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On the instability of Herman rings

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Introduction

Let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $\mathrm{End}(\bar{\mathbb{C}})$ be the space of analytic maps (endomorphisms) of $\bar{\mathbb{C}}$ into itself with degree >1 endowed with the C^0 topology (analytic in this paper will always mean complex analytic). It is well known that these maps are given by rational functions. Denote by $A(r_1, r_2)$ the annulus

$$A(r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}.$$

A *Herman ring* of $f \in \mathrm{End}(\bar{\mathbb{C}})$ is a connected component U of the complement of the Julia set $J(f)$ of f homeomorphic to an annulus and such that there exists $n \geqq 1$ satisfying $f^n(U) = U$. In this case it can be proved that if φ: $A(r_1, r_2) \to U$ is a conformal representation, the map $\varphi^{-1} f^n \varphi$: $A(r_1, r_2) \hookleftarrow$ is a rotation i.e. there exists $\theta \in \mathbb{R}$ such that

$$(\varphi^{-1} f^n \varphi)(x) = e^{2\pi\theta i} z$$

for all $z \in A(r_1, r_2)$. By elementary reasons, the number θ is independent of the conformal representation. It is called the rotation, number of the Herman ring. The minimum $n \geqq 1$ such that $f^n(U) = U$ is called the period of U.

Herman rings are one of the five possible dynamical behaviours in the complement of the Julia set (Sullivan [3]). It can be easily shown that of the other four, three of them don't appear for maps in an open and dense subset of $\mathrm{End}(\bar{\mathbb{C}})$. The purpose of this paper is to prove the same property for Herman rings.

Theorem. *The set of maps* $f \in \mathrm{End}(\bar{\mathbb{C}})$ *without Herman rings contains an open and dense subset of* $\mathrm{End}(\bar{\mathbb{C}})$.

This result is a step in the approach proposed in [2] to solve the fundamental conjecture on the generic dynamics of rational maps.

Conjecture I. *For a dense set of maps* $f \in \mathrm{End}(\bar{\mathbb{C}})$, *the Julia set of* f *is hyperbolic*, i.e. *there exist* $K>0$, $\lambda>1$ *such that*

$$|(f^n)'(z)| \geqq K\lambda^n$$

for all $z \in J(f)$ *and* $n \geqq 1$.

The results in [2], together with the theorem above, show that this conjecture is implied by the following one:

Conjecture II. *For a dense set of maps* $f \in \mathrm{End}(\bar{\mathbb{C}})$ *the Julia set of* f *has no invariant line fields.*

Recall that an *invariant line field* on the Julia set of f is a pair (A, L) where A (called the support of the line field) is an invariant Borel set (i.e. $f^{-1}(A) = A$) with *positive* Lebesgue measure contained in $J(f)$, and L is a map that to every $z \in A$ associates a one dimensional subspace $L(z) \subset T_z\bar{\mathbb{C}}$ such that $L(f(z)) = f'(z) L(z)$ for a.e. $z \in A$. In particular, if the Lebesgue measure of $J(f)$ is zero, there are no invariant line fields on $J(f)$. Since when $J(f)$ is hyperbolic its Lebesgue measure is zero, it follows that Conjectures I and II are equivalent. Obviously, a proof that $J(f)$ has Lebesgue measure zero for a dense set of maps $f \in \mathrm{End}(\bar{\mathbb{C}})$, would imply Conjecture II. However, to prove Conjecture II could be easier. One may start with an invariant line field (A, L) and, using some dense hypothesis on the map, prove that the Lebesgue measure of A is zero, thus reaching a contradiction. The existence of the line field should play a role in obtaining the distortion inequalities on which proofs that invariant sets have measure zero usually rely. This approach is motivated by a successful similar idea introduced by Sullivan to study the Ahlfors conjecture, that states that the limit set of a finitely generated Kleinian group has measure zero. The conjecture remains open but in [3] Sullivan proved that there are no invariant line fields on the limit set; a fact that suffices to replace the Ahlfors conjecture in several relevant applications.

The results in [3], plus the theorem above, show that the following conjecture also implies Conjecture I:

Conjecture III. *For every* $f \in \mathrm{End}(\bar{\mathbb{C}})$, *either the Lebesgue measure of* $J(f)$ *is zero or* $J(f) = \bar{\mathbb{C}}$ *and* f *is ergodic*, i.e. *for every Borel set* $A \subset \bar{\mathbb{C}}$ *such that* $f^{-1}(A) \subset A$, *either* A *or* A^c *has measure zero.*

However this conjecture is stronger than the first one and has an intrinsic interest that goes far beyond its application to prove Conjecture I.

