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On the rate of mixing of Axiom A flows

Mark Pollicott

Institut des Hautes Études Scientifiques, 35, route de Chartres, F-91440 Bures-sur-Yvette, France

§ 0. Introduction

Axiom A systems were originally introduced by Smale in his seminal paper on dynamical systems [28]. One of their main purposes was to generalise Anosov systems (both diffeomorphisms and flows). Perhaps the most significant feature of this generalisation was that it further divorced the purely dynamical aspects of the system from the underlying geometry of the manifold. Even in such generality remarkably powerful results can still be obtained for Axiom A diffeomorphisms. For example, the rate of mixing of an Axiom A diffeomorphism is always exponential ([2], p. 38) and the zeta function for the diffeomorphism is rational [16]. However, for Axiom A flows the corresponding results are not always valid. For example, the rate of mixing for Axiom A flows need not be exponentially fast [22] and the zeta functions for these flows need not be meromorphic in the entire complex plane [12].

The purpose of this paper is to actually relate the rate of mixing of an Axiom A flow to the meromorphic domain of its zeta function. In particular, we shall give necessary conditions for exponential mixing (which we also refer to as exponential decay of correlations). We shall also exhibit examples of Axiom A flows for which the rate of mixing can be chosen to be arbitrarily slow. In particular, this answers a question of Bowen regarding the possibility of polynomial rates of mixing for Axiom A flows ([3], p. 31). Following the program advanced by Bowen and Ruelle [6] our first step is to reduce the problem to the case of suspended flows using the powerful and useful symbolic dynamics of Bowen [4]. The rate of decay of the so-called correlation function is then reflected in the analytic domain of its Fourier transform. (In particular, for exponential decay this is governed by the Paley-Wiener theorem ([15], p. 174)). Our approach is to relate domain of the Fourier Transform (for the case of the suspended flow) to the spectrum of an associated Ruelle operator acting on an appropriate Banach space. We then use previous work by the author to relate this spectrum to the domain of the zeta function for the flow (for both suspended flows and Axiom A flows) [20].

We should point out that Parry and the author originally studied this zeta function in connection with Prime Orbit Theorems for Axiom A flows [19].

In Sect. 1 we introduce our principle tool, the Ruelle operator. In Sect. 2 we recall some results on suspended flows and in Sect. 3 we introduce the zeta function and correlation functions for these flows. In the fourth section we prove our main result in the context of suspended flows. In the fifth section we introduce Axiom A flows and rephrase our results in this stronger setting. In Sect. 6 we give a simple counter-example to show that in general there is no order of mixing common to the class of all Axiom A flows.

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§ 1. The Ruelle operator

The most important tool in our analysis of suspended flows (and consequently of Axiom A flows) will prove to be the Ruelle operator. Its usefulness in the study of subshifts of finite type (the discrete analogue of the suspended flows we shall be studying) was demonstrated by Bowen in [2]. The spectrum of this operator will be used in Sect. 4 as an intermediate step in relating the zeta function of a flow to its correlation function.

Let A be an aperiodic $k \times k$ matrix with entries 0 or 1 and define $\Sigma^+ = \left\{ x \in \prod_0^\infty \{1, 2, \dots, k\} \mid A(x_n, x_{n+1}) = 1,\ n \geqq 0 \right\}$. For any $0 < \theta < 1$ we can define a metric d^+ on Σ^+ by $d^+(x, y) = \theta^n$, where n is the largest positive integer for which $x_i = y_i$, $0 \leqq i \leqq n$. With respect to this metric Σ^+ is a compact zero-dimensional space. In fact, a basis for the corresponding topology is given by cylinders of the form

$$[x_0, \dots, x_{n-1}] = \{z \mid z_i = x_i,\ i = 0, \dots, n-1\}.$$

(This topology is exactly the induced topology on Σ^+ from the product topology on $\prod_0^\infty \{1, 2, \dots, k\}$.) The continuous map $\sigma\colon \Sigma^+ \to \Sigma^+$ given by $(\sigma x)_n = x_{n+1}$ is called a (one-sided) *subshift of finite type*. In fact, σ is trivially a bounded-one local homeomorphism. Subshifts of finite type were originally introduced by William Parry in [17]. They have proved extremely important in the study of Axiom A diffeomorphisms ([2], Chap. 3).

Given a continuous function $g\colon \Sigma^+ \to \mathbb{R}$ we can associate with it a real number $P(g)$, called the *pressure of* g, by $P(g) = \sup\{h(\mu) + \int g\, d\mu \mid \mu\ \sigma\text{-invariant probability measure}\}$ (where $h(\mu)$ is the entropy of $\sigma\colon \Sigma^+ \to \Sigma^+$ with respect to μ). There are other equivalent ways of defining pressure (cf. [31]).

The map $P\colon C(\Sigma^+) \to \mathbb{R}$ is easily seen to be Lipschitz continuous (with Lipschitz constant 1) and convex [32]. Those measures for which the supremum is attained are called the *equilibrium states* (or Gibbs states) for g. Let ω:

$\Sigma^+ \to \mathbb{R}$ be continuous and $c \in \mathbb{R}$ a constant then $P(g+\omega\sigma-\omega+c)=P(g)+c$ and it is obvious that g and $g+\omega\sigma-\omega+c$ share the same equilibrium states. For a given g there always exists at least one equilibrium state ([2], p. 25). If we further assume that $g\colon \Sigma^+ \to \mathbb{R}$ is Lipschitz (with respect to d^+, for some $0<\theta<1$) then g has a unique equilibrium state ([2], p. 27). (In the special case $g=0$, $P(0)$ is called the topological entropy of $\sigma\colon \Sigma^+ \to \Sigma^+$ and the unique equilibrium state is called the measure of maximum entropy or Parry measure).

For a fixed $0<\theta<1$, let F_θ denote the space of complex-valued Lipschitz functions and define a norm on this space by $\|g\| = \|g\|_\infty + \|g\|_\theta$, where $\| \ \|_\infty$ is the usual supremum norm on $C(\Sigma^+, \mathbb{C})$ and $\|g\|_\theta = \sup\left\{\frac{|g(x)-g(y)|}{d^+(x,y)} \,\middle|\, x \neq y\right\}$ i.e. $\|g\|_\theta$ is the smallest possible Lipschitz constant for g. With respect to this norm $(F_\theta, \| \ \|)$ is a Banach space ([23], p. 87).

