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On a free boundary problem for minimal surfaces

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For C^4-embedded manifolds $S \subset \mathbb{R}^3$ which are differmorphic to the standard sphere in $\mathbb{R}^3$ the existence of non-constant minimal surfaces bounded by S and intersecting S orthogonally along their boundaries is deduced.

Introduction

Let $B = \{w = (u, v) \in \mathbb{R}^2 / |w| < 1\}$ be the unit disc in $\mathbb{R}^2$ with boundary C, and let S be a closed surface in $\mathbb{R}^3$. A disc-type minimal surface spanning S by definition is a mapping $X \in C^2(B; \mathbb{R}^3) \cap C^1(\bar{B}; \mathbb{R}^3)$ such that

$$\Delta X = 0 \quad \text{in } B, \tag{1.1}$$

$$|X_u|^2 - |X_v|^2 = 0 = X_u \cdot X_v \quad \text{in } B, \tag{1.2}$$

$$X(C) \subset S, \tag{1.3}$$

$$X_n(w) \perp T_{X(w)} S, \quad \forall\, w \in C. \tag{1.4}$$

Here n is the outer normal to B, subscripts denote partial derivatives, $|\cdot|$ and $\cdot$ are the (Euclidean) norm and scalar product in $\mathbb{R}^3$, and $T_Y S$ for $Y \in S$ denotes the tangent space to S at Y. For brevity, problem (1.1)-(1.4) will be referred to as problem $P(S)$.

Free boundary problems of this and related types already were studied in the last century. In particular, the extensive investigations of H.A. Schwarz [18] should be mentioned here who applied the theory of elliptic integrals to these problems and among other results was able to completely describe the set of solutions of a famous problem posed by Gergonne in 1816.

In the first half of this century R. Courant proposed Dirichlet's Principle which he had so successfully applied to the solution of Plateau's problem as a

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means to approach $P(S)$ (cp. [4, 5]). He sought to characterize a minimal surface spanning a given manifold S as a stationary point of Dirichlet's integral

$$D(X)=\frac{1}{2}\int|\nabla X|^2\,dw \tag{1.5}$$

among comparison surfaces satisfying a suitably weakened form of the boundary condition (1.3), the conformality relations (1.2) and the transversality condition (1.4) taking the form of "natural" boundary conditions. Provided that he was able to prevent a minimizing sequence for Dirichlet's functional from degeneration by imposing a kind of "linking condition" on admissible functions, by weak lower semi-continuity of the Dirichlet integral Courant was able to produce a nontrivial (weak) solution of $P(S)$. In [5, p. 213ff.], this is illustrated for surfaces S of the type of the torus. Subsequently H. Lewy [15], W. Jäger [10], S. Hildebrandt and J.C.C. Nitsche [8, 9] established the regularity of the free trace for the solutions of $P(S)$ obtained in this manner. Finally, M. Grüter, S. Hildebrandt and J.C.C. Nitsche [7] also proved regularity along the free boundary for stationary points of D which were not necessarily absolute minima. Thus, problem $P(S)$ became fully accessible by variational methods.

However, if a "linking condition" is not available - as in the case of a convex compact surface S - the direct methods in the calculus of variations in general only produce the trivial constant solutions of $P(S)$, while any nonconstant solution necessarily is of non-minimum type. Instead a "mountain-pass-lemma" seems to be needed. But due to the nonlinearity of the boundary condition (1.3) the classical Palais-Smale condition (cp. e.g. [16]) does not seem to hold for $P(S)$, impeding the use of more refined variational techniques. Therefore for S as above the existence of nontrivial solutions of $P(S)$ in general had remained an open (cp. [5, p. 201], [13]; for more historical details we also refer the reader to [5, 6], and [13]).

In the following we establish:

Theorem 1.1. *For any embedded surface S of class C^4 and diffeomorphic to the unit sphere in $\mathbb{R}^3$ there exists a nonconstant minimal surface spanning S solving* (1.1)-(1.4).

Our approach relies on an adaption of a method developed by Sacks and Uhlenbeck [17] to prove existence of harmonic mappings from S^2 into S. In this problem a loss of compactness is encountered simular to $P(S)$, essentially due to the fact that in both cases nonlinearities arise which correspond to limiting exponents for Sobolev embeddings.

From this observation and in view of recent results for surfaces of prescribed constant mean currature [1, 20, 23, 24], and Yamabe's equation [2] it may be conjectured that a Palais-Smale type compactness condition (as in [21] or [22]) holds locally for $P(S)$ - and for the harmonic mapping problem - in a certain range of energies. For harmonic mappings this conjecture is strongly supported by the results of Sacks and Uhlenbeck [17].

Our results seem to extend to higher dimensions. Moreover, by analogy with the problem of closed geodesics on a closed compact surface[1] we conjec-

[1] In both cases the space of admissible functions may be identified with the space of closed curves on S

ture that in general there will be at least 3 distinct solutions of $P(S)$, and that analogous multiplicity results will hold in higher dimensions.

Indeed, if S is a quadrilateral in $\mathbb{R}^3$ (or more general: an $(n+1)$-lateral in $\mathbb{R}^n$) such results have recently been established by B. Smyth [19]. Finally, results of J.C.C. Nitsche [14] suggest that in general in $\mathbb{R}^3$ there will exists at most three distinct solutions of $P(S)$.

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2. Basic notations

The L^p-, $H^{m,p}$-, $C^{m,\lambda}$-spaces are defined as usual, $\|\cdot\|_{p;G}$ denoting the norm in $L^p(G)$ (cp. [1]). The domain G will sometimes be omitted in this notation. d denotes (total) derivative, ∇ stands for gradient, partial derivatives may be denoted by subscripts. $*$ signifies adjoint or duality. $\langle\cdot,\cdot\rangle_V$ is the dual paring between a vector space V and its dual. In case $V=\mathbb{R}^n$ we also write $\langle X, Y\rangle_{\mathbb{R}^n} = X\cdot Y$. $B_r(v; V)$ is the ball of radius r around v in V. In case $V=\mathbb{R}^2$ we omit V in this notation. Moreover, $B_1(0)=B$. A generic point of B is denoted by $w=(u, v)$. Sometimes (especially on $C=\partial B$) polar coordinates (r, ϕ) will be used. Capital letters X, Y, Z denote generic points in $\mathbb{R}^3$ or functions into $\mathbb{R}^3$.

