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Intersection cohomology complexes on a reductive group

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Let G be a reductive connected algebraic group over an algebraically closed field, and let u be a unipotent element of G. Let $A_G(u)$ be the group of components of the centralizer $Z_G(u)$. The group $A_G(u)$ acts naturally by permutations on the set of irreducible components of the variety of Borel subgroups containing u and Springer [20, 21] has shown that (with some restrictions on the characteristic) the irreducible representations of $A_G(u)$ appearing in this permutation representation, for various u (up to conjugacy), are in 1 −1 correspondence with the irreducible representations of the Weyl group. Note however that, in general, not all irreducible representations of $A_G(u)$ appear in this permutation representation. Our main interest in this paper is in understanding the missing representations. Let P be a parabolic subgroup of G with Levi decomposition $P=LU_P$, and let v be a unipotent element in L. Following Springer, we consider the variety

$$Y_{u,v}=\{gZ_L^0(v)\,U_P \mid g\in G, g^{-1}ug\in vU_P\}.$$

Then $\dim Y_{u,v}\leqq d=\frac{1}{2}(\dim Z_G(u)-\dim Z_L(v))$. (This is proved by Springer [21], with restrictions on characteristic; in the general case, it follows from results in § 1.) The group $Z_G(u)$ acts naturally on $Y_{u,v}$ by left translation. This induces an action of the finite group $A_G(u)$ on the finite set $S_{u,v}$ of irreducible components of dimension d of $Y_{u,v}$.

When P is a Borel subgroup and $v=1$, this is just the action considered earlier.

We say that an irreducible representation of $A_G(u)$ is cuspidal if it does not appear in the permutation representation $S_{u,v}$ for any P, v as above, with $P\neq G$.

It turns out that very few representations of $A_G(u)$ are cuspidal. More precisely:

If we fix a character χ of the group Γ of components of the centre of G, and if we are in good characteristic, then there is at most one pair (u,ρ) with u unipotent in G (up to conjugacy) such that ρ is an irreducible cuspidal representation of $A_G(u)$ on which Γ acts according to χ.

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If we take G to be almost simple, simply connected, the condition on G and χ that such a pair (u, ρ) exists is as follows:

Type A_n: χ is of order $n+1$
Type B_n: $\chi=1$, $2n+1\in\square$
$\chi\neq 1$, $2n+1\in\Delta$
Type C_n: $\chi=1$, $n\in\triangle$ and n even
$\chi\neq 1$, $n\in\triangle$ and n odd
Type D_n: $\chi=1$, $2n\in\square$ and $n/2$ even
$\chi\neq 1$, $\chi(\varepsilon)=1$, $2n\in\square$ and $n/2$ odd
$\chi(\varepsilon)\neq 1$, $2n\in\triangle$
Type E_6: $\chi\neq 1$
Type E_7: $\chi\neq 1$
Type E_8: $\chi=1$
Type F_4: $\chi=1$
Type G_2: $\chi=1$.

Here, $\square$ denotes the set $\{1,4,9,16,\ldots\}$, $\triangle$ denotes the set $\{1,3,6,10,15,\ldots\}$ and ε denotes the non-trivial element in the kernel of the natural map of Spin_{2n} onto SO_{2n}.

The classification in bad characteristic is different; see §15.

Given a pair (u, ρ) where u is a unipotent element in G (up to conjugacy) and ρ is an irreducible representation of $A_G(u)$, we define in §6 a triple (L, v, ρ'), up to G-conjugacy, where L is the Levi subgroup of a parabolic subgroup of G, v is a unipotent element in L and ρ' is a cuspidal representation of $A_L(v)$. Moreover, we show (Theorem 6.5) that the set of pairs (u, ρ) giving rise to a fixed triple (L, v, ρ') as above, may be naturally put into $1-1$ correspondence with the set of irreducible representations of the group of components of the normalizer of L (which is shown in §9 to be a Coxeter group). We call this the generalized Springer correspondence; it reduces to the correspondence described originally by Springer, in the case where L is a maximal torus. In this way, the classification of pairs (u, ρ) as above is reduced to the classification of cuspidal pairs.

In §§12 and 13, we determine in a combinatorial way this generalized Springer correspondence in the case of symplectic and special orthogonal groups in odd characteristic; this generalizes the main result of Shoji [15] on the usual Springer correspondence for these groups. Our approach is based on a variant of the notion of symbols in [6]. Recently, together with N. Spaltenstein, we have extended this result to the case of classical groups in characteristic two. Using the results of this paper, Spaltenstein has determined explicitly the generalized Springer correspondence for exceptional groups in arbitrary characteristic, in almost all cases.

In this paper we use extensively, just as in [9, 2], the intersection cohomology theory of Deligne-Goresky-MacPherson, (see [4, 1]). An important role in our proofs is played by a certain class of intersection cohomology complexes on G (see § 3). I believe that these are precisely the complexes whose existence was conjectured in [10, § 13]; if this is so, the further study of these complexes might lead to the complete computation of the character tables of finite Chevalley groups.

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0. Notations

Let $\bar{\mathbb{Q}}_l$ be an algebraic closure of the field of l-adic numbers. We shall consider constructible $\bar{\mathbb{Q}}_l$-sheaves on an algebraic variety over an algebraically closed field k, (l is a fixed prime, invertible in k). We shall call them constructible sheaves; a special case of these are the local systems. We shall often make use of the Deligne-Goresky-MacPherson intersection cohomology complex (see [4, 1]) of an irreducible variety X over k, with coefficients in a local system $\mathscr{E}$ over an open dense smooth subset of X; we shall denote it $IC(X,\mathscr{E})$. (It is an object in the bounded derived category $D_c^b(X,\bar{\mathbb{Q}}_l)$ of constructible sheaves on X.) We normalize it in such a way that

(0.1) *its cohomology sheaves $\mathscr{H}^i$ are zero for $i<0$, $\mathscr{H}^0$ extends $\mathscr{E}$ and*

$$\dim\operatorname{supp}\mathscr{H}^i<\dim X-i, \qquad \text{for } i>0.$$

We shall make use of the theory of perverse sheaves, for which the basic reference is [1]. Let K be a perverse sheaf (possibly shifted) on X, and assume that X is provided with an action of a connected algebraic group G. We say that K is G-equivariant if the following condition is satisfied. There exists an isomorphism α in the derived category between the pull-backs p_1^*K, p_2^*K (where $p_1: G\times X\to X$ is $(g,x)\mapsto x$ and $p_2: G\times X\to X$ is $(g,x)\mapsto gx$) such that the induced isomorphism between $i^*p_1^*K$ and $i^*p_2^*K$ is the identity $K\to K$; here, $i: X\to G\times X$ is $x\mapsto(e,x)$ so that $i^*p_1^*K=K$, and $i^*p_2^*K=K$. Then α is necessarily unique. (As Deligne told me, this follows from [1, Prop. 4.2.5].) It also follows that α satisfies the usual associativity condition. For a not necessarily perverse complex of constructible sheaves on X, the notion of G-equivariance is more delicate (Deligne); we shall not need it here. The notion of G-equivariant local system on X is obvious. If G is connected and its action on X is transitive, the G-equivariant local systems on X are in $1-1$ correspondence with the finite dimensional representations of the finite group of components of the isotropy group of a point $x \in X$.

All representations of finite groups are assumed to be in finite dimensional $\bar{\mathbb{Q}}_l$-vector spaces.

If X is an algebraic variety, $X' \subset X$ is a locally closed subvariety of X and $\mathscr{E}$ is a constructible sheaf on X, we shall write $H_c^i(X', \mathscr{E})$ for the cohomology with compact support of X' with coefficients in the restriction $\mathscr{E}|X'$. If $\mathscr{E}$ is a complex of sheaves with constructible cohomology sheaves on X, the same notation will be used for the hypercohomology with compact support of X' with respect to the restriction of that complex to X'.

The identity component of an algebraic group H will be denoted H^0, the centre of H is denoted $\mathscr{Z}_H$; its identity component is $\mathscr{Z}_H^0$. If H' is a subgroup of H and $h \in H$, we write $Z_{H'}(h)$ for the centralizer of h in H' and $N(H')$ or $N_H(H')$ for the normalizer of H' in H.

§ 1. Dimension estimates

1.1. Let G be a connected reductive algebraic group over an algebraically closed field k, and let $\mathscr{P}$ be a class of parabolic subgroups G. For $P \in \mathscr{P}$, we denote by U_P the unipotent radical of P, by $\bar{P}$ the reductive group P/U_P and by $\pi_P: P \to \bar{P}$ the natural projection. We assume given a conjugacy class C of G. We also assume given for each $P \in \mathscr{P}$, a $\bar{P}$-conjugacy class $C_{\bar{P}} \subset \bar{P}$ with the following property: for any $P_1, P_2 \in \mathscr{P}$ and any $g \in G$ with $P_2 = g P_1 g^{-1}$, we have $\pi_{P_2}^{-1}(C_{\bar{P}_2}) = g \pi_{\bar{P}_1}^{-1}(C_{\bar{P}_1}) g^{-1}$. (Thus it is enough to specify a conjugacy class in $\bar{P}$ for some P and then we have automatically a conjugacy class in $\bar{P}'$ for any $P' \in \mathscr{P}$.)

Let

$$Z = \{(g, P_1, P_2) \in G \times \mathscr{P} \times \mathscr{P} \mid g \in \pi_{P_1}^{-1}(\mathscr{Z}_{\bar{P}_1}^0 C_{\bar{P}_1}) \cap \pi_{P_2}^{-1}(\mathscr{Z}_{\bar{P}_2}^0 C_{\bar{P}_2})\},$$

$$Z' = \{(g, P_1, P_2) \in G \times \mathscr{P} \times \mathscr{P} \mid g \in \pi_{P_1}^{-1}(C_{\bar{P}_1}) \cap \pi_{P_2}^{-1}(C_{\bar{P}_2})\}.$$

We have a partition $Z = \bigcup_{\mathcal{O}} Z_{\mathcal{O}}$, according to the G-orbits $\mathcal{O}$ on $\mathscr{P} \times \mathscr{P}$; the piece $Z_{\mathcal{O}}$ is the subset of Z defined by the condition that $(P_1, P_2) \in \mathcal{O}$. We define in the same way a partition $Z'_{\mathcal{O}} = \bigcup_{\mathcal{O}} Z'_{\mathcal{O}}$. A G-orbit $\mathcal{O}$ is said to be *good* if, for $(P_1, P_2) \in \mathcal{O}$, there exists a common Levi subgroup for P_1 and P_2; otherwise, $\mathcal{O}$ is said to be *bad*.

We shall denote by ν_G the number of positive roots of G and we set $\bar{\nu} = \nu_{\bar{P}}$ ($P \in \mathscr{P}$); we also denote $\bar{r} = \dim \mathscr{Z}_{\bar{P}}^0$, for $P \in \mathscr{P}$. Let $\bar{c} = \dim C_{\bar{P}}$, for $P \in \mathscr{P}$, and $c = \dim C$.

The following result is well known in the case where $\mathscr{P}$ is the set of Borel subgroups, (see [20], [18, II 2.6]). Springer [21, 4.2] proved (a) and (b) also for arbitrary $\mathscr{P}$, but with some restrictions on the characteristic of k.

1.2. **Proposition.** (a) *Given* $P \in \mathscr{P}$ *and* $\bar{g} \in C_{\bar{P}}$, *we have* $\dim(C \cap \pi_P^{-1}(\bar{g})) \leqq \frac{1}{2}(c - \bar{c})$.

(b) *Given* $g \in C$, *we have* $\dim\{P \in \mathscr{P} | g \in \pi_P^{-1}(C_{\bar{P}})\} \leqq \left(\nu_G - \frac{c}{2}\right) - \left(\bar{\nu} - \frac{\bar{c}}{2}\right)$.

(c) *If* $d = 2\nu_G - 2\bar{\nu} + \bar{c} + \bar{r}$, *then* $\dim Z_{\mathcal{O}} \leqq d$ *if* $\mathcal{O}$ *is good, and* $\dim Z_{\mathcal{O}} < d$ *if* $\mathcal{O}$ *is bad. Hence* $\dim Z \leqq d$.

(d) *If* $d' = 2\nu_G - 2\bar{\nu} + \bar{c}$, *then* $\dim Z'_{\mathcal{O}} \leqq d'$ *for all* $\mathcal{O}$. *Hence* $\dim Z' \leqq d'$.

Proof. In the case where $\mathscr{P} = \{G\}$, the proposition is trivial. Therefore, we may assume that $\mathscr{P}$ is a class of proper parabolic subgroups of G and that the proposition is already known when G is replaced by a group of strictly smaller dimension.

We can map $Z_{\mathcal{O}}$ and $Z'_{\mathcal{O}}$ to $\mathcal{O}$, by $(g, P_1, P_2) \to (P_1, P_2)$. We see that proving (c) and (d) for $Z_{\mathcal{O}}, Z'_{\mathcal{O}}$ is the same as proving that for a fixed $(P', P'') \in \mathcal{O}$, we have

(c′) $$\dim\{\pi_{P'}^{-1}(\mathscr{Z}_{\bar{P}'}^0 C_{\bar{P}'}) \cap \pi_{P''}^{-1}(\mathscr{Z}_{\bar{P}''}^0 C_{\bar{P}''})\} \leqq 2\nu_G - 2\bar{\nu} + \bar{c} + \bar{r} - \dim \mathcal{O}$$

(d′) $$\dim(\pi_{P'}^{-1}(C_{\bar{P}'}) \cap \pi_{P''}^{-1}(C_{\bar{P}''})\} \leqq 2\nu_G - 2\bar{\nu} + \bar{c} - \dim \mathcal{O}$$

with strict inequality in (c′) if $\mathcal{O}$ is bad.

Choose Levi subgroups L' of P' and L'' of P'' such that L', L'' contain a common maximal torus. Then $P' \cap L''$ is a parabolic subgroup of L'' with unipotent radical $U_{P'} \cap L''$ and Levi subgroup $L' \cap L''$; $P'' \cap L'$ is a parabolic subgroup of L' with unipotent radical $U_{P''} \cap L'$ and Levi subgroup $L' \cap L''$. An element in $P' \cap P''$ can be written both in the form $x \cdot u$ ($x \in L'$, $u \in U_{P'}$) and in the form $y \cdot v$ ($y \in L''$, $v \in U_{P''}$). It is easy to see that there are unique elements $z \in L' \cap L''$, $u'' \in L' \cap U_{P''}$, $u' \in L'' \cap U_{P'}$, such that $x = zu''$, $y = zu'$. Hence (c′) is equivalent to

(c″) $$\begin{aligned} &\dim\{(u, v, u'', u', z) \in U_{P'} \times U_{P''} \times (U_{P''} \cap L') \times (U_{P'} \cap L'') \times (L' \cap L'') | \\ &\quad u''u = u'v,\ zu'' \in \mathscr{Z}_{L'}^0 C_{\bar{P}'},\ zu' \in \mathscr{Z}_{L''}^0 C_{\bar{P}''}\} \\ &\leqq 2\nu_G - 2\bar{\nu} + \bar{c} + \bar{r} - \dim \mathcal{O}, \end{aligned}$$

(with strict inequality for bad $\mathcal{O}$) and (d′) is equivalent to the inequality (d″) obtained from (c″) by dropping $\mathscr{Z}_{L'}^0$, $\mathscr{Z}_{L''}^0$, $\bar{r}$. (We identity $L' = \bar{P}'$, $L'' = \bar{P}''$, and thus we regard $C_{\bar{P}'} \subset L'$, $C_{\bar{P}''} \subset L''$.) When $(u', u'') \in (U_{P'} \cap L'') \times (U_{P''} \cap L')$ is fixed, the variety $\{(u, v) \in U_{P'} \times U_{P''} | u''u = u'v\}$ is isomorphic to $U_{P'} \cap U_{P''}$: if we set $\tilde{u} = u'^{-1} u'' u u''^{-1} \in U_{P'}$, $\tilde{v} = v u''^{-1} \in U_{P''}$, this variety becomes $\{(\tilde{u}, \tilde{v}) \in U_{P'} \times U_{P''} | \tilde{u} = \tilde{v}\}$.

Since $\dim(U_{P'} \cap U_{P''}) = 2\nu_G - 2\bar{\nu} - \dim \mathcal{O}$, we see that (c″), (d″) are equivalent to:

(c‴) $$\begin{aligned} &\dim\{(u'', u', z) \in (U_{P''} \cap L') \times (U_{P'} \cap L'') \times (L' \cap L'') | \\ &\quad zu'' \in \mathscr{Z}_{L'}^0 C_{\bar{P}'},\ zu' \in \mathscr{Z}_{L''}^0 C_{\bar{P}''}\} \leqq \bar{c} + \bar{r} \end{aligned}$$

(d‴) $$\dim\{(u'',u',z)\in(U_{P''}\cap L')\times(U_{P'}\cap L'')\times(L'\cap L'')\mid zu''\in C_{\bar P'}, uu'\in C_{\bar P''}\}\leqq\bar c$$

with strict inequality in (c‴) for bad $\mathcal{O}$.