Notice that we have a "filtration" of spaces F_θ indexed by $0<\theta<1$, in the sense that for $0<\theta<\theta'<1$, $\|g\|_{\theta'} \leqq \|g\|_\theta$ (if $g \in F_\theta$) and $F_\theta \subseteq F_{\theta'}$.

We define the *Ruelle operator* $L_f\colon F_\theta \to F_\theta$ $(f \in F_\theta)$ by $(L_f g)(x) = \sum_{\sigma y = x} g(y) \exp f(y)$, where the summation is over the finite set of $y \in \Sigma^+$ for which $\sigma y = x$.

We summarize in the following proposition the main features of the spectrum of $L_f\colon F_\theta \to F_\theta$ that we will need later.

Proposition 1. (i) (Ruelle). *If $u \in F_\theta$ is real-valued then* $\exp P(u)$ *is a simple eigenvalue of $L_u\colon F_\theta \to F_\theta$. Furthermore,* $\exp P(u)$ *is the unique eigenvalue of maximum modulus and the corresponding eigenfunction $h>0$ is strictly positive. In addition there exists a unique probability measure ρ such that $L_u^* \rho = \exp P(u) \rho$ and the σ-invariant probability measure μ given by $d\mu/d\rho = h$ is the unique equilibrium state for f. (Here L_u^* can be interpreted by $\int g \, d(L_u^* \rho) = \int L_u g \, d\rho$, $g \in C(\Sigma^+)$)* [3, 23, 33].

(ii) (Pollicott). *If $f = u + iv \in F_\theta$ is complex-valued then the spectrum of $L_f\colon F_\theta \to F_\theta$ consists of two components:*

(a) *The disc $\{z \mid |z| \leqq \theta \exp P(u)\}$;*
(b) *Isolated eigenvalues in the annulus $\theta \exp P(u) < |z| \leqq \exp P(u)$* [20].

(The type of spectrum described in (ii) is often called *quasi-compact*).

If we assume that $u\colon \Sigma^+ \to \mathbb{R}^+$ satisfies $u \in F_\theta$ and $P(u)=0$ then by part (i) of the proportion $L_u h = h$, for some $h>0$. Thus if we write $\omega = \log h$ then we have the following

Proposition 2. *For a real-valued function $u \in F_\theta$, $P(u)=0$, there exists a continuous function $\omega\colon \Sigma^+ \to R$ such that for $u' = u + \omega - \omega\sigma$ we have $u'<0$, $L_{u'} \mu = \mu$ (where μ is the unique equilibrium state for u and u')* (cf. [33]).

Remark. If we look at the larger space $C(\Sigma^+)$ then the spectrum of $L_u\colon C(\Sigma^+) \to C(\Sigma^+)$ consists of the entire disc $|z| \leqq \exp P(u)$ (cf. [29]). By considering a case with more stringent conditions Ruelle has shown an analogous operator to be nuclear [24].

§ 2. Suspended flows

Suspended flows are simply constructed continuous flows which will later prove to be useful models of Axiom A flows which embody much of the dynamics of the original system. In this section we recall some of the principal results on suspended flows. We first need to replace our endomorphism σ: $\Sigma^+ \to \Sigma^+$ of Sect. 1 by a homeomorphism (in some sense its "natural extension" (cf. [10], p. 239)). As before we assume that A is a $k \times k$ aperiodic matrix with entries 0 or 1 and define $\Sigma = \left\{ x \in \prod_{-\infty}^{+\infty} \{1, 2, \ldots, k\} \mid A(x_n, x_{n+1}) = 1,\ n \in Z \right\}$. For this space we define a metric d (for a given $0 < \theta < 1$) by $d(x, y) = \theta^{2n}$, where n is the largest positive integer for which $x_i = y_i$, $-n \leqq i \leqq n$.

The homeomorphism σ: $\Sigma \to \Sigma$ defined by $(\sigma x)_n = x_{n+1}$ is called the *(the two-sided) subshift of finite type* (we have now used σ to denote both the two-sided subshift and the one-sided subshift).

We can define the pressure and equilibrium states for a continuous function g: $\Sigma \to \mathbb{R}$ in complete analogy with the case of the one-sided subshift as follows. We define the *pressure* of g by $P(g) = \sup\{h(m) + \int g\, dm \mid m$ σ-invariant probability measure$\}$ (where $h(m)$ is the entropy of σ: $\Sigma \to \Sigma$ with respect to the invariant measure m). These measures for which the supremum is attained are called *equilibrium states* for g. As before, the addition of $\omega\sigma - \omega + c$ to g (where ω: $\Sigma \to \mathbb{R}$ is continuous and c constant) changes the pressure by c and does not change the equilibrium states. Also, if g is Lipshitz with respect to the d-metric then g has a unique equilibrium state ([2], p. 25).

For a strictly positive Lipschitz function r: $\Sigma \to \mathbb{R}^+$ we define a new space $\Sigma^r = \{(x, q) \in \Sigma \times \mathbb{R}^+ \mid 0 \leqq q \leqq r(x)\}$ where we identify $(x, r(x))$ and $(\sigma x, 0)$. We then define the suspended flow σ^r: $\Sigma^r \to \Sigma^r$ locally by $\sigma_t^r(x, q) = (x, q + t)$, taking into account the identifications. This can be interpreted intuitively as flowing vertically under the graph of r over Σ.

The flow σ_t^r: $\Sigma^r \to \Sigma^r$ is called (topologically) *weak-mixing* if there is no non-trivial solution to $F\sigma_t^r = e^{iat} F$, with $F \in C(\Sigma^r)$ and $a > 0$. An equivalent condition is that closed orbit periods should not be integer multiples of a single constant value [5]. We shall restrict our attention to weak-mixing flows. The case of not weak-mixing flows essentially reduces to that for Axiom A diffeomorphisms [5].

The σ^r-invariant probability measures on Σ^r all take the form $\mu \times l / \int r\, d\mu$, where l is Lebesgue measure on the real line and μ is a σ-invariant probability measure on Σ. We define the metric on Σ^r locally as the product of d and the usual metric on $\mathbb{R}$ (cf. [7]).