For $\alpha\geqq 1$ let $H^{1,2\alpha}=H^{1,2\alpha}(B;\mathbb{R}^3)$ with trace spaces $H^{1-\frac{1}{2\alpha},2\alpha} = H^{1-\frac{1}{2\alpha},2\alpha}(C;\mathbb{R}^3)$, the norm in $H^{1-\frac{1}{2\alpha},2\alpha}$ being

$$\|X\|^{2\alpha}_{1-\frac{1}{\alpha},2\alpha} = \int_C |X|^{2\alpha} dw + \int_C\int_C \frac{|X(w)-X(w)|^{2\alpha}}{|w-w'|^{2\alpha}}\, dw\, dw'. \tag{2.1}$$

Note that the embedding $H^{1,2\alpha}(B;\mathbb{R}^3)\to H^{1-\frac{1}{2\alpha},2\alpha}(C;\mathbb{R}^3)$ is continuous. Conversely, for any $\alpha\geqq 1$ there exists a continuous linear extension operator η_α from $H^{1-\frac{1}{2\alpha},2\alpha}$ into $H^{1,2\alpha}$. Moreover, from (2.1) we easily obtain the following

Lemma 2.1. *Let* $\alpha\geqq 1$. i) *If* $X, Y\in H^{1-\frac{1}{2\alpha},2\alpha}\cap L^\infty(C)$, *then* $X\cdot Y\in H^{1-\frac{1}{2\alpha},2\alpha}\cap L^\infty(C)$ *and*

$$\|X\cdot Y\|_{1-\frac{1}{2\alpha},2\alpha} \leqq \|X\|_\infty \|Y\|_{1-\frac{1}{2\alpha},2\alpha} + \|Y\|_\infty \|X\|_{1-\frac{1}{2\alpha},2\alpha}.$$

ii) *If* $X\in H^{1-\frac{1}{2\alpha},2\alpha}(C;\mathbb{R}^n)$ and $\sigma\in C^1(\mathbb{R}^n)$, *then*

$$\|\sigma\circ X\|_{1-\frac{1}{2\alpha},2\alpha} \leqq \|\sigma\circ X\|_{2\alpha} + \|(d\sigma)\circ X\|_\infty \|X\|_{1-\frac{1}{2\alpha},2\alpha}.$$

Also let $H_0^{1,2\alpha}$ denote the closure of $C_0^\infty(B;\mathbb{R}^3)$ in $H^{1,2\alpha}$. $⨍_A$ will denote the mean over a set A. The letters o, O are the standard Landau symbols, $\delta_{i,j}$ is the Kronecker symbol. The letter c denotes a generic constant, occasionally numbered for clarity. A summation convention is used.

Let S be an embedded surface of class C^4 differmorphic to the standard sphere in $\mathbb{R}^3$.

Note that this implies

(S_1) There is a uniform lower bound $\rho_S > 0$ on the injectivity radius of the exponential map on TS, the tangent manifold to S.

(S_2) There exists a constant $\delta_S > 0$ such that for any point $x \in \mathbb{R}^3$ at distance from S $\operatorname{dist}(X, S) \leqq \delta_S$ there is a unique point $Y = pr_S(X) \in S$ such that $\operatorname{dist}(X, S) = |X - Y|$.

By (S_2) the projection pr_S of a δ_S-neighborhood U of S onto S and the reflexion $R: U \to U$, $R(X) = 2pr_S(X) - X$, are well defined. Moreover, let $G: S \to S^2$ be the Gauss map on S assigning to any point on S the unique outer normal vector. Since $S \in C^4$, $G \in C^3$.

By the relations

$$R(X) = X + 2\operatorname{dist}(X, S) G(X) \tag{2.2}$$

if X lies in the bounded component of $\mathbb{R}^3 \setminus S$,

$$R(X) = X - 2\operatorname{dist}(X, S) G(X) \tag{2.3}$$

else, also $R \in C^3$.

3. The perturbed problem

We embed $P(S)$ in a one-parameter family of problems $(P_\alpha(S))_{\alpha \geqq 1}$, such that each $P_\alpha(S)$ for $\alpha > 1$ corresponds to a variational problem for which the Palais-Smale condition is satisfied. Applying the minimax-principle [16] we then obtain saddle-type solutions X_α for $P_\alpha(S)$, $\alpha > 1$. Finally, a non constant solution of $P(S) = P_1(S)$ is obtained on passing to the limit $\alpha \to 1$ in an appropriate sequence of such surfaces X_α.

For $\alpha > 1$ let

$$M_\alpha = \{X \in H^{1,2\alpha} \mid X(C) \subset S\},$$
$$E_\alpha(X) = \tfrac{1}{2} \int_B [(1 + |\nabla x|^2)^\alpha - 1] \, dw.$$

M_α is a reflexive Banach manifold with tangent space $T_X M_\alpha$ at a point $X \in M_\alpha$ given by

$$T_X M_\alpha = \{\varphi \in H^{1,2\alpha} \mid \varphi(w) \in T_{X(w)} S, \ \forall \, w \in C\}.$$

Similary we may define for $\alpha = 1$

$$M = M_1 = \{X \in H^{1,2} \cap C^0(\bar{B}; \mathbb{R}^3) \mid X(C) \subset S\}, \qquad E(X) = \tfrac{1}{2} \int_B |\nabla X|^2 \, dw.$$

M is a non-reflexive Banach-manifold with tangent spaces

$$T_X M = \{\varphi \in H^{1,2} \cap C^0(\bar{B}; \mathbb{R}^3) \mid \varphi(w) \in T_{X(w)} S, \ \forall \, w \in C\}.$$

For simple notation let

$$T_\alpha = H^{1,2\alpha} \cap C^0(\bar{B}; \mathbb{R}^3), \qquad \alpha \geqq 1.$$