Let us consider the variety in (d‴). Note that the projection π_3 of that variety on the z-coordinate is a union of finitely many conjugacy classes $\hat C_1\cup\hat C_2\cup\ldots\cup\hat C_n$ in the reductive group $L'\cap L''$. (By the finiteness of the number of unipotent classes in a reductive group [8], it is enough to show that the semisimple part z_s of z can take only finitely many values up to conjugacy in $L'\cap L''$; but z_s is conjugate in $L'\cap L''$ to one of the elements in the finite set obtained by intersecting the set of semisimple parts of elements in $C_{\bar P'}\subset L'$ with a fixed maximal torus in $L'\cap L''$.) The inverse image under π_3 of a point $z\in\hat C_i$ is a product of two varieties of the type considered in (a) but for a smaller group (G replaced by L' or L'') hence by the induction hypothesis it has dimension $\leqq\frac12(\bar c-\dim\hat C_i)+\frac12(\bar c-\dim\hat C_i)$. Hence $\dim\pi_3^{-1}(\hat C_i)\leqq\bar c$. Since this is true for each i $(1\leqq i\leqq n)$, we see that the variety in (d‴) has dimension $\leqq\bar c$.

A similar proof applies for (c‴). We denote by $\tilde\pi_3$ the projection of the variety in (c‴) on the z-coordinate. The image of $\tilde\pi_3$ is the intersection of two sets: $\mathscr{Z}^0_{L'}(\hat C'_1\cup\ldots\cup\hat C'_n)$ and $\mathscr{Z}^0_{L''}(\hat C''_1\cup\ldots\cup\hat C''_m)$, where $\hat C'_1,\ldots,\hat C'_n$, $\hat C''_1,\ldots,\hat C''_m$ are a finite set of conjugacy classes in $L'\cap L''$. (The same argument as for (c‴).)

Hence the image of $\tilde\pi_3$ is $\bigcup_{i,j}(\mathscr{Z}^0_{L'}\hat C'_i\cap\mathscr{Z}^0_{L''}\hat C''_j)$. Note that $\mathscr{Z}^0_{L'},\mathscr{Z}^0_{L''}$ are contained in the centre of $L'\cap L''$. It follows that the image of $\tilde\pi_3$ is of the form $(\mathscr{Z}^0_{L'}\cap\mathscr{Z}^0_{L''})\tilde C_1\cup\ldots\cup(\mathscr{Z}^0_{L'}\cap\mathscr{Z}^0_{L''})\tilde C_t$, for a finite set of conjugacy classes $\tilde C_1,\ldots,\tilde C_t$ in $L'\cap L''$. Now the same proof as in case (d‴) (using (a) for a smaller group) shows that the variety in (c‴) has dimension $\leqq\bar c+\dim(\mathscr{Z}^0_{L'}\cap\mathscr{Z}^0_{L''})$. Since $\dim(\mathscr{Z}^0_{L'}\cap\mathscr{Z}^0_{L''})\leqq\bar r$, with strict inequality if $\mathcal{O}$ is bad, we see that (c‴) is proved. Hence (c) and (d) are proved (assuming the induction hypothesis).

We now show that (b) is a consequence of (d). Let $Z'(C)$ be the subset of Z' defined by $Z'(C)=\{(g,P_1,P_2)\in Z'\mid g\in C\}$. If $Z'(C)$ is empty then, clearly, the variety in (b) is empty and (b) follows. Hence, we may assume that $Z'(C)$ is non-empty. From (d), we have $\dim Z'(C)\leqq d'$. We map $Z'(C)$ onto C by the projection on the g-factor. Each fibre of this map is a product of two copies of the variety in (b). It follows that the variety in (b) has dimension equal to $\frac12(\dim Z'(C)-\dim C)\leqq\frac12(d'-c)=\nu_G-\bar\nu+\frac{\bar c}{2}-\frac{c}{2}$ and (b) is proved.

Finally we show by a well-known argument that (a) is a consequence of (b). Consider the variety $\{(g,P)\in C\times\mathscr{P}\mid g\in\pi_P^{-1}(C_{\bar P})\}$. By projecting it to the g-coordinate, and using (b), we see that it has dimension $\leqq\nu_G-\bar\nu+\frac{c}{2}+\frac{\bar c}{2}$. If we project it to the P-coordinate, each fibre will be isomorphic to the variety $C\cap\pi_P^{-1}(C_{\bar P})$, ($P\in\mathscr{P}$ fixed). Hence $\dim(C\cap\pi_P^{-1}(C_{\bar P}))\leqq\nu_G-\bar\nu+\frac{c}{2}+\frac{\bar c}{2}-\dim\mathscr{P}=\frac{c+\bar c}{2}$. Now $C\cap\pi_P^{-1}(C_{\bar P})$ maps onto $C_{\bar P}$ (via π_P) and each fibre is the variety in (a). Hence the variety in (a) has dimension $\leqq\frac{c+\bar c}{2}-\bar c=\frac{c-\bar c}{2}$. The proposition is proved.

1.3. *Remarks.* (i) In the case where $\mathcal{O}$ is good, the variety in (c‴) above is $\{z\in L' | z\in \mathscr{Z}^0_{L'} C_{\bar{P}'} \cap \mathscr{Z}^0_{L'} C_{\bar{P}''}\}$, since $L=L'$. This variety (and hence $Z_{\mathcal{O}}$) is empty unless

$$\mathscr{Z}^0_{L'} C_{\bar{P}'} = \mathscr{Z}^0_{L'} C_{\bar{P}''}.$$

It this equality is satisfied, then $Z_{\mathcal{O}}$ has dimension equal to d.

(ii) The inequality in 1.2(b) can be reformulated as follows. If we fix $P_0\in\mathscr{P}$ and a conjugacy class $C_0\subset \bar{P}_0$ of dimension $\bar{c}$ then, for any $g\in C$, we have

$$\text{(1.3.1)} \qquad \dim\{xP_0\in G/P_0 | x^{-1}gx\in\pi_{\bar{P}_0}^{-1}(C_0)\} \leqq \left(\nu_G-\frac{c}{2}\right)-\left(\bar{\nu}-\frac{\bar{c}}{2}\right).$$

We shall need also the following variant of this inequality:

$$\text{(1.3.2)} \qquad \dim\{xP_0\in G/P_0 | x^{-1}gx\in\pi_{\bar{P}_0}^{-1}(\mathscr{Z}^0_{\bar{P}_0} C_0)\} \leqq \left(\nu_G-\frac{c}{2}\right)-\left(\bar{\nu}-\frac{\bar{c}}{2}\right).$$

This follows from (1.3.1) by observing that, for given g, there exist finitely many conjugacy classes $C_1, C_2, \dots, C_s$ in $\bar{P}_0$, of dimension $\bar{c}$, such that

$$x^{-1}gx\in\pi_{\bar{P}_0}^{-1}(\mathscr{Z}^0_{\bar{P}_0} C_0) \Rightarrow x^{-1}gx\in\pi_{\bar{P}_0}^{-1}(C_1\cup\dots\cup C_s).$$

§ 2. Cuspidal local systems

2.1. Let S be a subset of G such that S is the inverse image of a conjugacy class in $G/\mathscr{Z}^0_G$ under the natural map $G\to G/\mathscr{Z}^0_G$. Then S is a locally closed smooth subvariety of G of dimension equal to $\dim(S/\mathscr{Z}^0_G)+\dim(\mathscr{Z}^0_G)$.

2.2. **Proposition.** *Let $\mathscr{P}$, $C_{\bar{P}}\subset\bar{P}$ $(\forall P\in\mathscr{P}), \bar{\nu}, \bar{c}$ be as in 1.1. Let $\mathscr{E}$ be a local system on S. Let $\delta=\dim(S/\mathscr{Z}^0_G)-\dim(C_{\bar{P}})$, $(P\in\mathscr{P})$.*

(a) *For any $P\in\mathscr{P}$ and any $\bar{g}\in C_{\bar{P}}$, we have $\dim(\pi_{\bar{P}}^{-1}(\bar{g})\cap S)\leqq\frac{1}{2}\delta$, hence $H^i_c(\pi_{\bar{P}}^{-1}(\bar{g})\cap S,\mathscr{E})=0$ for $i>\delta$.*

(b) *The following conditions are equivalent:*

(2.2.1) *For any $P\in\mathscr{P}$ and any $\bar{g}\in C_{\bar{P}}$, we have $H^\delta_c(\pi_{\bar{P}}^{-1}(\bar{g})\cap S,\mathscr{E})=0$.*

(2.2.2) *For any $P\in\mathscr{P}$, any $\bar{g}\in C_{\bar{P}}$ and any irreducible component D of $\pi_{\bar{P}}^{-1}(\bar{g})\cap S$ of dimensional equal to $\frac{1}{2}\delta$, the restriction of $\mathscr{E}^*$ to some (or any) smooth open dense subset of D has no global sections $\neq 0$. (Here $\mathscr{E}^*$ is the local system dual to $\mathscr{E}$.)*

Proof. (a) It is clear that all elements in $\pi_{\bar{P}}^{-1}(\bar{g})$ have semisimple part in a fixed conjugacy class in G. Hence $\pi_{\bar{P}}^{-1}(\bar{g})$ is contained in the union of finitely many conjugacy classes in G. It is then enough to show that for any conjugacy class C of G such that $C\subset S$, we have $\dim(\pi_{\bar{P}}^{-1}(\bar{g})\cap C)\leqq\frac{1}{2}\delta$. This follows from 1.2(a) since $\dim C=\dim(S/\mathscr{Z}^0_G)$.

(b) Let D_0 be a smooth open dense subset of D. Then $H^\delta_c(D,\mathscr{E})\cong H^\delta_c(D_0,\mathscr{E})$ and the last space has dimension equal to the dimension of the space of global sections of $\mathscr{E}^*$ on D_0 (by Poincaré duality for D_0). It remains to note that (by (a)) $H_c^{\delta}(\pi_P^{-1}(\bar{g})\cap S, \mathscr{E})$ is the direct sum of the spaces $H_c^{\delta}(D, \mathscr{E})$ where D runs over the irreducible components of dimension $\frac{1}{2}\delta$ of $\pi_P^{-1}(\bar{g})\cap S$.

2.3. A one dimensional local system $\mathscr{S}$ on a torus T_1 is said to be *tame* if there exists an integer $m \geqq 1$ such that m is invertible in k and such that the inverse image of $\mathscr{S}$ under $z \to z^m$: $T_1 \to T_1$ is the constant sheaf $\bar{\mathbb{Q}}_l$ on T_1.

A constructible sheaf $\mathscr{E}$ on G is said to admit a *central character* if there exists a one dimensional tame local system $\mathscr{S}$ on $\mathscr{Z}_G^0$ with the following property. Let $\mathscr{S}_1$ be a one-dimensional tame local system on the torus G/G_{der} (G_{der} = derived group of G) such that the inverse image of $\mathscr{S}_1$ under the composition $\mathscr{Z}_G^0 \hookrightarrow G \to G/G_{\text{der}}$ is $\mathscr{S}$. Then we require that there exists a constructible sheaf $\mathscr{E}_1$ on $G/\mathscr{Z}_G^0$ such that $\mathscr{E}$ is isomorphic to the inverse image of $\mathscr{S}_1 \boxtimes \mathscr{E}_1$ under the natural map $G \to (G/G_{\text{der}}) \times (G/\mathscr{Z}_G^0)$.

We say that $\mathscr{S}$ is the central character of $\mathscr{E}$. It is uniquely determined by $\mathscr{E}$, if $\mathscr{E} \neq 0$. We shall use the same terminology for a constructible sheaf on a locally closed subvariety X of G, stable by multiplication by $\mathscr{Z}_G^0$: we identify it with the constructible sheaf on G which extends it by 0 on $G-X$. If G is semisimple, then any constructible sheaf on G admits a central character.

2.4. **Definition.** *Let $S \subset G$ be as in* 2.1 *and let $\mathscr{E}$ be a G-equivariant irreducible local system on S. We say that $\mathscr{E}$ is a cuspidal local system or that $(S, \mathscr{E})$ is a cuspidal pair for G if the conditions* (a), (b) *below are satisfied:*

(a) *$\mathscr{E}$ admits a central character.*

(b) *For any $\mathscr{P} \neq \{G\}$ and any $C_P \subset \bar{P}$ ($\forall P \in \mathscr{P}$) as in* 1.1, *the pair $(S, \mathscr{E})$ satisfies the equivalent conditions* (2.2.1), (2.2.2).

2.5. Let $S \subset G$ be as in 2.1 and let $\bar{S}$ be the image of S in $G/\mathscr{Z}_G^0$. From the definitions it follows easily that a G-equivariant irreducible local system $\mathscr{E}$ on S is cuspidal if and only if there exists a $(G/\mathscr{Z}_G^0)$-equivariant irreducible cuspidal local system $\mathscr{E}_1$ on $\bar{S}$ and a one dimensional tame local system $\mathscr{S}_1$ on G/G_{der} such that $\mathscr{E}$ is isomorphic to the pull back of $\mathscr{S}_1 \boxtimes \mathscr{E}_1$ under the natural map $G \to (G/G_{\text{der}}) \times (G/\mathscr{Z}_G^0)$.

Hence, if $(S, \mathscr{E})$ is cuspidal, then $\mathscr{E}$ is associated to a representation of the fundamental group of S which factors through a finite quotient. (The analogous property is true for $\mathscr{S}_1$ since it is tame, and for $\mathscr{E}_1$, since $G/\mathscr{Z}_G^0$ acts transitively on $\bar{S}$.) It follows that the dual $\mathscr{E}^*$ is again associated to a representation of the fundamental group of S which factors through a finite quotient. Hence the criterion (2.2.1) for $\mathscr{E}$ can be reformulated as the condition that $\mathscr{E}^*$ restricted to an open dense smooth subset of D has no constant direct summands $\neq 0$. This condition is clearly self dual. It follows that $(S, \mathscr{E}^*)$ is again cuspidal.

2.6. **Definition.** *An element $g \in G$ (or its conjugacy class) is said to be isolated if the connected centralizer of g_s (the semisimple part of g) is not contained in a Levi subgroup of a proper parabolic subgroup of G.*

It is known [8] that $G/\mathscr{Z}_G^0$ contains only finitely many isolated classes. A set $S \subset G$ as above is said to be isolated if some (or equivalently any) of its elements is isolated. It follows that there are only finitely many S which are isolated.

2.7. **Proposition.** *If $(S, \mathscr{E})$ is a cuspidal pair, then S is isolated.*

Proof. Let $g \in S$ and assume that $Z_G^0(g_s) \subset L$ where L is a Levi subgroup of a proper parabolic subgroup P of G. Let $\phi: U_P \to gU_P$ be the map given by $\phi(v) = vgv^{-1}$. (We have $vgv^{-1} \in gU_P$ since $\pi_P(vgv^{-1}) = \pi_P(g)$.) The map ϕ is injective: if $vgv^{-1} = v'gv'^{-1}$ then $v^{-1}v' \in Z_G(g)$; it follows that $v^{-1}v' \in Z_G(g_s)$ and, since $v^{-1}v' \in U_P$, we have $v^{-1}v' \in Z_G(g_s) \cap U_P = Z_G^0(g_s) \cap U_P \subset L \cap U_P = \{e\}$ hence $v = v'$. Since $\dim U_P = \dim gU_P$, it follows that $\phi(U_P)$ is dense in U_P; on the other hand, $\phi(U_P)$ is an orbit of the unipotent group U_P acting on the affine variety G, hence it is closed in G. Hence ϕ is a bijection of U_P onto gU_P. This implies that gU_P is contained in S, hence $\mathscr{E}$ is locally constant $\neq 0$ on gU_P; moreover, the restriction of $\mathscr{E}$ to gU_P is equivariant for the action of U_P on gU_P given by conjugation. This action of U_P on gU_P has trivial isotropy (since ϕ is bijective) hence the restriction of $\mathscr{E}$ to gU_P is a constant sheaf, $\neq 0$. It follows that $H_c^{2e}(gU_P, \mathscr{E}) \neq 0$ where $e = \dim(gU_P)$. Since the connected centralizer of g_s in L coincides with that in G, it follows that the connected centralizer of g in L coincides with that in G. Hence $\dim(\text{class of } g \text{ in } G) - \dim(\text{class of } g \text{ in } L) = \dim(G) - \dim(L) = 2\dim(U_P) = 2e$. It follows that $(S, \mathscr{E})$ cannot be cuspidal. The proposition is proved.

The following result gives further restrictions on S for which $(S, \mathscr{E})$ can be cuspidal.

2.8. **Proposition.** *Let $(S, \mathscr{E})$ be a cuspidal pair for G, let g be an element of S and let $H = Z_G(g)$. Then $H^0/\mathscr{Z}_G^0$ is unipotent.*

Proof. Let L be the centralizer in G of a maximal torus of H^0 and let P be a parabolic subgroup of G for which L is a Levi subgroup. Let $D = \{vgv^{-1} \mid v \in U_P\}$. Then, as we shall see in Lemma 2.9, D is an irreducible component of $cl_G(g) \cap gU_P$ of dimension equal to $e = \frac{1}{2}(\dim cl_G(g) - \dim cl_L(g))$. (Here, $cl_G(g)$, $cl_L(g)$ denote the conjugacy class of g in G, L.) The restriction of $\mathscr{E}$ to D is U_P-equivariant, for the (transitive) action of U_P given by conjugation, and the isotropy group of g is $Z_{U_P}(g)$. According to Spaltenstein [17], the group $Z_{U_P}(g)$ is connected. This implies that $\mathscr{E}|D$ is a constant sheaf, $\neq 0$ hence $H_c^{2e}(D, \mathscr{E}) \neq 0$. From the definition of a cuspidal pair, it then follows that $P = G$ and the proposition follows. (In an earlier version of this paper, this proposition was proved only in good characteristic. I am indebted to Spaltenstein for showing me his result on the connectedness of $Z_{U_P}(g)$, which allowed me to drop the hypothesis of good characteristic.)

We now state the following lemma, which has been used in the previous proposition.