We can now introduce pressure and equilibrium states for flows. Let F: $\Sigma^r \to \mathbb{R}$ be continuous then by analogy with the case in Sect. 1 we define the *pressure* of F by $P(F) = \sup\{h(m) + \int F\, dm \mid m$ σ^r-invariant probability measure$\}$ (where $h(m)$ is the entropy of σ_1^r: $\Sigma^r \to \Sigma^r$ with respect to m). If F is Hölder continuous then the above supremum is attained at exactly one measure m, called the *equilibrium measure* (or Gibbs measure) for F [6]. In the special case $F = 0$, $P(0)$ is called the *(topological) entropy* of σ^r, which is denoted $h(\sigma^r)$, and m is called the measure of maximal entropy.

There is a simple relationship between pressure and equilibrium states defined for the two systems σ^r: $\Sigma^r \to \Sigma^r$ and σ: $\Sigma \to \Sigma$ which was developed by Bowen and Ruelle [6]. Given a Hölder continuous function F: $\Sigma^r \to \mathbb{R}$ we can define f: $\Sigma \to \mathbb{R}$ by $f(x) = \int_0^{r(x)} F(x,t)\,dt$. The unique equilibrium state m for F then takes the form $m = \mu \times l / \int r\,d\mu$ where μ is the unique equilibrium state for $f - P(F) r \in F_\theta$ (For an appropriate choice of $0 < \theta < 1$) [6]. Furthermore $P(f - P(F) r) = 0$ [6]. The topological entropy of the flow $h(\sigma^r)$ occurs as the unique zero of the homeomorphism $\mathbb{R} \to \mathbb{R}$, $t \to P(-tr)$. Having related the suspended flow to the two-sided shift we can now relate the two-sided shift to the one-sided shift. Let π: $\Sigma \to \Sigma^+$ be the continuous surjection defined by $(\pi(x))_n = x_n$, $n \geqq 0$. Sinai proved the following [27].

Proposition 3. *Given a Lipschitz function* g_0: $\Sigma \to \mathbb{R}$ *(with respect to d) there exists* $\omega_0 \in C(\Sigma)$ *and* $g \in F_\theta$ *such that* $g_0 = g\pi + \omega_0 \sigma - \omega_0$ [27] (cf. [2], p. 10).

In particular, g and g_0 have the same pressure and equilibrium state. By Proposition 2 we can replace g by g' such that $L_{g'} 1 = 1$. Thus for our Hölder continuous function F: $\Sigma^r \to \mathbb{R}$ if we take $g_0 = f - P(F) r$ then $L^*_{g'} \mu = \mu$, where $m = \mu \times l / \int r\,d\mu$ is the unique equilibrium state for F. (Here we have made use of the obvious one-one correspondence between σ-invariant measures on Σ and Σ^+).

§ 3. Zetafunctions and correlation functions for suspended flows

We now recall the definitions of zeta functions and correlation functions and summarise their properties in preparation for the theorem in the next section.

Let σ^r_t: $\Sigma^r \to \Sigma^r$ be a suspended flow. It follows from the definitions that a closed σ-orbit $\{x, \sigma x, \ldots, \sigma^{n-1} x\}$ corresponds to a closed σ^r-orbit of period $r^n(x) = r(x) + r(\sigma x) + \ldots + r(\sigma^{n-1} x)$ i.e. the sums of the heights of r above points in the σ-orbit. Let F: $\Sigma^r \to \mathbb{R}$ be a Hölder continuous function following Ruelle ([23], p. 173) we define a *zeta function* for σ^r and F by

$$\zeta(s,F) = \prod_\tau \left(1 - \exp \int_0^{\lambda(\tau)} (F(\phi_t x_\tau) - s)\,dt\right)^{-1}$$

where the Euler product is over all σ^r-closed orbits τ of least period $\lambda(\tau)$, where $x_\tau \in \tau$. The product converges and $\zeta(s,F)$ is well-defined for $\mathscr{R}(s) > P(F)$. If $f(x) = \int_0^{r(x)} F(x,t)\,dt$ then $\zeta(s,F)$ can be rewritten as $\zeta(s,F) = \exp - \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} \exp(f^n - s r^n)(x)$ ([23], p. 173). By Proposition 2 we can associate with the Hölder continuous functions r, f: $\Sigma \to \mathbb{R}$ functions $r', f' \in F_\theta$ for which $r = r'\pi + \omega_1 \sigma - \omega_1$ and $f = f'\pi + \omega_2 \sigma - \omega_2$. The main results on $\zeta(s,F)$ we shall need are contained in the following proposition (cf. [20]).

Proposition 4. *Let* F: $\Sigma^r \to \mathbb{R}$ *be Hölder continuous then* $\zeta(s,F)$ *can be extended meromorphically to the strip* $P(F) \geqq \mathscr{R}(s) > P(F) - \delta$ *where:* (a) *the constant* $\delta > 0$

satisfies $P(f'-(P(F)-\delta)r')=C(\theta,h(\sigma))>0$; (b) *the poles* s_n *in this strip correspond to* 1 *being an eigenvalue for* $L_{f'-s_n r'}$: $F_\theta \to F$.

(The map $\mathbb{R}\to\mathbb{R}$, $t\to P(f'-(P(F)-t)r')$ is a homeomorphism.) Thus this proposition describes the poles for the zeta function in terms of the corresponding Ruelle operator (at least in a strip $P(F)\geqq\mathscr{R}(s)>P(F)-\delta$). If σ_t^r: $\Sigma^r\to\Sigma^r$ is a weak-mixing flow then $\zeta(s,F)$ has exactly one (simple) pole on $\mathscr{R}(s)=P(F)$, at $s=P(F)$ (This was a simple argument involving the spectrum of the Ruelle operator (cf. [19])).

We now turn to the idea of the "rate of mixing" for weak-mixing suspended flows. Given two Hölder continuous functions A, B: $\Sigma^r\to\mathbb{R}$ and an equilibrium state m on Σ^r (corresponding to a third Hölder continuous function F: $\Sigma^r\to\mathbb{R}$, say) it is known that if σ^r is (topologically) weak-mixing then $\int A\sigma_t^r\cdot B\,dm\to\int A\,dm\cdot\int B\,dm$ as $t\to+\infty$ [6]. This simply means that σ_t^r: $\Sigma^r\to\Sigma^r$ is mixing with respect to m [5]. Given σ^r, A and B we define the corresponding *correlation function* by $\rho_{A,B}(t)=\int A\sigma_t^r\cdot B\,dm-\int A\,dm\cdot\int B\,dm$. We will be primarily interested in relating the speed with which $\rho_{A,B}(t)$ tends to zero to the analytic domain of $\zeta(s,F)$.