We now derive a convenient representation of dE_α: $M \to T^*M_\alpha$. First define the projection $pr_\alpha(X;\cdot)$: $T_\alpha \to T_X M_\alpha \subset T_\alpha$ by letting

$$pr_\alpha(X;\varphi)=\varphi-\eta_\alpha([G(X)\langle G(X),\varphi\rangle_{\mathbb{R}^3}]|_C).$$

Lemma 3.1. *For any $\alpha \geqq 1$ the mapping pr_α: $M_\alpha \times T_\alpha \to TM_\alpha$ is Lipschitz continuous and satisfies the estimate: If (X_m) is a sequence in M_α such that $X_m \to X$ weakly in $H^{1,2\alpha}$ and uniformly as $m \to \infty$ then*

$$\|(X_m-X)-pr_\alpha(X_m;X_m-X)\|_{1,2\alpha} \to 0 \qquad (m\to\infty).$$

Proof. i) Lipschitz continuity follows from Lemma 2.1 and the continuity properties of η_α. ii) Clearly $X \in M_\alpha$. We may assume that $\|X_m - X\|_\infty < \rho_S$. Let $w \in C$, $T :=$ the geodesic distance from $X_m(w)$ to $X(w)$ on S, $\{Y_t\}_{0 \leqq t \leqq T}$ the unique geodesic line joining $Y_0 = X_m(w)$ with $Y_T = X(w)$ in the geodesic ball $B_{\rho_S}(Y_0)$ on S, parameterized by arc length.

Since $\left\langle G(Y_0), \frac{d}{dt} Y_t \Big|_{t=0} \right\rangle_{\mathbb{R}^3} = 0$, we have

$$G(Y_0)\langle G(Y_0), Y_0 - Y_T\rangle_{\mathbb{R}^3} = -\int_0^T \int_0^t G(Y_0)\left\langle G(Y_0), \frac{d^2}{dt^2} Y_t \right\rangle_{\mathbb{R}^3} dt\, ds$$
$$=: \sigma(Y_0, Y_T).$$

Note that since $S \in C^3$, by well-known properties of the exponential map, σ is differentiable with respect to $(Y_0, Y_T) \in S \times S$ satisfying $|Y_0 - Y_T| < \rho_S$, and in fact

$$|d\sigma(Y_0, Y_T)| \leqq c |Y_0 - Y_T|$$

with a constant c depending only on S.

Hence by continuity of η_α and Lemma 2.1

$$\begin{aligned}
&\|(X_m-X)-pr_\alpha(X_m;X_m-X)\|_{1,2\alpha}^{2\alpha} \\
&\quad = \|\eta_\alpha([G(X_m)\langle G(X_m), X_m - X\rangle_{\mathbb{R}^3}]|_C)\|_{1,2\alpha}^{2\alpha} \\
&\quad \leqq c\,\|G(X_m)\langle G(X_m), X_m - X\rangle_{\mathbb{R}^3}\|_{1-\frac{1}{2\alpha},2\alpha}^{2\alpha} \\
&\quad = c\,\|\sigma\circ(X_m, X)\|_{1-\frac{1}{2\alpha},2\alpha}^{2\alpha} \\
&\quad \leqq c\,\|X_m - X\|_\infty^{2\alpha}\cdot(\|X_m|_C\|_{1-\frac{1}{2\alpha},2\alpha}^{2\alpha} + \|X|_C\|_{1-\frac{1}{2\alpha},2\alpha}^{2\alpha}) \\
&\quad \to 0 \qquad (m\to\infty). \quad \text{qed}
\end{aligned}$$

Now uniformly represent $T_X M_\alpha = pr_\alpha(X; T_\alpha)$, and define $d_s E_\alpha$: $M \to T_\alpha^*$ by letting

$$\langle d_s E_\alpha(X), \varphi\rangle_{T_\alpha} = \langle dE_\alpha(X), pr_\alpha(X;\varphi)\rangle_{T_X M_\alpha}.$$

From these representations of TM_α *and* dE_α it is easy to deduce:

Lemma 3.2. *M_α, E_α are of class $C^{1,1}$.*

Moreover, we have

Lemma 3.3. *E_α satisfies the Palais-Smale condition on M_α: If (X_m) is a sequence in M_α such that $E_\alpha(X_m) \leqq c$ and $d_S E_\alpha(X_m) \to 0$ strongly in T_α^* as $m \to \infty$, then a subsequence of (X_m) converges strongly in M_α.*

Proof. By uniform boundedness $\|\nabla X_m\|_{2\alpha}^{2\alpha} \leqq E_\alpha(X_m) \leqq c$ and $\|X_m|_C\|_\infty \leqq \sup_{X \in S}|X|$, the family (X_m) is uniformly bounded in $H^{1,2\alpha}$. Hence, and by Sobolev's embedding theorem we may assume that $X_m \to X$ weakly in $H^{1,2\alpha}$ and uniformly on $\bar{B}$ as $m \to \infty$, where $X \in M_\alpha$. But then, as always letting $o(1)$ denote quantities that tend to zero as $m \to \infty$,

$$\begin{aligned} c^{-1}\|X_m - X\|_{1,2\alpha}^{2\alpha} &\leqq \int_B (\nabla X_m(1+|\nabla X_m|^2)^{\alpha-1} \\ &\quad - \nabla X(1+|\nabla X|^2)^{\alpha-1})(\nabla X_m - \nabla X)\,dw + o(1) \\ &= \int_B (1+|\nabla X_m|^2)^{\alpha-1}\,\nabla X_m \cdot \nabla(X_m - X)\,dw + o(1) \\ &= \langle d_S E_\alpha(X_m), X_m - X\rangle_{T_\alpha} \\ &\quad + \langle dE_\alpha(X_m), (X_m - X) - pr_\alpha(X_m; X_m - X)\rangle_{H^{1,2\alpha}} + o(1) \\ &\leqq c\,\|(X_m - X) - pr_\alpha(X_m; X_m - X)\|_{1,2\alpha} + o(1) \to 0 \ (m \to \infty), \end{aligned}$$

by Lemma 3.1. qed.

