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Inequalities defining orbit spaces*

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Summary. The orbit space of a representation of a compact Lie group has a natural semialgebraic structure. We describe explicit ways of finding the inequalities defining this structure, and we give some applications.

§ 0. Introduction

(0.1) The most ancient prototype of our main theorem is the following: A polynomial x^2+bx+c with real coefficients has real roots if and only if $b^2-4c\geqq 0$. Sylvester generalized this result to the case of a polynomial of degree n. We recall his results below, and we show how they fit into a more general framework.

Let $f(x)=x^n-b_1x^{n-1}+\ldots+(-1)^n b_n$ be a real monic polynomial with roots $a_1,\ldots,a_n$. Recall that $b_j=\sigma_j(a_1,\ldots,a_n)$ where the σ_j are the elementary symmetric functions. Let $\mathrm{Van}=(\mathrm{Van}_{ij})=(a_j^{i-1})$ be the Vandermonde matrix of the a_j, and set $\mathrm{Bez}=\mathrm{Van}\cdot\mathrm{Van}^t$. Then Bez (the "Bezoutiant") has ij entry ψ_{i+j-2}, where $\psi_k=\sum_j a_j^k$. The ψ_k are polynomials $\psi_k(b_1,\ldots,b_n)$ in the b_j, so we consider $\mathrm{Bez}=\mathrm{Bez}(b_1,\ldots,b_n)$ as a function of the b_j.

(0.2) **Theorem** (Sylvester, see [14]). (1) *The roots of f are all real if and only if* Bez *is positive semidefinite.*

(2) *The rank of* Bez *equals the number of distinct roots of f, and its signature equals the number of distinct real roots.*

We will use Sylvester's theorem to show that the orbit space $\mathbf{R}^n/S_n$ of $\mathbf{R}^n$ by the action of the symmetric group S_n is isomorphic to $\{b\in\mathbf{R}^n\colon \mathrm{Bez}(b)$ is positive semidefinite$\}$. Our main theorem gives a similar description of the orbit space of an arbitrary representation of a compact Lie group.

(0.3) Let W be a real representation space of the compact Lie group K. By a result of Hilbert and Hurwitz (see [18] p. 274), the graded algebra $\mathbf{R}[W]^K$ of

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K-invariant polynomial functions on W is finitely generated. Let $p_1, \ldots, p_m$ generate $\mathbf{R}[W]^K$, let I be their ideal of relations in $\mathbf{R}[y_1, \ldots, y_m]$, and let $Z \subseteq \mathbf{R}^m$ be the corresponding irreducible real algebraic set. Let $p=(p_1, \ldots, p_m)$: $W \to \mathbf{R}^m$. Then p^* gives an isomorphism of $\mathbf{R}[Z]$ with $\mathbf{R}[W]^K$. Since p is a polynomial mapping, its image $X \subseteq Z$ is a semialgebraic set.

Give W and X their classical topologies, and give the orbit space W/K the quotient topology. The following result shows that X is, essentially, just W/K.

(0.4) **Proposition** (see [16]). *The mapping p is proper, and it induces a homeomorphism $\bar{p}$: $W/K \to X$.*

(0.5) Our main theorem gives explicit inequalities describing X. We will give some classical examples where the inequalities are known, but first we recall the following:

Let C be a real symmetric matrix. We write $C \geqq 0$ to indicate that C is positive semidefinite.

(0.6) **Proposition.** *Let C be a real symmetric matrix. Then $C \geqq 0$ if and only if $C_\alpha \geqq 0$ for all α, where $\{C_\alpha\}$ is the set of determinants of principal* (i.e. *symmetric*) *minors of C.*

(0.7) *Example.* Let $K=S_n$ act on $W=\mathbf{R}^n$ as usual. Then $\mathbf{R}[W]^K = \mathbf{R}[\sigma_1, \ldots, \sigma_n]$, and we set $p=(\sigma_1, \ldots, \sigma_n)$: $W \to \mathbf{R}^n$. Since the σ_i are algebraically independent, $Z=\mathbf{R}^n$. By (0.2.1), we have $X=\{z \in \mathbf{R}^n\colon \operatorname{Bez}(z) \geqq 0\}$, and using (0.6) we get inequalities describing X.

(0.8) *Example.* Let W be the space of $n \times q$ real matrices, and let $K=O(n) = O(n, \mathbf{R})$ act by left multiplication. Then W is just q copies of the standard representation of K on $\mathbf{R}^n$. By classical invariant theory, the K-invariants are generated by the inner products of the various copies of $\mathbf{R}^n$, i.e. of the columns of our $n \times q$ matrices. Thus we can define

$$p\colon W \to \mathrm{Sym}_q$$
$$A \mapsto A^t A$$

where Sym_q denotes the space of real symmetric $q \times q$ matrices. It is an easy exercise to show that X is the set of all matrices $B \in \mathrm{Sym}_q$ such that:

(0.8.1) $$\operatorname{rank} B \leqq n.$$

(0.8.2) $$B \geqq 0.$$

Then Z is defined by (0.8.1), i.e. by the condition that the determinants of all $(n+1) \times (n+1)$ minors of B are zero. The inequalities for X are obtained from (0.8.2) using Proposition (0.6)

(0.9) We formulate the main theorem: Let (,) denote a K-invariant inner product on W as well as the dual inner product on W^*. The differentials dp_i: $W \to W^*$ are K-equivariant, and the functions $w \mapsto (dp_i(w), dp_j(w))$ give an $m \times m$ symmetric matrix valued function $\widetilde{\mathrm{Grad}}(w)$ with entries in $\mathbf{R}[W]^K$. There is a unique matrix valued function Grad on Z such that $\widetilde{\mathrm{Grad}}(w) = \mathrm{Grad}(p(w))$ for all $w \in W$.

(0.10) **Main Theorem.** $X=\{z\in Z\colon \mathrm{Grad}(z)\geqq 0\}$.

(0.11) *Remarks.* (1) Abud and Sartori [1] and [2] investigated the problem of finding inequalities defining X as a subset of $\mathbf{R}^m$, and they seem to have been the first to have realized that the inequalities of (0.10) hold on X.

(2) Let $K=S_n$ and $W=\mathbf{R}^n$ as in Example (0.7). Choose generators $p_j=1/j\,\psi_j$ of $\mathbf{R}[W]^K$ (see (0.1)). Then $\widetilde{\mathrm{Grad}}=(\psi_{i+j-2})$, so Theorem (0.2.1) follows from the main theorem. See also (5) below.

(3) In Example (0.8) the main theorem gives a redundant set of inequalities. We will recover the inequalities (0.8.2) exactly using a version of the main theorem proved in § 4.

(4) Choose an orthonormal basis $x_1, \ldots, x_n$ of W relative to (,). Then, relative to these co-ordinates, $\widetilde{\mathrm{Grad}}$ is the matrix of inner products of the gradients of the p_j; equivalently, $\widetilde{\mathrm{Grad}}=JJ^t$ where $J=(\partial p_i/\partial x_j)$ is the Jacobian of p. Note that J generalizes the Vandermonde matrix of the symmetric group case (c.f. (0.1)).

(5) The sets X and Z and our descriptions of them depend upon our choice of generators for $\mathbf{R}[W]^K$, but not in a serious way: Let $\mathscr{Z}$ denote the variety of real maximal ideals of $R[W]^K$ and let $\mathscr{X}=\pi(W)$, where $\pi\colon W\to\mathscr{Z}$ is dual to the inclusion $\mathbf{R}[W]^K\subseteq\mathbf{R}[W]$. Then Z and $\mathscr{Z}$ are canonically isomorphic, and inequalities defining X as a subset of Z, thought of as inequalities involving elements of $\mathbf{R}[\mathscr{Z}]=\mathbf{R}[W]^K$, define $\mathscr{X}$ as a subset of $\mathscr{Z}$. Hence changing the choice of generators may change the inequalities, but not the set they describe. In § 4 we will see other methods to get inequalities describing X (and $\mathscr{X}$).