§ 4. Exponential decay of the correlation function for suspended flows

In this section we shall relate the decay of correlations $\rho_{A,B}(t)$ to the zeta function for the flow. We shall achieve this by relating the poles of the Fourier transform $\hat\rho_{A,B}(s)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ist}\rho_{A,B}(t)\,dt$ to the spectrum of the corresponding Ruelle operator and then invoking Proposition 4. Our main result is the following

Theorem 1. *Let* σ^r: $\Sigma^r\to\Sigma^r$ *be a weak-mixing flow and* m *the unique equilibrium state for a Hölder continuous function* F: $\Sigma^r\to\mathbb{R}$ *then of the following statements* (b) *is a necessary condition for* (a)

a) $\rho_{A,B}(t)\to 0$ *exponentially fast for every Hölder continuous* A,B: $\Sigma^r\to\mathbb{R}$

b) $\zeta(s,F)$ *has an analytic extension to a domain* $\mathscr{R}(s)>P(F)-\varepsilon$ (*for some* $\varepsilon>0$), *except for a simple pole at* $s=P(F)$.

Proof. The main idea is to relate the poles for the zeta function $\zeta(s,F)$ and the Fourier transform $\hat\rho_{A,B}(s)$ and to use the Paley and Wiener Theorem ([15], p. 174). For σ^r we have

$$A\sigma_t^r(x,u)=\sum_{n=0}^{\infty}\left\{\int_{0^+}^{r(\sigma^n x)} A(\sigma^n x,v)\,\delta(u+t-v-r^n(x))\,dv\right\}.$$

(All but one term in the summation will be zero. However, this will prove to be a useful representation for reasons of "book-keeping"). By replacing A by $A-\int A\,dm$ we can assume $\int A\,dm=0$. Thus,

$$\rho_{A,B}(t)=\int A\sigma_t^r\cdot B\,dm$$

$$=\frac{1}{\int r\,d\mu}\int\left\{\sum_{n=0}^{\infty}\int_{0^+}^{r(x)} B(x,u)\left[\int_{0^+}^{r(\sigma^n x)} A(\sigma^n x,v)\,\delta(u+t-v-r^n(x))\,dv\right]du\right\}d\mu,\qquad t>0,$$

where we have assumed $m = \mu \times l / \int r \, d\mu$. The next few steps have much in common with [22]. We can take the Fourier transform of $\rho_{A,B}(t)$ (where we can assume $\mathscr{I}(s) > 0$ if necessary to guarantee existence) and write the following:

$$\hat{\rho}_{A,B}(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ist} \rho_{A,B}(t) \, dt$$

$$= \frac{1}{2\pi \int r \, d\mu} \sum_{n=0}^{\infty} e^{isr^n(x)} \left[\int_0^{r(x)} B(x,u) e^{ius} du \right] \cdot \left[\int_0^{r(\sigma^n x)} A(\sigma^n x, v) e^{ivs} dv \right] d\mu(x).$$

(For simplicity we take $\rho_{A,B}(t) = 0$, $t < 0$.)

For convenience we denote $\bar{B}(s,x) = \int_0^{r(x)} B(x,u) e^{-ius} du$, and $\bar{A}(s,x) = \int_0^{r(x)} A(x,v) e^{ivs} dv$.

We can legitimately make two simplifying assumptions:

(i) We can replace $r: \Sigma \to \mathbb{R}^+$ by $r': \Sigma^+ \to \mathbb{R}$, where $r = r'\pi + \omega\sigma - \omega$ as in Proposition 3 (and the ensueing comment), since the corresponding flows σ^r and $\sigma^{r'\pi}$ conjugate.

(ii) We can assume that $A, B: \Sigma^r \to \mathbb{R}$ are independent of x_i, $i < 0$, by first approximating by functions depending on only finitely many Σ co-ordinates.

If $g_0 = f_0 - P(F) r'$ then we can construct $g' = f' - P(F) r' \in F_\theta$ by Proposition 2 and Proposition 3 satisfying $L^*_{g'} \mu = \mu$. We can therefore write

$$\hat{\rho}(s) = \frac{1}{2\pi \int r \, d\mu} \sum_{n=0}^{\infty} \int L^n_{g'} \{ e^{isr'^n(x)} \bar{B}(s,x) \cdot \bar{A}(s, \sigma^n x) \} \, d\mu(x)$$

$$= \frac{1}{2\pi \int r \, d\mu} \sum_{n=0}^{\infty} \int \bar{A}(s,x) (L^n_{g'+isr'} \underline{B})(s,x) \, d\mu(x).$$

By Proposition 1 we can write $L^n_{g'+isr'} = \sum_{j=1}^{k} \lambda_j^n P_j + U_1$ where P_j: $F_\theta \to F_\theta$ is the eigenprojection correspond to the eigenvalue λ_j and $\lim_n \|U_n\|^{1/n} \leqq \theta' \exp P(g' - \mathscr{I}(s) r')$ for a given $\theta < \theta' < 1$. Therefore,

$$\hat{\rho}(s) = \frac{1}{2\pi \int r \, d\mu} \sum_{j=1}^{k} \int \bar{A}(s,x) \cdot P_j \bar{B}(s,x) \, d\mu \left(\sum_{n=0}^{\infty} \lambda_j^n \right)$$

$$+ \frac{1}{2\pi \int r \, d\mu} \sum_{n=0}^{\infty} \int \bar{A}(s,x) \cdot U_n \bar{B}(s,x).$$

If the condition $P(f - (P(F) + \mathscr{I}(s)) r) < C(\theta', h(\sigma))$ is satisfied then the second part converges to an analytic function. If λ_j is an isolated simple eigenvalue then $s \to \lambda_j$, P_j are both analytic and the first part of the above expression converges to a finite sum of meromorphic functions with poles whenever $s \to \lambda_j = 1$. (For repeated eigenvalues the sum of terms corresponding to the multiple eigenvalues combine to give a meromorphic function). We know from Proposition 4 that this is exactly the condition that $P(F) - is$ should give rise to a pole for $\zeta(s, F)$. Thus $\hat{\rho}_{A,B}(s)$ has an analytic extension to a strip $|\mathscr{I}(s)| \leqq \varepsilon$ if and only if $\zeta(s, F)$ has an analytic extension to the domain $\mathscr{R}(s) > P(F) - \varepsilon$, except for a simple pole at $s=P(F)$. (The pole at $s=0$ for $\hat{\rho}(s)$, corresponding to $P(F)\to\lambda_1=1$, vanishes since the corresponding "residue" is zero under the assumption $\int A\,dm=0$.) The theorem then follows from the Paley and Wiener theorem ([15], p. 174) relating the exponential decay of a function to the analytic continuation of its Fourier transform.

