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The error norm of Gauss-Lobatto quadrature formulae for weight functions of Bernstein-Szegö type*

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Summary. In certain spaces of analytic functions the error term of the Gauss-Lobatto quadrature formula relative to a (nonnegative) weight function is a continuous linear functional. Here we compute the norm of the error functional for the Bernstein-Szegö weight functions consisting of any of the four Chebyshev weights divided by an arbitrary quadratic polynomial that remains positive on $[-1, 1]$. The norm can subsequently be used to derive bounds for the error functional. The efficiency of these bounds is illustrated with some numerical examples.

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1. Introduction

Consider the Gauss-Lobatto quadrature formula for the (nonnegative) weight function w on $[-1, 1]$,

$$\int_{-1}^{1} f(t)w(t)\,dt = \lambda_0 f(-1) + \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) + \lambda_{n+1} f(1) + R_n(f)\,, \tag{1.1}$$

where $\tau_\nu = \tau_\nu^{(n)}$ are the zeros of the nth degree (monic) orthogonal polynomial $\pi_n^{(L)}(\cdot\,; w^{(L)})$ relative to the weight function $w^{(L)}(t) = (1 - t^2)w(t)$. It is known that all weights of (1.1) are positive, and that (1.1) has degree of exactness $d = 2n + 1$, i.e., $R_n(f) = 0$ for all $f \in \mathbb{P}_{2n+1}$ (see [2, Sect. 2.1.1]).

Let f be a holomorphic function in $C_r = \{z \in \mathbb{C} : |z| < r\}$, $r > 1$. Then f can be written as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in C_r\,. \tag{1.2}$$

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Define

$$X_r = \{f : f \text{ holomorphic in } C_r \text{ and } |f|_r < \infty\}, \tag{1.3}$$

where

$$|f|_r = \sup\{|a_k| r^k : k \in \mathbb{N}_0 \text{ and } R_n(t^k) \neq 0\} \tag{1.4}$$

is a seminorm on X_r. The error term R_n is a continuous linear functional on $(C[-1, 1], \|\cdot\|_\infty)$. The continuity of R_n, together with the uniform convergence of the series (1.2) on $[-1, 1]$, implies

$$R_n(f) = \sum_{k=0}^{\infty} a_k R_n(t^k),$$

which, in view of (1.4), gives

$$|R_n(f)| \leqq \left[\sum_{k=0}^{\infty} \frac{|R_n(t^k)|}{r^k}\right] |f|_r. \tag{1.5}$$

Since $|R_n(t^k)| \leqq 2\|w\|_1$, the series in (1.5) converges, and R_n is a bounded linear functional on $(X_r, |\cdot|_r)$. The $\|R_n\|$ can be used to obtain the best possible estimates of the type (1.5) for R_n, that is, if $f \in X_R$, then for every $r \in (1, R]$,

$$|R_n(f)| \leqq \|R_n\| \, |f|_r, \tag{1.6}$$

and consequently

$$|R_n(f)| \leqq \inf_{1 < r \leqq R} (\|R_n\| \, |f|_r). \tag{1.7}$$

If the weight function w satisfies the additional hypothesis

$$\frac{w(t)}{w(-t)} \text{ is nondecreasing on } (-1, 1), \tag{1.8$_\text{i}$}$$

or

$$\frac{w(t)}{w(-t)} \text{ is nonincreasing on } (-1, 1), \tag{1.8$_\text{d}$}$$

then $\|R_n\|$ is given by convenient representations. We derive those in Sect. 2.

Here we consider weight functions of Bernstein-Szegö type

$$w^{(\pm 1/2)}(t) = \frac{(1 - t^2)^{\pm 1/2}}{\rho(t)}, \quad -1 < t < 1, \tag{1.9}$$

$$w^{(\pm 1/2, \mp 1/2)}(t) = \frac{(1 - t)^{\pm 1/2}(1 + t)^{\mp 1/2}}{\rho(t)}, \quad -1 < t < 1, \tag{1.10}$$

where $\rho(t)$ is a polynomial of exact degree 2 that remains positive on $[-1, 1]$ (see [9, Sect. 2.6]). In Sect. 3, we compute explicitly $\|R_n\|$ for (1.9), (1.10), when either (1.8_i) or (1.8_d) holds. The quality of bounds (1.7), for the error term of the corresponding Gauss-Lobatto formula, is demonstrated with a few numerical examples in Sect. 4.

2. Some general results for the error norm

The following theorem will be useful in the subsequent development, but it is also important in its own right.

Theorem 2.1. *Consider the Gauss-Lobatto formula* (1.1) *for the weight function* w *on* $[-1, 1]$.