(0.12) Sketch of the proof of (0.10): Let K, W, etc. be as in (0.9). Let $V=W\otimes_{\mathbf{R}}\mathbf{C}$, and consider p as a mapping of V to $\mathbf{C}^m$. Let $G=K_{\mathbf{C}}$ be the complexification of K (see [6]). Then G is reductive, and $\mathbf{C}[V]^G\simeq\mathbf{R}[W]^K\otimes_{\mathbf{R}}\mathbf{C}$ is generated by the p_j. Let (,) denote the G-invariant bilinear forms on V and V^* extending the K-invariant bilinear forms on W and W^*. Let $z\in Z$. Using results of Kempf-Ness [11] and Dadok-Kac [7] we find a point $v=w_1+iw_2$ in $V=W\oplus iW$ with the following properties: the orbit Gv is closed, $p(v)=z$, the isotropy group $G_v=(K_v)_{\mathbf{C}}$, and $\bar{v}=kv$ for some $k\in K$. Using Luna's slice theorem [12] we compute $\mathrm{span}_{\mathbf{R}}\{dp_j(v)\}$, and we find that the linear functional $\lambda(v')\colon=(v', iw_2)$ is in this space.

Suppose now that $\mathrm{Grad}(z)\geqq 0$. Then $(\lambda,\lambda)\geqq 0$ (see (0.13) below). But $(\lambda,\lambda)=(iw_2, iw_2)=-(w_2,w_2)\leqq 0$. Hence $w_2=0$ and $v=w_1\in W$, i.e. $z\in X$.

We have used the following:

(0.13) *Remark.* Let β be a symmetric bilinear form on a real vector space U. Let $\{u_1,\ldots,u_r\}$ span U, and let A be the matrix with ij entry $\beta(u_i,u_j)$. Then $\mathrm{rank}\,A=\mathrm{rank}\,\beta$ and signature $A=$ signature β. In particular, $\beta\geqq 0$ (i.e. β is positive semidefinite) if and only if $A\geqq 0$.

(0.14) In §§ 1–2 we recall results of Luna, Kempf-Ness and Dadok-Kac. In § 3 we prove the main theorem. We determine when $X=Z$ and, more generally, we describe the boundary of X in Z. In §§ 4–5 we discuss other methods to obtain inequalities defining X, e.g. using spaces of covariants. In § 6 we discuss some ideas for generalizing our results. In § 7 we discuss the equivariant version of Hilbert's 17th Problem for real reductive groups. We establish positive results in the compact case and we give some counterexamples in the non-compact case.

§ 1. Differentials

We derive a simple consequence of Luna's slice theorem.

(1.1) Let S be a complex affine G-variety, where G is a reductive complex algebraic group. Then $\mathbf{C}[S]^G$ is finitely generated, and we let S/G denote the associated affine variety. Let $\pi\colon S\to S/G$ denote the canonical map. Then π is surjective, and each fiber of π contains a unique closed orbit (c.f. [12]).

(1.2) We specialize to the case of a representation space V of G. There is an associated representation of $\mathfrak{g}$ on V, where $\mathfrak{g}$ is the Lie algebra of G. Let $v\in V$, and let $T_v(Gv)$ denote the tangent space at v to the orbit Gv. Then $T_v(Gv)=\mathfrak{g}v$, where $\mathfrak{g}v=\{Av\colon A\in\mathfrak{g}\}$.

(1.3) *Definitions.* Set $\Delta(v)=\{\lambda\in V^*\colon \lambda$ is G_v-invariant and annihilates $\mathfrak{g}v\}$. $D(v)=\{df(v)\colon f\in\mathbf{C}[V]^G\}$.

(1.4) *Remarks.* (1) Suppose that $f_1,\dots,f_s$ generate $\mathbf{C}[V]^G$. Then the chain rule for differentiation shows that the $df_i(v)$ generate $D(v)$.

(2) Let $f\in\mathbf{C}[V]^G$. Then f is constant on Gv, hence $df(v)$ annihilates $\mathfrak{g}v$. If $g\in G$, then $df(v)=d(f\circ g)(v)=df(gv)\circ g$, hence $df(v)$ is G_v-invariant. Thus $D(v)\subseteq\Delta(v)$.

(1.5) **Proposition.** *Suppose that Gv is closed. Then $D(v)=\Delta(v)$.*

Proof. By Matsushima's theorem, $H=G_v$ is reductive (see [12] or Remark (2.2)). There is an H-invariant subspace N of V complementary to $\mathfrak{g}v$ and a unique H-stable decomposition $N=N^H\oplus N_1$. Restriction to N clearly gives an isomorphism of $\Delta(v)$ with $(N^*)^H\simeq(N^H)^*$. We show that the image of $D(v)\subseteq\Delta(v)$ is already $(N^*)^H$, hence $D(v)=\Delta(v)$.

Let $\psi\colon N\to V$ send $n\in N$ into $v+n$. Then ψ^* maps $\mathbf{C}[V]^G$ to $\mathbf{C}[N]^H$. Let I (resp. J) be the ideal in $\mathbf{C}[V]^G$ (resp. $\mathbf{C}[N]^H$) of functions vanishing at v (resp. 0). Then ψ^* induces $\delta\psi\colon I/I^2\to J/J^2$. Luna's slice theorem [12] shows that $\delta\psi$ is an isomorphism, as follows: Let

$$\phi\colon G\times_H N\to V$$
$$[g,n]\mapsto g(v+n)$$

where $G\times_H N$ is the usual twisted product. Then $\phi([e,0])=v$, and ϕ is étale at $[e,0]$. Since ϕ is equivariant, it induces a morphism $\phi/G\colon (G\times_H N)/G\to V/G$ which sends the point $[e,0]^\#$ corresponding to $[e,0]$ to the point $v^\#$ corresponding to v. Luna shows that ϕ/G is étale at $[e,0]^\#$, which implies that ϕ/G induces an isomorphism of the tangent (and cotangent) spaces at $[e,0]^\#$ and $v^\#$. But the cotangent space at $v^\#$ is isomorphic to I/I^2 and the cotangent space of $N/H\simeq(G\times_H N)/G$ at $[e,0]^\#$ is isomorphic to J/J^2, and the isomorphism induced by ϕ/G is easily seen to be $\delta\psi$.

Since $\delta\psi$ is an isomorphism, $\psi^*(I)+J^2=J$, and

$$\{d(h\circ\psi)(0)\colon h\in\mathbf{C}[V]^G\}=\{df(0)\colon f\in\mathbf{C}[N]^H\}.$$

But the latter space clearly equals $(N^*)^H$, and the former space is the image of $D(v)$ in $(N^*)^H$ via restriction to N. Hence $D(v)=\Delta(v)$. □

(1.6) *Remark.* The analogue of Proposition (1.5) for compact groups has been noticed before; Sartori [15] gave a proof, and there is also a proof buried in [16].

§ 2. Kempf-Ness theory

Let V be a complex representation space of the compact Lie group K, and let $G=K_{\mathbf{C}}$ denote the complexification of K. Let $\langle\ ,\ \rangle$ be a K-invariant hermitian form on V with associated norm $\|\ \|$. Let $v\in V$ and consider the function F_v: $G\to\mathbf{R}^+$ sending g to $\|gv\|^2$.

(2.1) **Theorem.** *Let* V, etc. *be as above.*

(1) F_v *has a critical point if and only if* Gv *is closed.*
(2) *All critical points of* F_v *occur at minima.*

Assume that $F_v(e)$ *is a minimum. Then*

(3) $Kv=\{v'\in Gv\colon \|v'\|=\|v\|\}$.
(4) $G_v=(K_v)_{\mathbf{C}}$.