Remark. For specific choices of A and B residues calculated for the poles of the zeta function may in fact be zero (the trivial examples are $A=0$ or $B=0$, or $s=P(F)$ if $\int A\,dm=0$, etc.). However, this will not be the general case.

Corollary 1.1. *$\hat{\rho}_{A,B}(s)$ always has a meromorphic extension to some strip $|\mathscr{I}(s)|\leqq\varepsilon$. Furthermore, if $\sigma_t^r\colon \Sigma^r\to\Sigma^r$ is weak-mixing then $\hat{\rho}_{A,B}(s)$ is analytic on the line $\mathscr{I}(s)=0$.*

Remark. The above theorem gives that the poles for $\hat{\rho}_{A,B}(s)$ generally occur as poles for $\zeta(s,F)$ (under translation by $P(F)$ and rotation by i). Consider the special case when there are only finitely many poles $s_1,\dots,s_n$ with $P(F)>\mathscr{R}(s_i)>P(F)-\varepsilon$. Then these can be interpreted as "resonances" against a background of otherwise exponential decay of order $e^{-\varepsilon t}$ i.e. there exist $C_1,\dots,C_n$ such that $\rho_{A,B}(t)=\sum_{j=1}^{n} C_j e^{-\varepsilon_j t}\cos t_j$, where $s_j=P(F)-\varepsilon_j+it_j$, $j=1,\dots,n$. (Here we have made use of the symmetry $\zeta(s,F)=\overline{\zeta(\bar{s},F)}$ which makes $s,\bar{s}$ poles.)

Remark. The shift $\sigma\colon\Sigma^+\to\Sigma^+$ induces a map $U\colon F_\theta\to F_\theta$ by $Uf=f\sigma$. If $L\colon F_\theta\to F_\theta$ satisfies $L^*\mu=\mu$ then $L\colon L^2(\mu)\to L^2(\mu)$ is the dual of $U\colon L^2(\mu)\to L^2(\mu)$ and by Proposition 1 $L^n g=\sum_{j=1}^{k}\frac{1}{\rho_j^n}P_j(g)+U_n g$, where ρ_j is a pole for the zetafunction $\zeta(z)=\exp-\sum_{n=1}^{\infty}\frac{z^n}{n}\operatorname{Card}\{x\mid\sigma^n x=x\}$ and $P_j\colon F_\theta\to F_\theta$ is an eigenprojection associated with $1/\rho_j(\|U_n\|^{1/n}\leqq 1-\varepsilon$, for some $\varepsilon>0$, all large n).

The flow $\phi_t\colon\Sigma^r\to\Sigma^r$ induces a map $U^r\colon\mathscr{H}^\alpha\to\mathscr{H}^\alpha$ for which the $L^2(m)$ dual is given by

$$(L^t G)(x,v)=\sum_{j=1}^{k}\frac{e^{ivs_j}P_j\left\{\int_0^{r(x)} G(x,u)\,e^{-ius_j}\,du\right\}}{2\pi\lambda_j'(s_j)\exp(P(F)-s_j)t}+\langle\text{error term}\rangle$$

when there are only finitely many poles for $\zeta(s,F)$ with $P(F)-\varepsilon<\mathscr{R}(s_j)<P(F)$. (Here $\mathscr{H}^\alpha$ are Hölder continuous functions with exponent $\alpha>0$).

Remark. Using Bochner's theorem [18], the group of unitary operators $U_t\colon L^2(m)\to L^2(m)$ define a spectral measure on R by $\rho(t)=\int F\,\sigma_t^r\cdot F\,dm=\int_{-\infty}^{+\infty} e^{it}\,d\sigma_F(\lambda)$, $-\infty<t<+\infty$. Therefore, we can formally write

$$\hat{\rho}(s)=\frac{1}{2\pi}\int_0^\infty \rho(t)\,e^{ist}\,dt=\int_{-\infty}^{+\infty}\int_0^\infty e^{it(\lambda+s)}\,dt\,d\sigma_F(\lambda)=\int_{-\infty}^{+\infty}\frac{d\sigma_F(\lambda)}{i(\lambda+\sigma)}.$$

Since this flow is bernoulli it has countable Lebesgue spectrum and in particular σ_F has no atoms on R.

§ 5. Axiom A flows

Let M be a compact manifold and let $\phi_t: M \to M$ be a C^1-flow. A compact invariant set Λ containing no fixed points is called *hyperbolic* if the tangent bundle restricted to Λ can be written as the Whitney sum of three $D\phi$-invariant continuous sub-bundles $T_\Lambda M = E + E^s + E^u$, where E is the one-dimensional bundle tangent to the flow and there are constants $C, \lambda > 0$ such that

(a) $\|D\phi_t(v)\| \leqq C e^{-\lambda t} \|v\|$ for $v \in E^s$, $t \geqq 0$

(b) $\|D\phi_{-t}(v)\| \leqq C e^{-\lambda t} \|v\|$ for $v \in E^u$, $t \geqq 0$.

A hyperbolic set Λ is called *basic* if

(i) the periodic orbits of $\phi_t|_\Lambda$ are dense in Λ

(ii) $\phi_t|_\Lambda$ is topologically transitive

(iii) there is an open set $U \supseteq \Lambda$ with $\bigcap_{t=-\infty}^{+\infty} \phi_t U$.

The *non-wandering set* Ω is defined by $\Omega = \{x \in M \mid$ for all neighbourhoods $V \ni x$, all $t_0 > 0$, exists $t > t_0$ with $\phi_t(V) \cap V \neq \phi\}$.