(*a*) *If* w *satisfies* (1.8_{i}), *then*

$$\|R_n\| = \frac{r}{(r^2-1)\pi_n^{(L)}(r)} \int_{-1}^{1} \frac{\pi_n^{(L)}(t)}{r-t} w^{(L)}(t)\,dt \,. \tag{2.1_{i}}$$

(*b*) *If* w *satisfies* (1.8_{d}), then

$$\|R_n\| = \frac{r}{(r^2-1)\pi_n^{(L)}(-r)} \int_{-1}^{1} \frac{\pi_n^{(L)}(t)}{r+t} w^{(L)}(t)\,\mathrm{d}t \,. \tag{2.1_{d}}$$

Proof. Substituting $f(t) = (1-t^2)t^k$, $k \in \mathbb{N}_0$, in (1.1), we find

$$R_n(t^k - t^{k+2}) = \int_{-1}^{1} t^k w^{(L)}(t)\,dt - \sum_{\nu=1}^{n} \lambda_\nu(1-\tau_\nu^2)\tau_\nu^k \,. \tag{2.2}$$

The right-hand side of (2.2) is the error term $R_n^{G(L)}(t^k)$ of the Gauss formula for the weight function $w^{(L)}$, thus,

$$R_n(t^k - t^{k+2}) = R_n^{G(L)}(t^k) \,. \tag{2.3}$$

(a) If w satisfies (1.8_{i}), the same does $w^{(L)}$. Then

$$R_n^{G(L)}(t^k) \geqq 0, \quad k \in \mathbb{N}_0 \tag{2.4_{i}}$$

(see [5, p. 1172]), and (2.3) yields

$$R_n(t^k) \geqq R_n(t^{k+2}) \,,$$

from which it follows by induction that

$$R_n(t^k) \leqq 0, \quad k \in \mathbb{N}_0 \,. \tag{2.5_{i}}$$

We now derive (2.1_{i}). First, (1.5) gives

$$\|R_n\| \leqq \sum_{k=0}^{\infty} \frac{|R_n(t^k)|}{r^k} \,. \tag{2.6}$$

Letting

$$\phi(z) = \frac{1}{z-r} = -\sum_{k=0}^{\infty} \frac{z^k}{r^{k+1}}, \quad z \in C_r \,, \tag{2.7}$$

we find, by virtue of (2.5_{i}),

$$|R_n(\phi)| = \left[\sum_{k=0}^{\infty} \frac{|R_n(t^k)|}{r^k}\right] |\phi|_r \,,$$

which, in conjunction with (2.6), yields

(2.8) $$\|R_n\| = \sum_{k=0}^{\infty} \frac{|R_n(t^k)|}{r^k}.$$

Moreover, we have

(2.9) $$\|R_n^{G(L)}\| = \sum_{k=0}^{\infty} \frac{|R_n^{G(L)}(t^k)|}{r^k}$$

(see [1, p. 536]). From (2.9), by means of (2.4_i), (2.3), (2.5_i) and (2.8), we find

(2.10) $$\|R_n\| = \frac{\|R_n^{G(L)}\|}{r^2 - 1},$$

which, together with (1.7_i) in [6], implies (2.1_i).

(b) If w satisfies (1.8_d), the same is true for $w^{(L)}$. In this case

(2.4_d) $$(-1)^k R_n^{G(L)}(t^k) \geqq 0, \quad k \in \mathbb{N}_0$$

(see [5, p. 1173]), hence (2.3) gives

$$(-1)^k R_n(t^k) \geqq (-1)^k R_n(t^{k+2}),$$

from which we get by induction

(2.5_d) $$(-1)^k R_n(t^k) \leqq 0, \quad k \in \mathbb{N}_0 .$$

Then considering

$$\psi(z) = -\frac{1}{r+z} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{z^k}{r^{k+1}}, \quad z \in C_r,$$

in place of (2.7), using (2.4_d), (2.5_d), and proceeding as in (a), we obtain (2.1_d). □

3. The error norm for weight functions of Bernstein-Szegö type

First we recall the following:

Proposition 3.1. ([4, Proposition 2.1]). *A real polynomial ρ of exact degree 2 satisfies $\rho(t) > 0$ for $-1 \leqq t \leqq 1$ if and only if it has the form*

(3.1) $$\rho(t) = \rho(t;\alpha,\beta,\delta) = \beta(\beta - 2\alpha)t^2 + 2\delta(\beta - \alpha)t + \alpha^2 + \delta^2$$

with

(3.2) $$0 < \alpha < \beta, \quad \beta \neq 2\alpha, \quad |\delta| < \beta - \alpha .$$

Lemma 3.2. ([6, Lemma 2.2]). (*a*) *Consider the weight functions $w^{(\pm 1/2)}$, with ρ given by* (3.1). *Then $w^{(\pm 1/2)}(t)/w^{(\pm 1/2)}(-t)$ is strictly increasing on* $(-1, 1)$ if

(3.3_1) $$\beta - 2\alpha > 0, \quad \beta(\beta - 2\alpha) \leqq \alpha^2 + \delta^2, \quad \delta < 0,$$

or

(3.3_2) $$\beta - 2\alpha < 0, \quad \delta < 0,$$

equal to 1 *if* $\delta = 0$, *and strictly decreasing on* $(-1, 1)$ *if*

$$(3.4_1) \qquad \beta - 2\alpha > 0, \quad \beta(\beta - 2\alpha) \leqq \alpha^2 + \delta^2, \quad \delta > 0\,,$$

or

$$(3.4_2) \qquad \beta - 2\alpha < 0, \quad \delta > 0\,.$$

(*b*) *Consider the weight functions* $w^{(\pm 1/2, \mp 1/2)}$, *with* ρ *given by* (3.1). *Then* $w^{(1/2, -1/2)}(t)/w^{(1/2, -1/2)}(-t)$ *is strictly decreasing on* $(-1, 1)$ *if either* (3.4_1) *or* (3.4_2) *holds, or* $\delta = 0$, *and* $w^{(-1/2, 1/2)}(t)/w^{(-1/2, 1/2)}(-t)$ *is strictly increasing on* $(-1, 1)$ *if either* (3.3_1) *or* (3.3_2) *holds, or* $\delta = 0$.