2.9. **Lemma.** *Let P be a parabolic subgroup of G and let L be a Levi subgroup of P. Let g be an element of L and let $cl_G(g)$, $cl_L(g)$ denote the conjugacy class of g in G, L. Let $D = \{vgv^{-1} \mid v \in U_P\}$ and let $V = \{xP \in G/P \mid x \in Z_G(g)\}$. Then*

(a) *D is an irreducible component of $cl_G(g) \cap gU_P$ of dimension equal to $\frac{1}{2}(\dim cl_G(g) - \dim cl_L(g))$.*

(b) *V is an irreducible variety of dimension $(\nu_G - \frac{1}{2}\dim cl_G(g)) - (\nu_L - \frac{1}{2}\dim cl_L(g))$.*

(c) $Z_G^0(g) \cap P = Z_P^0(g)$.

Proof. (a) An element $l\cdot v\in L\cdot U_P$ is in $Z_P(g)$ if and only if $lvg=glv$, i.e. if $(lg)(g^{-1}vg)=(gl)v$ i.e. if $lg=gl$ and $g^{-1}vg=v$. Thus $Z_P(g)=Z_L(g)\,Z_{U_P}(g)$. It follows that

$$\dim Z_P(g)=\dim Z_L(g)+\dim Z_{U_P}(g).$$

Now let P' be a parabolic subgroup having L as a Levi subgroup such that $P'\cap P=L$. Then we have also

$$\dim Z_{P'}(g)=\dim Z_L(g)+\dim Z_{U_{P'}}(g).$$

Consider the map $Z_P^0(g)\times Z_{P'}^0(g)\to Z_G^0(g)$ defined by $(g_1,g_2)\mapsto g_1g_2$. The pairs (g_1,g_2), (g_1',g_2') are mapped to the same element in $Z_G^0(g)$ if and only if $g_1'=g_1g_0$, $g_2'=g_0^{-1}g_1'$, where $g_0\in Z_P^0(g)\cap Z_{P'}^0(g)$. Note also that $Z_L^0(g)\subset Z_P^0(g)\cap Z_{P'}^0(g)\subset Z_L(g)$. It follows that

$$\dim Z_G^0(g)=\dim Z_P^0(g)+\dim Z_{P'}^0(g)-\dim(Z_P^0(g)\cap Z_{P'}^0(g))+\delta,\qquad (\delta\geqq 0)$$

hence

$$\dim Z_G^0(g)=\dim Z_P^0(g)+\dim Z_{P'}^0(g)-\dim Z_L^0(g)+\delta$$

$$=\dim Z_{U_P}^0(g)+\dim Z_{U_{P'}}^0(g)+\dim Z_L^0(g)+\delta. \tag{2.9.1}$$

We have $\dim Z_{U_P}(g)=\dim U_P-\dim D$ and similarly $\dim Z_{U_{P'}}(g)=\dim U_{P'}-\dim D'$, where $D'=\{v'gv'^{-1}\mid v'\in U_{P'}\}$. Introducing this into (2.9.1), it follows that

$$\dim Z_G^0(g)=\dim U_P+\dim U_{P'}+\dim Z_L^0(g)-\dim D-\dim D'+\delta$$

hence

$$\dim D+\dim D'=\dim cl_G(g)-\dim cl_L(g)+\delta. \tag{2.9.2}$$

Now D is contained in $cl_G(g)\cap gU_P$ and is closed (it is an orbit of a unipotent group on an affine variety); by 1.2(a), any irreducible component of $cl_G(g)\cap gU_P$ has dimension $\leqq\frac{1}{2}(\dim cl_G(g)-\dim cl_L(g))$. It follows that $\dim D\leqq\frac{1}{2}(\dim cl_G(g)-\dim cl_L(g))$ and similarly $\dim D'\leqq\frac{1}{2}(\dim cl_G(g)-\dim cl_L(g))$. Comparing with (2.9.2) it follows that $\delta=0$ and that D (resp. D') is an irreducible component of $cl_G(g)\cap gU_P$ (resp. of $cl_G(g)\cap gU_{P'}$) of dimension equal to $\frac{1}{2}(\dim cl_G(g)-\dim cl_L(g))$.

(b) From the proof of (a) we see that $\dim Z_G(g)=\dim Z_P(g)+\dim Z_{U_{P'}}(g)$, hence $\dim V=\dim(Z_G(g)/Z_G(g)\cap P)=\dim Z_G(g)-\dim Z_P(g)=\dim Z_{U_{P'}}(g)$.

On the other hand, we have

$$(\nu_G-\tfrac{1}{2}\dim cl_G(g))-(\nu_L-\tfrac{1}{2}\dim cl_L(g))=\tfrac{1}{2}(\dim Z_G(g)-\dim Z_L(g))$$

$$=\tfrac{1}{2}(\dim Z_{U_P}(g)+\dim Z_{U_{P'}}(g)). \tag{2.9.3}$$

Hence to prove (b) it is enough to prove the equality $\dim Z_{U_P}(g)=\dim Z_{U_{P'}}(g)$. This follows from the equality $\dim D=\dim D'$ in (a) and from the equality

$$\dim U_P=\dim U_{P'}. \tag{2.9.4}$$

(c) Let $T=\mathscr{Z}_L^0$ and let $H=Z_G(g)$. Then T is a torus contained in H^0, hence $Z_{H^0}(T)$ is connected. We have $L=Z_G(T)$. It follows that $L\cap H^0=Z_{H^0}(T)$ is connected, hence $L\cap H^0$ is contained in $Z_L^0(g)$.

The group $Z_P(g)\cap Z_G^0(g)$ contains $Z_P^0(g)$ as a normal subgroup of finite index. Let g_0 be a fixed element of $Z_P(g)\cap Z_G^0(g)$. From the proof of (a), we have $\delta=0$ and hence the set A of products $g_1\cdot g_2$ $(g_1\in Z_P^0(g), g_2\in Z_{P'}^0(g))$ is constructible dense in $Z_G^0(g)$. Hence the left translate g_0A of A is again constructible dense in $Z_G^0(g)$ and therefore it must meet A. It follows that $g_0g_1g_2=g'_1g'_2$ for some $g_1,g'_1\in Z_P^0(g)$, $g_2,g'_2\in Z_{P'}^0(g)$. Set $\tilde{g}_0=g_1'^{-1}g_0g_1$. Then $\tilde{g}_0\in Z_P(g)\cap Z_{P'}(g)\cap H^0$, hence $\tilde{g}_0\in L\cap H^0\subset Z_L^0(g)\subset Z_P^0(g)$. Thus $g_1'^{-1}g_0g_1\in Z_P^0(g)$. Since $g'_1,g_1\in Z_P^0(g)$, it follows that $g_0\in Z_P^0(g)$. Since $g_0\in Z_P(g)\cap Z_G^0(g)$ was arbitrary, we see that $Z_P(g)\cap Z_G^0(g)=Z_P^0(g)$. This completes the proof of (c).

2.10. In order to classify the cuspidal pairs for G, it is sufficient to classify the cuspidal pairs for $G/\mathscr{Z}_G^0$. Indeed, the cuspidal pairs for G are obtained by pulling back the cuspidal pairs of $G/\mathscr{Z}_G^0$ under the natural map $G\to G/\mathscr{Z}_G^0$ and then tensoring by a one dimensional local system on G obtained by pulling back under $G\to G/G_{\mathrm{der}}$ a one dimensional tame local system on G/G_{der}.

Assume now that G is semisimple, and let $\pi\colon \tilde{G}\to G$ be its simply connected covering. Assume that the cuspidal pairs for $\tilde{G}$ are already classified. (There are only finitely many of them, by 2.7.) Let Γ be the kernel of π (a finite abelian group). Let $(\tilde{S},\tilde{\mathscr{E}})$ be a cuspidal pair for $\tilde{G}$. Then $\tilde{\mathscr{E}}$ is $\tilde{G}$-equivariant, hence in particular Γ-equivariant; Γ acts trivially on $\tilde{S}$, hence it acts on each stalk of $\tilde{\mathscr{E}}$. We require that Γ acts trivially on $\tilde{\mathscr{E}}$. Let $S=\pi(\tilde{S})$ and let $\mathscr{E}$ be the direct image of $\tilde{\mathscr{E}}$ under the finite covering $\tilde{S}\to S$. Let $\tilde{g}\in\tilde{S}$, $g=\pi(\tilde{g})$, let $\tilde{\rho}$ be the irreducible representation of $Z_{\tilde{G}}(\tilde{g})/Z_{\tilde{G}}^0(\tilde{g})$ corresponding to $\tilde{\mathscr{E}}$, and let ρ be the representation of $Z_G(g)/Z_G^0(g)$ induced by the representation of the image of $Z_{\tilde{G}}(\tilde{g})/Z_{\tilde{G}}^0(\tilde{g})\to Z_G(g)/Z_G^0(g)$ defined by $\tilde{\rho}$, ($\tilde{\rho}$ factors through that image). Then $\mathscr{E}$ is the G-equivariant local system on S corresponding to ρ. It decomposes as a direct sum of irreducible local systems in the same way as ρ decomposes as a direct sum of irreducible representations.

If $\mathscr{E}_1$ is any irreducible direct summand of $\mathscr{E}$, then $(S,\mathscr{E}_1)$ is a cuspidal pair for G, and all cuspidal pairs for G are obtained in this way.

Hence the question of classifying the cuspidal pairs of a reductive group G is reduced to the case where G is semisimple, simply connected. We can further reduce to the case where S is a unipotent class, as follows.

Assume that G is semisimple, simply connected, let $S\subset G$ be an isolated conjugacy class, (see 2.6), let $g\in S$, and $\mathscr{E}$ be the G-equivariant local system on S corresponding to the irreducible representation ρ of $Z_G(g)/Z_G^0(g)$. Let $g=su=us$ be the Jordan decomposition of g with s semisimple and u unipotent in $H=Z_G(s)=Z_G^0(s)$. We have a canonical isomorphism $Z_H(u)/Z_H^0(u)\xrightarrow{\sim} Z_G(g)/Z_G^0(g)$, hence ρ may be regarded as an irreducible representation of $Z_H(u)/Z_H^0(u)$ so that it gives rise to an irreducible H-equivariant local system $\mathscr{E}_1$ on S_1 ($=H$-conjugacy class of u). Then

(2.10.1). $(S,\mathscr{E})$ is a cuspidal pair for $G\Leftrightarrow(S_1,\mathscr{E}_1)$ is a cuspidal pair for H.

The proof of this statement is based on the consideration of the map $f\colon \pi_{\bar{P}}^{-1}(\bar{g})\cap S\to\pi_{\bar{P}}^{-1}(\bar{g}_s)\cap$class of s, defined by taking semisimple parts, and the corresponding Leray spectral sequence for $\mathscr{E}$. (Here P is a proper parabolic subgroup of G, $\bar{g}$ is an element of $\bar{P}$, $\bar{g}_s$ is its semisimple part; we assume that $\pi_{\bar{P}}^{-1}(\bar{g})\cap S\neq\emptyset$.) We omit further details.

§ 3. Admissible local systems

3.1. We shall define a partition of the reductive connected group G, into finitely many locally closed, smooth, irreducible pieces, stable by conjugation.

For $g \in G$, we denote by g_s the semisimple part of g and by $H_G(g)$ the centralizer in G of the connected centre of $Z_G^0(g_s)$. Alternatively, $H_G(g)$ could be defined as the smallest closed subgroup of G with contains $Z_G^0(g_s)$ and is the Levi subgroup of some parabolic subgroup of G.

The pieces in our partition of G are parametrized by pairs (L, S_1) (up to G-conjugacy) where L is a closed subgroup of G which is the Levi subgroup of some parabolic subgroup G and S_1 is a subset of L which is the inverse image of an isolated conjugacy class in $L/\mathscr{Z}_L^0$ under the natural map $L \to L/\mathscr{Z}_L^0$. (It is clear that there are only finitely many G-conjugacy classes of such pairs.) The piece corresponding to (L, S_1) is defined by

$$Y = Y_{(L,S_1)} = \text{union of all conjugacy classes in } G \text{ which meet } (S_1)_{\text{reg}},$$
$$\text{where } (S_1)_{\text{reg}} = \{g \in S_1 \mid H_G(g) = L\} = \{g \in S_1 \mid Z_G^0(g_s) \subset L\}.$$

Then Y is a locally closed smooth irreducible subvariety of G, stable by conjugation; the map $g \to H_G(g)$ defines a locally trivial fibration $Y \to$ variety of all conjugates of L, all of whose fibres are isomorphic to $\bigcup_{n \in N(L)/L} n(S_1)_{\text{reg}} n^{-1}$. (The last variety is a disjoint union of finitely many copies of $(S_1)_{\text{reg}}$, and $(S_1)_{\text{reg}}$ is open dense in S_1.) It follows that

$$\dim Y = 2\nu_G - 2\nu_L + \dim S_1.$$

It is clear that Y depends only on the G-conjugacy class of (L, S_1).

If g is an arbitrary element of G, then g is contained in a unique piece Y as above: Y is associated to (L, S_1) where $L = H_G(g)$ and S_1 is the L-conjugacy class of g, times $\mathscr{Z}_L^0$. Hence the Y's form a finite partition of G.

3.2. On each piece $Y = Y_{(L, S_1)}$ in our partition we define a class of G-equivariant local systems as follows. Let $\mathscr{E}_1$ be an L-equivariant irreducible local system on S_1 such that the pair $(S_1, \mathscr{E}_1)$ is cuspidal for L, (see 2.4); in particular, $\mathscr{E}_1$ admits a central character $\mathscr{S}_1$ (with respect to L), where $\mathscr{S}_1$ is a one dimensional tame local system on $\mathscr{Z}_L^0$. We consider the pull back $\hat{\mathscr{E}}_1$ of $\mathscr{E}_1 | (S_1)_{\text{reg}}$ to $\hat{Y} = \{(g, x) \in Y \times G \mid x^{-1} g x \in (S_1)_{\text{reg}}\}$ under the map $\hat{Y} \to (S_1)_{\text{reg}}$, $(g, x) \mapsto x^{-1} g x$. It is clear that $\hat{\mathscr{E}}_1$ is an irreducible $G \times L$-equivariant local system on $\hat{Y}$ for the action of $G \times L$ on $\hat{Y}$ given by (g_0, l_0): $(g, x) \mapsto (g_0 g g_0^{-1}, g_0 x l_0^{-1})$. (Note that $\mathscr{E}_1 | (S_1)_{\text{reg}}$ is irreducible since $(S_1)_{\text{reg}}$ is open in S_1, and $\hat{\mathscr{E}}_1$ is irreducible on $\hat{Y}$ since $\hat{Y}$ is isomorphic to $(S_1)_{\text{reg}} \times G$.) Now L acts freely on $\hat{Y}$ by right multiplication on the x-coordinate and the orbit space $\hat{Y}/L$ is $\tilde{Y} = \{(g, xL) \in Y \times (G/L) \mid x^{-1} g x \in (S_1)_{\text{reg}}\}$. Since $\hat{\mathscr{E}}_1$ is L-equivariant, it is the inverse image under $\hat{Y} \to \tilde{Y}$ of a well defined, G-equivariant local system $\tilde{\mathscr{E}}_1$ on $\tilde{Y}$, which is necessarily irreducible. We now take the direct image $\pi_* \tilde{\mathscr{E}}_1$ of $\tilde{\mathscr{E}}_1$ under the map $\pi: \tilde{Y} \to Y$, $\pi(g, xL) = g$. This map is a finite unramified covering which is a principal fibration with group $\mathscr{W}_{S_1} =$ stabilizer of S_1 in $\mathscr{W} = N(L)/L$. It follows that $\pi_* \tilde{\mathscr{E}}_1$ is a local system on Y (necessarily G-

equivariant); the dimension of a stalk of $\pi_*\tilde{\mathscr{E}}_1$ is equal to $|\mathscr{W}_{S_1}|$ times the dimension of a stalk of $\mathscr{E}_1$. It is easy to see that $\pi_*\tilde{\mathscr{E}}_1$ admits a central character $\mathscr{S}$ (with respect to G), where $\mathscr{S}$ is the restriction of $\mathscr{S}_1$ to the subgroup $\mathscr{Z}_G^0$ of $\mathscr{Z}_L^0$.

3.3. **Definition.** *An irreducible local system $\mathscr{E}$ on $Y = Y_{(L,S_1)}$ is said to be admissible, if it is a direct summand of $\pi_*(\tilde{\mathscr{E}}_1)$ where $(S_1, \mathscr{E}_1)$ is a cuspidal pair (for L), as above.*

It follows automatically that $\mathscr{E}$ is G-equivariant and that $\mathscr{E}$ admits a central character.

In the case where $L = G$ and $(S_1, \mathscr{E}_1)$ is a cuspidal pair for G, the set S_1 is automatically isolated in L (see 2.7) hence $Y = Y_{(L,S_1)}$ is defined; we have $Y = S_1$. Our definition for $\pi_*(\mathscr{E}_1)$ leads just to $\mathscr{E}_1$; we see that in this case, the admissible local systems on Y are just the $\mathscr{E}_1'$ such that $(S_1, \mathscr{E}_1')$ is a cuspidal pair (for G). In the case where $L \neq G$, $\pi_*(\tilde{\mathscr{E}}_1)$ above is not necessarily irreducible.