There also exists a more direct approach to prove Conjecture I. It is based on the fact that when every periodic orbit is hyperbolic and there are no critical points in the Julia set, then this set is hyperbolic. Since the hyperbolicity of the periodic orbits is a generic property, it remains to prove that the absence of critical points in the Julia set is also a generic property. Suppose that $J(f)$ contains a critical point p. By elementary properties of $J(f)$, the set $\bigcup_{n \geqq 0} f^{-n}(\{p\})$ is dense in $J(f)$. In particular, it accumulates at p. Then one could try to find $g \in \mathrm{End}(\bar{\mathbb{C}})$ arbitrarily near to f, such that p is a critical point of g and $p \in \bigcup_{n \geqq 0} g^{-n}(\{p\})$. This means that p is a critical periodic point of g. In particular it is an attractive periodic point. Since the attractive periodic points of f survive the perturbation and appear (moving its positions a little) as attractive periodic points of g, the net result of the perturbation is to increase the number of attractive periodic points. However, in an open and dense subset of End($\bar{\mathbb{C}}$), this number is locally constant [2]. Then, the conclusion would be that elements of this open and dense set cannot have critical points in the Julia set.

The proof of the theorem is given in the next section. It is based on the fact that an open and dense set of maps $f \in \text{End}(\bar{\mathbb{C}})$ is structurally stable [2, 3]. We shall show that the structural stability of f implies that f has a certain property that we shall call infinitesimal stability. This means that given an infinitesimal variation G of f (i.e. G is the derivative at $w=0$ of a map H: $D \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, where D is the unit disk, such that $H(0, z) = f(z)$ for all $z \in \bar{\mathbb{C}}$) there exists a continuous vectorfield V on $\bar{\mathbb{C}}$ such that

$$f'(z)\, V(z) - V(f(z)) = G(z) \tag{1}$$

for all $z \in \bar{\mathbb{C}}$. What we shall do is to prove that a map with a Herman ring is not structurally stable. As an example of the method we shall use, consider a map $f \in \text{End}(\bar{\mathbb{C}})$ with a fixed Herman ring U that contains the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ as an invariant circle where f acts as the rotation $z \to e^{i\theta} z$. Such a behaviour is impossible because it would force f to be a rotation but it is a good heuristical example. Then, at points of S^1 Eq. (1) becomes:

$$e^{i\theta}\, V(z) - V(e^{i\theta} z) = G(z). \tag{2}$$

Now take $J: D \times \bar{\mathbb{C}}$ defined by $H(w, z) = (l + w) f(z)$. Then $G(z) = f(z)$ is an infinitesimal variation of f. Since $f(z) = e^{i\theta} z$, Eq. (2) implies that there exists a continuous vectorfield V on $\bar{\mathbb{C}}$ such that for all $z \in S^1$

$$e^{i\theta}\, V(z) - V(e^{i\theta} z) = e^{i\theta} z. \tag{3}$$

However it is quite easy to show (intergrating this equation on S^1) that there is no continuous $V: S^1 \to \mathbb{C}$ satisfying (3). Then, the map f is not infinitesimally stable. In the general case the basic idea is the same but first we have to transform the map, using a special change of coordinates, in a map that has a Herman ring that contains an invariant circle that almost coincides with the unit circle and where f acts almost as a rotation. More precisely, if $f \in \text{End}(\bar{\mathbb{C}})$ has a Herman ring U that contains an invariant circle γ, we shall construct a quasiconformal homeomorphism $g: \mathbb{C} \hookleftarrow$ such that $gfg^{-1} \in \text{End}(\bar{\mathbb{C}})$ and $g(\gamma)$, which is an invariant circle of gfg^{-1}, is near to S^1. By a result of Sullivan [3], if f is structurally stable, then gfg^{-1} is structurally stable. Using the invariant circle $g(\gamma)$ we shall prove that gfg^{-1} is not infinitesimally stable. Hence it is not structurally stable and then f is not structurally stable.

This idea seems essentially analytical. However it evolved from a formally different geometric idea that can be used to prove the theorem for *fixed* Herman rings (i.e. having period $n=1$) but that we couldn't extend to the general case. Since this geometric approach throws light on the nature of the proof of the theorem, we shall now briefly explain it. Suppose that $f \in \text{End}(\bar{\mathbb{C}})$