Now we compute the error norm for the weight functions (1.9), (1.10), when either (1.8_i) or (1.8_d) holds. Let $R_n^{(\pm 1/2)}(\cdot) = R_n(\cdot\,; w^{(\pm 1/2)})$, $R_n^{(\pm 1/2, \mp 1/2)}(\cdot) = R_n(\cdot\,; w^{(\pm 1/2, \mp 1/2)})$.

Theorem 3.3. *Consider the weight function* $w^{(-1/2)}$, *with* ρ *given by* (3.1). *Let* $\tau = r - \sqrt{r^2 - 1}$. *We have*

$$(3.5) \quad \| R_n^{(-1/2)} \| = \frac{8\pi r \tau^{2n+2}}{[(\beta - 2\alpha)\tau^2 + 2\delta\tau + \beta][(\beta - 2\alpha)\tau^2(1 - \tau^{2n-2}) + 2\delta\tau(1 - \tau^{2n}) + \beta(1 - \tau^{2n+2})]\sqrt{r^2 - 1}}, \quad n \geqq 1\,,$$

if either (3.3_1) *or* (3.3_2) *holds, and the same formula* (3.5), *with* δ *replaced by* $-\delta$, *if either* (3.4_1) *or* (3.4_2) *holds. Also*, (3.5) *is true when* $\delta = 0$.

Proof. If $w^{(-1/2)}$ satisfies either (1.8_i) or (1.8_d), then from (2.10) (cf. Theorem 2.1), in view of $w^{(L)(-1/2)}(t) = (1 - t^2)w^{(-1/2)}(t) = w^{(1/2)}(t)$, we get

$$(3.6) \qquad \| R_n^{(-1/2)} \| = \frac{\| R_n^{G(1/2)} \|}{r^2 - 1}\,,$$

where $R_n^{G(1/2)}$ is the error term of the Gauss formula for the weight function $w^{(1/2)}$. Theorem 3.3 follows from (3.6) and Theorem 3.2 in [6]. □

Remark 1. Since $0 < \tau < 1$, it is easy to show that $(\beta - 2\alpha)\tau^2 + 2\delta\tau + \beta > 0$ if either (3.3_1) or (3.3_2) holds. The remaining expression in the denominator of (3.5) is also positive. This follows from (3.18) below, with $m = n$, and the fact that $\pi_n^{(1/2)}(r) > 0$ (all zeros of $\pi_n^{(1/2)}$ are contained in $(-1, 1)$). Hence, the right-hand side of (3.5) is positive.

Theorem 3.4. *Consider the weight function* $w^{(1/2)}$, *with* ρ *given by* (3.1). *Let* $\tau = r - \sqrt{r^2 - 1}$. *We have*

$$(3.7) \quad \| R_n^{(1/2)} \| = \frac{8\pi r \tau^{2n+4}(\tau^2 - 2\gamma_1\tau - 4\gamma_2)\sqrt{r^2 - 1}}{[(\beta - 2\alpha)\tau^2 + 2\delta\tau + \beta](4\gamma_2\tau^2\omega_n + 2\gamma_1\tau\omega_{n+1} - \omega_{n+2})}, \quad n \geqq 1\,,$$

where

$$(3.8)\quad \begin{aligned} \gamma_1 &= \frac{\alpha\delta}{[(\beta-\alpha)^2-\delta^2]n^2+(\beta^2-\alpha^2-\delta^2)n+\alpha\beta}, \\ \gamma_2 &= \frac{[(\beta-\alpha)^2-\delta^2](n+1)^2+(\beta^2-\alpha^2-\delta^2)(n+1)+\alpha\beta}{4\{[(\beta-\alpha)^2-\delta^2]n^2+(\beta^2-\alpha^2-\delta^2)n+\alpha\beta\}}, \\ \omega_n &= (\beta-2\alpha)\tau^2(1-\tau^{2n-2})+2\delta\tau(1-\tau^{2n})+\beta(1-\tau^{2n+2}), \end{aligned}$$

if either (3.3_1) *or* (3.3_2) *holds, and the same formulae* (3.7), (3.8), *with* δ *replaced by* $-\delta$, *if either* (3.4_1) *or* (3.4_2) *holds. Also,* (3.7), (3.8) *are true when* $\delta = 0$.