3.4. Let $N_{\mathscr{E}_1}$ be the set of all $n \in N(L)$ such that $nS_1n^{-1} = S_1$ and such that the automorphism $g \to ngn^{-1}$ of S_1 can be lifted to $\mathscr{E}_1$. Then $N_{\mathscr{E}_1} \supset L$; we set $\mathscr{W}_{\mathscr{E}_1} = N_{\mathscr{E}_1}/L$. Given $w \in \mathscr{W}_{\mathscr{E}_1}$, we consider the corresponding coset $N_{\mathscr{E}_1,w}$ in $N_{\mathscr{E}_1}$. On the product $N_{\mathscr{E}_1,w} \times S_1$ we have two local systems $\mathscr{E}_1', \mathscr{E}_1''$. The first one, $\mathscr{E}_1'$, is the inverse image of $\mathscr{E}_1$ under the map $(n,g) \to ngn^{-1}$; the second one, $\mathscr{E}_1''$, is the inverse image of $\mathscr{E}_1$ under the map $(n,g) \to g$. Let $\mathscr{A}_{\mathscr{E}_1,w}$ be the space of all homomorphisms of $\mathscr{E}_1''$ into $\mathscr{E}_1'$ inducing identity on the base $N_{\mathscr{E}_1,w} \times S_1$. Then $\mathscr{A}_{\mathscr{E}_1,w}$ is a one-dimensional $\bar{\mathbb{Q}}_l$-vector space, since $\mathscr{E}_1', \mathscr{E}_1''$ are isomorphic (use the L-equivariance of $\mathscr{E}_1$ and the definition of $N_{\mathscr{E}_1}$) and irreducible. Let $\mathscr{A}_{\mathscr{E}_1} = \bigoplus_w \mathscr{A}_{\mathscr{E}_1,w}$ (w runs over $\mathscr{W}_{\mathscr{E}_1}$). There is a natural $\bar{\mathbb{Q}}_l$-algebra structure on $\mathscr{A}_{\mathscr{E}_1}$ such that $\mathscr{A}_{\mathscr{E}_1,w} \cdot \mathscr{A}_{\mathscr{E}_1,w'} = \mathscr{A}_{\mathscr{E}_1,ww'}$. Indeed, an element of $\mathscr{A}_{\mathscr{E}_1,w}$ gives rise to a system of homomorphisms $f_n: (\mathscr{E}_1)_g \to (\mathscr{E}_1)_{ngn^{-1}}$ defined for each $g \in S_1$ and each $n \in N_{\mathscr{E}_1,w}$. ($(\mathscr{E}_1)_g$ is the stalk of $\mathscr{E}_1$ at g.) This system satisfies $f_n \circ f_l^0 = f_{nln^{-1}}^0 \circ f_n$ for all $n \in N_{\mathscr{E}_1,w}$ and all $l \in L$, where $f_l^0: (\mathscr{E}_1)_g \to (\mathscr{E}_1)_{lgl^{-1}}, (l \in L)$, define the L-equivariant structure of $\mathscr{E}_1$. Similarly, an element of $\mathscr{A}_{\mathscr{E}_1,w'}$ gives rise to a system of homomorphisms $f_n': (\mathscr{E}_1)_g \to (\mathscr{E}_1)_{ngn^{-1}}$, $(\forall g \in S_1, \forall n \in N_{\mathscr{E}_1,w'})$ such that $f_n' \circ f_l^0 = f_{nln^{-1}}^0 \circ f_n'$ for all $n \in N_{\mathscr{E}_1,w'}$ and all $l \in L$. We now define $f_n'' = f_{n_1} \circ f_{n_2}': (\mathscr{E}_1)_g \to (\mathscr{E}_1)_{ngn^{-1}}$ for any $g \in S_1$, $n \in N_{\mathscr{E}_1,ww'}$, where $n_1 \in N_{\mathscr{E}_1,w}$, $n_2 \in N_{\mathscr{E}_1,w'}$ are such that $n = n_1 n_2$; this is in dependent of the choice of n_1, n_2 and it corresponds to a unique element of $\mathscr{A}_{\mathscr{E}_1,ww'}$, the product of the two elements in $\mathscr{A}_{\mathscr{E}_1,w}, \mathscr{A}_{\mathscr{E}_1,w'}$. The algebra $\mathscr{A}_{\mathscr{E}_1}$ has a unit element; it lies in $\mathscr{A}_{\mathscr{E}_1,1}$.

3.5. **Proposition** *The local system $\pi_*(\tilde{\mathscr{E}}_1)$ on Y is semisimple. Its endomorphism algebra is naturally isomorphic to the algebra $\mathscr{A}_{\mathscr{E}_1}$. Any endomorphism of $\pi_*(\tilde{\mathscr{E}}_1)$ is automatically G-equivariant.*

Consider the element of $\mathscr{A}_{\mathscr{E}_1,w}$ corresponding to the system of homomorphisms $f_n: (\mathscr{E}_1)_g \to (\mathscr{E}_1)_{ngn^{-1}}$, $(\forall g \in S_1, \forall n \in N_{\mathscr{E}_1,w})$, as above. For each $n \in N_{\mathscr{E}_1,w}$, we consider the system of homomorphisms $\hat{f}_n: (\hat{\mathscr{E}}_1)_{g,x} \to (\hat{\mathscr{E}}_1)_{g,xn^{-1}}$, $(\forall (g,x) \in \hat{Y})$, defined as $f_n: (\mathscr{E}_1)_{x^{-1}gx} \to (\mathscr{E}_1)_{nx^{-1}gxn^{-1}}$. (We may identify $(\hat{\mathscr{E}}_1)_{g,x} = (\mathscr{E}_1)_{x^{-1}gx}$, $(\hat{\mathscr{E}}_1)_{g,xn^{-1}} = (\mathscr{E}_1)_{nx^{-1}gxn^{-1}}$.) This gives rise, for each $(g, xL) \in \tilde{Y}$, to a homomorphism

$\tilde{f}_n: (\tilde{\mathscr{E}}_1)_{g,xL} \to (\tilde{\mathscr{E}}_1)_{g,xn^{-1}L}$. (We may identify $(\tilde{\mathscr{E}}_1)_{g,xL}$ with the space of global sections of the restriction of $\hat{\mathscr{E}}_1$ to the fibre of $\hat{Y} \to \tilde{Y}$ over (g, xL); similarly, for $(\tilde{\mathscr{E}}_1)_{g,xn^{-1}L}$.) The system of homomorphism $\tilde{f}_n$ is clearly independent of the choice of n, $(n \in N_{\mathscr{E}_1,w})$; we denote it $\tilde{f}$. Since the stalk $\pi_*(\tilde{\mathscr{E}}_1)_g$, $(g \in Y)$, is the direct sum $\bigoplus_{xL} (\tilde{\mathscr{E}}_1)_{g,xL}$ (sum over all xL such that $(g, xL) \in \tilde{Y}$), it follows that $\tilde{f}$ gives rise to an endomorphism of $\pi_*(\tilde{\mathscr{E}}_1)_g$, for each $g \in Y$, and this comes from a unique endomorphism of the local system $\pi_*(\tilde{\mathscr{E}}_1)$. This defines a linear map $\mathscr{A}_{\mathscr{E}_1,w} \to \mathrm{End}_G(\pi_*(\tilde{\mathscr{E}}_1))$ (G-equivariant endomorphisms) hence to a linear map $\mathscr{A}_{\mathscr{E}_1} \to \mathrm{End}_G(\pi_*(\tilde{\mathscr{E}}_1))$ which is clearly an injective algebra homomorphism preserving the unit element. We now show that $\dim \mathrm{End}(\pi_*(\tilde{\mathscr{E}}_1)) \leqq |\mathscr{A}_{\mathscr{E}_1}|$. First note that the local system $\tilde{\mathscr{E}}_1$ is irreducible. (It is enough to show that $\hat{\mathscr{E}}_1$ is irreducible; since $\tilde{Y} \approx (S_1)_{\mathrm{reg}} \times G$ and $\tilde{\mathscr{E}}_1 \approx \mathscr{E}_1 \otimes 1$, it is enough to show that $\mathscr{E}_1 | (S_1)_{\mathrm{reg}}$ is irreducible; but this follows from the fact that $\mathscr{E}_1$ is irreducible on S_1 and $(S_1)_{\mathrm{reg}}$ is open in S_1.) Since $\pi: \tilde{Y} \to Y$ is a principal bundle with group $\mathscr{W}_{S_1}$, it follows that $\pi_*(\tilde{\mathscr{E}}_1)$ is semisimple and that the dimension of its endomorphism algebra is equal to the number of $w \in \mathscr{W}_{S_1}$ such that the map $\tilde{Y} \to \tilde{Y}$ defined by w can be lifted to $\tilde{\mathscr{E}}_1$. Given such w, and a representative $n \in N(L)$ for w, it follows that the map $(g, x) \mapsto (g, xn^{-1})$: $\hat{Y} \to \hat{Y}$ can be lifted to $\hat{\mathscr{E}}_1$ and hence the map $g \mapsto ngn^{-1}: (S_1)_{\mathrm{reg}} \to (S_1)_{\mathrm{reg}}$ can be lifted to $\mathscr{E}_1 | (S_1)_{\mathrm{reg}}$. Since $(S_1)_{\mathrm{reg}}$ is open dense in S_1 it follows that the map $g \mapsto ngn^{-1}: S_1 \to S_1$ can be lifted to $\mathscr{E}_1$. Hence $w \in \mathscr{W}_{\mathscr{E}_1}$. Thus $\dim \mathrm{End}(\pi_*(\tilde{\mathscr{E}}_1)) \leqq |\mathscr{W}_{\mathscr{E}_1}|$. It follows that

$$\mathscr{A}_{\mathscr{E}_1,w} \xrightarrow{\sim} \mathrm{End}_G(\pi_*(\tilde{\mathscr{E}}_1)) = \mathrm{End}(\pi_*(\tilde{\mathscr{E}}_1)).$$

The proposition is proved.

3.6. *Remark.* If we choose basis elements $b_w \in \mathscr{A}_{\mathscr{E}_1,w}$ for each $w \in \mathscr{W}_{\mathscr{E}_1}$, we have $b_w b_{w'} = \lambda_{w,w'} b_{ww'}$, where $\lambda_{w,w'} \in \bar{\mathbb{Q}}_l^*$ is a 2-cocycle of $\mathscr{W}_{\mathscr{E}_1}$. Thus, the algebra $\mathscr{A}_{\mathscr{E}_1}$ is the group algebra of $\mathscr{W}_{\mathscr{E}_1}$, twisted by a 2-cocycle. (In 9.2 it is shown that this cocycle is trivial in an interesting special case.)

3.7. Let us denote by $\mathscr{A}_{\mathscr{E}_1}^{\vee}$ the set of isomorphism classes of irreducible $\mathscr{A}_{\mathscr{E}_1}$-modules. Given a semisimple object M of some abelian category such that M is an $\mathscr{A}_{\mathscr{E}_1}$-module, we shall write $M_\rho = \mathrm{Hom}_{\mathscr{A}_{\mathscr{E}_1}}(\rho, M)$, $(\rho \in \mathscr{A}_{\mathscr{E}_1}^{\vee})$, and we have $M = \bigoplus_{\rho \in \mathscr{A}_{\mathscr{E}_1}^{\vee}} (\rho \otimes M_\rho)$ with $\mathscr{A}_{\mathscr{E}_1}$ acting only on the ρ-factor and where M_ρ is again in our abelian category. (We assume chosen representatives for each isomorphism class of representations of $\mathscr{A}_{\mathscr{E}_1}$.)

In particular, we have $\pi_*(\tilde{\mathscr{E}}_1) = \bigoplus_{\rho \in \mathscr{A}_{\mathscr{E}_1}^{\vee}} \rho \otimes \pi_*(\tilde{\mathscr{E}}_1)_\rho$ where $\pi_*(\tilde{\mathscr{E}}_1)_\rho$ are admissible local systems on Y, and all admissible local systems on Y are obtained in this way.

§ 4. Admissible complexes

4.1. Let $\mathscr{E}$ be an admissible local system on $Y = Y_{(L,S_1)} \subset G$, see 3.3. We consider the intersection cohomology complex (see [4, 1] and (0.1)), of the closure $\bar{Y}$ with coefficients in $\mathscr{E}$; we denote it $IC(\bar{Y}, \mathscr{E})$. It is a G-equivariant complex (for the action of G on $\bar{Y}$ given by conjugation) and all its cohomology sheaves admit the same central character as $\mathscr{E}$. The complexes $IC(\bar{Y},\mathscr{E})$, for various $Y,\mathscr{E}$ as above, are called the *admissible complexes* of G. Let us write $\mathscr{E}=\pi_*(\tilde{\mathscr{E}}_1)_{\rho'}$, $(\rho'\in\mathscr{A}_{\mathscr{E}_1}^\vee)$ as in 3.7. (Here $(S_1,\mathscr{E}_1)$ is a cuspidal pair for L.) By the definition of intersection cohomology complexes we have

(4.1.1.) $$\operatorname{End} IC(\bar{Y},\pi_*(\tilde{\mathscr{E}}_1))=\operatorname{End}\pi_*(\tilde{\mathscr{E}}_1)=\mathscr{A}_{\mathscr{E}_1}$$

(the last equality follows from 3.5). Hence we have

(4.1.2) $$IC(\bar{Y},\pi_*(\tilde{\mathscr{E}}_1))=\bigoplus_{\rho\in\mathscr{A}_{\mathscr{E}_1}^\vee}\rho\otimes IC(\bar{Y},\pi_*(\tilde{\mathscr{E}}_1))_\rho$$

where $IC(\bar{Y},\pi_*(\tilde{\mathscr{E}}_1))_\rho=IC(\bar{Y},\pi_*(\tilde{\mathscr{E}}_1)_\rho)$ is an admissible complex. From 3.5 it follows that

(4.1.3) All endomorphisms of $IC(\bar{Y},\pi_*(\tilde{\mathscr{E}}_1))$ are automatically G-equivariant.

4.2. We shall now give a construction of $IC(\bar{Y},\pi_*(\tilde{\mathscr{E}}_1))$ which is similar to that in [9].

Choose a parabolic subgroup $P\subset G$ having L as a Levi subgroup. Let

$$X=\{(g,xP)\in G\times(G/P)\mid x^{-1}gx\in\bar{S}_1\cdot U_P\}.$$

(Here $\bar{S}_1$ denotes the closure of S_1; note that the subset $\bar{S}_1\cdot U_P$ of P is stable by P-conjugacy, so that the definition makes sense.) We define a map $\phi\colon X\to G$ by $\phi(g,xP)=g$. It is G-equivariant for the action of G on X given by $g_0\colon(g,xP)\to(g_0gg_0^{-1},g_0xP)$ and the action of G on itself, by conjugation.

4.3. **Lemma.** (a) *X is an irreducible variety of dimension* $=\dim Y$.

(b) *ϕ is proper and $\phi(X)=\bar{Y}$ (=the closure of Y in G).*

(c) *The map $(g,xL)\to(g,xP)$ is an isomorphism $\gamma\colon\tilde{Y}\xrightarrow{\sim}\phi^{-1}(Y)$, (see 3.2).*

Proof. (a) The second projection $X\to G/P$ has all its fibres isomorphic to $\bar{S}_1\cdot U_P$ hence X is irreducible of dimension equal to $\dim(G/P)+\dim S_1+\dim U_P=\dim Y$.

(b) The proof of this assertion is standard, but for the convenience of the reader we give it here. Let $X'=\{(g,xP)\mid x^{-1}gx\in P\}$. Then $X\subset X'$. We show that X is closed in X'. Note that X and X' can be regarded as sets of P-orbits on $\hat{X},\hat{X}'$ where $\hat{X}=\{(g,x)\in G\times G\mid x^{-1}gx\in\bar{S}_1\cdot U_P\}$, $\hat{X}'=\{(g,x)\mid x^{-1}gx\in P\}$ and P acts (freely) by right translation on the x-coordinate. Hence it is enough to show that $\hat{X}$ is closed in $\hat{X}'$. By the change of coordinates $g'=x^{-1}gx$, the inclusion $\hat{X}\subset\hat{X}'$ becomes $G\times(\bar{S}_1\cdot U_P)\subset G\times P$ which clearly has closed image.

Thus X is closed in X'. It is well known that the map $X'\to G$ given by $(g,xP)\to g$ is proper. Since ϕ is the restriction of that map to a closed subset, it is also proper. It is clear that $Y\subset\phi(X)$. Since ϕ is proper, $\phi(X)$ must be closed hence $\bar{Y}\subset\phi(X)$. We have $\dim Y=\dim\bar{Y}\leqq\dim\phi(X)\leqq\dim(X)=\dim Y$. Since $\bar{Y}$ and $\phi(X)$ are both irreducible and closed, it follows that $\bar{Y}=\phi(X)$.