Proof. All but (4) can be found in Kempf-Ness [11], and (4) is in Dadok-Kac [7]. We review the main points of the proofs of (3) and (4).

Let $\mathfrak{k}$ (resp. $\mathfrak{k}_v$) denote the Lie algebra of K (resp. K_v). Recall that $\mathfrak{g}=\mathfrak{k}\oplus i\mathfrak{k}$. Let $Y\in i\mathfrak{k}$, and consider the function $a(s)=\|\exp(sY)v\|^2$; $s\in\mathbf{R}$. There is a maximal toral subalgebra $\mathfrak{t}\subseteq\mathfrak{k}$ such that $Y\in i\mathfrak{t}$, and using the corresponding weight decomposition of V one easily shows the following (see [7]):

(*) Either $Yv=0$ or $a''(s)>0$ for all s.

Let $g\in G$. Then g can be uniquely written in the form $k\exp(Y)$ where $k\in K$ and $Y\in i\mathfrak{k}$ ([10] Ch. IX). If $\|gv\|=\|v\|$, then $a(1)=a(0)$ where $a(s)$ is defined above. But $a'(0)=0$ since $F_v(e)$ is a minimum, and it follows that $a''(s)=0$ for some s with $0<s<1$. By (*), $Y\in i\mathfrak{k}_v$, hence $gv=kv$ proving (3). If $gv=v$, then $k\in K_v$, so $G_v=K_v\exp(i\mathfrak{k}_v)=(K_v)_{\mathbf{C}}$ yielding (4). Part (2) follows from similar considerations, and (1) is an application of the Hilbert-Mumford criterion. □

(2.2) *Remark.* Note that (4) above implies Matsushima's theorem.

(2.3) **Corollary.** *Let* $v\in V$. *The following are equivalent*:

(1) $\langle\mathfrak{k}v, v\rangle=0$.
(2) $\langle\mathfrak{g}v, v\rangle=0$.
(3) *There is an* $f\in\mathbf{C}[V]^G=\mathbf{C}[V]^K$ *such that* $df(v)(v')=\langle v', v\rangle$ *for all* $v'\in V$.

Proof. Suppose that (1) (or equivalently (2)) holds. Then the function F_v has a critical point at $e\in G$, hence Gv is closed and $G_v=(K_v)_{\mathbf{C}}$. The linear functional

$v' \mapsto \langle v', v\rangle$ vanishes on $\mathfrak{g}v$ and is K_v-invariant, hence also G_v-invariant. Proposition (1.5) then gives (3). Conversely, if (3) holds, then $\langle \mathfrak{g}v, v\rangle = df(v)(\mathfrak{g}v)=0$ (c.f. (1.4)). □

(2.4) *Remarks.* (1) The question of whether or not (2.3.1) implies (2.3.3) has been raised in the mathematical physics literature (see [9]).

(2) The subset M of V satisfying the equivalent conditions of Corollary (2.3) is real algebraic, K-stable, and $M/K \simeq V/G$.

We now specialize to the case where $V = W \otimes_{\mathbf{R}} \mathbf{C} = W \oplus iW$, and W is a representation space of K. Then $\mathbf{C}[V]^G \simeq \mathbf{R}[W]^K \otimes_{\mathbf{R}} \mathbf{C}$ and the quotient variety V/G has a real structure. If $p=(p_1, \ldots, p_m)$: $W \to \mathbf{R}^m$ is as in (0.3), then the p_j, considered now as polynomials on V, generate $\mathbf{C}[V]^G$. The image $p(V) \subseteq \mathbf{C}^m$ is isomorphic to V/G (see (1.1)). Recall that $p(W) = X \subseteq Z$, where $Z = p(V) \cap \mathbf{R}^m$ is (isomorphic to) the real points of V/G. We assume that our K-invariant hermitian form $\langle\ ,\ \rangle$ restricts to a K-invariant inner product $(\ ,\ )$ on W.

(2.5) **Proposition.** *Let* W, K, etc. *be as above. Let* $z \in Z$ *and choose* $v \in V$ *such that* Gv *is closed,* $p(v)=z$ *and* $\|v\| \leqq \|gv\|$ *for all* $g \in G$. *Then*

(1) $\bar{v} = kv = k^{-1}v$ *for some* $k \in K$.
(2) $G_{\bar{v}} = G_v = \bar{G}_v = (K_v)_{\mathbf{C}}$.
(3) *If* $z \in X$, *then* $v \in W$.

Proof. The polynomials p_j are real, so $p(\bar{v}) = \overline{p(v)} = p(v)$. Now $G\bar{v} = \bar{G}\bar{v} = \overline{Gv}$ is closed, so both v and $\bar{v}$ lie on closed orbits and have the same image in V/G. Thus $G\bar{v} = Gv$, and since $\|v\| = \|\bar{v}\|$, we have $\bar{v} = kv$ for some $k \in K$. Write $v = w_1 + iw_2$ where w_1, $w_2 \in W$. Then $kw_1 = w_1$, $kw_2 = -w_2$, hence $k^{-1}v = \bar{v}$ and we have proved (1). Part (2) follows from (2.1.4) and the equalities $K_{w_1} \cap K_{w_2} = K_v = K_{\bar{v}}$. Finally, if $w \in W$, then $\langle \mathfrak{k}w, w\rangle = (\mathfrak{k}w, w) = 0$, so Gw is closed and $\|w\| \leqq \|gw\|$ for all $g \in G$. If $p(v) = p(w) \in X$, then we must have $v \in Kw \subseteq W$. □

(2.6) *Remark.* The set of isotropy groups K_v where Gv is closed and $\|v\| \leqq \|gv\|$ for all $g \in G$ is the same as the set of isotropy groups K_w, $w \in W$ (see [17] § 5).

(2.7) *Definitions.* Let v and k be as above. Set

$$D_{\mathbf{R}}(v) = \operatorname{span}_{\mathbf{R}}\{dp_j(v); j=1, \ldots, m\}.$$
$$\Delta_{\mathbf{R}}(v) = \{\lambda \in \Delta(v) \colon \lambda \circ k = \bar{\lambda}\}.$$

(2.8) **Corollary.** *Let* v, k, etc. *be as in* (2.5). *Then* $D_{\mathbf{R}}(v) = \Delta_{\mathbf{R}}(v)$.

Proof. For any j and $x \in V$, $dp_j(v)(kx) = dp_j(k^{-1}v)(x) = dp_j(\bar{v})(x) = \overline{dp_j(v)}(x)$, hence $D_{\mathbf{R}}(v) \subseteq \Delta_{\mathbf{R}}(v)$. Now note that complex conjugation (resp. composition with k) gives a conjugate linear (resp. linear) isomorphism between $\Delta(v)$ and $\Delta(\bar{v})$. Composing these isomorphisms we get a conjugate linear isomorphism $\tau \colon \Delta(v) \to \Delta(v)$ whose fixed points are $\Delta_{\mathbf{R}}(v)$. By (1.4.1) and (1.5), $\Delta(v) \simeq D_{\mathbf{R}}(v) \otimes_{\mathbf{R}} \mathbf{C}$, hence the τ fixed points in $\Delta(v)$ are $D_{\mathbf{R}}(v)$. □

§ 3. The Main Theorem

(3.1) Let W, K, V, G, p and X be as in (2.5). Let $(\ ,\ )$ be a K-invariant inner product on W, and let the symbol $(\ ,\ )$ also denote the corresponding symmetric non-degenerate forms on W^*, V and V^*. We have a matrix valued function Grad on $p(V)$, where $\mathrm{Grad}(p(v)) = (dp_i(v), dp_j(v))$, $v \in V$. On Z, the entries of Grad are real, and our main theorem (0.10) states that $X = \{z \in Z: \mathrm{Grad}(z) \geqq 0\}$.