Proof. Assume that either (3.3_1) or (3.3_2) holds. Then by Lemma 3.2(a), $w^{(1/2)}(t)/w^{(1/2)}(-t)$ is strictly increasing on $(-1, 1)$, and applying (2.1_i) in Theorem 2.1(a),

$$(3.9)\qquad \| R_n^{(1/2)} \| = \frac{r}{(r^2-1)\pi_n^{(L)(1/2)}(r)} \int_{-1}^{1} \frac{\pi_n^{(L)(1/2)}(t)}{r-t} w^{(L)(1/2)}(t)\,dt ,$$

where $w^{(L)(1/2)}(t) = (1-t^2)w^{(1/2)}(t)$, and $\pi_n^{(L)(1/2)}$ is the nth degree (monic) orthogonal polynomial relative to $w^{(L)(1/2)}$. Using Christoffel's theorem (see [9, Sect. 2.5]) we can express $\pi_n^{(L)(1/2)}$ in terms of $\pi_m^{(1/2)}$, the (monic) orthogonal polynomials relative to the weight function $w^{(1/2)}$,

$$(3.10)\qquad (t^2-1)\pi_n^{(L)(1/2)}(t) = \text{constant}\cdot \begin{vmatrix} \pi_n^{(1/2)}(t) & \pi_{n+1}^{(1/2)}(t) & \pi_{n+2}^{(1/2)}(t) \\ \pi_n^{(1/2)}(-1) & \pi_{n+1}^{(1/2)}(-1) & \pi_{n+2}^{(1/2)}(-1) \\ \pi_n^{(1/2)}(1) & \pi_{n+1}^{(1/2)}(1) & \pi_{n+2}^{(1/2)}(1) \end{vmatrix}.$$

The $\pi_m^{(1/2)}$ have been explicitly computed in [4, Eq. (3.9)],

$$(3.11)\quad \pi_m^{(1/2)}(t) = \frac{1}{2^m}\left[U_m(t) + \frac{2\delta}{\beta}U_{m-1}(t) + \left(1-\frac{2\alpha}{\beta}\right)U_{m-2}(t)\right], \quad m \geqq 1,$$

where U_k is the kth degree Chebyshev polynomial of the second kind. This can be defined by

$$(3.12)\qquad U_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}, \quad k = 0, 1, 2, \ldots .$$

From (3.11) we find, by means of (3.12),

$$(3.13)\quad \begin{aligned} \pi_m^{(1/2)}(-1) &= \frac{(-1)^m}{2^{m-1}\beta}[(\beta-\alpha-\delta)m+\alpha], \\ \pi_m^{(1/2)}(1) &= \frac{1}{2^{m-1}\beta}[(\beta-\alpha+\delta)m+\alpha]. \end{aligned}$$

Expanding the determinant in (3.10), and using (3.13), we obtain, after an elementary but tedious computation,

$$(3.14)\quad \pi_n^{(L)(1/2)}(t) = \frac{1}{t^2-1}[\pi_{n+2}^{(1/2)}(t) - \gamma_1\pi_{n+1}^{(1/2)}(t) - \gamma_2\pi_n^{(1/2)}(t)], \quad n \geqq 1,$$

where γ_1, γ_2 are given by (3.8). Then the integral on the right-hand side of (3.9) takes the form

$$(3.15)\qquad \int_{-1}^{1} \frac{\pi_n^{(L)(1/2)}(t)}{r-t} w^{(L)(1/2)}(t)\,dt = \gamma_2 \int_{-1}^{1} \frac{\pi_n^{(1/2)}(t)}{r-t} w^{(1/2)}(t)\,dt + \gamma_1 \int_{-1}^{1} \frac{\pi_{n+1}^{(1/2)}(t)}{r-t} w^{(1/2)}(t)\,dt - \int_{-1}^{1} \frac{\pi_{n+2}^{(1/2)}(t)}{r-t} w^{(1/2)}(t)\,dt\,.$$

We recall from [6, Eq. (3.17)] that

$$(3.16)\qquad \int_{-1}^{1} \frac{\pi_m^{(1/2)}(t)}{r-t} w^{(1/2)}(t)\,dt = \frac{\pi\tau^{m+1}}{2^{m-2}\beta[(\beta-2\alpha)\tau^2+2\delta\tau+\beta]},\quad m \geqq 1\,.$$

Thus (3.15) gives

$$(3.17)\qquad \int_{-1}^{1} \frac{\pi_n^{(L)(1/2)}(t)}{r-t} w^{(L)(1/2)}(t)\,dt = \frac{\pi\tau^{n+1}(4\gamma_2+2\gamma_1\tau-\tau^2)}{2^n\beta[(\beta-2\alpha)\tau^2+2\delta\tau+\beta]}\,.$$

Also, from (3.11), by means of

$$U_k(r) = \frac{1-\tau^{2k+2}}{2\tau^{k+1}\sqrt{r^2-1}},\quad k = 0, 1, 2, \ldots$$

(see [8, p. 10]), we get

$$(3.18)\quad \pi_m^{(1/2)}(r) = \frac{(\beta-2\alpha)\tau^2(1-\tau^{2m-2})+2\delta\tau(1-\tau^{2m})+\beta(1-\tau^{2m+2})}{2^{m+1}\beta\tau^{m+1}\sqrt{r^2-1}}\,.$$

Then (3.14) yields

$$(3.19)\qquad \pi_n^{(L)(1/2)}(r) = \frac{\omega_{n+2}-2\gamma_1\tau\omega_{n+1}-4\gamma_2\tau^2\omega_n}{2^{n+3}\beta\tau^{n+3}(r^2-1)^{3/2}}\,,$$

where ω_n is given by (3.8). Combining (3.9) with (3.17) and (3.19) we obtain (3.7).