(c) It is easy to see that γ maps $\tilde{Y}$ injectively into $\phi^{-1}(Y)$. We now show that $\gamma(\tilde{Y})=\phi^{-1}(Y)$. Consider an element in $\phi^{-1}(g)$ for $g\in(S_1)_{\text{reg}}$; it is of the form (g,xP) where $x^{-1}gx\in\bar{S}_1\cdot U_P$. Let us identify L with P/U_P and let

$\pi_P: P \to L$ be the natural projection. If we denote $z = \pi_P(x^{-1}gx)$, we have $z \in \bar{S}_1$. Under the projection $L \to L/\mathscr{Z}_L^0$, S_1 is mapped to a single conjugacy class C and $\bar{S}_1$ to the closure $\bar{C}$ of that conjugacy class. The set of elements in $L/\mathscr{Z}_L^0$ whose semisimple part is in a fixed conjugacy class is closed, hence the semisimple part of any element in $\bar{C}$ is equal to the semisimple part of some element in C. This implies that the semisimple part of any element in $\bar{S}_1$ is equal to the semisimple part of some element in S_1 times an element $\zeta \in \mathscr{Z}_L^0$; since $\mathscr{Z}_L^0 S_1 = S_1$, we may assume that $\zeta = 1$. In particular, $z_s = \pi_P(x^{-1}g_s x)$ is equal to g'_s for some $g' \in S_1$. Since S_1 is isolated in L, we have $H_L(g') = L$ for any $g' \in S_1$. Since $H_L(g')$ depends only on g'_s, and since $z_s = g'_s$, it follows that $H_L(z_s) = L$. Now let L' be a Levi subgroup of P containing $x^{-1}g_s x$. We consider the isomorphism $L' \xrightarrow{\sim} L$ obtained as a composition $L' \xrightarrow{\sim} P/U_P \xleftarrow{\sim} L$. Under this isomorphism, $x^{-1}g_s x \in L'$ corresponds to $z_s \in L$. From $H_L(z_s) = L$, it follows that $H_{L'}(x^{-1}g_s x) = L'$, hence $H_G(x^{-1}g_s x) \supset L'$, hence $H_G(g_s) \supset xL'x^{-1}$. This, combined with $H_G(g) = H_G(g_s) = L$ (since $g \in (S_1)_{\text{reg}}$) implies $L \supset xL'x^{-1}$. Since L, L' have the same dimension, it follows that $L = xL'x^{-1}$. The Levi subgroups L, L' of P are also conjugate by an element of P: we have $L = p^{-1}L'p$, $p \in P$. Let $x' = xp$. Then $x'^{-1}Lx' = L$ and $x'^{-1}gx' \in \bar{S}_1 \cdot U_P$. Since $g \in L$, we have also $x'^{-1}gx' \in L$ hence $x'^{-1}gx' \in \bar{S}_1 U_P \cap L = \bar{S}_1$. On the other hand, from $g \in S_1$, we have $x'^{-1}gx' \in x'^{-1}S_1 x'$. Thus, $x'^1 S_1 x'$ meets $\bar{S}_1$. The image of $x'^{-1}S_1 x'$ under $L \to L/\mathscr{Z}_L^0$ is a single conjugacy class of the same dimension as the image of S_1; it meets the image of $\bar{S}_1$ which is the union of the image of S_1 with finitely many conjugacy classes of smaller dimension. It follows that $x'^{-1}S_1 x'$ and S_1 have the same image in $L/\mathscr{Z}_L^0$, hence $x'^{-1}S_1 x' = S_1$. Since $x' \in N(L)$, we must have also $x'^{-1}(S_1)_{\text{reg}} x' = (S_1)_{\text{reg}}$. From $g \in (S_1)_{\text{reg}}$, it now follows that $x'^{-1}gx' \in x'^{-1}(S_1)_{\text{reg}} x'$ hence $x'^{-1}gx' \in (S_1)_{\text{reg}}$. This means that $(g, x'L) \in \tilde{Y}$. Since $x'P = xP$, it follows that $\phi^{-1}(g) \subset \gamma(\tilde{Y})$. Here g was an arbitrary element of $(S_1)_{\text{reg}}$. Since any element of Y is conjugate in G to an element of $(S_1)_{\text{reg}}$, it follows that $\phi^{-1}(Y) \subset \gamma(\tilde{Y})$. Hence γ is a bijection between $\tilde{Y}$ and $\phi^{-1}(Y)$. The proof of the fact that the inverse of this bijection is algebraic is easy and will be omitted.

4.4. The variety $\bar{S}_1$ is statified into finitely many smooth strata: the orbits of $\mathscr{Z}_L^0 \times L$ with $\mathscr{Z}_L^0$ acting by translation and L by conjugation. There is a unique open stratum: S_1. Taking the inverse images of these strata under the map $\hat{X} \to \bar{S}_1$, $(g, x) \mapsto$ (projection of $x^{-1}gx \in \bar{S}_1 \cdot U_P$ onto the $\bar{S}_1$-factor), we get a stratification $\hat{X} = \bigcup_\alpha \hat{X}_\alpha$ of $\hat{X}$ into smooth strata (α runs through the strata of $\bar{S}_1$). The stratum $\hat{X}_{\alpha_0}$ (α_0 = stratum S_1) is open dense. The strata $\hat{X}_\alpha$ are P-invariant for the (free) P action on $\hat{X}$ given by right translation on the x-coordinate. Hence their images X_α in $X = \hat{X}/P$ form a stratification of X into finitely many smooth strata, with X_{α_0} open dense. We consider the L-equivariant local system $\mathscr{E}_1$ on S_1; we take its inverse image under $\hat{X}_{\alpha_0} \to S_1$ and we get a $G \times P$-equivariant local system on $\hat{X}_{\alpha_0}$, which is necessarily the inverse image under $\hat{X}_{\alpha_0} \to \hat{X}_{\alpha_0}/P = X_{\alpha_0}$ of a G-equivariant local system on X_{α_0}, which will be denoted $\bar{\mathscr{E}}_1$. ($\bar{\mathscr{E}}_1$ extends to X_{α_0} the local system on $\phi^{-1}(Y) \cong \tilde{Y}$ which was denoted as $\tilde{\mathscr{E}}_1$ in 3.2.) Similarly, the L-equivariant complex $IC(\bar{S}_1, \mathscr{E}_1)$ on $\bar{S}_1$ gives rise by taking inverse image under $\hat{X} \to \bar{S}_1$ to a $G \times P$-equivariant complex on $\hat{X}$, which is the inverse image under $\hat{X}\to\hat{X}/P=X$ of a well defined G-equivariant complex on X, which is just $IC(X,\bar{\mathscr{E}}_1)$.

Let us denote $K=IC(X,\bar{\mathscr{E}}_1)$. We consider the complex $\phi_! K$ on $\bar{Y}$, where ϕ is as in 4.2.

4.5. **Proposition.** *$\phi_! K$ is canonically isomorphic to $IC(\bar{Y},\pi_*\tilde{\mathscr{E}}_1)$.*

Using 4.3(c) and the fact that $K|\phi^{-1}(Y)\cong\bar{\mathscr{E}}_1|\phi^{-1}(Y)\cong\tilde{\mathscr{E}}_1$, we see that the restriction of $\phi_! K$ to Y is just $\pi_*\tilde{\mathscr{E}}_1$. Since ϕ is proper (4.3(b)), it is enough to prove the following assertion, (see [4] and (0.1)):

(4.5.1) For any $i>0$, we have $\dim\operatorname{supp}\mathscr{H}^i(\phi_! K)<\dim Y-i$ (where $\mathscr{H}^i$ denotes the i-th cohomology sheaf of $\phi_! K$)

and also the analogous assertion in which K is replaced by $K^*=IC(X,\bar{\mathscr{E}}_1^*)$, with $\mathscr{E}_1^*=$local system dual to $\mathscr{E}_1$. (Note that $(S_1,\mathscr{E}_1^*)$ is again a cuspidal pair for L). We shall only prove the assertion (4.5.1) for K; the proof for K^* is identical.

If $g\in\bar{Y}$, the stalk $\mathscr{H}_g^i(\phi_! K)$ at g is equal to $H^i(\phi^{-1}(g),K)$ (hypercohomology of $\phi^{-1}(g)$ with coefficients in the restriction of K to $\phi^{-1}(g)$). We can stratify $\phi^{-1}(g)$ by $\phi^{-1}(g)_\alpha=\phi^{-1}(g)\cap X_\alpha$. If $H^i(\phi^{-1}(g),K)\neq0$, then there exists α such that $H_c^i(\phi^{-1}(g)_\alpha,K)\neq0$. Hence it is enough to prove:

(4.5.2) For any $i>0$ and any α, we have $\dim\{g\in\bar{Y}\,|\,H_c^i(\phi^{-1}(g)_\alpha,K)\neq0\}<\dim\bar{Y}-i$.

Assume first that $\alpha\neq\alpha_0$ and let $S_{1,\alpha}$ be the stratum of $\bar{S}_1$ corresponding to α. If $H_c^i(\phi^{-1}(g)_\alpha,K)\neq0$, we see from the hypercohomology spectral sequence for K on $\phi^{-1}(g)_\alpha$, that we can write $i=j_1+j_2$ where $j_2\leqq2\dim\phi^{-1}(g)_\alpha$ and $\mathscr{H}^{j_1}(K|\phi^{-1}(g)_\alpha)\neq0$. The last condition implies that the j_1-th cohomology sheaf on K (on X) is non-zero on X_α. From (0.1) it follows that $j_1<\dim X-\dim X_\alpha=\dim S_1-\dim S_{1,\alpha}$. Therefore we have $i<2\dim\phi^{-1}(g)_\alpha+\dim S_1-\dim S_{1,\alpha}$ and hence it is enough to show that $\dim\{g\in\bar{Y}|\dim\phi^{-1}(g)_\alpha>(i/2)-((\dim S_1-\dim S_{1,\alpha})/2)\}<\dim\bar{Y}-i$. If this is violated for some $i>0$, it would follow that the set of triples

$$\{(g,xP,x'P)\in\bar{Y}\times(G/P)\times(G/P)|x^{-1}gx\in S_{1,\alpha}U_P,x'^{-1}gx'\in S_{1,\alpha}U_P\}$$

has dimension $>\dim Y-i+i-(\dim S_1-\dim S_{1,\alpha})=2\nu_G-2\nu_L+\dim S_{1,\alpha}$. This contradicts 1.2(c).

Next, we assume that $\alpha=\alpha_0$. If $H_c^i(\phi^{-1}(g)_{\alpha_0},K)\neq0$, then $i\leqq2\dim\phi^{-1}(g)_{\alpha_0}$ since K restricted to $\phi^{-1}(g)_{\alpha_0}$ is just a local system. Hence it is enough to show that $\dim\{g\in\bar{Y}^1|\dim\phi^{-1}(g)_{\alpha_0}\geqq i/2\}<\dim\bar{Y}-i$, $(i>0)$. If this is violated for some $i>0$, it would follow that the space of triples

(4.5.3) $$\{(g,xP,x'P)\in\bar{Y}\times(G/P)\times(G/P)|x^{-1}gx\in S_1U_P,x'^{-1}gx'\in S_1U_P\}$$

has some irreducible component of dimension $\geqq2\nu_G-2\nu_L+\dim S_1$, whose projection to $\bar{Y}$, has dimension $<\dim\bar{Y}$.

From 1.2(c) it follows that this component contains the subset of (4.5.3) defined by the condition $x^{-1}x'\in PnP$ where n is a fixed element of $N(L)$;

moreover, the remark in 1.3(i) shows that n must satisfy $nS_1n^{-1}=S_1$ (otherwise, the component is empty). If $g\in(S_1)_{\mathrm{reg}}$, then (g, P, nP) belongs to our component hence g belongs to its projection to $\bar{Y}$. Since this projection is G-equivariant and $Y=\bigcup_{g_0\in G} g_0(S_1)_{\mathrm{reg}}g_0^{-1}$, it follows that the projection of our component to $\bar{Y}$ contains all of Y and hence it has dimension equal to $\dim\bar{Y}$. This is a contradiction. The proposition is proved.

§ 5. Sheaves on the variety of semisimple classes

5.1. Let $\sigma: G\to A$ be the Steinberg map [23], where A is the affine variety whose points are the semisimple classes of G and σ is the morphism which attaches to $g\in G$ the conjugacy class of g_s. (A may be identified with the quotient of a maximal torus by the corresponding Weyl group.) Let

$$A_Y=\sigma(\bar{Y})=\sigma(\bar{S}_1\,U_P)=\sigma(\bar{S}_1)=\sigma(S_1)\subset A.$$

(We have used the fact that the semisimple part $(lu)_s$ of $l\cdot u$ ($l\in L$, $u\in U_P$) is conjugate under an element in U_P to l_s. This can be seen as follows. Consider a Levi subgroup $\tilde{L}$ of P containing $(lu)_s$. Then $\tilde{L}$ is conjugate to L under an element $v\in U_P$. Thus $v(lu)_sv^{-1}\in L$. Under $\pi_P: P\to P/U_P$, the elements $v(lu)_sv^{-1}$, l_s are mapped to the same element. Since π_P restricted to L is injective, we have $v(lu)_sv^{-1}=l_s$, as asserted.)

Let $(A_Y)_{\mathrm{reg}}=\sigma((S_1)_{\mathrm{reg}})$. Then A_Y is an irreducible closed subvariety of A of dimension equal to $\dim(\mathscr{Z}_L^0)$, and $(A_Y)_{\mathrm{reg}}$ is an open dense subset of A_Y.

5.2. In addition to $S_1\subset L\subset P$ and $\mathscr{E}_1$ on S_1, we consider another set of data $S_1'\subset L'\subset P'$, $\mathscr{E}_1'$ on S_1', of the same kind. In particular, $(S_1', \mathscr{E}_1')$ is a cuspidal pair for L'. We shall denote the various objects associated to $S_1', L', P', \mathscr{E}_1'$ by a prime attached to the notation for the corresponding object for $S_1, L, P, \mathscr{E}_1$. For example, we have $\phi': X'\to\bar{Y}'$, $K'=IC(X', \bar{\mathscr{E}}_1')$.

Consider the fibre product $Z=X\times_G X'=\{((g, xP), (g', x'P'))\in X\times X' \mid g=g'\}$; we assume that $\bar{Y}\cap\bar{Y}'\neq\emptyset$, for otherwise Z is empty. Let $\tilde{\sigma}: Z\to A_Y\cap A_{Y'}$ be the composition of the map $X\times_G X'\to\bar{Y}\cap\bar{Y}'$ defined by ϕ on the first coordinate or ϕ' on the second coordinate, with the restriction of σ to $\bar{Y}\cap\bar{Y}'$. We form the external tensor product $K\boxtimes K'$ (a complex of sheaves on Z). Let

$$d_0=2\nu_G-\nu_L-\nu_{L'}+\tfrac{1}{2}(\dim(S_1/\mathscr{Z}_L^0)+\dim(S_1'/\mathscr{Z}_{L'}^0)).$$

For $a\in A_Y\cap A_{Y'}$, we set $Z^a=\tilde{\sigma}^{-1}(a)\subset Z$. Given a stratum α of $\bar{S}_1$ and a stratum α' of $\bar{S}_1'$, we set $Z^a_{\alpha,\alpha'}=Z^a\cap(X_\alpha\times_G X'_{\alpha'})$ where X_α are as in 4.4. Then the $Z^a_{\alpha,\alpha'}$ form a partition of Z^a into locally closed pieces with $Z^a_{\alpha_0,\alpha_0'}$ open in Z^a, (α_0, α_0' are the open strata in $\bar{S}_1, \bar{S}_1'$).

5.3. **Lemma.** (a) $\dim Z^a_{\alpha,\alpha'}\leq d_0-\tfrac{1}{2}(\dim(S_1/\mathscr{Z}_L^0)-\dim(S_{1,\alpha}/\mathscr{Z}_L^0))$
$-\tfrac{1}{2}(\dim(S_1'/\mathscr{Z}_{L'}^0)-\dim(S'_{1,\alpha'}/\mathscr{Z}_{L'}^0))$

(b) $\dim Z^a\leq d_0$

(c) *The natural map*

$$H_c^{\delta'}(Z^a, K\boxtimes K')\leftarrow H_c^{\delta'}(Z^a_{\alpha_0,\alpha_0'}, K\boxtimes K')=H_c^{\delta'}(Z^a_{\alpha_0,\alpha_0'}, \bar{\mathscr{E}}_1\boxtimes\bar{\mathscr{E}}_1')$$

is an isomorphism for $\delta'>2d_0$, and is surjective for $\delta'=2d_0$. It is an isomorphism for $\delta'=2d_0$, if $L=L', S_1=S'_1$ and if a is such that $Z^a_{\alpha,\alpha'}$ is empty whenever exactly one of α, α' is equal to α_0.

Proof. (a) We have a natural map defined by ϕ or ϕ' from $Z^a_{\alpha,\alpha'}$ to $\sigma^{-1}(a)$. Since $\sigma^{-1}(a)$ is a union of finitely many conjugacy classes it is enough to estimate the dimension of the subset of $Z^a_{\alpha,\alpha'}$ lying over a fixed conjugacy class C in $\sigma^{-1}(a)$. Consider the natural map of this subset to C. All fibres of this map are isomorphic to a product of two varieties of the kind appearing in 1.2(b). Hence, by 1.2(b), each of these fibres has dimension at most

$$(\nu_G-\tfrac{1}{2}\dim(C))-(\nu_L-\tfrac{1}{2}\dim(S_{1,\alpha}/\mathscr{Z}^0_L))+(\nu_G-\tfrac{1}{2}\dim(C))$$
$$-(\nu_{L'}-\tfrac{1}{2}\dim(S'_{1,\alpha'}/\mathscr{Z}^0_{L'}))$$

and (a) follows.

(b) follows immediately from (a). To prove (c), it is enough to show that for any $(\alpha,\alpha')\neq(\alpha_0,\alpha'_0)$, we have $H^{\delta'}_c(Z^a_{\alpha,\alpha'}, K\boxtimes K')=0)$ for $\delta'\geqq 2d_0$ and (under the assumptions in the last sentence of the lemma) even for $\delta'=2d_0-1$. Assume that this cohomology group is non-zero, under our assumptions.