Definition 3.4. *A surface $X \in M_\alpha$ is called a critical point of E_α* iff $d_S E_\alpha(X) = 0$, *its value $E_\alpha(X)$ then is called critical.*

By Lemmata 3.2, 3.3 Lusternik-Schnirelman theory may be applied to problems $P_\alpha(S)$ in order to obtain non-constant critical points of E_α for $\alpha > 1$. To define a suitable class of subsets of M_α which is invariant under continuous deformations of M_α introduce polar coordinates (ϕ, θ) on S^2. Then $S^2 \cong C \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $C \times \{-\frac{\pi}{2}\}$, $C \times \{\frac{\pi}{2}\}$ collapsed to points. Let $\sigma\colon S \to S^2$ be the diffeomorphism in the assumptions on S. Then any continuous mapping $p\colon [-\frac{\pi}{2}\ \frac{\pi}{2}] \to M_\alpha$ such that $p(-\frac{\pi}{2})$, $p(\frac{\pi}{2})$ are constant maps induces a mapping $\tilde{p}\colon S^2 \to S^2$ by letting

$$\tilde{p}(\phi, \theta) = \sigma(p(\theta)(e^{i\phi})). \tag{3.1}$$

Endowing the space of mappings $S^2 \to S^2$ with the C^0-topology set

$$P_\alpha = \{p \in C^0([-\tfrac{\pi}{2}, \tfrac{\pi}{2}]; M_\alpha) \mid p(-\tfrac{\pi}{2}) \equiv \text{const}, p(\tfrac{\pi}{2}) \equiv \text{const}, \tilde{p} \in C^0(S^2; S^2)$$
$$\text{is homotopic to the identity on } S^2\}.$$

Since (3.1) for $\tilde{p} = id/S^2$ defines a path $p \in C^1$, clearly $P_\alpha \neq \emptyset$ for any $\alpha \geqq 1$.

Proposition 3.5. *For any $\alpha > 1$ there exists a critical point $X_\alpha \in M_\alpha$ of E_α characterized by the condition*

$$E_\alpha(X_\alpha) = \inf_{p \in P} \sup_{X \in \mathrm{im}(p)} E_\alpha(X) =: \beta_\alpha.$$

Proof. Since P_α is invariant under deformations of M_α along integral curves of ∇E_α the result follows in a standard way. (Cp. e.g. [3, p. 42f], [16, p. 190].) qed.

Remark 3.6. Note that for all $X \in M_\alpha$, all $\alpha' \in [1, \alpha]$ we have $E_{\alpha'}(X) \leqq E_\alpha(X)$, and $P_\alpha \subset P_{\alpha'}$. Hence for all $\alpha' \leqq \alpha$ we have $\beta_{\alpha'} \leqq \beta_\alpha$.

Taking a comparison path p of class C^1 we thus obtain a uniform upper bound for the critical values β_α: $\beta_\alpha \leqq \beta_\infty < \infty$.

Moreover,

Lemma 3.7. *There exists a number* $\beta_0 > 0$ *such that for any* $\alpha \geqq 1$ *we have* $\beta_\alpha \geqq \beta_0$.

Proof. By Remark 3.6 we may restrict our attention to $\alpha = 1$. Replacing an arbitrary curve $p \in P$, by replacing $X = p(t)$ by the harmonic extension of $X|_C$ does not change $\tilde{p}$ and reduces energy. Hence it suffices to give a lower bound for $\sup_{X \in \operatorname{im}(p)} E(X)$ if $p \in P$ consists of harmonic surfaces X.

For a harmonic surface $X \in M$ $\operatorname{dist}(X, S) = \sup_{w \in B} \operatorname{dist}(X(w), S)$ is assumed at an interior point $w \in B$. By conformal reparametrization we may achieve that $w = 0$, without changing $E(X)$. By the mean value property

$$X(0) = \fint_C X(w)\, dw,$$

and since $X(C) \subset S$, we may estimate

$$\begin{aligned} \operatorname{dist}(X, S) &= \operatorname{dist}(X(0), S) \leqq \fint_C |X(0) - X(w')|\, dw' \\ &\leqq \fint_C \fint_C |X(w) - X(w')|\, dw\, dw' \\ &\leqq c_1 \left(\fint_C \fint_C \frac{|X(w) - X(w')|^2}{|w - w'|^2}\, dw\, dw' \right)^{\frac{1}{2}} \\ &\leqq c_2 (E(X))^{\frac{1}{2}}. \end{aligned}$$

Now $p([-\frac{\pi}{2}, \frac{\pi}{2}] \times B)$ cannot lie in a δ_S-neighborhood of S. Else we could continuously project $\operatorname{im}(p)$ onto S via pr_S defined in §2. Then applying the homotopy $\{p_r\}_{0 \leqq r \leqq 1}$, $p_r(\theta, w) = pr_S(p(\theta, rw))$ we obtain a homotopy of $\tilde{p} \sim \mathrm{id}|_{S^2}$ with a mapping that takes S^2 into a line. Since such a mapping is homotopic to a constant mapping in $C^0(S^2; S^2)$, while $\mathrm{id}|_{S^2}$ is not, a contradiction results. Hence we obtain

$$\inf_{p \in P} \sup_{X \in \operatorname{im}(p)} \operatorname{dist}(X, S) \geqq \delta_S,$$

and therefore $\beta_1 \geqq (c_2^{-1} \delta_S)^2 =: \beta_0 > 0$. qed.

In Lemma 5.1 we show that the numbers $(\beta_\alpha)_{\alpha > 1}$ in fact are bounded from below by the energy of a non-constant minimal surface spanning S. For convex surfaces S the next result therefore provides a better bound for β_α.