Proof of (0.10). Define $\langle x, y \rangle := (x, \bar{y})$ for $x, y \in V$. Then $\langle\ ,\ \rangle$ is a K-invariant hermitian form on V. Let $z \in Z$, and choose $v \in V$ so that Gv is closed, $p(v) = z$, and $\|v\| \leqq \|gv\|$ for all $g \in G$. By Corollary (2.3), the linear form $\lambda_1(v') := \langle v', v \rangle = (v', \bar{v})$ lies in $D(v)$. Since the function $v \mapsto (v, v)$ is G-invariant, the form $\lambda_2(v') := (v', v)$ also lies in $D(v)$. Define $\lambda = \frac{1}{2}(\lambda_2 - \lambda_1)$, and write $v = w_1 + i w_2$ where $w_1, w_2 \in W$. Then $\lambda(v') = (v', i w_2)$. By (2.5.1) and (2.8), $\lambda \in \Delta_{\mathbf{R}}(v) = D_{\mathbf{R}}(v)$, and $(\lambda, \lambda) = (i w_2, i w_2) = -(w_2, w_2) \leqq 0$. Suppose that $\mathrm{Grad}(z) \geqq 0$. Then $(\lambda, \lambda) \geqq 0$ by (0.13), and we must have $w_2 = 0$. Hence $v = w_1 \in W$ and $z = p(w_1) \in X$. □

Let z, v, etc. be as above, but do not assume that $\mathrm{Grad}(z) \geqq 0$. Let $k \in K$ be as in (2.5), so that $kv = \bar{v}$. Note that $\Delta(v)$ (equivalently, $\Delta_{\mathbf{R}}(v)$) is stable under the action of k if and only if $\Delta(v)$ (equivalently, $\Delta_{\mathbf{R}}(v)$) is stable under complex conjugation. If K is finite, then $\Delta(v) = (V^*)^{K_v}$ is k-stable.

(3.2) **Proposition.** *Let z, v, k* etc. *be as above. Assume that $\Delta(v)$ is k-stable* (e.g. *K is finite*). *Let $\Delta_{\mathbf{R}}^{+}(v)$* (resp. *$\Delta_{\mathbf{R}}^{+}(v)$*) *denote the* $+1$ (resp. -1) *eigenspace of k acting on $\Delta_{\mathbf{R}}(v)$. Then*

(1) rank $\mathrm{Grad}(z) = \dim \Delta(v)$.

(2) signature $\mathrm{Grad}(z) = \dim \Delta_{\mathbf{R}}^{+}(v) - \dim \Delta_{\mathbf{R}}^{-}(v)$.

Proof. If $\lambda \in \Delta_{\mathbf{R}}^{+}(v)$, then $\bar{\lambda} = \lambda$, i.e. λ is real valued on W. Thus $(\ ,\ )$ is positive definite on $\Delta_{\mathbf{R}}^{+}(v)$, and similarly $(\ ,\ )$ is negative definite on $\Delta_{\mathbf{R}}^{-}(v)$. Hence $(\ ,\ )$ is nondegenerate on $\Delta_{\mathbf{R}}(v)$, and (1) and (2) follow from Remark (0.13). □

(3.3) *Example.* It can happen that rank $\mathrm{Grad}(z) < \dim \Delta(v)$: Let $K = O(n)$ act diagonally on $W = \mathbf{R}^n \oplus \mathbf{R}^n$, $n \geqq 2$. Set $\alpha = x \cdot x$, $\beta = y \cdot y$ and $\gamma = x \cdot y$ for $(x, y) \in \mathbf{R}^n \oplus \mathbf{R}^n$. Then $\mathbf{R}[W]^K = \mathbf{R}[\alpha, \beta, \gamma]$ and we may take $p = (\alpha\, \beta, \gamma)$: $W \to Z = \mathbf{R}^3$ (c.f. (0.8)). Let $z = (z_1, z_2, z_3) \in \mathbf{R}^3$ and choose $v \in V$ as in (2.5). Then one easily calculates that rank $dp = 3$ (resp. rank $\mathrm{Grad}(z) = 3$) if and only if $z_1 z_2 - z_3^2 \neq 0$ (resp. $(z_1 + z_2)(z_1 z_2 - z_3^2) \neq 0$).

(3.4) *Remark.* Let $K = S_n$, $W = \mathbf{R}^n$ and $p = (\sigma_1, \ldots, \sigma_n)$ as in Example (0.7). We use Proposition (3.2) to establish Sylvester's result (0.2.2): Let $z \in \mathbf{R}^n$, and choose v and k as in (2.5). Write $z = (z_1, \ldots, z_n)$ and $v = (v_1, \ldots, v_n)$. Recall that the v_j are the roots of the polynomial $f(x) = x^n - z_1 x^{n-1} + \ldots + (-1)^n z_n$.

Note that $\Delta(v)^* = V^{K_v} = \{y \in V: y_i = y_j \text{ if } v_i = v_j\}$. Thus $\dim \Delta(v)$ is the number of distinct roots of f, which by (3.2.1) equals rank $\mathrm{Grad}(z)$, where $\mathrm{Grad}(z) = (d\sigma_i(v), d\sigma_j(v))$. The signature of $\mathrm{Grad}(z)$ is $\dim \Delta_{\mathbf{R}}^{+}(v) - \dim \Delta_{\mathbf{R}}^{-}(v)$. Since $kv = \bar{v}$, we see that $\dim \Delta_{\mathbf{R}}^{-}(v)$ equals one-half the number of distinct non-real roots of f. This suffices to establish (0.2.2), with $\mathrm{Bez}(z)$ replaced by $\mathrm{Grad}(z)$. Now $\mathrm{Bez}(z) = (dq_i(v), dq_j(v)) = (\psi_{i+j-2}(z))$ (see (0.1)), where $q_i(v) = (1/i) \sum v_j^i$, and

Grad(z) and Bez(z) have the same rank and signature by Remarks (0.13) and (1.4).

As noted by the referee, similar considerations give a method of counting positive roots of f (also due to Sylvester, see [14]): Let $K'=(\pm 1)^n \rtimes S_n$ act on $W=\mathbf{R}^n$ in the standard way. Then $\mathbf{R}[W]^{K'}$ is generated by $\{p'_i\}$ or $\{q'_i\}$ where $p'_i(x_1, \ldots, x_n)=\sigma_i(x_1^2, \ldots, x_n^2)$ and $q'_i(x_1, \ldots, x_n)=(1/2i)\sum x_j^{2i}$; $i=1, \ldots, n$. Let Bez$'(z)$ denote the matrix with ij entry $\psi_{i+j-1}(z)$. Then Bez$'(z)=(dq'_i(v), dq'_j(v))$ if $z_i=p'_i(v)$; $i=1, \ldots, n$. One establishes as above that the rank of Bez$'(z)$ equals the number of distinct non-zero roots of f, and that its signature is the difference between the number of distinct positive and negative real roots.

(3.5) We now determine when $X=Z$, and we determine the interior of X as a subset of Z.

(3.6) **Lemma.** *The following are equivalent:*

(1) $X=Z$.

(2) *There is no* $k\in K$ *with eigenvalue* -1 *on* W.

Proof. Assume (1), and let $w\in W$ and $k\in K$ with $kw=-w$. We may assume that the first element p_1 of $p=(p_1, \ldots, p_m)$ is the function $x\mapsto(x, x)$, where (,) is a K-invariant inner product on W. Since $kw=-w$, every polynomial invariant of odd degree vanishes at w. Hence $p(iw)=-p(w)\in Z=X$. There is thus $w'\in W$ with $p(w')=p(iw)$. But $p_1(w')\geqq 0$ while $p_1(iw)\leqq 0$. Hence $w=0$, proving (2).