The flow ϕ is called *Axiom A* if Ω is a disjoint union of a finite number of basic sets and hyperbolic fixed points. We shall *always* consider ϕ to be restricted to a basic set Λ containing more than one closed orbit.

By analogy with the suspended flow we define a zeta function for $\phi_t: \Lambda \to \Lambda$ and a Hölder continuous function $F: \Lambda \to \mathbb{R}$, by

$$\zeta(s, F) = \prod_\tau \left(1 - \exp \int_0^{\lambda(\tau)} (F(\phi_t x_\tau) - s)\, dt\right)^{-1},$$

where the product is over all closed ϕ-orbits τ of least period $\lambda(\tau)$, and $x_\tau \in \tau$.

Let $F: \Lambda \to \mathbb{R}$ be continuous then as for σ^r we define the *pressure* of F by $P(F) = \sup\{h(m) + \int F\, dm \mid m\ \phi\text{-invariant probability measure}\}$ (where $h(m)$ is the entropy of $\phi_1: \Lambda \to \Lambda$ with respect to m). The measures where this supremum is attained are again called the *Equilibrium states* for F. If F is Hölder continuous then there is exactly one equilibrium state for F [6]. If $F = 0$ then $h(\phi) = P(0)$ is called the (topological) entropy of ϕ and the unique equilibrium state is called the measure of maximal entropy.

Let $A, B: \Lambda \to \mathbb{R}$ be Hölder continuous then we can define $\rho_{A,B}(t) = \int A\phi_t \cdot B\, dm - \int A\, dm \cdot \int B\, dm$ to be the *correlation function* for ϕ, m, A and B. As for σ^r we call $\phi_0: \Lambda \to \Lambda$ *weak-mixing* if there is no non-trivial solution to $F\phi_t = e^{iat} F$, $F \in C(\Lambda)$, $a > 0$. If ϕ is weak-mixing then $\rho_{A,B}(t) \to 0$ as $t \to +\infty$.

The connection between Axiom A flows and suspended flows is the following powerful result of Bowen [4, 5].

Proposition 5 (Bowen). *Let ϕ be an Axiom A flow (restricted to a basic set) of entropy $h(\phi)$.*

(i) *There exists a suspended flow $\sigma_t^r: \Sigma^r \to \Sigma^r$ and a Lipschitz, surjective, bounded-one map $p: \Sigma^r \to \Lambda$ such that $p\sigma_t^r = \phi_t p$. Furthermore, if m is an equilibrium state for $Fp: \Sigma^r \to R$ then $p^* m$ is an equilibrium state for $F: \Lambda \to \mathbb{R}$ and p is* *an isomorphism with respect to these two measures. (In particular, ϕ and σ^r have the same entropy and ϕ is weak-mixing if and only if σ^r is weak-mixing.)*

(ii) *There exists $\varepsilon_0>0$ and a function $\psi(s)$ which is non-zero and analytic for $\mathscr{R}(s)>P(F)-\varepsilon_0$ and satisfies $\zeta(s, Fp)=\psi(s)\,\zeta(s, F)$.*

Using the above proposition, Theorem 1 takes the following form for Axiom A flows.

Theorem 2. *Let $\phi_t\colon \Lambda\to\Lambda$ be a weak-mixing Axiom A flow (restricted to a basic set) then the Fourier transform $\hat{\rho}_{A,B}(s)$ has a meromorphic extension to a strip $|\mathscr{I}(s)|\leqq\varepsilon$, which is analytic on the real line. Furthermore $\rho_{A,B}(t)$ tends to zero exponentially fast (for all Hölder continuous functions $A, B\colon \Lambda\to\mathbb{R}$) only if $\zeta(s,F)$ has an analytic extension to some strip $R(s)>P(F)-\varepsilon$, except for the simple pole at $s=P(F)$.*

If we write $\zeta(s)=\zeta(s,0)=\prod_\tau(1-\exp-s\,\lambda(\tau))^{-1}$ (cf. [3], p. 31) then a special case of the theorem is the following

Corollary 2.1. *If m is the measure of maximal entropy for the weak-mixing flow ϕ then $\rho_{A,B}(t)$ tends to zero exponentially fast as t increases (for all Hölder continuous $A, B\colon \Lambda\to\mathbb{R}$)* only if *$\zeta(s)$ has an analytic extension to a strip $h(\phi)-\varepsilon<\mathbb{R}(s)<h(\phi)$.*

Remark. We now know from Theorem 2 and [20] that of the following three conditions (iii) is implied by either (i) or (ii).

(i) Correlations decay exponentially fast (with respect to the measure of maximum entropy).

(ii) Prime Orbit theorems have exponential error terms.

(iii) $\zeta(s)$ has an analytic extension to a strip.

§ 6. Geodesic flows

A special case of an Axiom A flow is a geodesic flow on (the unit tangent) bundle of a) compact surface of constant negative curvature [1]. This special class of flows have been studied at length [11, 14, 30]. In this section we relate our results to existing knowledge on geodesic flows.

Let Γ be a discrete subgroup of $PSL(2, R)=SL(2, R)/\{\pm I\}$ such that $PSL(2,\mathbb{R})/\Gamma$ is compact. We then define the flow on the compact manifold $M=PSL(2,\mathbb{R})/\Gamma$ by $\phi_t\colon M\to M$, $\phi_t(g\Gamma)=g_t g\Gamma$ where $g_t=\begin{pmatrix} e^t & 0\\ 0 & e^{-t}\end{pmatrix}$. The flow is (topologically) weak-mixing and of entropy 1 and the measure of maximum entropy is precisely the Riemann measure [10]. For these geodesic flows the zeta function $\zeta(s)=\prod_\tau(1-\exp-s\lambda(\tau))^{-1}$ is known to have a non-zero meromorphic extension to the enitire complex plane with the only possible poles in $\mathscr{R}(s)>1/2$ lying at points $1>s_1>\ldots>s_k$ on the real line [14]. This analysis is based on the important Selberg trace formula.