Proposition 3.8. *Assume S is a convex surface and $\kappa > 0$ a bound for the sectional curvature of S. If X is a non-constant solution of $P(S)$ we have*

$$E(X) \geqq \pi / \kappa^2.$$

Proof. By Lemma 1 of [11] for any $\varepsilon>0$ we may extend G to $\mathbb{R}^3$ such that $\|dG\|_\infty \leqq \kappa+\varepsilon$. Now note that the set $G\circ X(C)$ cannot be contained in a hemisphere $\{Y\in S^2 \mid Y\cdot Y_0>0\}$. Else by the Hopf maximum principle and since $X\not\equiv$ const.

$$\langle d_S E(X), Y_0\rangle_T=\int\left\langle \frac{\partial}{\partial n}X, Y_0\right\rangle_{\mathbb{R}^3}-\left\langle \frac{\partial}{\partial n}X, G(X)\right\rangle_{\mathbb{R}^3}\langle G(X), Y_0\rangle_{\mathbb{R}^3}dw$$
$$=-\int_C\left|\left\langle \frac{\partial}{\partial n}X, G(X)\right\rangle_{\mathbb{R}^3}\right|\langle G(X), Y_0\rangle_{\mathbb{R}^3}dw<0.$$

Hence, introducing polar coordinates on C, by (1.2):

$$2\pi\leqq\int_0^{2\pi}\left|\frac{d}{d\phi}G\circ X\right|d\phi\leqq\|dG\|_\infty\int_0^{2\pi}\left|\frac{\partial}{\partial\phi}X\right|d\phi$$
$$=\|dG\|_\infty\int_C\left\langle\frac{\partial}{\partial n}X, G(X)\right\rangle_{\mathbb{R}^3}dw$$
$$=\|dG\|_\infty\int_B\langle\nabla X, dG(X)\,\nabla X\rangle_{\mathbb{R}^6}dw$$
$$\leqq 2\|dG\|_\infty^2 E(X)\leqq 2(\kappa+\varepsilon)^2E(X).$$

Letting $\varepsilon\to 0$ the claim follows. qed.

4. Regularity properties of the perturbed problems

Lemma 4.1. *For any $\alpha\in]1,\frac{3}{2}[$ any critical point $X\in M_\alpha$ of E_α belongs to the class $H^{2,q}(B;\mathbb{R}^3)$, for any $q<\infty$, and is a classical solution of the equation*

$$-\nabla[(1+|\nabla X|^2)^{\alpha-1}\nabla X]=0 \tag{4.1}$$

satisfying the orthogonality condition (1.4) *pointwise on C.*

Proof. i) Since for any $\varphi\in H_0^{1,2\alpha}$

$$\langle dE_\alpha(X),\varphi\rangle_{H^{1,2\alpha}}=\langle d_S E_\alpha(X),\varphi\rangle_{T_\alpha}=0,$$

X is a weak solution of (4.1). But then by Hölder continuity and [12, Theorem 1.11.1] $X\in H^{2,2}_{\mathrm{loc}}(B;\mathbb{R}^3)$, and higher interior regularity follows as in [17, Prop. 2.3].

ii) To obtain boundary regularity we use the reflection principle to extend X as a solution of a quasilinear elliptic system to a region containing B. The problem of boundary regularity thereby is reduced to the problem of interior regularity for equations of the type encountered also in [17]. Since X is Hölder continuous in $\bar{B}$ there exists $\underline{r}<1$ such that for $\underline{r}<|w|\leqq 1$ $\operatorname{dist}(X(w),S)<\delta_S$, where δ_S is the constant in condition (S_2). Let $\bar{r}=\underline{r}^{-1}$ and define

$$\tilde{X}(w)=R(X(w/|w|^2)),\qquad \text{if } 1<|w|<\bar{r},$$

R being the reflection in S. Since $R^2=\mathrm{id}$, clearly $X(w)=R(\tilde{X}(w/|w|^2))$ for $\underline{r}<|w|<1$; thus as in [8, p. 261f] we obtain that for all $\varphi\in H_0^{1,2\alpha}(B_{\bar{r}}(0)\setminus B;\mathbb{R}^3)$

$$\frac{d}{d\varepsilon}\int_B (1+|\nabla R((\tilde{X}+\varepsilon\varphi)(w/|w|^2))|^2)^\alpha dw|_{\varepsilon=0}=0,$$

or equivalently, that $\tilde{X}$ weakly solves the system

$$\begin{aligned}&-\nabla[(1+|w|^4\tilde{g}_{ij}(\tilde{X})\nabla\tilde{X}^i\cdot\nabla\tilde{X}^j)^{\alpha-1}\tilde{g}_{ij}(\tilde{X})\nabla\tilde{X}^j]\\ &+\tfrac{1}{2}(1+|w|^4\tilde{g}_{ij}(\tilde{X})\nabla\tilde{X}^i\nabla\tilde{X}^j)^{\alpha-1}\frac{\partial}{\partial X^i}\tilde{g}_j\,(\tilde{X})\nabla\tilde{X}^j\nabla\tilde{X}^k=0,\qquad 1\leqq i\leqq 3,\end{aligned}\tag{4.2}$$

with

$$\tilde{g}_{ij}(X)=\left\langle\frac{\partial}{\partial X^i}R(X),\frac{\partial}{\partial X^j}R(X)\right\rangle_{\mathbb{R}^3}$$

and summation over repeated indices.

Note that by (2.2), (2.3) $\tilde{g}\in C^2$ and

$$\tilde{g}_{ij}(X)=\delta_{ij}+0(\operatorname{dist}(X,S)).\tag{4.3}$$

Now extend X to $B_{\bar{r}}(0)$ by letting $X(w)=\tilde{X}(w)$ for $|w|\in]1,\bar{r}[$, and define a new matrix function g: $B_{\bar{r}}(0)\times\mathbb{R}^3\to\mathbb{R}^{3\times 3}$ by letting

$$g_{ij}(w,X)=\begin{cases}\delta_{ij}, & \text{if } w\in\bar{B}\\ \tilde{g}_{ij}(X), & \text{if } w\notin B.\end{cases}$$

Also let

$$a(w)=\begin{cases}1, & \text{if } w\in\bar{B}\\ |w|^4, & \text{if } w\notin B.\end{cases}$$