On the other hand, if $X\neq Z$, choose $z\in Z-X$ and let $v=w_1+iw_2$ and $k\in K$ be as in (2.5). Then $w_2\neq 0$ and $kw_2=-w_2$. □

(3.7) **Proposition.** *Assume that the representation of* K *on* W *is faithful. Then* $X=Z$ *if and only if* K *is a finite group of odd order.*

Proof. Any torus in K contains elements of order 2, and such elements have eigenvalue -1 on W. If K is finite of even order, then K contains elements of order 2. Hence $X=Z$ implies that K is finite of odd order. Conversely, if K has odd order, then no elements have eigenvalue -1, and $X=Z$. □

(3.8) *Examples.* Let $K=\mathbf{Z}/n$ act on $W=\mathbf{R}^2$ by rotations. Then $\mathbf{R}[W]^K\simeq\mathbf{R}[a, b, c]/(a^n=b^2+c^2)$, where $a=|z|^2$, $b=\operatorname{Re} z^n$ and $c=\operatorname{Im} z^n$; $z\in\mathbf{C}\simeq W$. If n is odd, then the inequality $a\geqq 0$ is forced by the equation $a^n=b^2+c^2$, and $X=Z$. If n is even, we must add the inequality $a\geqq 0$ to describe X.

Using (3.7) we easily determine the interior of X in Z: Let $x=p(w)$. Let N_w denote a K_w-complement to $T_w(Kw)$ in W. The representation (N_w, K_w) of K_w on N_w is called the *slice representation* at w. The usual differentiable slice theorem (see [17] §1) says that the germ of the semialgebraic set N_w/K_w at the orbit 0 is analytically isomorphic to W/K near Kw, and this goes also for the germs of their Zariski closures. Hence we have:

(3.9) **Corollary.** *Let* x, w, K_w, *and* N_w *be as above. Then* x *is in the interior of* X *if and only if the image of* K_w *in* $GL(N_w)$ *is a finite group of odd order.*

(3.10) **Corollary.** *Suppose that* K *is finite and acts effectively on* W. *Then the boundary of* X *in* Z *is the image of* $\bigcup_{k\in\mathcal{T}} W^k$, *where* $\mathcal{T}\subseteq K$ *is the set of elements of order 2.*

(3.11) It is well-known that, up to isomorphism, there are only finitely many slice representations (N_i, K_i) $i=0,\dots,r$ which occur in W. We get a corresponding stratification $X=\coprod X_i$. There is a unique type of slice representation (N_0, K_0) where K_0 acts trivially on N_0. The corresponding orbits and isotropy groups are said to be *principal.* Moreover, X_0 is exactly the set of smooth points of X (Bierstone [3]). Hence

(3.12) **Corollary.** *The interior of X in Z consists entirely of smooth points if and only if the images of the non-principal isotropy groups K_i in $GL(N_i)$, $i=1,\dots,r$, are not finite of odd order.*

In (3.12) we can even restrict ourselves to "subprincipal" slice representations ([17] § 11). If Z is smooth, i.e. $\mathbf{R}[W]^K$ is a regular ring, then the interior of X in Z is X_0, and the conditions above on the K_i are automatically satisfied!

(3.13) Let Z' denote the points of Z where Grad has maximal rank. (Z' consists of principal orbits of the representation of $G=K_{\mathbf{C}}$ on V, see [17] § 5). Then Z' has finitely many components, and the signature of Grad is constant on each of them. In general, the signature of Grad does not separate these components and all possible signatures do not occur.

(3.14) *Example.* Take for K and W two copies of the usual representation of $\{\pm 1\}$ on $\mathbf{R}$. Then $Z \simeq \mathbf{R}^2$ and $Z'=(\mathbf{R}^2\text{-coordinate axes})$. The matrix Grad has signature 0 in two quadrants.

(3.15) *Example.* Let $W=\mathbf{R}^2$ and K the dihedral group of order 6. Let a, b, and c be as in Example (3.8) (with $n=3$). Then the K-invariants are generated by a and b, and X is defined by the inequality $a^3 \geqq b^2$. The matrix Grad has signature 2 where $a^3 > b^2$ and signature 0 where $a^3 < b^2$, but it never has signature -2.

§ 4. Using covariants to detect real points

(4.1) Let K be a compact Lie group and let $G=K_{\mathbf{C}}$. Let S be a complex affine G-variety and let $\pi\colon S \to S/G$ be the canonical map. We suppose that the action $G \times S \to S$ is defined over $\mathbf{R}$. (The real structure on G comes from its representation as $K_{\mathbf{C}}$.) Then S has a conjugation σ such that $\sigma(gs)=\bar{g}\sigma(s)$; $s\in S$, $g\in G$. Let $S(\mathbf{R})$ denote the real (i.e. σ-fixed) points of S, and let $\mathbf{R}[S]$ denote the elements f of $\mathbf{C}[S]$ such that $f\circ\sigma=\bar{f}$. The quotient variety S/G has a real structure induced from that of S.

(4.2) *Problem.* Let $z\in(S/G)(\mathbf{R})$. When is z in $\pi(S(\mathbf{R}))$?

Our main theorem solves this problem when $G \times S \to S$ is a representation.

(4.3) **Proposition.** *Let S be as above.*

(1) *There is a representation space W of K and a G-equivariant embedding α: $S \to V = W \otimes_{\mathbf{R}} \mathbf{C}$ so that $\alpha(\sigma s)=\overline{\alpha(s)}$ for all $s\in S$.*

(2) *Let $z\in(S/G)(\mathbf{R})$ and let O denote the closed orbit in S above z. Then O is defined over $\mathbf{R}$, and z lies in $\pi(S(\mathbf{R}))$ if and only if $O(\mathbf{R}) \neq \emptyset$.*

Proof. There is a K-stable finite dimensional subspace $U \subseteq \mathbf{R}[S]$ such that U generates $\mathbf{R}[S]$. Set $\alpha(s)(f)=f(s)$ for $s\in S$, $f\in U\otimes_{\mathbf{R}}\mathbf{C}$. Then $\alpha: S\to U^*\otimes_{\mathbf{R}}\mathbf{C}$ satisfies (1) with $W=U^*$. Part (2) follows from (1) and Proposition (2.5). □

(4.4) *Remarks*. (1) In case K is real reductive (but not necessarily compact), Proposition (4.3) still holds, except that O becomes a finite union of closed orbits. The proof requires a result of Birkes [4] (c.f. Luna [13]).

(2) Let V be as in (4.3.1), and identify S with $\alpha(S)\subseteq V$. Then (4.3.2) shows that $\rho(W)\cap(S/G)(\mathbf{R})=\rho(S(\mathbf{R}))$, where $\rho\colon V\to V/G$ is the canonical map. Hence our main theorem solves Problem (4.2), but the criterion is not intrinsic to S.

(4.5) By Proposition (4.3), we may reduce Problem (4.2) to the case where S is homogeneous, which we assume for the rest of this section. We use covariants to try to detect whether or not $S(\mathbf{R})\neq\emptyset$.

Let L be a real K-representation space with invariant inner product $(\ ,\ )$. Let $M=L\otimes_{\mathbf{R}}\mathbf{C}$, and extend $(\ ,\ )$ to a (G-invariant) complex bilinear form on M. Let $\mathrm{Mor}_G(S,M)$ denote the space of G-equivariant morphisms from S to M, and let $\mathrm{Mor}_G(S,M)_{\mathbf{R}}$ denote the subset of morphisms f satisfying $f(\sigma s)=\overline{f(s)}$, $s\in S$. If $t\in S(\mathbf{R})$, then $f\mapsto f(t)$ gives an isomorphism of $\mathrm{Mor}_G(S,M)_{\mathbf{R}}$ with L^{K_t}, hence the constant functions $s\mapsto(f(s), f(s))$, $f\in\mathrm{Mor}_G(S,M)_{\mathbf{R}}$ are all non-negative. We determine when the condition $(\ ,\ )\geqq 0$ on $\mathrm{Mor}_G(S,M)_{\mathbf{R}}$ forces $S(\mathbf{R})\neq\emptyset$.