Collet, Epstein, and Gallavotti have obtained the result that the decay of the correlation function on a surface of curvature -1 is of order $t^b \exp - t/2$ [9]. In particular, this means that our analysis relating $\zeta(s)$ to $\hat{\rho}_{A,B}(s)$ is invalid for $\mathscr{R}(s) \leqq s_1$, else we would be unable to have a pole at $s = s_1$, for the zeta function. In principle, therefore, we can expect to get an upper bound on s_1. However, in the Selberg approach s_1 has a simple geometric interpretation. Let H^+ be the Lobachevsky upper half plane with the Poincaré metric $ds^2 = (dx^2 + dy^2)/y^2$. (This is the usual metric giving the space curvature $\kappa = -1$). The subgroup $\Gamma \subseteq PSL(2, \mathbb{R})$ defines isometries of H^+ as linear fractional transformations i.e. for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ then g: $H^+ \to H^+$ by $g(z) = (az+b)/(cz+d)$ and H^+/Γ becomes a compact Riemann surface with universal covering space H^+ and covering group Γ. Let m be the Riemann measure on H^+/Γ and let $-\Delta$ be the Laplace-Beltrani operator acting on $L^2(H^+/\Gamma)$ (which is itself determined by the Poincaré metric). The spectrum of Δ is unbounded and takes the form $0 < \lambda_1 < \ldots < \lambda_k < 1/4 \leqq \lambda_{k+1} < \ldots$. The poles $s_1, \ldots, s_k$ for $\zeta(s)$ and the eigenvalues $\lambda_1, \ldots, \lambda_k$ for $-\Delta$ are related by $\lambda_j = s_j(1 - s_j)$, $j = 1, \ldots, k$.

It has been shown by Schoen, Wolpert and Yau [26] that for any compact surface of genus g and curvature $\kappa = -1$ there exists a "universal" constant $C = C(g)$ such that $\lambda_1 \geqq C l_1$ where l_1 is the length of the shortest geodesic on the surface.

We show below how this relates to our estimates.

Proposition 6. *Let l_1, l_2 be the shortest and longest distances respectively accross the fundamental domain for Γ then $\lambda_1 \geqq \alpha(1-\alpha)$ where $\alpha = l_1/l_2 \cdot [l_1/(l_1 + \log 4g)]$*

Sketch proof. (1) Following Bowen and Series [8, 25] we can construct an expanding endomorphism T: $S^1 \to S^1$ from the action of the generators of Γ on the Poincaré disc. The geodesic flow can be represented as a suspended (semi) flow over this transformation, where we take the suspending function to be $r = \log |T'|$: $S^1 \to \mathbb{R}^+$.

(2) There exists a markov partition for T: $S^1 \to S^1$ which enables it to be modeled by a (one-sided) subshift of finite type. Furthermore, r becomes Lipschitz provided $\theta > \exp - l_1$, and also $|r| \leqq l_2$. The hypothesis in Proposition 4 require that $P(-\sigma r) < \frac{|\log \theta|^2}{|\log \theta| + h(T)}$, where $s = \sigma + it$. Since $\sigma \to P(-\sigma r)$ is convex and $\frac{dP}{d\sigma}(-\sigma r)|_{\sigma=\sigma_0} = -\int r \, d\mu$ (where μ is the unique equilibrium state for $-\sigma_0 r$) ([23], p. 99) this is certainly satisfied if $|h(\sigma^r) - \sigma| < \alpha$ i.e. $\sigma > \sigma_1 = 1 - \alpha$.

(3) The zeta functions for the geodesic flow ϕ and the suspended (semi)-flow σ^r satisfy: $\zeta_\phi(s)/\zeta_{\sigma^r}(s)$ is non-zero and analytic for $\mathscr{R}(s) > 0$ (cf. [21]). Thus recalling the result of Collet, Epstein and Gallavotti [9] and Theorem 2 we see that we must have $\sigma_1 \geqq s_1$ to avoid an error term of order $e^{-(1-s_1)t}$. Therefore $s_1 \leqq 1 - \alpha$.

(4) From the Selberg theory we can relate s_1 to λ_1 and finally deduce that $\lambda_1 \geqq \alpha(1-\alpha)$.

§ 7. Other rates of decay of correlations

In [22] Ruelle produced an example of a flow for which the correlation function does not decay exponentially. The author produced the same example to show that $\zeta(s)$ cannot always be extended analytically to a strip $h-\varepsilon<\mathscr{R}(s)<h$, for some $\varepsilon>0$. (The connection between the two is now obvious from Theorem 1 and Theorem 2. In fact no flow formed by suspending a locally constant function $r\colon \Sigma\to\mathbb{R}$ will have exponential decay [20]). In this section we show that by refining this simple example flows can be constructed which decay slower than any predetermined rate. In particular, Axiom A flows need not mix at a polynomial rate, answering a question of Bowen ([3], p. 31).

Example. Let $\Sigma=\prod_{-\infty}^{+\infty}\{0,1\}$ and $r\colon \Sigma\to\mathbb{R}^+$ by

$$r(x)=\begin{cases}1 & \text{if} \quad x_0=0\\ \alpha & \text{if} \quad x_0=1.\end{cases}$$

Let $0<\beta<1$ be a "badly approximable" number in the sense that there exists $\delta>0$ such that $|\beta-p/q|\geqq\delta/q^2$ for all $p,q\in\mathbb{Z}^+$. (For example, $\beta=1/\sqrt{2}$) [13]. Define $F=\chi_{\Sigma\times[0,\beta]}$ and $G=\chi_{\Sigma\times[0,1/2]}$ and consider $\rho(t)=\int F\sigma_t^r\cdot G\,dm$, where m is equilibrium state. (Strictly speaking F is not Hölder continuous but this can be easily overcome.) For clarity we initially assume that q is even and $\alpha=p/q$. Consider the case where $t=k/q$, $k\in Z^+$, then $F\sigma_t^r=\chi_u$, where U is a union of sets of the form $C\times\left[a/q,\dfrac{a+1}{q}\right]$ or $C\times[a/q,\ a/q+\varepsilon]$ (where $\beta=b/q+\varepsilon$ with $\delta/q^2\leqq\varepsilon\leqq 1/q-\delta/q^2$ and C is a cylinder in Σ).