Now assume that either (3.4_1) or (3.4_2) holds. If $R_n^{(1/2)(-)}$ is the error functional corresponding to the weight function $w^{(1/2)}(-t)$, using (1.1) we can show that $R_n^{(1/2)(-)}(t^k) = (-1)^k R_n^{(1/2)}(t^k)$, $k \in \mathbb{N}_0$. By Lemma 3.2(a), $w^{(1/2)}(t)/w^{(1/2)}(-t)$ is strictly decreasing on $(-1, 1)$, hence $w^{(1/2)}(-t)/w^{(1/2)}(t)$ is strictly increasing on $(-1, 1)$, and (2.8) implies that $\|R_n^{(1/2)}\| = \|R_n^{(1/2)(-)}\|$. Also, $w^{(1/2)}(-t;\alpha,\beta,\delta) = w^{(1/2)}(t;\alpha,\beta,-\delta)$, with α, β, $-\delta$ satisfying either (3.3_1) or (3.3_2). Therefore, $\|R_n^{(1/2)(-)}\|$, and consequently $\|R_n^{(1/2)}\|$, is given by (3.7), (3.8), with δ replaced by $-\delta$.

Finally, it is clear that (3.7), (3.8) remain true when $\delta = 0$. □

Remark 2. As in Remark 1, $(\beta-2\alpha)\tau^2+2\delta\tau+\beta > 0$ if either (3.3_1) or (3.3_2) holds. Here the remaining expression in the denominator of (3.7) is negative (cf. (3.19)), but it can be shown that so is $\tau^2-2\gamma_1\tau-4\gamma_2$ for $0 < \tau < 1$. Thus, the right-hand side of (3.7) is positive.

Theorem 3.5. *Consider the weight functions* $w^{(\pm 1/2, \mp 1/2)}$, *with* ρ *given by* (3.1). *Let* $\tau = r - \sqrt{r^2 - 1}$. *We have*

(3.20)
$$\| R_n^{(1/2, -1/2)} \| = \frac{8\pi r \tau^{2n+3}(\tau + 2\gamma)}{[(\beta - 2\alpha)\tau^2 - 2\delta\tau + \beta](2\gamma\tau\bar{\omega}_n + \bar{\omega}_{n+1})} \left(\frac{r+1}{r-1}\right)^{1/2}, \quad n \geqq 1,$$

where

(3.21)
$$\gamma = \frac{(\beta - \alpha + \delta)(n+1) + \alpha}{2[(\beta - \alpha + \delta)n + \alpha]},$$
$$\bar{\omega}_n = (\beta - 2\alpha)\tau^2(1 - \tau^{2n-2}) - 2\delta\tau(1 - \tau^{2n}) + \beta(1 - \tau^{2n+2}),$$

if either (3.4_1) *or* (3.4_2) *holds, or* $\delta = 0$. *The* $\| R_n^{(-1/2, 1/2)} \|$ *is given by the same formulae* (3.20), (3.21), *with* δ *replaced by* $-\delta$, *if either* (3.3_1) *or* (3.3_2) *holds, or* $\delta = 0$.

Proof. First we compute $\| R_n^{(1/2, -1/2)} \|$. If (3.4_1) or (3.4_2) holds, or $\delta = 0$, then by Lemma 3.2(b), $w^{(1/2, -1/2)}(t)/w^{(1/2, -1/2)}(-t)$ is strictly decreasing on $(-1, 1)$, and using (2.1_d) in Theorem 2.1(b),

(3.22)
$$\| R_n^{(1/2, -1/2)} \| = \frac{r}{(r^2 - 1)\pi_n^{(L)(1/2, -1/2)}(-r)} \cdot \int_{-1}^{1} \frac{\pi_n^{(L)(1/2, -1/2)}(t)}{r + t} w^{(L)(1/2, -1/2)}(t)\,dt,$$

where $w^{(L)(1/2, -1/2)}(t) = (1 - t^2) w^{(1/2, -1/2)}(t) = (1 - t) w^{(1/2)}(t)$, and $\pi_n^{(L)(1/2, -1/2)}$ is the nth degree (monic) orthogonal polynomial relative to $w^{(L)(1/2, -1/2)}$. From Christoffel's theorem we can get a formula for $\pi_n^{(L)(1/2, -1/2)}$ in terms of $\pi_m^{(1/2)}$,

$$(t - 1)\pi_n^{(L)(1/2, -1/2)}(t) = \text{constant} \cdot \begin{vmatrix} \pi_n^{(1/2)}(t) & \pi_{n+1}^{(1/2)}(t) \\ \pi_n^{(1/2)}(1) & \pi_{n+1}^{(1/2)}(1) \end{vmatrix}.$$

Expanding the determinant above, and using the second equation in (3.13), we find