By embedding S as in (4.3) and applying Proposition (2.5) we obtain the following:

(4.6) **Proposition.** *There is a point $t\in S$ and $k\in K$ such that*

(1) $\sigma(t)=kt=k^{-1}t$.

(2) $G_t=(K_t)_{\mathbf{C}}$.

(3) *If $S(\mathbf{R})\neq\emptyset$, then $t\in S(\mathbf{R})$.*

(4.7) **Proposition.** *Let t,k* etc. *be as above. Then $(\ ,\ )\geqq 0$ on* $\mathrm{Mor}_G(S,M)_{\mathbf{R}}$ *if and only if*

(1) *k acts trivially on L^{K_t}, or*

(2) $S(\mathbf{R})\neq\emptyset$.

Proof. Let $f\in\mathrm{Mor}_G(S,M)_{\mathbf{R}}$. Then $f(t)\in M^{G_t}=M^{K_t}=L^{K_t}\otimes_{\mathbf{R}}\mathbf{C}$. Moreover, $kf(t)=f(kt)=f(\sigma t)=\overline{f(t)}$. If k acts trivially on L^{K_t}, then $f(t)=\overline{f(t)}\in L^{K_t}$, and $(f,f)\geqq 0$. Hence (1) (or (2)) implies that $(\ ,\ )\geqq 0$.

Suppose that (1) and (2) fail. Then $k\notin K_t$ while $k^2\in K_t$. Thus k, acting on L^{K_t}, has order 2, and there is $0\neq l\in L^{K_t}$ with $kl=-l$. Define $f\colon S\to M$ by $f(gt)=igl$. Since $K_t\subseteq K_l$ and $(K_t)_{\mathbf{C}}=G_t$, we see that f is a well-defined G-morphism. Moreover, $f(\sigma(gt))=f(\bar{g}kt)=i\bar{g}kl=\overline{igl}=\overline{f(gt)}$, and $f\in\mathrm{Mor}_G(S,M)_{\mathbf{R}}$. But $(f,f)=-(l,l)<0$, so $(\ ,\ )$ fails to be positive semidefinite. □

(4.8) **Corollary.** *Suppose that every element (or just every element of order 2) of* $\mathrm{Norm}_K(K_t)/K_t$ *acts non-trivially on L^{K_t}. If $(\ ,\ )\geqq 0$ on $\mathrm{Mor}_G(S,M)_{\mathbf{R}}$, then $S(\mathbf{R})\neq\emptyset$.*

Let W, K, etc. be as in (0.3). Let $f_1,\dots,f_r$ be $\mathbf{R}[W]^K$-module generators of $\mathrm{Mor}_G(V,M)_{\mathbf{R}}$. Then there is a symmetric matrix valued function Cov on Z such that $\mathrm{Cov}(z)=(f_i(v), f_j(v))$ whenever $p(v)=z$.

(4.9) **Theorem.** *Let* K_i, $i=0,\ldots,r$; *represent the conjugacy classes of isotropy groups occuring in* W. *Suppose that* $\mathrm{Norm}_K(K_i)/K_i$ *acts faithfully on* L^{K_i} *for every* i. *Then* $X=\{z\in Z\colon \mathrm{Cov}(z)\geqq 0\}$.

Proof. Clear using Remark (2.6) and the fact that $\mathrm{Mor}_G(V,M)$ restricts onto $\mathrm{Mor}_G(Gv,M)$ for every closed orbit Gv. □

(4.10) *Examples.* We consider a point $z\in Z$, and we choose $v\in V$ as in Proposition (2.5). We see when the condition $\mathrm{Cov}(z)\geqq 0$ forces $z\in X$.

(1) Suppose that $K_v=\{e\}$, and suppose that L is a faithful representation of K. Then $\mathrm{Cov}(z)\geqq 0$ forces $z\in X$. This applies, in particular, if K is an adjoint group and L its adjoint representation.

(2) Suppose that $K_v=K_l$ for some $l\in L$. Then $\mathrm{Norm}_K(K_l)/K_l$ acts faithfully on L^{K_l}, hence $\mathrm{Cov}(z)\geqq 0$ implies $z\in X$. In particular, $\mathrm{Cov}(z)\geqq 0$ implies $z\in X$ if $L=W$, but this already follows from our main theorem!

(3) Let W and K be as in Example (0.8), i.e. $K=O(n)$ acting on $n\times q$ matrices. Let L be the usual representation of K on $\mathbf{R}^n$. The isotropy groups one sees in W are all conjugate to $O(n-r)$, $0\leqq r\leqq n$, so the hypotheses of Theorem (4.9) hold. The condition $\mathrm{Cov}(z)\geqq 0$ that one gets is exactly (0.8.2).

(4) Let $W=W_1\oplus W_2$ where $\dim W_1=3$, $\dim W_2=5$, and $K=SO(3)$ acts irreducibly on W_1 and W_2. Let $L=W_1$. The principal orbits in W have trivial stabilizer, so by (1), $\mathrm{Cov}(z)\geqq 0$ implies $z\in X$ if z is principal. Now the principal isotropy groups of W_2 are conjugate to $H=\mathbf{Z}/2\oplus\mathbf{Z}/2$, and $W_1^H=\{0\}$. This implies that there are no non-zero K-equivariant polynomial maps from W_2 to W_1. Hence $\mathrm{Cov}(z)=0$ for z in the Zariski closure C of the image of $\{0\}\times W_2$ in Z. Thus the condition $\mathrm{Cov}(z)\geqq 0$ implies $z\in X$ if z is principal, but not if $z\in C-X$.

§ 5. Infinitely many inequalities

Let $G\times S\to S$ be as in (4.5), i.e. S is homogeneous. Let $\rho\colon \mathbf{C}[S]\to\mathbf{C}[S]^G\simeq\mathbf{C}$ be projection onto the invariants. We may realize ρ by integration over K:

$$\rho(f)=\int_K f(ks)\,dk,$$

where $s\in S$ is arbitrary.

(5.1) **Proposition.** *The following are equivalent:*

(1) $S(\mathbf{R})\neq\emptyset$.

(2) $\rho(f^2)\geqq 0$ *for all* $f\in\mathbf{R}[S]$.

Proof. 1. Clearly (1) implies (2). Assume (2). Choose W as in (4.3) so that S embeds in $V=W\otimes_{\mathbf{R}}\mathbf{C}$. Let $L=W$ (so $M=V$), and let $e_1,\ldots,e_n$ be an orthonormal basis of L relative to a K-invariant inner product. Let $f\in\mathrm{Mor}_G(S,M)_{\mathbf{R}}$. Write $f=\sum f_i e_i$ where $f_i\in\mathbf{R}[S]$. By (2), $(f,f)=\sum\rho(f_i^2)\geqq 0$, and then $S(\mathbf{R})\neq\emptyset$ by (4.10.2).

Proof 2. Assume (2). Let $I=\{f\in\mathbf{R}[S]\colon \rho(f^2)=0\}$. Let $f\in I$ and let $h\in\mathbf{R}[S]$. Then $\rho((rh+f)^2)\geqq 0$ for all $r\in\mathbf{R}$, and it follows that $\rho(hf)=0$. Hence 0

$=\rho(h^2 f\cdot f)=\rho((hf)^2)$, and $hf\in I$. One similarly shows that I is an additive subgroup of $\mathbf{R}[S]$. Hence I is an ideal, and I is clearly K-invariant. If $I\neq 0$, then I must equal $\mathbf{R}[S]$. But $\rho(1^{\Delta})=\rho(1)=1\neq 0$. Thus $I=0$.

