K} h^{p+1})$, i.e.

$$\tilde{\Lambda}(h) := \{x \mid \operatorname{dist}(x, \Lambda(h)) \leqq \bar{K} h^{p+1}\}$$

and $\tilde{U}(h) := U_0 - B(\bar{K} h^{p+1})$, i.e.

$$\tilde{U}(h) := \{x \mid |x - y| \leqq \bar{K} h^{p+1} \quad \text{implies} \quad y \in U_0\}.$$

There is a h_0 such that for $h < h_0$ the inclusion $\tilde{\Lambda}(h) \subset \tilde{U}(h)$ holds. Set $\tilde{U}_0 := \tilde{U}(h_0)$ and now Theorem 3.2 follows immediately from the estimate $|y_n - y_n^*| \leqq \bar{K} h^{p+1}$. □

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