has a Herman ring U with period $n \geqq 1$ and let $\varphi: A(r_1, r_2) \to U$ be a conformal representation. It is easy to prove that the curves $\gamma = \varphi(\{z \mid |z| = r\})$, for $r_1 < r < r_2$ are invariant under f^n and independent of the conformal representation. This family of curves is called the canonical foliation of U. Moreover observe that if α is a simple closed curve contained in U, then $f_n(\alpha) \cap \alpha \neq \emptyset$. Now we claim that a structurally stable $f \in \mathrm{End}(\overline{\mathbb{C}})$ cannot have a Herman ring whose canonical foliation contains a convex curve $\gamma \subset \mathbb{C}$. In fact, if f is structurally stable every nearby $g \in \mathrm{End}(\overline{\mathbb{C}})$ contains a fixed Herman ring U_g satisfying $\gamma \subset U_g$ (but obviously γ doesn't need to belong to the canonical foliation of U_g). Without lose of generality we can suppose that the origin of $\mathbb{C}$ is in the interior of γ. Take $g = (1-\varepsilon) f$ with $0 < \varepsilon < 1$. Then $g(\gamma) = (1-\varepsilon) f(\gamma) = (1-\varepsilon)\gamma$. Then, since γ is convex and contains the origin in its interior, we obtain $\gamma \cap g(\gamma) = \emptyset$ contradicting the intersection property stated above. Since g can be taken arbitrarily near to f, taking ε small, this proves that f is not structurally stable thus completing a proof by contradiction of our claim. However this proof is not immediatly extendable to rings with period $n > 1$ because the calculation of $g^n(\gamma)$ is not straightforward as in the previous case. Anyhow, in the case of a fixed Herman ring, the assumption of the existence of a convex closed curve γ in its canonical foliation can be avoided. The idea is to produce a quasiconformal homeomorphism $h = \overline{\mathbb{C}} \hookleftarrow$ such hfh^{-1} is a rational map and the ring $h(U)$ has large modulus. This means that there is a conformal representation $\varphi: A(r_1, r_2) \to h(U)$ with r_1 very small and r_2 very large. Without lose of generality we can assume that $\varphi(1) = 1$ and that 0 and ∞ belong to different connected components of the complement of $h(U)$. According to Theorem I.6, that we shall prove in Section II, such a conformal representation is very near to the identity map or the inversion $z \to z^{-1}$ in the annulus $A(1/2, 2)$ if r_1 is very small and r_2 very large. Then $\varphi(\{z \mid |z| = 1\})$ is convex. Hence hfh^{-1} is not structurally stable and then f is not structurally stable.

Proof of the theorem. First we shall recall some basic results about quasiconformal structures and mappings, analytic motions, structural stability and quasiconformal conjugacies necessary for the proof of the theorem.

If $U \subset \overline{\mathbb{C}}$ is an open set, a *quasiconformal structural* on U is a function $\mu: U \to \mathbb{C}$ with

$$\|\mu\|_\infty < 1.$$

Associated to μ we have a field of ellipses that to a.e. $z \in U$ associates the ellipse

$$E_\mu(z) = \{e^{i\theta} + \mu e^{-i\theta} \mid \theta \in \mathbb{R}\}.$$

The set of quasiconformal structures on U is denoted $Qc(U)$.

If $U \subset \overline{\mathbb{C}}$ and $V \subset \overline{\mathbb{C}}$ are open sets, we say that a continuous map $f: U \to V$ is *quasiconformal* if it satisfies the following conditions:

a) For a.e. $z \in U$, f is differentiable at z and its derivative is an orientation preserving isomorphism.

b) The function $K(z) = \|f'(z)\| \cdot \|(f')^{-1}(z)\|$ satisfies $\|K\|_\infty < +\infty$.

c) f is absolutely continuous on lines, i.e. the real and imaginary parts of the restriction of f to a line parallel to the real or imaginary axis are absolutely continuous. This, together with (a) and (b), implies that the Lebesgue measure of a Borel set $A \subset U$ is zero if and only if the measure of $f(A)$ is zero [1].

Every quasiconformal map $f: U \to V$ induces a map $f_*: Qc(V) \to Qc(U)$ defined as follows. Suppose that $\mu \in Qc(V)$. Using (a), (b) and (c) it is easy to prove that there exists a unique $\nu \in Qc(U)$ such that the ellipse $f'(z) E\ (z)$ is homothetic to $E_\mu(f(z))$. Define $f_*(\mu) = \nu$.

The following is the well known measurable Riemann mapping theorem due to Morrey. For a proof see [1].

Theorem I.1. [1] *For every $\mu \in Qc(\bar{\mathbb{C}})$ there exists a quasiconformal homeomorphism $g: \bar{\mathbb{C}} \hookleftarrow$ such that $g_*(\mu) = 0$.*

An *analytic motion* of $\bar{\mathbb{C}}$ is a map $\varphi: W \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, where W is a connected complex manifold, such that, denoting $\varphi_\lambda(\cdot) = \varphi(\lambda, \cdot)$, the following properties are satisfied:

a) $\varphi_\lambda: \bar{\mathbb{C}} \hookleftarrow$ is injective for all $\lambda \in W$.

b) For all $z \in \mathbb{C}$, the map $W \ni w \to \varphi_\lambda(w) \in \bar{\mathbb{C}}$ is analytic.

c) There exists $\lambda_0 \in W$ such that φ_{λ_0} is the identity.