From the hypercohomology spectral sequence for the restriction of $K\boxtimes K'$ to $Z^a_{\alpha,\alpha'}$, we see that we can write $\delta'=i+j+j'$, where $i\leqq 2\dim Z^a_{\alpha,\alpha'}$ and where both restrictions $\mathscr{H}^j(K)|X_\alpha, \mathscr{H}^{j'}(K')|X_{\alpha'}$ are not identically zero. From (a) it follows that

$$i\leqq 2d_0+\dim(S_{1,\alpha}/\mathscr{Z}^0_0)-\dim(S_1/\mathscr{Z}^0_L)+\dim(S'_{1,\alpha'}/\mathscr{Z}^0_{L'})-\dim(S'_1/\mathscr{Z}^0_{L'}).$$

Moreover, by the definition of K, K', we must have

(5.3.1) $$j\leqq\dim(X)-\dim(X_\alpha)=\dim(S_1/\mathscr{Z}^0_L)-\dim(S_{1,\alpha}/\mathscr{Z}^0_L)$$
$$j'\leqq\dim(X')-\dim(X'_{\alpha'})=\dim(S'_1/\mathscr{Z}^0_{L'})-\dim(S'_{1,}{}^{\alpha'}/\mathscr{Z}^0_{L'})$$

and at least one of these inequalities must be strict since $(\alpha,\alpha')\neq(\alpha_0,\alpha_0)$; it follows that $\delta'=i+j+j'<2d_0$. Under the assumptions in the last sentence of the lemma, both inequalities (5.3.1) are strict and it follows that $\delta'=i+j+j'\leqq 2d_0-2$. This proves (c).

5.4. Let $\mathscr{P}$ (resp. $\mathscr{P}'$) be the variety of all conjugates of P (resp. of P'). For each locally closed G-invariant subset E of $\mathscr{P}\times\mathscr{P}'$, ($G$ acts diagonally), we consider the subset $Z^E_{\alpha_0,\alpha'_0}$ of $X_{\alpha_0}\times_G X'_{\alpha'_0}$ consisting of all pairs (g, xP), $(g', x'P')$ in $X_{\alpha_0}\times X'_{\alpha'_0}$, such that $g=g'$, $(xPx^{-1}, x'P'x'^{-1})\in E$. Let $\mathscr{T}_E$ be the $(2d_0)$-th cohomology sheaf of the direct image with compact support of $K\boxtimes K'|Z^E_{\alpha_0,\alpha'_0}$ $(=\bar{\mathscr{E}}_1\boxtimes\bar{\mathscr{E}}'_1|Z^E_{\alpha_0,\alpha'_0})$ under the restriction of $\tilde{\sigma}$ to $Z^E_{\alpha_0,\alpha'_0}$; it is a constructible sheaf on A_Y.

Let $E_{\leqq i}$ (resp. E_i) be the union of all G-orbits on $\mathscr{P}\times\mathscr{P}'$ of dimension $\leqq i$ (resp. $=i$). From 5.3(b) and the cohomology exact sequence associated to the partition of a space into an open and a closed subspace we get an exact sequence of sheaves

(5.4.1) $$0\to\mathscr{T}_{E_i}\to\mathscr{T}_{E_{\leqq i}}\to\mathscr{T}_{E_{\leqq i-1}}\to 0.$$

We also see that

(5.4.2) $$\mathcal{T}_{E_i} = \bigoplus_{\mathcal{O}} \mathcal{T}_{\mathcal{O}}$$

where $\mathcal{O}$ runs through the set of G-orbits $\mathcal{O} \subset \mathcal{P} \times \mathcal{P}'$ of dimension i.

When $E = \mathcal{P} \times \mathcal{P}'$, we shall set $\mathcal{T}_E = \mathcal{T}$. We have $\mathcal{T} = \mathcal{T}_{E_i}$, for large i. Let $\tilde{\mathcal{T}}$ be the $(2d_0)$-th cohomology sheaf of $\tilde{\sigma}_!(K \boxtimes K')$, ($\tilde{\sigma}$ is defined in 5.2). It is a constructible sheaf on $A_Y \cap A_{Y'}$.

The imbedding of $X_{\alpha_0} \times_G X'_{\alpha'_0}$ into Z (as an open subset) gives rise to a natural map of sheaves $\mathcal{T} \to \tilde{\mathcal{T}}$ over $A_Y \cap A_{Y'}$. At level of stalks, it is the map in 5.3(c) for $\delta' = 2d_0$. From 5.3(c) we see that

(5.4.3) *The natural map of sheaves $\mathcal{T} \to \tilde{\mathcal{T}}$ is surjective.*

(5.4.4) *Definition. A constructible sheaf $\mathcal{E}$ on an irreducible variety V is said to be perfect if* (a) *there exists an open dense smooth subset $V_0 \subset V$ such that $\mathcal{E} \mid V_0$ is locally constant and $\mathcal{E} = IC(V, \mathcal{E} \mid V_0)$, and* (b) *the support of any nonzero constructible subsheaf of $\mathcal{E}$ is dense in V.*

(In particular, the complex $IC(V, \mathcal{E} \mid V_0)$ is reduced to a single sheaf.) For example, if $\pi\colon V' \to V$ is a finite morphism with V' smooth and if $\mathcal{E}'$ is a locally constant sheaf on V', then $\mathcal{E} = \pi_* \mathcal{E}'$ is a perfect sheaf on V. Also,

(5.4.5) if $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ is an exact sequence of constructible sheaves on V, with $\mathcal{E}_1$ and $\mathcal{E}_3$ perfect, then $\mathcal{E}_2$ is perfect.

5.5. **Theorem.** (a) *If the pairs (L, S_1), (L', S'_1) are not conjugate under an element in G, then $\mathcal{T} = 0$.*

(b) *If $L = L'$, $S_1 = S'_1$ so that $Y = Y'$, $A_Y = A_{Y'}$ then $\mathcal{T}$ is a perfect sheaf on A_Y.*

(c) *The natural map of sheaves $\mathcal{T} \to \tilde{\mathcal{T}}$ (see* (5.4.3)) *is an isomorphism.*

Before starting the proof, we shall need some preliminary results.

5.6. **Lemma.** *Assume that $\mathcal{O} \subset \mathcal{P} \times \mathcal{P}'$ is a G-orbit with following property: for some (or any) $(\tilde{P}, \tilde{P}') \in \mathcal{O}$, $\tilde{P}$ and $\tilde{P}'$ do not have a common Levi subgroup. Then $\mathcal{T}_{\mathcal{O}} = 0$.*

Proof. We must prove that, for any $a \in A_Y \cap A_{Y'}$, we have

$$H_c^{2d_0}(Z^a \cap Z^{\mathcal{O}}_{\alpha_0, \alpha'_0}, \bar{\mathcal{E}}_1 \boxtimes \bar{\mathcal{E}}'_1) = 0.$$

Using the inequality $\dim(Z^a \cap Z^{\mathcal{O}}_{\alpha_0, \alpha'_0}) \leq d_0$ (5.4(a)) and the fibration $Z^a \cap Z^{\mathcal{O}}_{\alpha_0, \alpha'_0} \to \mathcal{O}$, we see that it is enough to show that for any $a \in A_Y \cap A_{Y'}$ and any $x, x' \in G$ such that $(xPx^{-1}, x'P'x'^{-1}) \in \mathcal{O}$, we have $H_c^{2d_0 - 2\dim \mathcal{O}}(V^a, j^*(\mathcal{E}_1 \otimes \mathcal{E}'_1)) = 0$ where

$$V^a = \{g \in \sigma^{-1}(a) \mid x^{-1} g x \in S_1 U_P, x'^{-1} g x' \in S'_1 U_{P'}\}$$

and $j\colon V^a \to S_1 \times S'_1$ is defined by $j(g) = (S_1\text{-component of } x^{-1} g x,\ S'_1\text{-component of } x'^{-1} g' x')$. Let $\tilde{P} = xPx^{-1}$, $\tilde{P}' = x'P'x'^{-1}$. Choose Levi subgroups $\tilde{L}$ of $\tilde{P}$ and $\tilde{L}'$ of $\tilde{P}'$ such that $\tilde{L}, \tilde{L}'$ contain a common maximal torus. We have $g \in \tilde{P} \cap \tilde{P}'$. As in the proof of 1.2, we write uniquely $g = z\tilde{u}'u = z\tilde{u}v$, where $z \in \tilde{L} \cap \tilde{L}'$, $\tilde{u}' \in \tilde{L}$

$\cap U_{\tilde{P}'}$, $\tilde{u}\in\tilde{L}'\cap U_{\tilde{P}}$, $u\in U_{\tilde{P}}$, $v\in U_{\tilde{P}'}$. Let $\tilde{S}_1$ be the image of $x^{-1}S_1x\subset\tilde{P}$ under $\tilde{P}\to\tilde{L}$ and let $\tilde{S}'_1$ be the image of $x'^{-1}S'_1x'\subset\tilde{P}'$ under $\tilde{P}'\to\tilde{L}'$. Then V^a can be described as

$$\{(u,v,\tilde{u}',\tilde{u},z)\in U_{\tilde{P}}\times U_{\tilde{P}'}\times(\tilde{L}\cap U_{\tilde{P}'})\times(\tilde{L}'\cap U_{\tilde{P}})\times(\tilde{L}\cap\tilde{L}')|\\ \tilde{u}'u=\tilde{u}v,\ z\tilde{u}'\in\tilde{S}_1\cap\sigma^{-1}(a),\ z\tilde{u}\in\tilde{S}'_1\cap\sigma^{-1}(a)\}.$$

This if fibred over

$$\bar{V}^a=\{(\tilde{u}',\tilde{u},z)\in(\tilde{L}\cap U_{\tilde{P}'})\times(\tilde{L}'\cap U_{\tilde{P}})\times(\tilde{L}\cap\tilde{L}')|\\ z\tilde{u}'\in\tilde{S}_1\cap\sigma^{-1}(a),\ z\tilde{u}\in\tilde{S}'_1\cap\sigma^{-1}(a)\}$$

with all fibres $\approx U_{\tilde{P}}\cap U_{\tilde{P}'}$, (see the proof of 1.2); note that $\dim(U_{\tilde{P}}\cap U_{\tilde{P}'})=2\nu_G-\nu_L-\nu_{L'}-\dim\mathcal{O}$. We reduced to showing that

$$H^r_c(\bar{V}^a,\bar{j}^*(\mathcal{F}_1\otimes\mathcal{F}'_1))=0$$

where $r=\dim(S_1/\mathcal{Z}^0_L)+\dim(S'_1/\mathcal{Z}^0_{L'})$, $\bar{j}\colon\bar{V}^a\to\tilde{S}_1\times\tilde{S}'_1$ is defined by $\bar{j}(\tilde{u}',\tilde{u},z)=(z\tilde{u}',z\tilde{u})$, $\mathcal{F}_1$ is the local system on $\tilde{S}_1$ obtained from $\mathcal{E}_1$ via the isomorphism $s\mapsto pr_{\tilde{L}}(x^{-1}sx)$: $S_1\xrightarrow{\sim}\tilde{S}_1$ and $\mathcal{F}'_1$ is the local system on $\tilde{S}'_1$ obtained in a similar way from $\mathcal{E}'_1$. Let $\pi_3\colon\bar{V}^a\to\tilde{L}\cap\tilde{L}'$ be the projection of $\bar{V}^a$ on the z-coordinate. The image $\pi_3(\bar{V}^a)$ is a union of finitely many conjugacy classes $\hat{C}_1,\hat{C}_2,\ldots,\hat{C}_n$ in $\tilde{L}\cap\tilde{L}'$ (compare the proof of 1.2), and, since $\dim\bar{V}^a\leqq\frac{1}{2}r$, it is enough to show that for any $z\in\hat{C}_i$, we have $H_c^{r-2\dim\hat{C}_i}(\pi_3^{-1}(z),\bar{j}^*(\mathcal{F}_1\otimes\mathcal{F}'_1))=0$.

Now $\pi_3^{-1}(z)$ is a product $D\times D'$ where D is the set of all $\tilde{l}\in(\tilde{L}\cap\tilde{P}')\cap\tilde{S}_1$ whose image under $\tilde{L}\cap P'\to\tilde{L}\cap\tilde{L}'$ is equal to z and D' is the set of all $\tilde{l}'\in(\tilde{L}'\cap\tilde{P})\cap\tilde{S}'_1$ whose image under $\tilde{L}'\cap\tilde{P}\to\tilde{L}\cap\tilde{L}'$ is equal to z. Moreover, $\bar{j}^*(\mathcal{F}_1\otimes\mathcal{F}'_1)$ on $\pi_3^{-1}(z)$ corresponds to the tensor product of $\mathcal{F}_1|D$ with $\mathcal{F}'_1|D'$. Since $2\dim D\leqq d_1=\dim(S_1/\mathcal{Z}^0_L)-\dim(\hat{C}_i)$, $2\dim D'\leqq d'_1=\dim(S'_1/\mathcal{Z}^0_{L'})-\dim(\hat{C}_i)$, (see 1.2(a)), and $d_1+d'_1=r-2\dim\hat{C}_i$, we are reduced to showing that

$$H^{d_1}_c(D,\mathcal{F}_1)\otimes H^{d'_1}_c(D',\mathcal{F}'_1)=0.$$

We now make use of our assumption on $\mathcal{O}$. It implies that either $\tilde{L}\cap\tilde{P}'$ is a proper parabolic subgroup of $\tilde{L}$ or that $\tilde{L}'\cap\tilde{P}$ is a proper parabolic subgroup of $\tilde{L}'$. In the first case, we have $H^{d_1}_c(D,\mathcal{F}_1)=0$, since $(\tilde{S}_1,\mathcal{F}_1)$ is a cuspidal pair for $\tilde{L}$; in the second case, we have $H^{d'_1}_c(D',\mathcal{F}'_1)=0$, since $(\tilde{S}'_1,\mathcal{F}'_1)$ is a cuspidal pair for $\tilde{L}'$. (See 2.2(b).) The lemma is proved.

5.7. We next consider the sheaf $\mathcal{T}_{\mathcal{O}}$ in the case where $\mathcal{O}\subset\mathcal{P}\times\mathcal{P}'$ has the following property: for some (or any) $(\tilde{P},\tilde{P}')\in\mathcal{O}$, $\tilde{P}$ and $\tilde{P}'$ have a common Levi subgroup. In this case, we may assume that $L=L'$. We can find $n\in N(L)$ such that $(P,nP'n^{-1})\in\mathcal{O}$. In the case where S_1, nS'_1n^{-1} are disjoint, the argument in the previous lemma shows that $\mathcal{T}_{\mathcal{O}}=0$. (Indeed, if we carry out that argument with $x=1$, $x'=n^{-1}$, we see that $\pi_3^{-1}(z)$ is empty for any $z\in\hat{C}_i$.) We assume now that S_1, nS'_1n^{-1} are not disjoint. It follows that $S_1=nS'_1n^{-1}$, and hence $Y=Y'$ and $A_Y=A_{Y'}$. In this case, the proof of the previous lemma shows that $\mathcal{T}_{\mathcal{O}}=R^r\bar{\sigma}_!(\mathcal{E}_1\otimes n^*(\mathcal{E}'_1))(-r')$ where $\bar{\sigma}\colon S_1\to A_Y$ is the restriction of σ, $r=2\dim(S_1/\mathcal{Z}^0_L)$, $(-r')$ denotes Tate twist by $-r'=-(\nu_G-\nu_L)$, and $n^*(\mathcal{E}'_1)$ is the inverse image of $\mathscr{E}'_1$ (on S'_1) under the map $g \mapsto n^{-1}gn$: $S_1 \to S'_1$. (We take in the proof of the previous lemma: $x=1$, $x'=n^{-1}$, $\tilde{P}=P$, $\tilde{P}'=nP'n^{-1}$, $\tilde{L}=\tilde{L}'=L$; then $\bar{V}^a = S_1 \cap \sigma^{-1}(a)$.)

Let $[S_1]$ be the set of L-conjugacy classes of elements which are semisimple parts of elements in S_1. It is a homogeneous $\mathscr{Z}_L^0$-space ($\mathscr{Z}_L^0$ acts by left multiplication) with finite isotropy group. We can hence regard $[S_1]$ as a smooth algebraic variety and we can factorize $\bar{\sigma}=\bar{\bar{\sigma}} \cdot \pi$ where $\pi: S_1 \to [S_1]$ is the obvious map, in such a way that π and $\bar{\bar{\sigma}}: [S_1] \to A_Y$ are morphisms. Let $\mathscr{F} = \mathscr{E}_1 \otimes n^*(\mathscr{E}'_1)$ and let $[\mathscr{F}] = R^r \pi_!(\mathscr{F})$, (a constructible sheaf on $[S_1]$). We show that $[\mathscr{F}]$ is a local system on $[S_1]$. Since $\mathscr{E}_1, n^*(\mathscr{E}'_1)$ admit a central character (with respect to L), it follows that there exists an integer m (invertible in k) such that $\mathscr{F}$ is $\mathscr{Z}_L^0$-equivariant for the action of $\mathscr{Z}_L^0$ on S_1 given by $z: g \mapsto z^m g$. This action of $\mathscr{Z}_L^0$ induces an action of $\mathscr{Z}_L^0$ on $[S_1]$ such that $\pi: S_1 \to [S_1]$ is $\mathscr{Z}_L^0$-equivariant. It follows that $[\mathscr{F}]$ is $\mathscr{Z}_L^0$-equivariant. Since this action of $\mathscr{Z}_L^0$ on $[S_1]$ (which depends on m) is transitive, it follows that $[\mathscr{F}]$ is a local system, as asserted. Now $\bar{\bar{\sigma}}$ is a finite (ramified) covering and $\mathscr{T}_{\mathcal{O}} = \bar{\bar{\sigma}}_* [\mathscr{F}]$. Since $[\mathscr{F}]$ is a local system on $[S_1]$ and $[S_1]$ is smooth it follows that (see 5.2.1):

(5.7.1) $\mathscr{T}_{\mathcal{O}}$ is a perfect sheaf on A_Y.