(3.23)
$$\pi_n^{(L)(1/2, -1/2)}(t) = \frac{1}{t - 1}[\pi_{n+1}^{(1/2)}(t) - \gamma \pi_n^{(1/2)}(t)], \quad n \geqq 1,$$

where γ is given by (3.21). Then the integral on the right-hand side of (3.22) takes the form

(3.24)
$$\int_{-1}^{1} \frac{\pi_n^{(L)(1/2, -1/2)}(t)}{r + t} w^{(L)(1/2, -1/2)}(t)\,dt = \gamma \int_{-1}^{1} \frac{\pi_n^{(1/2)}(t)}{r + t} w^{(1/2)}(t)\,dt - \int_{-1}^{1} \frac{\pi_{n+1}^{(1/2)}(t)}{r + t} w^{(1/2)}(t)\,dt.$$

We have $w^{(1/2)}(-t; \alpha, \beta, \delta) = w^{(1/2)}(t; \alpha, \beta, -\delta)$, and using (3.11), (3.12) we can show that $\pi_m^{(1/2)}(-t; \alpha, \beta, \delta) = (-1)^m \pi_m^{(1/2)}(t; \alpha, \beta, -\delta)$. Therefore, replacing t by $-t$ in the left-hand side of (3.16) yields

$$\int_{-1}^{1} \frac{\pi_m^{(1/2)}(t)}{r + t} w^{(1/2)}(t)\,\mathrm{d}t = \frac{(-1)^m \pi \tau^{m+1}}{2^{m-2}\beta[(\beta - 2\alpha)\tau^2 - 2\delta\tau + \beta]}, \quad m \geqq 1,$$

in view of which (3.24) gives

$$(3.25)\quad \int_{-1}^{1} \frac{\pi_n^{(L)(1/2,-1/2)}(t)}{r+t} w^{(L)(1/2,-1/2)}(t)\,dt = \frac{(-1)^n \pi \tau^{n+1}(\tau+2\gamma)}{2^{n-1}\beta[(\beta-2\alpha)\tau^2 - 2\delta\tau + \beta]}.$$

Also, from (3.18) we get

$$\pi_m^{(1/2)}(-r) = \frac{(-1)^m[(\beta-2\alpha)\tau^2(1-\tau^{2m-2}) - 2\delta\tau(1-\tau^{2m}) + \beta(1-\tau^{2m+2})]}{2^{m+1}\beta\tau^{m+1}\sqrt{r^2-1}},$$

which, inserted into (3.23), yields

$$(3.26)\qquad \pi_n^{(L)(1/2,-1/2)}(-r) = \frac{(-1)^n(2\gamma\tau\bar{\omega}_n + \bar{\omega}_{n+1})}{2^{n+2}\beta\tau^{n+2}(r+1)\sqrt{r^2-1}},$$

where $\bar{\omega}_n$ is given by (3.21). Combining (3.22), (3.25) and (3.26) we derive (3.20).

We now compute $\| R_n^{(-1/2,1/2)} \|$ assuming that either (3.3_1) or (3.3_2) holds, or $\delta = 0$. Let $R_n^{(-1/2,1/2)(-)}$ be the error functional corresponding to the weight function $w^{(-1/2,1/2)}(-t)$. Similarly as in the proof of Theorem 3.4, we can show that $\| R_n^{(-1/2,1/2)} \| = \| R_n^{(-1/2,1/2)(-)} \|$. Since $w^{(-1/2,1/2)}(-t;\alpha,\beta,\delta) = w^{(1/2,-1/2)}(t;\alpha,\beta,-\delta)$, with α, β, $-\delta$ satisfying either (3.4_1) or (3.4_2), or $\delta = 0$, $\| R_n^{(-1/2,1/2)(-)} \|$, and consequently $\| R_n^{(-1/2,1/2)} \|$, is given by (3.20), (3.21), with δ replaced by $-\delta$. □

Remark 3. Similarly as in Remark 2, we can show that the right-hand side of (3.20) is positive.

Remark 4. Let $\Gamma = \partial C_r = \{z \in \mathbb{C} : |z| = r\}$, $r > 1$. If f is a function holomorphic in C_r and continuous on $\bar{C}_r$, then we can obtain a bound for the error term of (1.1) of the form

$$(3.27)\qquad |R_n(f)| \leqq \frac{l(\Gamma)}{2\pi} \max_{|z|=r} |K_{n+2}(z)| \max_{|z|=r} |f(z)|$$

(cf. (1.8) in [3]), where $l(\Gamma) = 2\pi r$ is the length of Γ and

$$(3.28)\qquad K_{n+2}(z) = R_n\left(\frac{1}{z-\cdot}\right)$$

is known as the "kernel". If w satisfies (1.8_i) or (1.8_d), Gautschi has shown in [3, Eqs. (2.1) and (2.4)] that

$$(3.29)\qquad \max_{|z|=r} |K_{n+2}(z)| = \frac{1}{r^2-1} \max_{|z|=r} |K_n^{G(L)}(z)|,$$

where $K_n^{G(L)}$ is the "kernel" of the Gauss formula for the weight function $w^{(L)}$. Moreover, by (3.26) in [6],

$$(3.30)\qquad \max_{|z|=r} |K_n^{G(L)}(z)| = \frac{\| R_n^{G(L)} \|}{r}.$$

It is now immediate from (3.29), (3.30) and (2.10) that

(3.31) $$\max_{|z|=r} |K_{n+2}(z)| = \frac{\| R_n \|}{r} .$$

Then (3.27) implies

(3.32) $$|R_n(f)| \leqq \| R_n \| \max_{|z|=r} |f(z)| ,$$

and if f is holomorphic in C_R,

(3.33) $$|R_n(f)| \leqq \inf_{1<r<R} \left(\| R_n \| \max_{|z|=r} |f(z)| \right) .$$