The next theorem is the λ-lemma of [2]:

Theorem I.2. *If $\varphi: W \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is an analytic motion, then φ is continuous and for all $\lambda \in W$, $\varphi_\lambda: \bar{\mathbb{C}} \hookleftarrow$ is a quasiconformal homeomorphism.*

We shall also need the following easy corollary:

Corollary. *If $\varphi: W \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is an analytic motion, then, for every $\lambda_1 \in W$, the vectorfield*

$$\bar{\mathbb{C}} \ni z \to \frac{\partial \varphi}{\partial \lambda}(\lambda_1, z) \in T_z \bar{\mathbb{C}}$$

Proof. Take $\lambda_1 \in W$ and consider the vectorfield $V(z) = (\partial\varphi/\partial\lambda)(\lambda_1, z)$. To prove the continuity of V take a point $z \in \bar{\mathbb{C}}$ and a sequence $z_n \to z$. Define a sequence of analytic functions $F_n: W \to \bar{\mathbb{C}}$ by $F_n(\lambda) = \varphi_\lambda(z_n)$ and define $F: W \to \bar{\mathbb{C}}$ by $F(z) = \varphi_\lambda(z)$. By I.2 φ is continuous. This obviously implies that $F_n \to F$ when $n \to +\infty$ uniformely on compact sets. Then $F'(\lambda_1) = \lim_{n \to +\infty} F_n'(\lambda_1)$. Since $F'(\lambda_1) = V(z)$ and $F_n'(\lambda_1) = V(z_n)$ this concludes the proof of the corollary.

We say that $f \in \mathrm{End}(\bar{\mathbb{C}})$ is *structurally stable* if there exists a neighborhood W of f and an analytic motion $\varphi: W \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that

$$\varphi_f = \text{Identity}$$

and

$$\varphi_g^{-1} f \varphi_g = g$$

for all $g \in W$.

A *variation* of $f \in \mathrm{End}(\bar{\mathbb{C}})$ is an analytic map $F: D \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, where $D = \{z \in C \mid |z| < 1\}$ and

$$F(0, z) = f(z)$$

for all $z \in D$. The analytic map $G: \bar{\mathbb{C}} \to T\bar{\mathbb{C}}$ defined as

$$G(z) = \frac{\partial F}{\partial \lambda}(0, z) \tag{1}$$

is called an *infinitesimal variation* of f. We say that $f\in \mathrm{End}(\bar{\mathbb{C}})$ is *infinitesimally stable* if for every infinitesimal variation G of f there exists a continuous vectorfield V on $\bar{\mathbb{C}}$ such that:

$$f'(z)\,V(z)-V(f(z))=G(z). \tag{2}$$

Theorem I.3. *Structurally stable maps are infinitesimally stable.*

To prove this theorem take a structurally stable $f\in \mathrm{End}(\bar{\mathbb{C}})$ and a variation $F: D\times\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ of the map f as above. By the definition of structural stability there exists an analytic motion $\varphi: D\times\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ such that:

$$\varphi_{\lambda_0}=\text{Identity}$$

and

$$\varphi_\lambda(f(z))=F(\lambda,\varphi_\lambda(z)) \tag{3}$$

for all $\lambda\in D$ and $z\in\bar{\mathbb{C}}$. Defining the vectorfield V on $\bar{\mathbb{C}}$ as

$$V(z)=\frac{\partial\varphi}{\partial\lambda}(\lambda_0,z) \tag{4}$$

and derivating (3) with respect to λ at $\lambda=\lambda_0$, we obtain:

$$f'(z)\,V(z)\,V(z)=V(f(z))+\frac{\partial F}{\partial\lambda}(\lambda_0,z).$$

Then, if G is the infinitesimal variation given by (1), it follows that Eq. (2) is satisfied by V given by (4), which by Theorem I.2 is continuous.

Theorem I.4. [2, 3] *The set of structurally stable maps* $f\in \mathrm{End}(\bar{\mathbb{C}})$ *is an open and dense subset of* $\mathrm{End}(\bar{\mathbb{C}})$.

Theorem I.5. [3] *If* $f\in \mathrm{End}(\bar{\mathbb{C}})$ *is structurally stable and* $g\in \mathrm{End}(\bar{\mathbb{C}})$ *is quasiconformally equivalent to* f (i.e. *there exists a quasiconformal homeomorphism* h: $\bar{\mathbb{C}}\hookleftarrow$ *such that* $h^{-1}fh=g$) *then* g *is structurally stable.*

The following theorem will be proved in the next section.

Theorem I.6. *Given two sequences:*

$$\dots r_2<r_1<1<R_1<R_2<\dots$$

such that $R_n\to+\infty$ *and* $r_n\to 0$ *and a sequence of injective analytic maps* φ_n: $A(r_n,R_n)\to\mathbb{C}$ *such that:*

a) $\varphi_n(1)=1$ *for all* n;

b) *For all* n, *the origin is contained in the bounded connected component of the complement of* $\varphi_n(A(r_n,R_n))$;

c) *For all* n, *the winding number with respect to the origin of the curve* $\theta\to\varphi_n(e^{i\theta})$ *is* 1, *then* $\{\varphi_n\}$ *converges uniformely to the identity on compact subsets of* $\mathbb{C}-\{0\}$.