Then by (4.1), (4.2) X is a weak solution of the equation

$$\begin{aligned}&-\nabla[(1+a(w)g_{kl}(w,X)\nabla X^k\nabla X^l)^{\alpha-1}g_{ij}(w,X)\nabla X^j]\\ &\quad+\tfrac{1}{2}(1+a(w)g_{kl}(w,X)\nabla X^k\nabla X^l)^{\alpha-1}\frac{\partial}{\partial X^i}g_j\,(w,X)\nabla X^j\nabla X\;=0,\\ &\quad 1\leqq i\leqq 3,\end{aligned}\tag{4.4}$$

in $B_{\bar{r}}(0)\setminus C$. But for $\varphi\in H_0^{1,2}(B_{\bar{r}}(0);\mathbb{R}^3)$ the boundary integrals

$$\begin{aligned}&\int_C(1+|\nabla X|^2)^{\alpha-1}\left\langle\frac{\partial}{\partial r}X,\varphi\right\rangle_{\mathbb{R}^3}dw\\ &\quad-\int_C(1+a(w)\,g_{kl}(w,\tilde{X})\nabla\tilde{X}^k\nabla\tilde{X}^l)^{\alpha-1}g_{ij}(w,\tilde{X})\frac{\partial}{\partial r}\tilde{X}^j\varphi^i dw\\ &\quad=2\int_C(1+|\nabla X|^2)^{\alpha-1}\left[\left\langle\frac{\partial}{\partial r}X,\varphi\right\rangle_{\mathbb{R}^3}-\left\langle\frac{\partial}{\partial r}X,G(X)\right\rangle_{\mathbb{R}^3}\langle G(X),\varphi\rangle_{\mathbb{R}^3}dw\right.\\ &\quad=2\langle d_S E_\alpha(X),\varphi\rangle_{T_\alpha}=0.\end{aligned}$$

Hence X in fact weakly solves equation (4.4) in $B_{\bar{r}}(0)$.

iii) To show the contended regularity from (4.4) next note that X satisfies a local Morrey-type condition

$$\int_{B_r(w_0)} (1+|\nabla X|^2)^\alpha dw \leqq c_0 r^\gamma \tag{4.5}$$

with constants $c_0, \gamma > 0$ which are uniform for all $w_0 \in B_{\frac{1+\bar{r}}{2}}(0)$, $r \leqq r_0 \leqq \frac{\bar{r}-1}{4}$, and like r_0 depend only on the modulus of continuity and energy $E_\alpha(X)$ of X. Indeed, letting w_0, r be as above choose a smooth cut-off function τ vanishing outside $B_{2r}(w_0)$ and identically 1 on $B_r(w_0)$. Define $\bar{X} = \fint_{B_{2r}(w_0)\setminus B_r(w_0)} X\,dw$ and insert $\varphi = (X-\bar{X})\tau^2$ into (4.4) to obtain:

$$\begin{aligned}
&\int_{B_{\bar{r}}(0)} (1+|\nabla X|^2)^{\alpha-1} |\nabla X|^2 \tau^2 dw \\
&\quad \leqq c_1 \int_{B_{\bar{r}}(0)} (1+|\nabla X|^2)^{\alpha-1} |X-\bar{X}|^2 |\nabla\tau|^2 dw \\
&\quad\quad + c_2 \int_{B_{\bar{r}}(0)} (1+|\nabla X|^2)^{\alpha-1} |\nabla X|^2 |X-\bar{X}|\, \tau^2 dw.
\end{aligned}$$

Choosing $r_0 \leqq \frac{\bar{r}-1}{4}$ sufficiently small we can achieve that $|X-\bar{X}| < \frac{1}{2c_2}$.

Hence by the Poincaré inequality

$$\begin{aligned}
\int_{B_r(w_0)} (1+|\nabla X|^2)^{\alpha-1} |\nabla X|^2 dw &\leqq c_3 \int_{B_{2r}(w_0)\setminus B_r(w_0)} (1+|\nabla X|^2)^\alpha dw \\
&\quad + c_3 \int_{B_{2r}(w_0)\setminus B_r(w_0)} \frac{|X-\bar{X}|^{2\alpha}}{r^{2\alpha}} dw \\
&\leqq c_4 \int_{B_{2r}(w_0)\setminus B_r(w_0)} (1+|\nabla X|^2)^\alpha dw
\end{aligned} \tag{4.6}$$

and (4.5) follows by hole-filling: Add c_4 times the left-hand side to (4.6), divide by (c_4+1) and iterate.

iv) Now introduce polar coordinates (r, ϕ) on $B_R(0)$. For $h \neq 0$ let

$$X_h = \frac{1}{|h|}[X(r, \phi + r^{-1}h) - X(r, \phi)]$$

be the difference quotient in angular direction. Also let τ be a smooth cut-off function having support in a ball $B_{2r}(w_0)$ centered at a point $w_0 \in C$ and identically 1 on $B_r(w_0)$. We shall assume that $2r < \bar{r} - 1$. Inserting the testing function

$$\varphi = (X_h \tau^2)_{-h}$$

into (4.5) after some standard estimates and rearranging of terms we arrive at the inequality

$$\begin{aligned}
&\int_{B_{\bar{r}}(0)} (1+|\nabla X|^2)^{\alpha-1} |\nabla X_h|^2 \tau^2 dw \\
&\quad \leqq c_1 \cdot \int_{B_{\bar{r}}(0)} (1+|\nabla X|^2)^{\alpha-1} |\nabla X|^2 |X_h|^2 \tau^2 dw \\
&\quad\quad + c_2 \cdot \int_{B_{\bar{r}}(0)} (1+|\nabla X|^2)^{\alpha-1} (|\nabla X|^2 + |X_h|^2) |\nabla\tau|^2 dw \\
&\quad\quad + c_3 \cdot \int_{B_{\bar{r}}(0)\setminus B} (1+a\tilde{g}_{lm}(X)\nabla X^l \nabla X^m)^{\alpha-1} \frac{\partial}{\partial X^i} \tilde{g}_j(X) \nabla X^j \nabla X^k (X_h\tau^2)_{-h} dw.
\end{aligned} \tag{4.7}$$