Therefore, the $\max_{|z|=r} |K_{n+2}(z)|$ for the weight functions (1.9), (1.10), when either (1.8_{i}) or (1.8_{d}) holds, has also been computed.

Remark 5. We could not compute the norm of the error functional of the Gauss-Radau quadrature formula relative to the weight functions (1.9), (1.10), when either (1.8_{i}) or (1.8_{d}) holds. The sign of the error functional, when applied to the monomials t^k, $k \in \mathbb{N}_0$, does not seem to follow a pattern like the one described by (2.4_{i})–(2.4_{d}) or (2.5_{i})–(2.5_{d}), which is essential in the derivation of Theorem 2.1. Consider, e.g., the weight function

$$w_0^{(-1/2)}(t) = (1+\mu)^2 \frac{(1-t^2)^{-1/2}}{(1+\mu)^2 - 4\mu t^2}, \quad -1 < t < 1, \quad 0 < |\mu| < 1 ,$$

which is of the form (1.9), with ρ given by (3.1), and $\alpha = 1$, $\beta = 2/(1+\mu)$, $\delta = 0$. Clearly, $w_0^{(-1/2)}(t)/w_0^{(-1/2)}(-t) = 1$ on $(-1, 1)$ for all $0 < |\mu| < 1$. If R_n^R is the error term of the Gauss-Radau formula for $w_0^{(-1/2)}$, we have found numerically that when $0 < \mu < 1$, the sign of $R_n^R(t^k)$ does not follow the same pattern for all $k \in \mathbb{N}_0$.

4. Examples

All computations in this section were performed on a MicroVAX II computer in quad precision (machine precision approximately 33 decimal digits).

Example 4.1

$$\int_{-1}^{1} \frac{(1-t^2)^{-1/2} \cos t}{\beta(\beta - 2\alpha)t^2 + 2\delta(\beta - \alpha)t + \alpha^2 + \delta^2} dt ,$$

where α, β, δ satisfy (3.2).

The integral can be approximated by the Gauss-Lobatto formula (1.1) for the weight function $w^{(-1/2)}$, with ρ given by (3.1). Since (1.1) has degree of exactness

$d = 2n + 1$, and $f(z) = \cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$ is entire, one easily finds

$$|f|_r = \begin{cases} \frac{r^{2n+2}}{(2n+2)!}, & 1 < r \leqq \sqrt{(2n+3)(2n+4)}, \\ \frac{r^{2n+2k+2}}{(2n+2k+2)!}, & \sqrt{(2n+2k+1)(2n+2k+2)} < r \leqq \sqrt{(2n+2k+3)(2n+2k+4)}, \\ & k = 1, 2, \ldots . \end{cases}$$

Thus, $f \in X_\infty$, and an error bound can be obtained from (1.7), with $\| R_n^{(-1/2)} \|$ given by (3.5).

Another error bound can be found if $|f|_r$ is estimated by $\max_{|z|=r} |f(z)|$ (see [7, Eq. (4.2)]). Then (1.7) takes the form

$$|R_n^{(-1/2)}(f)| \leqq \inf_{1<r<\infty} \left(\| R_n^{(-1/2)} \| \max_{|z|=r} |f(z)| \right), \tag{4.1}$$

which is the same as the bound obtained by Gautschi (cf. (3.33)). Using $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, we find, for $|z| = r$,

$$\begin{aligned} |f(z)| &= \tfrac{1}{2} |e^{-r\sin\theta} e^{ir\cos\theta} + e^{r\sin\theta} e^{-ir\cos\theta}| \\ &\leqq \tfrac{1}{2}(e^{r\sin\theta} + e^{-r\sin\theta}) = \cosh(r\sin\theta), \end{aligned}$$

from which it follows that

$$\max_{|z|=r} |f(z)| \leqq \cosh r .$$

Since for $\theta = \pi/2$, $|f(z)| = \cosh r$, we finally get

$$\max_{|z|=r} |f(z)| = \cosh r ,$$

hence (4.1) becomes

$$|R_n^{(-1/2)}(f)| \leqq \inf_{1<r<\infty} (\| R_n^{(-1/2)} \| \cosh r) . \tag{4.2}$$

Our results are shown in Table 1. (Numbers in parentheses indicate decimal exponents.) We have chosen two sets of values for α, β, δ. The first $\alpha = \sqrt{5}$, $\beta = 3 + \sqrt{5}$, $\delta = -1$ satisfies (3.3_1), while the second $\alpha = 2$, $\beta = 2 + \sqrt{3}$, $\delta = 1/\sqrt{3}$ satisfies (3.4_2). The infimum for each bound was attained at the value of r given before that bound. In the last column we give the modulus of the actual error. The true value of the integral was computed using the Gauss formula for the Chebyshev weight function of the first kind. Whenever the actual error is close to machine precision, the actual error could be larger than the error bound. In this case we enter "m.p." (for machine precision) in the last column.