To prove the theorem, we shall show that if $f\in \mathrm{End}(\bar{\mathbb{C}})$ has a Herman ring, then there exists $g\in \mathrm{End}(\bar{\mathbb{C}})$ quasiconformally equivalent to f, such that g is not infinitesimally stable. By Theorem I.3, g is not structurally stable. Hence, by I.5, f is not structurally stable. Therefore structurally stable maps don't have Herman rings. By I.4, this proves the theorem.

Suppose that $f \in \mathrm{End}(\bar{\mathbb{C}})$ has a Herman ring U with period m and rotation number θ. Denote:

$$C = \bigcup_{j=0}^{m} U_j, \qquad U_j = f^j(U),\ j = 0, \ldots, m.$$

Observe that $U_0 = U_m = U$. The property stated in the next lemma is technical but will play a fundamental role at the end of the proof. However it is unnecessary when $m=1$. Given a point $x \in \bar{\mathbb{C}}$ *and* $0 \leqq j < m$ define the number $\tau_j(x)$ as 1 or -1 according to whether the map $f \mid U_j$: $U_j \to U_{j+1}$ preserves or reverses orientation of the invariant ourves of the annulus U_j in the *plane* $\bar{\mathbb{C}} - \{x\}$, i.e., whether the clockwise orientation of the invariant curves in U_j in the plane $\bar{\mathbb{C}} - \{x\}$ is mapped in the clockwise orientation or not. Define

$$v_j(x) = \prod_{i=j}^{m-1} \tau_i(x).$$

Lemma I.7. *There exists* $x \in \bar{\mathbb{C}} - C$ *such that* $\sum_{j=0}^{m-1} v_j(x) \neq 0$.

Proof. We say that two points x' and x'' in $\bar{\mathbb{C}} - C$ are related by a j-crossing if x' and x'' are in different connected components of $\bar{\mathbb{C}} - U_j$ but in the same connected component of $\bar{\mathbb{C}} - U_i$ for all $i \neq j$, $0 \leqq i < m$. Suppose that x' and x'' are in this situation and fix in each U_i an invariant curve γ_i endowed with the clockwise orientation of the plane $\bar{\mathbb{C}} - \{x'\}$. Then in the plane $\bar{\mathbb{C}} - \{x''\}$ all the curves γ_i, $i \neq j$ are clockwise oriented and γ_j is oriented counterclockwise. Using this remark, it follows that, if $0 < j \leqq m-1$, we have:

$$\begin{aligned} \tau_j(x') &= -\tau_j(x'') \\ \tau_{j-1}(x') &= -\tau_j(x'') \\ \tau_i(x') &= \tau_i(x''), \quad 0 \leqq i \leqq m-1,\ i \neq j,\ i \neq j-1. \end{aligned}$$

This condition and the definition of the v_j's imply:

$$\begin{aligned} v_j(x') &= -v_j(x'') \\ v_i(x') &= v_i(x'') \quad 0 \leqq i \leqq m-1,\ i \neq j. \end{aligned}$$

Now choose two points x' and x'' in $\bar{\mathbb{C}} - C$ related by a 1-crossing and suppose that x' doesn't satisfy the required condition, i.e.:

$$\sum_{j=0}^{m-1} v_j(x') = 0.$$

By the relations above

$$\begin{aligned} \sum_{j=0}^{m-1} v_j(x'') &= v_1(x'') + \sum_{j \neq 1} v^j(x'') = -v_1(x') + \sum_{j \neq 1} v_j(x') \\ &= -2v_1(x') + \sum_{j=0}^{m-1} v_j(x') = -2v_1(x') \neq 0. \end{aligned}$$

This completes the proof of the lemma. Using it, choose x in the complement of C such that

$$\sum_{j=0}^{m-1} v_j(x) \neq 0.$$

Without loss of generality we can suppose that $x=\infty$. Hence:

$$\sum_{j=0}^{m-1} v_j(\infty) \neq 0.$$

Take a conformal representation:

$$\varphi\colon A(r,R)\to U$$

that maps clockwise oriented circles centered at the origin onto clockwise oriented invariant curves of U. We can suppose

$$1 \in U$$

and

$$\varphi(1)=1.$$

Take a sequence of quasiconformal homeomorphisms:

$$k_n\colon A(r/n, Rn)\to A(r,R)$$

satisfying:

$$k_n(1)=1$$

and commuting with the rotation of angle θ, i.e.