Now "integrate by parts" with respect to the difference quotient in ϕ in the last term to obtain

$$\left| \int_{B_{\bar r}(0)\setminus B} (1+a\tilde g_{lm}(X)\nabla X^l \nabla X^m)^{\alpha-1} \frac{\partial}{\partial X^i} \tilde g_{jk}(X)\nabla X^j \nabla X^k (X_h \tau^2)_{-h} dw \right|$$
$$\leqq \frac{1}{2c_3} \int_{B_{\bar r}(0)} (1+|\nabla X|^2)^{\alpha-1} |\nabla X_h|^2 \tau^2 dw$$
$$+c_4 \cdot \int_{B_{\bar r}(0)} (1+|\nabla X|^2)^{\alpha-1} |\nabla X|^2 |X_h|^2 \tau^2 dw.$$

To estimate the first term on the right in (4.7) as in [8] we use the self-reproducing property of Morrey-spaces [12, Lemma 5.4.1] to obtain from (4.5):

$$\int_{B_{\bar r}(0)} (1+|\nabla X|^2)^{\alpha-1} |\nabla X|^2 |X_h|^2 \tau^2 dw$$
$$\leqq c_5 \cdot c_0 \int_{B_{\bar r}(0)} |\nabla (X_h \tau)|^2 dw \cdot r^{\gamma/2}$$
$$\leqq c_6 \cdot r^{\gamma/2} \int_{B_{\bar r}(0)} |\nabla X_h|^2 \tau^2 dw + c_7 \cdot \int_{B_{\bar r}(0)} |X_h|^2 |\nabla \tau|^2 dw.$$

Choosing r sufficiently small we hance find that

$$\int_{B_{\bar r}(0)} |\nabla X_h|^2 \tau^2 dw \leqq c_8 \int_{B_{\bar r}(0)} (1+|\nabla X|^2)^{\alpha-1} (|\nabla X|^2 + |X_h|^2) |\nabla \tau|^2 dw.$$

Since $X_h \to \frac{1}{r} \cdot \frac{\partial}{\partial \phi} X$ in $H^{1,2\alpha}$ as $h \to 0$ the right hand side of this estimate remains uniformly bounded as $h \to 0$, whence $\frac{1}{r} X_\phi \in H^{1,2}_{\mathrm{loc}}(B_{\bar r}(0))$. But then from equation (4.1) upon solving for X_{rr} we obtain that $X \in H^{2,2}(B; \mathbb{R}^3)$ and hence also $X \in H^{2,2}_{\mathrm{loc}}(B_{\bar r}(0); \mathbb{R}^3)$.

v) To conclude the proof note that by the boundary condition (1.3) for X the matrix function $w \to g(w, X(w))$ is a.e. differentiable on $B_{\bar r}(0)$ and satisfies

$$|dg(\cdot, X(\cdot))| \leqq c|dX| + c.$$

Hence and since $X \in H^{2,2}_{\mathrm{loc}}$, differentiation is permitted in (4.4) and we obtain after multiplication with $(g_{ij}(X))^{-1}(1+a g_{ij}(X)\nabla X^i \nabla X^j)^{1-\alpha}$ that a.e. on $B_{\bar r}(0)$:

$$|\Delta X^k + 2(\alpha-1)(1+a g_{lm}(X)\nabla X^l \nabla X^m)^{-1} a g_{ij} \nabla X^i \nabla^2 X^j \nabla X^k| \leqq c|\nabla X|^2, \tag{4.8}$$

with a constant c depending only on S.

Since $X \in H^{2,2}_{\mathrm{loc}} \to H^{1,q}_{\mathrm{loc}}$, $\forall q < \infty$, from this differential inequality as in [17, Prop. 3.1] we infer that for $\alpha \in]1, \frac{3}{2}[$ $X \in H^{2,q}(B; \mathbb{R}^3)$, $\forall q < \infty$.

In particular $X \in C^1(\bar B; \mathbb{R}^3)$, and (1.4) follows in the pointwise sense from the condition $dE_\alpha(X) = O \in T^*_X M_\alpha$. qed

Remark 4.2. For later reference note that as a by-product of the preceding proof we obtain that any critical point $X \in M_\alpha$ of E_α may be extended in $H^{2,q}$ to a ball $B_{\bar{r}}(0)$ as a solution of the elliptic system (4.4) or the differential inequality (4.8), for any $q < \infty$, where $\bar{r} > 1$ depends only on the modulus of continuity of X (on $\bar{B}$), and the $H^{2,q}$-norm of X on $B_{\bar{r}}(0)$ for any $q < \infty$ may be estimated by a constant depending only on q, the modulus of continuity and the "energy" $E_\alpha(X)$ of X, uniformly as $\alpha \to 1$.

5. Passing to the limit

1. In this final section we use the regularity result of Section 4 to obtain a non-constant solution of $P(S)$. Consider a sequence $(\alpha_m)_{m\in\mathbb{N}}$, $\alpha_m > 1$, $\alpha_m \to 1$ $(m \to \infty)$, and let $X_m = X_{\alpha_m}$ be the solution of $P_{\alpha_m}(S)$ quaranteed by Proposition 3.5. By Remark 3.6 we may assume $X_m \to X \in M$ weakly in $H^{1,2}$. Next choose points $w_m \in \bar{B}$ and numbers $\mu_m > 0$ such that

$$|\nabla X_m(w_m)| = \sup_{w \in B} |\nabla X_m(w)| =: \mu_m^{-1},$$

and introduce a further sequence $(Y_m)_{m\in\mathbb{N}}$ by rescaling:

$$Y_m(w) = X_m(w_m + \mu_m w).$$

Lemma 5.1. i) *If* $\mu_m \geqq \mu > 0$, *then the sequence* (X_m) *converges to a non-constant solution* X *of* $P(S)$ *in* $C^1(\bar{B}; \mathbb{R}^3)$.