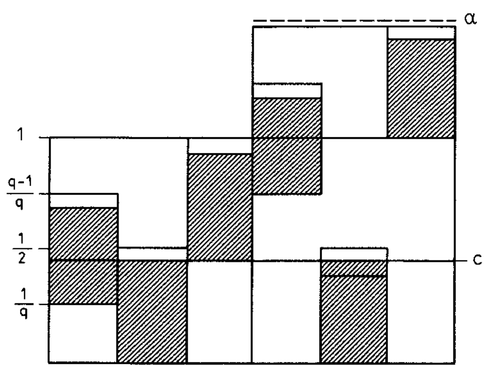

Fig. 1

In particular, for a given k we can assume that there are sufficiently many elements of U of the second form (i.e. the proportion of sets of the form $C\times[a/q,a/q+\varepsilon]\gg 1/q$) that $|\rho(k/q)-\rho(k/q-\varepsilon)|\gg\delta/q\gg\delta/q^3\gg 1$ (as k increases). Otherwise we need only change k by at most q for this conditions to hold. In practice we have to take α irrational in order to get a weak-mixing flow. However, we can choose α to be "well-approximable" in the sense that for a monotonically increasing function $f\colon \mathbb{Z}^+ \to \mathbb{R}^+$ we choose α such that there is an infinite sequence of rational approximates p_k/q_k with $|\alpha - p_k/q_k| \leqq 1/f(q_k)$ [13]. In particular we can make $\rho(t)$, with σ^r defined with either α or p_k/q_k, close for long initial periods depending on the closeness of the approximation. Thus we can choose α so that $\rho(t)$ contradicts any prescribed rate of decay.

Bowen has shown that any suspended flow is conjugate to some Axiom A flow restricted to a basic set. (In fact, in this case the flow could be chosen to be C^∞.)

Remark. It is interesting to ask what estimates on $\rho(s)$ can be obtained from less information on $\zeta(s)$. By generalising the above example we can constuct flows with poles $\sigma_n + i t_n$ satisfying $1 - \sigma_n \gg 1/t_n^\delta$ $(0 < \delta < 1)$. In this case it is possible to show that $\int_0^t \rho(u)\,du/t = O(\exp - \eta \log\log t)$, for some $\eta > 0$. However, such averages yield little useful information.

References

1. Arnold, V.I., Avez, A.: Ergodic problems of classical mechanics. New York: Benjamin 1968
2. Bowen, R.: Equilibrium states and the ergodic theory of Anosov diffeomorphisms. S.L.N. **470**. Berlin-Heidelberg-New York: Springer 1975
3. Bowen, R.: On Axiom A diffeomorphisms. Am. Math. Soc. Regional Conf. Proc. No. **35**, Providence 1978
4. Bowen, R.: Symbolic dynamics for hyperbolic flows. Am. J. Math. **95**, 429–459 (1973)
5. Bowen, R.: Periodic orbits for hyperbolic flows. Am. J. Math. **94**, 1–30 (1972)
6. Bowen, R., Ruelle, D.: The ergodic theory of Axiom A flows. Invent. Math. **29**, 181–202 (1975)
7. Bowen, R., Walters, P.: Expansive one-parameter flows. J. Differ. Equations **12**, 180–193 (1972)
8. Bowen, R., Series, C.: Markov maps associated with Fuchsian groups. Publ. Math. Inst. Hautes Etud. Sci. **50**, 153–170 (1979)
9. Collet, P., Epstein, H., Gallavotti, G.: Perturbations of geodesic flows on surfaces of constant negative curvature and their mixing properties. Commun. Math. Phys. **95**, 61–112 (1984)
10. Cornfeld, I.P., Fomin, S.V., Sinai, Y.G.: Ergodic Theory. Berlin-Heidelberg-New York: Springer 1982
11. Fomin, S.V., Gelfand, I.M.: Geodesic flows on manifolds of constant negative curvature. Transl. Am. Math. Soc. **1**, 49–65 (1955)
12. Gallavotti, G.: Funzioni zeta ed insiemi basilar. Accad. Lincei. Rend. Sc. fismat. e mat. **61**, 309–317 (1976)
13. Hardy, G.H., Wright, E.M.: An introduction to the theory of numbers. Oxford: O.U.P. 1983
14. Hejhal, D.A.: The Selberg trace formula and the Riemann zeta function. Duke Math. J. **43**, 441–482 (1976)
15. Katznelson, Y.: An introduction to harmonic analysis. Dover: New York 1976
16. Manning, A.: Axiom A diffeomorphisms have rational zeta functions. Bull. London Math. Soc. **3**, 215–220 (1971)
17. Parry, W.: Intrinsic Markov Chains. Trans. Am. Math. Soc., **112**, 55–65 (1964)
18. Parry, W.: Topics in Ergodic theory. Cambridge: C.U.P. 1981
19. Parry, W., Pollicott, M.: An analogue of the prime number theorem for closed orbits of Axiom A flows. Ann. Math. **118**, 573–591 (1983)
20. Pollicott, M.: Meromorphic extensions of generalised zeta functions. (Preprint)
21. Pollicott, M.: Asymptotic distribution of closed geodesics. (To appear in Isr. J. Math.)
22. Ruelle, D.: Flows which do not exponentially mix. C.R. Acad. Sci. Paris, **296**, 191–194 (1983)
23. Ruelle, D.: Thermodynamic formalism. Reading: Addison-Weley 1978

24. Ruelle, D.: Zeta functions for expanding maps and Anosov flows. Invent. Math. **34**, 23L-242 (1976)
25. Series, C.: Symbolic dynamics for geodesic flows. Acta Math. **146**, 103-128 (1981)
26. Schoen, R., Wolpert, S., Yau, S.T.: Geometric bounds on the low eigenvalues of a compact surface. In: Proc. Symp. Pure Math. **36**, (1980)
27. Sinai, Y.G.: Gibbs measures in ergodic. theory. Russ. Math. Surv. **27**, 21-69 (1972)
28. Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747-817 (1967)
29. Tuncel, S.: Conditional pressure and coding. Isr. J. Math. **39**, 101-112 (1981)
30. Venkov, A.B.: Spectral theory of automorphic functions, the Selberg zeta function, and some problems of analytic number theory and mathematical physics. Russ. Math. Surv. **34**, 79-153 (1979)
31. Walters, P.: A variational principle for the pressure of continuous transformations. Am. J. Math. **97**, 937-971 (1976)
32. Walters, P.: An introduction to ergodic theory. G.T.M. **79**, Berlin-Heidelberg-New York: Springer 1981
33. Walters, P.: Ruelle's operator theorem and g-measures. Trans. Am. Math. Soc. **214**, 375-387 (1975)

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