$$k_n(e^{i\theta} z)=e^{i\theta} k_n(z).$$

Now define a sequence of quasiconformal structures

$$\mu_n \in Qc(\bar{\mathbb{C}})$$

by:

$$\mu_n/U=(\varphi k_n^{-1})_*(0)$$

$$\mu_n/f^{-j}(U)=(f^j)_*(\mu_n), \quad j\geqq 0$$

$$\mu_n(z)=0, \quad z\notin \bigcup_{j\geqq 0} f^{-j}(U).$$

It is easy to check (using that k_n commutes with the rotation $z\to e^{i\iota}z$) that μ_n is f-invariant, i.e.:

$$f_*(\mu_n)=\mu_n. \tag{1}$$

Now take a quasiconformal homeomorphism $h_n\colon \bar{\mathbb{C}}\hookleftarrow$, whose existence is granted by Theorem I.1, such that

$$(h_n)_*(0)=\mu_n \tag{2}$$

$$h_n(0)=0 \tag{3}$$

$$h_n(1)=1 \tag{4}$$

$$h_n(\infty)=\infty. \tag{5}$$

Then, defining $g_n = h_n f h_n^{-1}$, we have

$$(g_n)_*(0) = 0 \tag{6}$$

because, by (1) and (2):

$$(g_n)_*(0) = (h_n^{-1})_* f_* (h_n)_*(0) = (h_n^{-1})_* f_*(\mu_n) = (h_n^{-1})_*(\mu_n) = 0.$$

But (6) implies that $g_n: \bar{\mathbb{C}} \hookleftarrow$ is analytic [1]. Then, to complete the proof of the theorem, it suffices to show that g_n is not infinitesimally stable for large values of n. Observe that $U_n = h_n(U)$ is a Herman ring of g_n with period m and rotation number θ. Moreover, by (5), the annuli $g_n^j(U_n)$, $j = 0, 1, \ldots, m-1$, are contained in $\mathbb{C}$. Suppose that G is an infinitesimal variation of g_n and consider the infinitesimal stability equation:

$$g_n'(z)\, V(z) - V(g_n(z)) = G(z). \tag{7}$$

We shall work with this equation restricted to $C_n = \bigcup_{j=0}^{m-1} g_n^j(U_n)$, and restricted to this set, g_n is invertible. Denote $g_n^{-1}: C_n \hookleftarrow$ the inverse of $g_n/C_n: C_n \hookleftarrow$. Define the vectorfield $\tilde{G}$ on C_n by:

$$\tilde{G} = G g_n^{-1}.$$

Then, on C_n, Eq. (7) can be written as:

$$g_n'(g_n^{-1}(z))\, V(g_n^{-1}(z)) - V(z) = \tilde{G}(z).$$

From this equation follows that

$$(g_n^m)'(g_n^{-m}(z))\, V(g_n^{-m}(z)) - V(z) = \sum_{j=0}^{m-1} (g_n^j)'(g_n^{-j}(z))\, \tilde{G}(g_n^{-j}(z)). \tag{8}$$

Now consider the map

$$\psi_n = h_n \varphi k_n : A(r/n, Rn) \to U_n.$$

This map is a bijection. Moreover it is analytic because

$$(\psi_n)_*(0) = (\psi k_n)_* (h_n)_*(0)$$

and, by (2) and the definition of μ_n on U:

$$(\psi_n)_*(0) = (\varphi k_n)_*(\mu_n) = 0.$$

Now define the vectorfield $\hat{V}$ on $A(r/n, Rn)$ by

$$\hat{V}(z) = (\psi_n^{-1})'(\psi_n(z))\, V(\psi_n(z))$$

and the conformal representation:

$$\Phi_n^{(j)}: A(r/n, Rn) \to g_n^{-j}(U_n)$$

as:

$$\Phi_n^{(j)} = g_n^{-j} \psi_n.$$

Then, applying $(\psi_n^{-1})'(\psi_n(z))$ to Eq. (8) evaluated at the point $\psi_n(z)$, we obtain:

$$(\psi_n^{-1} g_n^m \psi_n)'((\psi_n^{-1} g_n^{-m} \psi_n)(z))\, \hat{V}((\psi_n^{-1} g_n^{-m} \psi_n)(z)) - \hat{V}(z)$$
$$= \sum_{j=0}^{m-1} ((\Phi_n^{(j)})^{-1})'(\Phi_n^{(j)}(z))\, \tilde{G}(\Phi_n^{(j)}(z)). \tag{9}$$

Recalling that $(\psi_n^{-1} g_n^m \psi_n)(z) = e^{i\theta} z$ for $z \in A(r/n, Rn)$, Eq. (9) becomes:

$$e^{i\theta} \hat{V}(e^{-i\theta} z) - \hat{V}(z) = \sum_{j=0}^{m-1} ((\Phi_n^{(j)})^{-1})'(\Phi_n^{(j)}(z))\, \tilde{G}(\Phi_n^{(j)}(z)) \tag{10}$$

for all $z \in A(r/n, Rn)$. It is easy to see, using Fourier series, that:

$$\int_{S^1} \frac{1}{z^2} (e^{i\theta} H(e^{-i\theta} z) - H(z))\, dz = 0$$

for every continuous function $H: S^1 \to \mathbb{C}$. Hence, for all n:

$$\sum_{j=0}^{m-1} \int_{S^1} \frac{1}{z^2} ((\Phi_n^{(j)})^{-1}(\Phi_n^{(j)}(z))\, \tilde{G}(\Phi_n^{(j)}(z))\, dz = 0. \tag{11}$$