Suppose that $S(\mathbf{R})=\emptyset$. If S is irreducible over $\mathbf{R}$, then (see [14]) we can write -1 as a sum of squares of elements of $\mathbf{R}(S)$. Clearing denominators, we get an equation $-f^2=\sum h_i^2$ where f and the $h_i\in\mathbf{R}[S]$ and $f\neq 0$. (Even if S is reducible, we can clearly establish such an equation). Applying ρ and (2) we see that $\rho(f^2)=0$. Hence $I\neq 0$, a contradiction. □

§ 6. A general setting

(6.1) A general problem, encompassing the ones we have considered so far, is the following: Let $G\times S\to S$ be an action of an algebraic group on a variety S, where everything is defined over a field F. We suppose that there is a quotient variety S/G in some reasonable sense.

(6.2) *Problem.* When does an F-point of S/G lift to an F-point of S?

We explore an approach to this general problem which works in certain cases. Our results are incomplete, but our attempt points out some of the difficulties involved. The methods we use are a variation of those used in [14] for the classical groups.

For simplicity we assume that char $F=0$. Let $\mathfrak{k}$ be a semisimple Lie algebra over F, and let $\mathfrak{g}$ denote $\mathfrak{k}\otimes_F\bar{F}$ where $\bar{F}$ is the algebraic closure of F. Let Γ denote the group of Lie algebra automorphisms of $\mathfrak{g}$, and let G (the adjoint group) denote the identity component of Γ. We assume:

(*) Every component of Γ is defined over F and contains an F-point.

Let $\pi: S\to B$ be a principal G-bundle (in an étale sense) defined over F, where S and B are $\bar{F}$-varieties. Let $\tilde{S}=S\times_G\Gamma$, and set $E=S\times_G\mathfrak{g}$ and $E_0=B\times\mathfrak{g}$. There is an inclusion $S\to\tilde{S}$ arising from the inclusion $G\to\Gamma$. Note that E and E_0 are Lie algebra bundles over B. Let $\mathrm{Iso}(E_0,E)$ denote the variety of fiberwise $\bar{F}$-Lie algebra isomorphisms of E_0 and E. Then $\mathrm{Iso}(E_0,E)$ is a subvariety of the tensor product bundle $E_0^*\otimes E$, and a point of $\mathrm{Iso}(E_0,E)$ is an isomorphism $\gamma_b\colon \mathfrak{g}\to E_b$ for some $b\in B$.

Note that the fiber product $\tilde{S}\times_B E_0$ is isomorphic to $(S\times_G\Gamma)\times\mathfrak{g}$, and the natural action of Γ on $\mathfrak{g}$ gives us a map to E. Hence there is a canonical map $\alpha\colon\tilde{S}\to\mathrm{Iso}(E_0,E)$, and α is easily seen to be an isomorphism over F (c.f. [8], III § 4 no. 2). In fact, it suffices to check the case of a trivial bundle.

(6.3) **Theorem.** *Let b be an F-point of B. Then $b=\pi(s)$ for an F-point s in S if and only if the Lie algebras $\mathfrak{g}$ and E_b are F-isomorphic.*

Proof. Consider the inclusion $S\subseteq\tilde{S}\simeq\mathrm{Iso}(E_0,E)$. If s is F-rational and $\pi(s)=b$, then the image of s in $\mathrm{Iso}(E_0,E)$ is the required isomorphism. Conversely, an F-isomorphism of $\mathfrak{g}$ with E_b is an element $\tilde{s}$ of $\tilde{S}_b(F)$. By our assumption (*), there are F-rational points in every component of $\tilde{S}_b$, and one component is S_b. □

Note that the F-points $E_b(F)$ of E_b are an F-form of $\mathfrak{g}$, and the theorem says that b is the image of an F-point of S if and only if $E_b(F)$ and $\mathfrak{k}$ are isomorphic Lie algebras over F.

Now assume that $F=\mathbf{R}$. Our condition (*) eliminates several choices of $\mathfrak{k}$. For example, we cannot have $\mathfrak{k}=\mathfrak{k}_1\oplus\mathfrak{k}_2$ where $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are non-isomorphic real forms of the same complex Lie algebra $\mathfrak{g}_0$. In this case there is an element of Γ interchanging the copies of $\mathfrak{k}_1\otimes_{\mathbf{R}}\mathbf{C}\simeq\mathfrak{k}_2\otimes_{\mathbf{R}}\mathbf{C}\simeq\mathfrak{g}_0$ in $\mathfrak{g}\simeq\mathfrak{g}_0\oplus\mathfrak{g}_0$, but the component of Γ containing this element can have no real points.

Suppose that the signature of the killing form separates the various real forms of $\mathfrak{g}$ and that $\mathfrak{g}$ satisfies (*). Then since one can describe the set of real $m\times m$ matrices of given rank and signature as a union of sets given by explicit inequalities, one can explicitly describe the image of $S(\mathbf{R})$ in $B(\mathbf{R})$. Unfortunately, there are even some simple Lie algebras with distinct real forms satisfying (*) having killing forms of the same signature (e.g. $\mathfrak{g}=\mathsf{A}_{13}$).

§ 7. The equivariant Hilbert's 17th problem

(7.1) Let W be a real representation space of a real reductive group K. Then $\mathbf{R}[W]^K$ is finitely generated. Let $p=(p_1,\dots,p_m)$, X and Z be as in (0.1). Then $X=\operatorname{Im} p$ is closed [13], but p is not in general a proper mapping and the orbit space W/K is not usually Hausdorff.

We say that the representation of K on W has property (H) if there are positive polynomials $f_1,\dots,f_l\in\mathbf{R}[W]^K$ such that every $f\in\mathbf{R}(W)^K$ which is positive (i.e. $f(x)\geqq 0$ whenever $f(x)$ is defined) can be written in the form

(7.2)
$$\sum_{1\leqq i_1<\dots<i_j\leqq l} a_{i_1\cdots i_j} f_{i_1}\cdots f_{i_j}$$

where the $a_{i_1\cdots i_j}$ are sums of squares of elements of $\mathbf{R}(W)^K$. If $K=\{e\}$, then Artin's solution to Hilbert's 17th Problem shows that every positive f is a sum of squares. In the equivariant case, however, one is forced to allow representations of the form (7.2). For example, if $K=\{\pm 1\}$ acting by multiplication on $\mathbf{R}$, then the polynomial function x^2 is positive, but it is not a sum of squares of rational invariants.

The main point of Procesi [14] was to establish property (H) and to find explicit polynomials $f_1,\dots,f_l$ in the case of the standard representation of S_n on $\mathbf{R}^n$. We establish property (H) in the case that K is compact (this was conjectured in Bochnak-Efroymson [5]), and we give examples of representations of non-compact K where (H) fails.

(7.3) Let P be a closed semialgebraic subset of $\mathbf{R}^m$. We assume that the Zariski closure T of P is irreducible. We say that P is *elementary* if there are $f_1,\dots,f_l\in\mathbf{R}[T]$ so that $P=\{t\in T\colon f_i(t)\geqq 0\ i=1,\dots,l\}$. We say that P is *quasi-elementary* if there is an algebraic subset Y of T such that $\dim Y<\dim T$ and $P\cup Y$ is elementary.

(7.4) **Proposition.** *Let P and T be as above. Let $f_1,\dots,f_l\in\mathbf{R}[T]$, let $Q=\{t\in T\colon f_i(t)\geqq 0$ for all $i\}$, and suppose that $P\subseteq Q$. Then the following conditions are equivalent, and each implies that P is quasi-elementary.*

(1) *There is an algebraic set* Y *in* T *such that* $\dim Y < \dim T$ *and* $Q \cup Y = P \cup Y$.

(2) $\dim(Q-P) < \dim T$.

(3) *Every* $f \in \mathbf{R}(T)$ *which is positive on* P *can be written in the form* (7.2) *where the* $a_{i_1 \dots i_j}$ *are sums of squares in* $\mathbf{R}(T)$.