ii) *If* $\mu_m \to 0$, *there exists a non-constant solution* X_0 *of* $P(S)$ *such that*

$$E(X_0) + E(X) \leqq \liminf_{m\to\infty} E_{\alpha_m}(X_m). \tag{5.1}$$

Proof. i) In the first case the sequence (X_m) is equicontinuous, hence by Lemma 4.1 equibounded in $H^{2,q}(B; \mathbb{R}^3)$ for any $q < \infty$. Thus a subsequence (X_m) converges in $C^1(\bar{B}; \mathbb{R}^3)$ to X. Clearly, X satisfies (1.3), (1.4) and – passing to the limit $\alpha \to 1$ in (4.1) – is harmonic. Hence $dE(X) = O \in T_X M$. Moreover, by the estimate of Lemma 3.6 X is non-constant. To verify the conformality relations (1.2) let τ_ε, $|\varepsilon| < 1$, be any differentiable family of diffeomorphisms of $\bar{B}$ such that $\tau_0 = \mathrm{id}$. Since $\frac{d}{d\varepsilon}(X \circ \tau_\varepsilon)|_{\varepsilon=0} = \frac{d}{d\varepsilon}\tau_\varepsilon|_{\varepsilon=0} \cdot \nabla X \in T_X M$ from

$$O = \left\langle dE(X), \frac{d}{d\varepsilon}(X \circ \tau_\varepsilon)|_{\varepsilon=0} \right\rangle_{T_X M}$$
$$= \frac{d}{d\varepsilon} E(X \circ \tau_\varepsilon)|_{\varepsilon=0}$$

and the classical result [5, Chapter III.4] we infer (1.2).

ii) In the second case we distinguish two possibilities:

a) Either the sets $B^m = \{w \mid w_m + \mu_m w \in B\}$ exhaust all of $\mathbb{R}^2$, or

b) there exists a vector $w_0 \in \mathbb{R}^2$ such that after suitable rotation of B^m and translation by w_0 B^m is contained in the half-plane $\mathbb{R}^2_+ = \{v > 0\}$ and exhausts this set as $m \to \infty$.

a) In the event a) holds from the equation for Y_m

$$L(Y_m) := \Delta Y_m + 2(\alpha_m - 1)\frac{\nabla Y_m \nabla^2 Y_m \nabla Y_m}{\mu_m^2 + |\nabla Y_m|^2} = 0 \quad \text{in } B^m \tag{5.2}$$

(in symbolic notation) and arguments as in part i) of this proof, a subsequence (Y_m) C^1-locally converges to some vector $Y \in C^1(\mathbb{R}^2; \mathbb{R}^3)$. Passing to the limit $\alpha_m \to 1$ in (5.2) Y is harmonic. Moreover, since $|\nabla Y_m(0)| = |\nabla Y(0)| = 1$, Y is non-constant. But by the maximum principle for Eq. (4.1) the images of the surface Y_m are contained in the convex hull of S, hence bounded uniformly in m. Thus also Y is bounded. But then Y must be constant, a contradiction.

Hence we are left with case

b) Using Remark 4.2 we first note that the surfaces Y_m may be extended to domains $\tilde{B}_m$ as solution of systems like (4.4) satisfying a differential inequality like (cp. (4.8))

$$|L(Y_m)| \leqq c |\nabla Y_m|^2$$

with a constant c depending only on S, where the sets $(\tilde{B}^m)_{m \in \mathbb{N}}$ exhaust a half-plane $\{v > v_0\}$ for some $v_0 < 0$, and the sequence (Y_m) is locally uniformly bounded in $H^{2,q}$ on this set, for any $q < \infty$. Hence we may extract a subsequence (Y_m) that converges in $C^1(\overline{\mathbb{R}^2_+ \cap B_R(0)}; \mathbb{R}^3)$, for any $R < \infty$, with limit some non-constant function Y belonging to the class $H^{2,q}(\mathbb{R}^2_+ \cap B_R(0); \mathbb{R}^3)$ for any $R < \infty$. Moreover, $Y(\{v=0\}) \subset S$, $\frac{\partial}{\partial v} Y(u,0) \perp T_{Y(u,0)}S$ for all $u \in \mathbb{R}$, and – passing to the limit in (5.2) – $\Delta Y = 0$. Furthermore, for any k we have

$$\begin{aligned}
\int_{B^k} |\nabla Y|^2 dw + \int_B |\nabla X|^2 dw &\leqq \liminf_{m\to\infty} \int_{B^k} |\nabla Y_m|^2 dw + \int_B |\nabla X|^2 dw \\
&= \liminf_{m\to\infty} \Big(\int_{\{w_m\} + \mu_m B^k} |\nabla X_m|^2 dw + \int_{B \setminus (\{w_m\} + \mu_m B^k)} |\nabla X|^2 dw \Big) \qquad (5.3) \\
&\leqq \liminf_{m\to\infty} \int_B |\nabla X_m|^2 dw \leqq 2 \liminf_{m\to\infty} E_{\alpha_m}(X_m).
\end{aligned}$$

Passing to the limit $k \to \infty$ we infer that $|\nabla Y| \in L^2(\mathbb{R}^2_+)$. But then the harmonic function $f = 2 \cdot Y_u \cdot Y_v$ belongs to $C^2 \cap L^1(\mathbb{R}^2_+)$ and vanishes on $\{v=0\}$, and hence may be continued in $C^2 \cap L^1(\mathbb{R}^2)$ to a harmonic function on $\mathbb{R}^2$, simply taking $f(u, -v) = -f(u, v)$. It follows that $f \equiv 0$. Hence the harmonic conjugate of f, $g = |Y_u|^2 - |Y_v|^2$ is constant. Since also $g \in L^1$, $g \equiv 0$, proving conformality (1.2). Conformally reparameterizing Y we hence obtain a surface $X_0 \in H^{1,2}(B; \mathbb{R}^2)$ satisfying conditions (1.1)–(1.4) a.e. By the regularity result of Grüter, Hildebrandt, and Nitsche [7] X_0 also is a classical solution of $P(S)$. Moreover, by conformal invariance of the Dirichlet integral estimate (5.3) conveys to X_0, concluding the proof of Lemma 5.1 and this paper. qed

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