5.8. *Proof of Theorem* 5.5(a) *and* (b). We have $\mathscr{T} = \mathscr{T}_{E_i}$ for large i. With the assumptions of (a), we have seen that $\mathscr{T}_{\mathcal{O}}=0$ for all G-orbits on $\mathscr{P} \times \mathscr{P}'$; from (5.4.1), (5.4.2) it follows then by induction on i that $\mathscr{T}_{E_i}=0$ for all i, and in particular, that $\mathscr{T}=0$. With the assumption (b), we see from (5.7.1), (5.4.1), (5.4.2), (5.4.5), by induction on i that $\mathscr{T}_{E_i}$ is perfect on A_Y for all i. Since $\mathscr{T}=\mathscr{T}_{E_i}$ for large i, it follows that $\mathscr{T}$ is a perfect sheaf on A_Y.

Before proving 5.5(c), we prove a lemma.

5.9. **Lemma.** *Let $g \in (S_1)_{\mathrm{reg}}$, $h \in \bar{S}_1 U_P$ be two elements such that $h_s = x^{-1} g_s x$ for some $x \in G$. Then h is conjugate under an element in P to an element $\tilde{h} \in \bar{S}_1$ such that $H_G(\tilde{h}) = L$, (see* 3.1).

Proof. Much of the argument is the same as that in the proof of 4.3(c). We identify L with P/U_P and let $\pi_P: P \to L$ be the natural projection. Let $z = \pi_P(h) \in \bar{S}_1$. As in *loc. cit.* we see that $H_L(z_s)=L$. Let L' be a Levi subgroup of P containing h_s. Then $H_{L'}(h_s)=L'$, hence $H_G(h_s) \supset L'$ hence $H_G(g_s) = xH_G(h_s)x^{-1} \supset xL'x^{-1}$. This combined with $H_G(g)=H_G(g_s)=L$ implies $L \supset xL'x^{-1}$, hence $L=xL'x^{-1}$. The Levi subgroups L, L' of P are also conjugate by an element of P: we have $L=p^{-1}L'p$, $p \in P$. Let $x'=xp$. Then $x'^{-1}Lx'=L$ and $x'^{-1}g_s x' = p^{-1}h_s p = \tilde{h}_s$, where $\tilde{h}=p^{-1}hp \in \bar{S}_1 U_P$.

Let $\tilde{g}=x'^{-1}gx'$. Then $H_G(\tilde{g}_s)=L$ and $\tilde{g}_s=\tilde{h}_s$.

We have $\tilde{h} \in Z_G^0(\tilde{h}_s) = Z_G^0(\tilde{g}_s) \subset H_G(\tilde{g}_s)=L$, hence $\tilde{h} \in L \cap \bar{S}_1 U_P = \bar{S}_1$. We have $H_G(\tilde{h}) = H_G(\tilde{h}_s) = H_G(\tilde{g}_s) = L$. The lemma is proved.

5.10. *Proof of Theorem* 5.5(c). In case (a), we have $\mathscr{T}=0$, hence, by (5.4.3), we have also $\tilde{\mathscr{T}}=0$. In case (b), by (5.4.3), it is enough to show that the kernel of $\mathscr{T} \to \tilde{\mathscr{T}}$ is zero; if we show that the stalks of that kernel are zero at any point in $(A_Y)_{\mathrm{reg}}$, then the fact that $\mathscr{T}$ is perfect and part (b) of the definition of perfectness (5.4.4) would imply that the stalks of that kernel are zero everywhere.

Therefore, by 5.3(c), it is enough to prove that, for any $a \in (A_Y)_{\text{reg}}$, the variety $Z^a_{\alpha,\alpha'}$ is empty whenever exactly one of α, α' is equal to α_0. Thus, we must prove the following statement. If h, x, x' are elements of G such that h_s is conjugate to g_s for some $g \in (S_1)_{\text{reg}}$, $x^{-1}hx \in \bar{S}_1 U_P$, $x'^{-1}hx' \in S_1 U_{P'}$, then $x^{-1}hx \in S_1 U_P$.

Let $h_1 = x^{-1}hx$, $h'_1 = x'^{-1}hx'$. By 5.9, we can find $p \in P$, $p' \in P'$ such that $h_2 = p^{-1}h_1 p \in \bar{S}_1$, $h'_2 = p'^{-1}h'_1 p' \in \bar{S}_1$, $H_G(h_2) = L$, $H_G(h'_2) = L$. We have $h'_2 \in S_1 U_{P'} \cap \bar{S}_1 = S_1$. The elements h_2, h'_2 are conjugate under an element $n \in G$, which necessarily normalizes L: we have $h'_2 = nh_2n^{-1}$, $nLn^{-1} = L$. Then $n^{-1}S_1 n$ meets $\bar{S}_1$, and an argument in the proof of 4.3(c) shows that $n^{-1}S_1 n$ must be equal to S_1. It follows that $h_2 \in S_1$, hence $h_1 \in pS_1p^{-1} \subset S_1 U_P$, as desired. This completes the proof of Theorem 5.5.

5.11. **Proposition.** *With the assumption of* 5.5(b), *and assuming* $P = P'$, *we have a canonical isomorphism of sheaves over* A_Y: $\mathcal{T} \approx \bigoplus_w \mathcal{T}_{\mathcal{O}(w)}$; *here* w *runs over* $\mathcal{W}_{S_1}$ *and* $\mathcal{O}(w)$ *is the G-orbit on* $\mathcal{P} \times \mathcal{P}$ *which contains* $(P, nP'n^{-1})$ *where* $n \in N(L)$ *represents* w.

Proof. We have $\mathcal{T} = R^{2d_0}(\tilde{\sigma}_0)_!(\bar{\mathcal{E}}_1 \boxtimes \bar{\mathcal{E}}'_1)$ where $\tilde{\sigma}_0$ is the restriction of $\tilde{\sigma}: Z \to A_Y$ to the open subset $Z_0 = X_{\alpha_0} \times_G X_{\alpha_0}$ of Z. The inverse image $(\tilde{\sigma}_0)^{-1}(A_Y)_{\text{reg}}$ is the set $\{(g, xP, x'P') \in \bar{Y} \times (G/P) \times (G/P') \mid x^{-1}gx \in S_1 U_P,\ x'^{-1}gx' \in S_1 U_P,\ \sigma(g) \in \sigma((S_1)_{\text{reg}})\}$. From lemma 5.9, we see that this set is the same as

$$\{(g, xP, x'P') \in Y \times (G/P) \times (G/P') \mid x^{-1}gx \in S_1 U_P, x'^{-1}gx' \in S_1 U_P\}$$

which, by 4.3(c) is the same as the fibre product $\tilde{Y} \times_Y \tilde{Y}$. Hence the restriction of $\mathcal{T}$ to $(A_Y)_{\text{reg}}$ is the same as $R^{2d_0}(\tilde{\sigma}_1)_!(\tilde{\mathcal{E}}_1 \boxtimes \tilde{\mathcal{E}}'_1)$ where $\tilde{\sigma}_1: \tilde{Y} \times_Y \tilde{Y} \to (A_Y)_{\text{reg}}$ is the projection to Y, followed by the restriction of σ to Y. Consider the partition $Z_0 = \bigcup_{\mathcal{O}} Z_0^{\mathcal{O}}$ where $\mathcal{O}$ runs over all G-orbits on $\mathcal{P} \times \mathcal{P}'$. (Here $Z_0^{\mathcal{O}}$ consists of all pairs (g, xP), $(g', x'P')$ in $X_{\alpha_0} \times X_{\alpha_0}$ such that $g = g'$, $(xPx^{-1}, x'Px'^{-1}) \in \mathcal{O}$.) Then the pieces $Z_0^{\mathcal{O}}$ are locally closed in Z_0. However, the intersections $(\tilde{\sigma}_0)^{-1}(A_Y)_{\text{reg}} \cap Z_0^{\mathcal{O}}$ are both open and closed in $(\tilde{\sigma}_0)^{-1}(A_Y)_{\text{reg}}$ and are empty unless $\mathcal{O} = \mathcal{O}(w)$ for some $w \in \mathcal{W}_{S_1}$. (This follows from the fact that $\tilde{Y}$ is a principal $\mathcal{W}_{S_1}$-bundle over $\tilde{Y}$ and $\mathcal{W}_{S_1}$ is finite.) The direct image with compact support of $\tilde{\mathcal{E}}_1 \boxtimes \tilde{\mathcal{E}}'_1$ under the restriction of $\tilde{\sigma}_0$ to one of the pieces $(\tilde{\sigma}_0)^{-1}(A_y)_{\text{reg}} \cap Z_0^{\mathcal{O}}$ is easily seen to be just the restriction of $\mathcal{T}_{\mathcal{O}}$ to $(A_Y)_{\text{reg}}$. It follows that we have a canonical isomorphism $\mathcal{T} \approx \bigoplus_w \mathcal{T}_{\mathcal{O}(w)}$ over the open set $(A_Y)_{\text{reg}}$. Since $\mathcal{T}$ and $\bigoplus_w \mathcal{T}_{\mathcal{O}(w)}$ are perfect sheaves, this isomorphism extends uniquely to an isomorphism over the entire A_Y.

5.12. We preserve the assumptions of 5.11. The sheaf $\tilde{\mathcal{T}} = R^{2d_0}\tilde{\sigma}_!(K \boxtimes K')$ can be also regarded as $R^{2d_0}\sigma_!(\phi_!(K) \otimes \phi'_!(K'))$, since $\tilde{\sigma}$ factorizes as $Z \xrightarrow{(\phi,\phi')} \bar{Y} \xrightarrow{\sigma} A_Y$. The complex $\phi_!(K) \otimes \phi'_!(K')$ on $\bar{Y}$ has a natural structure of a module over the algebra $\mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}'_1}$. It follows that $\tilde{\mathcal{T}}$, which is $R^{2d_0}\sigma_!$ of this complex, inherits a natural action of $\mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}'_1}$. This action has the following property. Consider the summand $\mathcal{A}_{\mathcal{E}_1, w_1}$, $(w_1 \in \mathcal{W}_{\mathcal{E}_1})$, of $\mathcal{A}_{\mathcal{E}_1}$, the summand $\mathcal{A}_{\mathcal{E}'_1, w'_1}$,

$(w'_1 \in \mathscr{W}_{\mathscr{E}'_1})$ of $\mathscr{A}_{\mathscr{E}'_1}$ and the summand $\mathscr{T}_{\mathcal{O}(w)}$ of $\mathscr{T}=\tilde{\mathscr{T}}$ (see 5.11), $w \in \mathscr{W}_{S_1}$. Then

(5.12.1) $$(\mathscr{A}_{\mathscr{E}_1, w_1} \otimes \mathscr{A}_{\mathscr{E}'_1, w'_1}) \cdot \mathscr{T}_{\mathcal{O}(w)} = \mathscr{T}_{\mathcal{O}(w_1 w w_1'^{-1})}.$$

(By 5.5(b), it is enough to prove this for the restriction to $(A_Y)_{\text{reg}}$ where it follows from definitions.)

5.13. **Proposition.** *Consider two sets of data $(S_1 \subset L \subset P, \mathscr{E}_1)$ and $(S'_1 \subset L' \subset P', \mathscr{E}'_1)$ as above. Assume that $S_1 = \mathscr{Z}^0_L \cdot C_1$, $S'_1 = \mathscr{Z}^0_{L'} \cdot C'_1$ where C_1 (resp. C'_1) is a unipotent class of L (resp. L′), and that $\mathscr{E}_1$ (resp. $\mathscr{E}'_1$) has as central character the constant sheaf $\bar{\mathbb{Q}}_l$ on $\mathscr{Z}^0_L$ (resp. on $\mathscr{Z}^0_{L'}$). Let $\mathscr{T}=\tilde{\mathscr{T}}$ be the corresponding sheaf (5.5(c)) on $A_Y \cap A_{Y'}$ and let $\mathscr{T}_{a_0}=\tilde{\mathscr{T}}_{a_0}$ be its stalk at a_0 = conjugacy class of $1 \in G$. We regard $\tilde{\mathscr{T}}$ (and in particular $\tilde{\mathscr{T}}_{a_0}$) as a $\mathscr{A}_{\mathscr{E}_1} \otimes \mathscr{A}_{\mathscr{E}'_1}$-modules, as above.*

(a) *If the pairs $(S_1, L, \mathscr{E}_1)$, $(S'_1, L', \mathscr{E}'^*_1)$ are not conjugate by an element of G, then $\tilde{\mathscr{T}}_{a_0}=0$.*

(b) *Assume that $S_1=S'_1$, $L=L'$, $P=P'$, $\mathscr{E}'_1=\mathscr{E}^*_1$. Then the $\mathscr{A}_{\mathscr{E}_1} \otimes \mathscr{A}_{\mathscr{E}^*_1}$-module $\tilde{\mathscr{T}}_{a_0}$ is isomorphic to the $\mathscr{A}_{\mathscr{E}_1} \otimes \mathscr{A}^0_{\mathscr{E}_1}$-module $\mathscr{A}_{\mathscr{E}_1}$, (the module structure is given by left and right multiplication). Here $\mathscr{A}^0_{\mathscr{E}_1}$ is the algebra opposed to $\mathscr{A}_{\mathscr{E}_1}$; we identify $\mathscr{A}_{\mathscr{E}^*_1}$ with $\mathscr{A}^0_{\mathscr{E}_1}$ in a natural way by taking transpose maps.*

Proof. We may assume that $L=L'$, $S_1=S'_1$, (see 5.7), hence $C_1=C'_1$. For $w \in \mathscr{W}_{S_1}$, the stalk of $\mathscr{T}_{\mathcal{O}(w)}$ at a_0 is $H^r_c(C_1, \mathscr{E}_1 \otimes n^*_w(\mathscr{E}'_1))(-r')$, (notations as in 5.7); since $r=2\dim(C_1)$, this is a one dimensional $\bar{\mathbb{Q}}_l$-vector space if $n^*_w(\mathscr{E}'_1) \approx \mathscr{E}^*_1$ (as local systems on S_1, or equivalently, as local systems on C_1) and is zero otherwise. (Here n_w is a representative for w in $N(L)$.)

From 5.7, it now follows that $\tilde{\mathscr{T}}_{a_0}=0$ unless $n^*_w(\mathscr{E}'_1) \approx \mathscr{E}^*_1$ for some $w \in \mathscr{W}_{S_1}$ and (a) follows.

With the assumption of (b), we see that the stalk of $\mathscr{T}_{\mathcal{O}(w)}$ at a_0 is one dimensional if $w \in \mathscr{W}_{\mathscr{E}_1}$ and is zero otherwise. We denote $\mathscr{T}_{a_0, w}$ the stalk of $\mathscr{T}_{\mathcal{O}(w)}$ at a_0, for $w \in \mathscr{W}_{\mathscr{E}_1}$. Then, by 5.11, we have a direct sum decomposition

$$\tilde{\mathscr{T}}_{a_0} = \bigoplus_{w \in \mathscr{W}_{\mathscr{E}_1}} \mathscr{T}_{a_0, w}$$

into one dimensional $\bar{\mathbb{Q}}_l$-vector spaces. From (5.12.1) it follows that the $\mathscr{A}_{\mathscr{E}_1} \otimes \mathscr{A}_{\mathscr{E}^*_1}$-module structure of $\tilde{\mathscr{T}}_{a_0}$ satisfies

$$(\mathscr{A}_{\mathscr{E}_1, w_1} \otimes \mathscr{A}_{\mathscr{E}^*_1, w'_1}) \cdot \mathscr{T}_{a_0, w} = \mathscr{T}_{a_0, w_1 w w_1'^{-1}},$$

$(\forall w, w_1, w'_1 \in \mathscr{W}_{\mathscr{E}_1})$ and (b) follows from this.

§6. A generalization of Springer's correspondence

6.1. With the notations in 1.1 we consider the diagram

$$\begin{array}{ccc} D' = \{(g, P) \in C \times \mathscr{P} \mid \pi_P(g) \in C_P\} & \xrightarrow{f_1} & C \\ \downarrow{\scriptstyle f_2} & & \\ D = \{(\bar{g}, P) \mid P \in \mathscr{P}, \bar{g} \in C_P\} & & \end{array}$$

where $f_1(g, P) = g$, $f_2(g, P) = (\pi_P(g), P)$. The group G acts naturally on all three varieties in the diagram and its action is transitive on C and D. Let $d_1 = (\nu_G - \frac{1}{2}\dim C) - (\bar{\nu} - \frac{1}{2}\bar{c})$, $d_2 = \frac{1}{2}(\dim(C) - \bar{c})$. Then all fibres of f_1 have dimension $\leqq d_1$ and all fibres of f_2 have dimension $\leqq d_2$ (see 1.2(a), (b)). Moreover some (or all) fibres of f_1 have dimension equal to d_1 if and only if some (or all) fibres of f_2 have dimension equal to d_2. (Both conditions are equivalent to the condition that the space D' has dimension equal to $d_0 = (\nu_G - \bar{\nu}) + \frac{1}{2}(\dim C + \bar{c})$.) Now let $\mathscr{E}^{\cdot}$ be an irreducible G-equivariant local system on C and $\mathscr{E}^{\cdot\prime\prime}$ an irreducible G-equivariant local system on D. We shall need the following result.

(6.1.1) *The multiplicity of* $\mathscr{E}^{\cdot}$ *in the* G*-equivariant local system* $\mathscr{A}_1 = R^{2d_1}(f_1)_!(f_2^* \mathscr{E}^{\cdot\prime\prime})$ *on* C *is equal to the multiplicity of* $\mathscr{E}^{\cdot\prime\prime}$ *in the* G*-equivariant local system* $\mathscr{A}_2 = R^{2d_2}(f_2)_!(f_1^* \mathscr{E}^{\cdot})$ *on* D.