Proof. The equivalence of (1) and (2) follows from the fact that the dimension of a semialgebraic set and its Zariski closure are the same. See ([5] § 11) for the rest. □

(7.5) **Corollary.** *Let* W, K, X *and* Z *be as in* (7.1). *Suppose that* $\mathbf{R}(W)^K$ *is the quotient field of* $\mathbf{R}[W]^K$. *Then the representation of* K *on* W *has property* (H) *if and only if* X *is quasi-elementary.*

(7.6) *Remark.* One can easily show that $\mathbf{R}(W)^K$ is the quotient field of $\mathbf{R}[W]^K$ for all W if and only if all real characters χ: $K \to \mathbf{R}^*$ have finite image. In particular, this property holds if K is compact or if $\mathfrak{k}$ is semisimple.

(7.7) **Theorem** (see (0.6) and (0.10). *If* K *is compact, then* X *is elementary and the representation of* K *on* W *has property* (H).

(7.8) We now give a class of representations where (H) fails: Let $a, b \in \mathbf{Z}^+$, where $ab > 0$. Let $J_{a,b}$ denote the diagonal matrix with a 1's followed by b -1's on the diagonal. Define a bilinear form $(\ ,\ )_{a,b}$ by $(x, y)_{a,b} = x^t J_{a,b}\ y$ for $x, y \in \mathbf{R}^{a+b}$. Let $O(a, b)$ denote the corresponding orthogonal group. The invariants of $K = O(a, b)$ on an arbitrary number of copies of $\mathbf{R}^{a+b}$ are generated by the corresponding inner product invariants.

Identify $(a+b)\mathbf{R}^{a+b}$ with the space W of $(a+b) \times (a+b)$ matrices. Then, in analogy with Example (0.8), we may take

$$p: W \to \mathrm{Sym}_{a+b}$$
$$A \mapsto A^t J_{a,b} A.$$

Note that p is equivariant with respect to the action of $GL(a+b)$, where $h \in GL(a+b)$ sends $A \in W$ into Ah^t and $B \in \mathrm{Sym}_{a+b}$ into hBh^t. We will show that (H) fails for the representation of $K = O(a, b)$ on $W = (a+b)\mathbf{R}^{a+b}$ if $a+b \geqq 3$.

Let D_{a+b} denote the diagonal elements of Sym_{a+b}. We will often abbreviate Sym_{a+b} (resp. D_{a+b}) by Sym (resp. D). Let $X^0_{a,b}$ denote the non-singular elements of Sym with exactly b negative eigenvalues, let $X_{a,b}$ denote the closure of $X^0_{a,b}$, and set $D_{a,b} = D \cap X_{a,b}$.

(7.9) **Lemma.** *Let* $X_{a,b}$, etc. *be as above. Then*

(1) $\mathrm{Im}\, p = X_{a,b}$

(2) *If* $X_{a,b}$ *is quasi-elementary, then so is* $D_{a,b}$.

Proof. If $A \in W$ is non-singular, then $p(A) \in X^0_{a,b}$, so $\mathrm{Im}\, p \subseteq X_{a,b}$.
It is easy to see that $\mathrm{Im}\, p \supseteq D \cap X^0_{a,b}$. But every $GL(a+b)$-orbit in Sym intersects D, so $\mathrm{Im}\, p \supseteq X^0_{a,b}$, and (1) follows.

Suppose that there are $f_1, \dots, f_l \in \mathbf{R}[\mathrm{Sym}]$ and a proper algebraic subset Y of Sym such that $X_{a,b} \cup Y = \{A \in \mathrm{Sym}: f_i(A) \geqq 0$ for all $i\}$. Let $h_1, \dots, h_q \in GL(a+b)$. Since $X_{a,b}$ is $GL(a+b)$-invariant, we have that $X_{a,b} \cup (Y \cap (\bigcap_j h_j^{-1} Y)) = \{A \in \mathrm{Sym}:$

$f_i(A) \geqq 0$ and $f_i(h_j A h_j^t) \geqq 0$ for all $i, j\}$. Thus we may reduce to the case that Y is $GL(a+b)$-invariant. Then $Y \cap D \neq D$, hence $D_{a,b}$ is quasi-elementary. □

(7.10) **Lemma.** *Suppose that $D_{a,b}$ is quasi-elementary. Then $D_{a-1,b} \cup D_{a,b-1}$, $D_{a-1,b}$ and $D_{a,b-1}$ are quasi-elementary.*

Proof. We denote points A of D as pairs (A', y) where A' is the upper $(a+b-1) \times (a+b-1)$ submatrix of A and y the last diagonal entry. Let $f_1, \ldots, f_l \in \mathbf{R}[D]$ so that $D_{a,b} \cup Y = \{A: f_i(A) \geqq 0$ for all $i\}$, where Y is a proper algebraic subset of D. Let Γ_r denote $\{(A', y): y \det A' = r\}$, $r \in \mathbf{R}$. Choose r so that $(-1)^b r > 0$. Then $\dim \Gamma_r \cap D_{a,b} = a+b-1$, and we may assume that $\dim \Gamma_r \cap Y < a+b-1$. Choose a non-zero non-negative polynomial function h on D_{a+b-1} so that the functions $f_i'(A') := h(A') f_i(A', r/\det A')$ are polynomial. Then $\dim(Q' - \pi(D_{a,b} \cap \Gamma_r)) < a+b-1$ where $Q' = \{A': f_i'(A') \geqq 0$ for all $i\}$ and $\pi(A', y) = A'$. It follows that the closure of $\pi(D_{a,b} \cap \Gamma_r)$ is quasi-elementary. But the closure is easily seen to be $D_{a-1,b} \cup D_{a,b-1}$. Finally, $D_{a-1,b}$ and $D_{a,b-1}$ are quasi-elementary since the determinant function has different signs on the non-singular elements of $D_{a-1,b}$ and $D_{a,b-1}$. □

(7.11) **Lemma.** *Let $C = \{(x, y) \in \mathbf{R}^2$: $x \geqq 0$ or $y \geqq 0\}$. Then C is not quasi-elementary.*

Proof. Suppose that there are polynomials $f_1, \ldots, f_l$ such that $C \subseteq Q$ and $\dim(Q - C) \leqq 1$, where $Q = \{(x, y): f_i(x, y) \geqq 0$ for all $i\}$. Every point b of the boundary ∂C of C is then the limit of points where at least one of the f_i is strictly negative. Thus $\partial C \subseteq \bigcup_i \mathrm{Var}(f_i)$, where $\mathrm{Var}(f_i)$ is the zero set of f_i. Since ∂C contains $\{(x, y): x = 0, y \leqq 0\}$, there is an i such that $x | f_i$. Since f_i is non-negative in the first and second quadrants, f_i/x must be non-negative in the first quadrant and non-positive in the second. Thus f_i/x is zero on the positive y-axis, i.e. $x^2 | f_i$. One can replace f_i by f_i/x^2 in $\{f_1, \ldots, f_l\}$. This process can be continued indefinitely, which is absurd. □

(7.12) **Proposition.** *Suppose that $a+b \geqq 3$. Then $X_{a,b}$ is not quasi-elementary and the standard representation of $O(a, b)$ on $(a+b)\mathbf{R}^{a+b}$ does not have property (H).*

Proof. If $X_{a,b}$ is quasi-elementary, then using Lemmas (7.9) and (7.10) one can show that $D_{2,0} \cup D_{1,1}$ or $D_{0,2} \cup D_{1,1}$ is quasi-elementary. But both these sets are isomorphic to the set C of Lemma (7.11). □

(7.13). *Remark.* The techniques above easily extend to show the following: Let $a+b \geqq 3$, $ab \neq 0$, and let $c \in \mathbf{Z}^+$. Then the representation of $O(a, b)$ on $c\mathbf{R}^{a+b}$ has property (H) if and only if $c \leqq \min\{a, b\}$.

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