Now observe that the functions $\Phi_n^{(j)}$, after being composed with suitable affine maps $z \to az + b$ and inversions $z \to z^{-1}$, satisfy the hypothesis of I.6. More precisely, we can choose numbers $\hat{\lambda}_n^{(j)}$, $\hat{\gamma}_n^{(j)}$ and maps $\hat{L}_n^{(j)}$ that are either the identity or inversions $z \to z^{-1}$, such that for all j the sequence

$$\varphi_n^{(j)} = \hat{\lambda}_n^{(j)} \hat{L}_n^{(j)} (\Phi_n^{(j)} + \hat{\gamma}_n^{(j)}) : A(r/n, Rn) \to \mathbb{C}$$

satisfies the hypothesis of I.6. Then, for all j:

$$\varphi_n^{(j)} \to I \tag{12}$$

when $n \to +\infty$, uniformely on compact subsets of $\mathbb{C} - \{0\}$. Setting $\lambda_n^{(j)} = (\hat{\lambda}_n^{(j)})^{-1}$, $L_n^{(j)} = (\hat{L}_n^{(j)})^{-1}$ and $\gamma_n^{(j)} = (\hat{L}_n^{(j)})^{-1}(-\hat{\gamma}_n^{(j)}/\hat{\lambda}_n^{(j)})$, we can write:

$$\Phi_n^{(j)} = \lambda_n^{(j)} L_n^{(j)} \varphi_n^{(j)} + \gamma_n^{(j)}. \tag{13}$$

Using (12) and (13), we can compute $((\Phi_n^{(j)})^{-1}(\Phi_n^{(j)}(z))$ and show it can be written in a neighborhood N of S^1 in the form:

$$((\Phi_n^{(j)})^{-1})'(\Phi_n^{(j)}(z)) = \frac{\varepsilon_n^{(j)}}{\lambda_n^{(j)}} S_n^{(j)}(z)\, z^{\delta(j,n)}$$

where for all j:

$$S_n^{(j)} \to 1 \tag{14}$$

uniformely on N when $n \to +\infty$, and $\varepsilon_n^{(j)}$ and $\delta(j,n)$ are respectively -1 and 2 or 1 and 0 according to whether $L_n^{(j)}$ is an inversion or the identity. Then each term of (11) is

$$\int_{S^1} \frac{1}{z^2}((\Phi_n^{(j)})^{-1})'(\Phi_n^{(j)}(z))\,\tilde{G}(\Phi_n^{(j)}(z))\,dz$$
$$= \varepsilon_n^{(j)} \int_{S^1} \frac{1}{z^2} \cdot \frac{1}{\lambda_n^{(j)}} S_n^{(j)}(z)\, z^{\delta(j,n)}\, \tilde{G}(\lambda_n^{(j)} L_n^{(j)} \varphi_n^{(j)}(z) + \gamma_n^{(j)})\,dz. \tag{15}$$

Now take the variation $F\colon D \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, where D is the unit disk, defined by:

$$f(w,z) = (1+w)\,g_n(z).$$

The corresponding infinitesimal variation is:

$$G(z) = \frac{\partial F}{\partial w}(0,z) = g_n(z)$$

and then

$$\tilde{G}(z) = z.$$

Replacing this $\tilde{G}$ in (15) we obtain:

$$\varepsilon_n^{(j)} \int_{S^1} \frac{1}{z^2} S_n^{(j)}(z)\, z^{\delta(j,n)} \left(L_n^{(j)} \varphi_n^{(j)}(z) + \frac{\gamma_n^{(j)}}{\lambda_n^{(j)}} \right) dz.$$

By (14), when $n \to +\infty$, this integral is approximated by:

$$\varepsilon_n^{(j)} \int_{S^1} \frac{1}{z^2} z^{\delta(j,n)} \left(L_n^{(j)} z + \frac{\gamma^{(j)}}{\lambda_n^{(j)}} \right) dz$$
$$= \varepsilon_n^{(j)} \int_{S^1} \frac{1}{z^2} z^{\delta(j,n)} L_n^{(j)} z\,dz + \varepsilon_n^{(j)} \int_{S^1} \frac{1}{z^2} z^{\delta(j,n)} \frac{\gamma_n^{(j)}}{\lambda_n^{(j)}}\,dz.$$

When $L_n^{(j)}$ is the identity, $\delta(j,n)$ and $\varepsilon_n^{(j)}$ are 0 and 1 and the value of this expression is $2\pi i$. When $L_n^{(j)}$ is an inversion, $\delta(j,n)$ and $\varepsilon_n^{(j)}$ are 2 and -1 and then the value is $-2\pi i$. Now observe that $L_n^{(j)}$ is the identity or an inversion according to whether

$$g_n^j\colon g_n^{-j}(U_n) \to U_n$$

preserves or reverses orientation of the invariant curves of the Herman ring (in the sense of the statement of Lemma I.7), and then $L_n^{(j)}$ is the identity or an inversion according to whether the number $v_j(\infty)$ (defined as in the statement of I.7) is 1 or -1. Therefore the sum at left in (11) is

$$2\pi i \sum_{j=0}^{m-1} v_j(\infty),$$

which it is $\neq 0$ by the way we placed the point ∞. Therefore the sum at left in (11) converges to a number $\neq 0$ when $n \to +\infty$ thus contradicting (11).