Example 4.2

$$\int_{-1}^{1} \frac{t^2(1-t^2)^{-1/2}}{(4+t^2)[\beta(\beta-2\alpha)t^2 + 2\delta(\beta-\alpha)t + \alpha^2 + \delta^2]} dt ,$$

where α, β, δ satisfy (3.2).

Table 1. Error bounds (1.7), (4.2) and actual error for Example 4.1

α	β	δ	n	r	Bound (1.7)	r	Bound (4.2)	Error
$\sqrt{5}$	$3+\sqrt{5}$	-1	2	$\sqrt{56}$	2.169(−5)	6.169	6.960(−5)	1.918(−5)
			5	$\sqrt{182}$	4.899(−13)	12.079	2.160(−12)	4.582(−13)
			10	$\sqrt{552}$	1.998(−28)	22.042	1.182(−27)	1.923(−28)
			15	$\sqrt{1122}$	8.267(−46)	32.028	5.883(−45)	m.p.
			20	$\sqrt{1892}$	1.505(−64)	42.022	1.226(−63)	m.p.
2	$2+\sqrt{3}$	$1/\sqrt{3}$	2	$\sqrt{56}$	4.236(−5)	6.162	1.357(−4)	3.771(−5)
			5	$\sqrt{182}$	9.597(−13)	12.075	4.229(−12)	9.016(−13)
			10	$\sqrt{552}$	3.921(−28)	22.039	2.319(−27)	3.785(−28)
			15	$\sqrt{1122}$	1.624(−45)	32.026	1.156(−44)	m.p.
			20	$\sqrt{1892}$	2.958(−64)	42.020	2.409(−63)	m.p.

As in the previous example, the integral can be approximated using the Gauss-Lobatto formula (1.1) for the weight function $w^{(-1/2)}$, with ρ given by (3.1). This time $f(z)=\frac{z^2}{4+z^2}=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{z^{2k}}{2^{2k}}$ is holomorphic in C_2, and

$$|f|_r=\frac{r^{2n+2}}{2^{2n+2}},\quad 1<r\leqq 2\,.$$

Thus, $f\in X_2$, and an error bound can be obtained from (1.7), with $\|R_n^{(-1/2)}\|$ given by (3.5).

Moreover, for $|z|=r$,

$$|f(z)|=\frac{r^2}{\sqrt{16+r^4+8r^2\cos 2\theta}}\,,$$

from which it follows that

$$\max_{|z|=r}|f(z)|=\frac{r^2}{4-r^2},\quad 1<r<2\,.$$

Hence, estimating $|f|_r$ by $\max_{|z|=r}|f(z)|$, we find another error bound

$$|R_n^{(-1/2)}(f)|\leqq \inf_{1<r<2}\left(\|R_n^{(-1/2)}\|\frac{r^2}{4-r^2}\right). \tag{4.3}$$

Our results are shown in Table 2. We have picked the same sets of values for α, β, δ as in Example 4.1. The true value of the integral was again computed using the Gauss formula for the Chebyshev weight function of the first kind.

In contrast to the previous example, bounds (1.7) and (4.3) overestimate the actual error by a few orders of magnitude when n is large. This happens because the infima in both bounds are attained at relatively small values of r. It is easy to see that $\tau\to 0$ as $r\to\infty$, while $\tau\to 1$ as $r\to 1$. Consequently, the magnitude of $\tau^{2n+2}/\sqrt{r^2-1}$ (cf. (3.5)), for n large, is substantially larger in the latter case than in the former, which accounts for the contrast between the two examples.

Table 2. Error bounds (1.7), (4.3) and actual error for Example 4.2

α	β	δ	n	r	Bound (1.7)	r	Bound (4.3)	Error
$\sqrt{5}$	$3+\sqrt{5}$	-1	2	2.000	4.753(−4)	1.736	4.357(−3)	1.408(−4)
			5	2.000	1.758(−7)	1.861	3.252(−6)	2.435(−8)
			10	2.000	3.354(−13)	1.923	1.146(−11)	1.309(−14)
			15	2.000	6.399(−19)	1.946	3.189(−17)	7.036(−21)
			20	2.000	1.221(−24)	1.959	8.000(−23)	3.782(−27)
2	$2+\sqrt{3}$	$1/\sqrt{3}$	2	2.000	9.268(−4)	1.737	8.536(−3)	2.727(−4)
			5	2.000	3.432(−7)	1.861	6.365(−6)	4.720(−8)
			10	2.000	6.547(−13)	1.923	2.239(−11)	2.537(−14)
			15	2.000	1.249(−18)	1.947	6.231(−17)	1.364(−20)
			20	2.000	2.383(−24)	1.959	1.563(−22)	7.330(−27)

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