Proof of Theorem I.6. Given two points p and q in $\overline{\mathbb{C}}$, we say that an analytic map $\varphi:(A(r,R)\to\overline{\mathbb{C}}$ *separates* p and q if they belong to different connected components of the complement of $\varphi(A(r,R))$. The proof of I.6 is an application of the following lemma.

Lemma. *For all* $0<r<R<\infty$ *and* $p\in\overline{\mathbb{C}}$, $q\in\overline{\mathbb{C}}$, *the family of analytic injective maps* $\varphi: A(r,R)\to\overline{\mathbb{C}}$ *that separate* p *and* q *is normal.*

Proof. We can suppose that $p=0$ and $q=\infty$. Let $\varphi_n: A(r,R)\to\mathbb{C}$ be a sequence of analytic injective maps separating 0 and ∞. Take

$$r_1<r<R<R_1. \tag{1}$$

Then $\varphi_n(A(r,R))$ cannot contain $A(r_1,R_1)$ because this would imply that $\varphi_n^{-1}/A(r_1,R_1)$ is an analytic injective map of $A(r_1,R_1)$ into $A(r,R)$ such that the bounded connected component of the complement of $\varphi_n^{-1}(A(r_1,R_1))$ contains the disk $\{z\,||z|<r\}$, and it is well known that this is impossible under Condition (1). Then for all n there exists $\alpha_n\in\mathbb{C}$ with

$$|\alpha_n|=r_1 \quad \text{or} \quad |\alpha_n|=\mathrm{R}_1 \tag{2}$$

and

$$\alpha_n\notin\varphi_n(A(r,R)). \tag{3}$$

Then, by (3), the sequence of functions

$$\alpha_n^{-1}\varphi_n:\ A(r,R)\to\mathbb{C}$$

doesn't take the value 1 and, since φ_n doesn't take the value 0, the same property holds for $\alpha_n^{-1}\varphi_n$. Therefore there exists a subsequence $\{\alpha_{n_j}^{-1}\varphi_{n_j}\}$ that converges uniformely on compacts sets of $A(r,R)$ either to ∞ or to an analytic map $\varphi: A(r,R)\to\mathbb{C}$. In the first case, since $|\alpha_n|\geqq r_1$ by (2), it follows that $\varphi_{n_j}\to\infty$ on compact subsets of $A(r,R)$. In the second case, since by (2) we can suppose that the sequence $\{\alpha_{n_j}\}$ converges to some $\alpha\in\mathbb{C}$, it follows that $\varphi_{n_j}\to\alpha\varphi$.

Now let us prove Theorem I.6. Let $\{\varphi_n\}$ be a sequence satisfying its hypothesis. Suppose that it is not true that $\varphi_n\to$Identity uniformely on compact subsets of $\mathbb{C}-\{0\}$. Then there exist $m_1<m_2<m_3<\dots$ such that for all m_i the sequence

$$\varphi_n: A(r_{m_i},R_{m_i})\to\mathbb{C}, \qquad n\geqq m_i,$$

doesn't converge uniformely on compact subsets of $A(r_{m_i},R_{m_i})$ to the identity. But by the hypothesis of I.6 every φ_n separates 0 and ∞. By the lemma this is a normal sequence for each m_i. Therefore, using the normality of these sequences and a standard diagonal procedure, there exists an analytic function

$$\varphi: \mathbb{C}-\{0\}=\bigcup_i A(r_{m_i},R_{m_i})\to\mathbb{C}$$

and a subsequence $\{\varphi_{n_j}\}$ such that for all i

$$\varphi_{n_j}/A(r_{m_i},R_{m_i})\to\varphi$$

when $j \to +\infty$, uniformely on compact subsets of $A(r_{m_i}, R_{m_i})$. Since $\varphi_n(1)=1$ for all n it follows that $\varphi(1)=1$, and since φ_n is injective for all n, the function φ is also injective or it is constant. But φ cannot be constant because by hypothesis (c) of Theorem I.6 the winding number with respect to zero of the arc $\theta \to \varphi(e^{i\theta})$ is $\neq 0$. Hence φ is injective. Then the origin and ∞ cannot be essential singularities. Hence φ is a rational injective function with $\varphi(1)=1$. This means that φ is either the identity or $z \to z^{-1}$. But hypothesis (c) of Theorem I.6 implies that the winding number with respect to the origin of the are $\theta \to \varphi(e^{i\theta})$ is one. Therefore φ is the identity.

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