It is enough to show that both multiplicities are equal to the dimension of the space of homomorphisms between the local systems $f_1^* \mathscr{E}^{\cdot}$, $f_2^* \mathscr{E}^{\cdot\prime\prime}$ restricted to D'_0, where D'_0 is an open smooth G-invariant subset of D' which meets exactly those irreducible components of D' which have dimension d_0. We may assume that D'_0 is nonempty. Let $D'_{0,i}$, $1 \leqq i \leqq n$, be its connected components. Choose a point $x'_i \in D'_{0,i}$ for each i and let x_i denote $f_1(x'_i)$. The local system $\mathscr{E}^{\cdot}$ on C corresponds to an irreducible representation γ of the fundamental group $\pi_1(C)$ at x_i, which factors through a finite quotient. The local system $f_2^* \mathscr{E}^{\cdot\prime\prime}$ on D'_0 corresponds to a representation δ_i of the fundamental group $\pi_1(D'_{0,i})$ at x'_i, which factors through a finite quotient, for each i.

Consider the natural homomorphism $h_i: \pi_1(D'_{0,i}) \to \pi_1(C)$ induced by f_1. Its kernel may be identified with the image in $\pi_i(D'_{0,i})$ of the fundamental group at x'_i of the fibre $D'_{0,i} \cap f_1^{-1}(x_i)$ and $\pi_1(C)/\mathrm{im}(h_i)$ may be identified with the set of connected components of this fibre. Let $h_i^* \gamma$ be the representation of $\pi_1(D'_{0,i})$ obtained by composing h_i and γ. Let $h_{i,!}\delta_i$ be the representation of $\pi_1(C)$ obtained by taking the coinvariants of δ_i with respect to the action of $\ker h_i$, regarding it as a representation of $\pi_1(D'_{0,i})/\ker h_i \cong \mathrm{im}(h_i)$ and then inducing it from $\mathrm{im}(h_i)$ to $\pi_1(C)$. Then $h_i^* \gamma$ corresponds to the local system $f_1^* \mathscr{E}^{\cdot}$ on $D'_{0,i}$ and it factors through a finite quotient; the direct sum of the $h_{i,!}\delta_i$ (over i) corresponds to the local system $\mathscr{A}_1$ over C and it also factors through a finite quotient.

By the Frobenius reciprocity formula for representations of finite groups, the multiplicity of γ in the $\pi_1(C)$-module $h_{i,!}\delta_i$ is equal to the dimension of the space of $\pi_1(D'_{0,i})$-homomorphisms between $h_i^* \gamma$ and δ_i. It follows that the multiplicity of $\mathscr{E}^{\cdot}$ in the local system $\mathscr{A}_1$ is equal to the dimension of the space of homomorphisms between the local systems $f_1^* \mathscr{E}^{\cdot}$, $f_2^* \mathscr{E}^{\cdot\prime\prime}$ on D'_0. An entirely similar argument shows that the multiplicity of $\mathscr{E}^{\cdot\prime\prime}$ in the local system $\mathscr{A}_2$ on D is equal to the dimension of the space of homomorphisms between the local systems $f_2^* \mathscr{E}^{\cdot\prime\prime}, f_1^* \mathscr{E}^{\cdot}$ on D'_0, and hence it is equal to the dimension of the space of homomorphisms between the local systems $f_1^* \mathscr{E}^{\cdot}, f_2^* \mathscr{E}^{\cdot\prime\prime}$ on D'_0 (since these local systems are semisimple). This completes the proof of (6.1.1).

6.2. Let $\mathscr{N}$ (or $\mathscr{N}_G$) be the set of all pairs $(C, \mathscr{E}^{\cdot})$ where C is a unipotent class in G and $\mathscr{E}^{\cdot}$ is an irreducible G-equivariant local system on C.

Let $\mathcal{N}^{(0)}$ (or $\mathcal{N}_G^{(0)}$) be the subset of $\mathcal{N}$ consisting of those $(C, \mathscr{E}^\bullet)$ for which $(\mathscr{Z}_G^0 \cdot C, 1 \boxtimes \mathscr{E}^\bullet)$ is a cuspidal pair (see 2.4); here $1 \boxtimes \mathscr{E}^\bullet$ is the inverse image of $\mathscr{E}^\bullet$ under $\mathscr{Z}_G^0 \cdot C \to C$.

From the definition of a cuspidal pair (2.4) we see that a pair $(C, \mathscr{E}^\bullet) \in \mathcal{N}$ is in $\mathcal{N}^{(0)}$ if and only for any parabolic subgroup $P \neq G$ and any unipotent class $C' \subset P/U_P$, the following condition is satisfied:

$$H_c^{\dim(C)-\dim(C')}(\pi_P^{-1}(\bar{g}) \cap C, \mathscr{E}^\bullet) = 0, \qquad \text{for all } \bar{g} \in C'.$$

Now let $(C, \mathscr{E}^\bullet)$ be an arbitrary element of $\mathcal{N}$. We can find a parabolic subgroup $P \subset G$ with the following two properties:

(a) For some unipotent element $\bar{g} \in P/U_P$ we have

$$H_c^{\dim(C)-\dim(C_1)}(\pi_P^{-1}(\bar{g}) \cap C, \mathscr{E}^\bullet) \neq 0,$$

(here C_1 is the conjugacy class of $\bar{g}$ in P/U_P).

(b) P is a minimal parabolic subgroup satisfying (a). (Indeed, property (a) is always satisfied for $P = G$, $\bar{g} \in C$.)

Choose C_1 and $\bar{g} \in C_1$ such that (a) above is satisfied, and let $\mathscr{E}_1^\bullet$ be an irreducible P/U_P-equivariant local system on C_1 which is a direct summand of the local system $R^{\dim(C)-\dim(C_1)}(f_2)_!(\mathscr{E}^\bullet)$ on C_1,

$$(f_2 \colon \pi_P^{-1}(C_1) \cap C \to C_1, f(g) = \pi_P(g)).$$

Choose a Levi subgroup L of P and identify L with P/U_P via π_P; we then regard C_1 as a unipotent class in L and $\mathscr{E}_1^\bullet$ as an L-equivalent local system on C_1.

6.3. Proposition. *The triple $(L, C_1, \mathscr{E}_1^\bullet)$ above the uniquely determined (up to G-conjugacy) by $(C, \mathscr{E})$. Moreover, $(C_1, \mathscr{E}_1^\bullet)$ is necessarily in $\mathcal{N}_L^{(0)}$.*

Proof. If $(C_1, \mathscr{E}_1^\bullet)$ is not in $\mathcal{N}_L^{(0)}$, then there would exist a parabolic subgroup $\tilde{P}$ of G such that $\tilde{P} \subsetneqq P$ and a unipotent element $\tilde{g} \in \tilde{P}/U_{\tilde{P}}$ such that $H_c^{\dim(C_1)-\dim(C_2)}(\pi_{\tilde{P}}^{-1}(\tilde{g}) \cap C_1, \mathscr{E}_1^\bullet) \neq 0$, where C_2 is the conjugacy class of $\tilde{g}$ in $\tilde{P}/U_{\tilde{P}}$. It would follow that

$$H_c^{\dim(C_1)-\dim(C_2)}(\pi_{\tilde{P}}^{-1}(\tilde{g}) \cap C_1, R^{\dim(C)-\dim(C_1)}(f_2)_!(\mathscr{E}^\bullet)) \neq 0.$$

Using the Leray spectral sequence for the map $f_2 \colon \pi_{\tilde{P}}^{-1}(\tilde{g}) \cap C \to \pi_{\tilde{P}}^{-1}(\tilde{g}) \cap C_1$ (the restriction of π_P), whose fibres have dimension $\leq \frac{1}{2}(\dim(C) - \dim(C_1))$, it follows that $H_c^{\dim(C)-\dim(C_2)}(\pi_{\tilde{P}}^{-1}(\tilde{g}) \cap C, \mathscr{E}^\bullet) \neq 0$ so that $\tilde{P}$ satisfies (a) in 6.2, contradicting the minimality of P. We have, therefore, proved that $(C_1, \mathscr{E}_1^\bullet)$ is in $\mathcal{N}_L^{(0)}$.

Let $S_1 = \mathscr{Z}_L^0 \cdot C_1$, $\mathscr{E}_1 = 1 \boxtimes \mathscr{E}_1^\bullet$ = inverse image of $\mathscr{E}_1^\bullet$ under $\mathscr{Z}_L^0 \cdot C_1 \to C_1$. Let $Y = Y_{(L,S_1)}$, (see 3.1) and let $\phi \colon X \to \bar{Y}$ (see 4.2) be defined with respect to P. From (6.1.1) and the definition of $(C_1, \mathscr{E}_1^\bullet)$ it follows that $\mathscr{E}^\bullet$ is a direct summand of $R^{2d} f_!(\bar{\mathscr{E}}_1)$ restricted to C, where $f \colon X_{\alpha_0} = \{(g, xP) \in \bar{Y} \times G/P \mid x^{-1} g x \in S_1 U_P\} \to \bar{Y}$ is the projection on the g-coordinate, $\bar{\mathscr{E}}_1$ is defined as in 4.4 and $d = (\nu_G - \frac{1}{2}\dim C) - (\nu_L - \frac{1}{2}\dim C_1)$.

Now let P' be another parabolic subgroup of G satisfying (a), (b) in 6.2 for $(C, \mathscr{E}^{\bullet})$, let L' be a Levi subgroup of P' and let $C'_1 \subset L'$ and $\mathscr{E}_1^{\bullet\prime}$ (on C'_1) be chosen in the same way as $(C_1, \mathscr{E}_1^{\bullet})$ was chosen starting from P. We must show that $(L'_1, C'_1, \mathscr{E}_1^{\bullet\prime})$ is G-conjugate to $(L, C_1, \mathscr{E}_1^{\bullet})$.

Just as before, we see that $\mathscr{E}^{\bullet}$ is a direct summand of the local system $R^{2d} f'_!(\bar{\mathscr{E}}'_1)|C$ where f', d', $S'_1, \mathscr{E}'_1$, $\bar{\mathscr{E}}'_1$, are defined just as f, d, S_1, $\mathscr{E}_1$, $\bar{\mathscr{E}}_1$, using P', L', C'_1, $\mathscr{E}_1^{\bullet\prime}$ instead of P, L, C_1, $\mathscr{E}_1^{\bullet}$. It follows that the local system $R^{2d} f_!(\bar{\mathscr{E}}_1) \otimes R^{2d'} f'_!(\bar{\mathscr{E}}_1'^{*})|C$ (* stands for dual) contains the constant sheaf $\bar{\mathbb{Q}}_l$ on C as a direct summand, and hence

$$H_c^{2\dim(C)}(C, R^{2d} f_!(\bar{\mathscr{E}}_1) \otimes R^{2d'} f'_!(\bar{\mathscr{E}}_1'^{*})|C) \neq 0. \tag{6.3.1}$$

From the Leray spectral sequence of the map

$$Z_{\alpha_0, \alpha'_0, C} = \{(g, xP, x'P') \in C \times (G/P) \times (G/P') | x^{-1} g x \in S_1 U_P, x'^{-1} g x' \in S'_1 U_{P'}\} \to C$$

(projection to g-coordinate) all of whose fibres have dimension $\leq d + d'$ and from (6.3.1) it follows that

$$H_c^{2d_0}(Z_{\alpha_0, \alpha'_0, C}, \bar{\mathscr{E}}_1 \boxtimes \bar{\mathscr{E}}_1'^{*}) \neq 0. \tag{6.3.2}$$

where $d_0 = d + d' + \dim C$. Now let $Z^{ao}_{\alpha_0, \alpha'_0}$ be defined in the same way as $Z_{\alpha_0, \alpha'_0, C}$ except that now g is allowed to be any unipotent element of G, (see 5.2). Then, when C varies, the $Z_{\alpha_0, \alpha'_0, C}$ form a partition of $Z^{ao}_{\alpha_0, \alpha'_0}$ into locally closed pieces of dimension $\leq d_0$ (5.3(b)); this, together with (6.3.2) shows that

$$H_c^{2d_0}(Z^{ao}_{\alpha_0, \alpha'_0}, \bar{\mathscr{E}}_1 \boxtimes \bar{\mathscr{E}}_1'^{*}) \neq 0.$$

This means that $\mathscr{T}_{a_0} \neq 0$ where $\mathscr{T}$ is defined in 5.4 (in terms of $(S_1, \mathscr{E}_1)$, $(S'_1, \mathscr{E}'_1)$) and $\mathscr{T}_{a_0}$ is defined in 5.13. From 5.13(a), it now follows that $(L, S_1, \mathscr{E}_1)$ and $(L', S'_1, \mathscr{E}'_1)$ are G-conjugate. Hence $(L, C_1, \mathscr{E}_1^{\bullet})$, $(L', C'_1, \mathscr{E}_1^{\bullet\prime})$ are G-conjugate and the proposition is proved.

6.4. The previous proposition shows that the construction in 6.2 gives a well defined map $\Phi: (C, \mathscr{E}^{\bullet}) \mapsto (L, C_1, \mathscr{E}_1^{\bullet})$ from the set $\mathscr{N}_G$ (see 6.2) to the set $\mathscr{M}_G$ consisting of all triples $(L, C_1, \mathscr{E}_1^{\bullet})$ (up to G-conjugacy) where L is a Levi subgroup of a parabolic subgroup of G, C_1 is a unipotent conjugacy class of L and $\mathscr{E}_1^{\bullet}$ is an irreducible L-equivariant local system on C_1 such that $(C_1, \mathscr{E}_1^{\bullet}) \in \mathscr{N}_L^0$. As we have seen in the proof of 6.3, if $(C, \mathscr{E}^{\bullet})$ is mapped by Φ to $(L, C_1, \mathscr{E}_1^{\bullet})$, then (with the notations in that proof), C must be contained in $\bar{Y} = \bar{Y}_{(L, S_1)}$ and $\mathscr{E}^{\bullet}$ must be a direct summand of $R^{2d} f_!(\bar{\mathscr{E}}_1)|C$, ($f, \bar{\mathscr{E}}_1$ are defined in terms of some parabolic subgroup $P \subset G$ having L as a Levi subgroup). Conversely, if C is a unipotent class of G contained in $\bar{Y}$ and $\mathscr{E}^{\bullet}$ is an irreducible G-equivariant local system on C, which is a direct summand of $R^{2d} f_!(\bar{\mathscr{E}}_1)|C$, $d = (\nu_G - \frac{1}{2}\dim C) - (\nu_L - \frac{1}{2}\dim C_1)$ then, from (6.1.1) we see that $\pi_P^{-1}(C_1) \cap C$ is non-empty and that $\mathscr{E}_1^{\bullet}$ is a direct summand of $R^{\dim(C) - \dim(C_1)}(f_2)_!(\mathscr{E}^{\bullet})$ where $f_2: \pi_P^{-1}(C_1) \cap C \to C_1$ is defined by $\pi_P: P \to L$. In particular, P satisfies condition (a) of 6.2 with respect to $(C, \mathscr{E}^{\bullet})$ and $\bar{g} \in C_1$. If P contained strictly a parabolic subgroup $\tilde{P}$ which still satisfies condition (a) of 6.2, we would get (by reversing an argument in the proof of 6.3) a contradiction with the fact that $(C_1, \mathscr{E}_1^{\cdot}) \in \mathscr{N}_L^{(0)}$. It follows that $(L, C_1, \mathscr{E}_1^{\cdot}) = \Phi(C, \mathscr{E}^{\cdot})$.

We can now state the main result of this chapter; it generalizes results of Springer [20] and Borho-MacPherson [2].

6.5. **Theorem.** *Let* $(L, C_1, \mathscr{E}_1^{\cdot}) \in \mathscr{M}_G$, *let* $(S_1, \mathscr{E}_1) = (\mathscr{Z}_L^0 \cdot C_1, 1 \boxtimes \mathscr{E}_1^{\cdot})$ *be the corresponding cuspidal pair for L, let P be a parabolic subgroup of G having L as a Levi subgroup, let* $\phi_! K$ *be the corresponding complex on* $\bar{Y} = \bar{Y}_{(L, S_1)}$.

(a) *For* $(C, \mathscr{E}^{\cdot}) \in \mathscr{N}$, *we have* $\Phi(C, \mathscr{E}^{\cdot}) = (L, C_1, \mathscr{E}_1^{\cdot})$ *if and only if* $C \subset \bar{Y}$ *and* $\mathscr{E}^{\cdot}$ *is a direct summand of* $R^{2d_C} f_!(\bar{\mathscr{E}}_1)|C$, *where f is the restriction of* ϕ *to* $X_{\alpha_0} \subset X$, $\bar{\mathscr{E}}_1$ *is defined as in* 4.4, *and* $d_C = (\nu_G - \frac{1}{2}\dim(C)) - (\nu_L - \frac{1}{2}\dim(C_1))$.

(b) *With notations in* (a), *the natural homomorphism*

$$R^{2d_C} f_!(\bar{\mathscr{E}}_1)|C \to \mathscr{H}^{2d_C}(\phi_!(K)|C$$

(given by the imbedding of X_{α_0} *into X as an open subset) is an isomorphism.*

(c) *For any* $\rho \in \mathscr{A}_{\mathscr{E}_1}^{\vee}$, *let* $(\phi_! K)_\rho$ *be defined by* $\phi_! K = \bigoplus_{\rho \in \mathscr{A}_{\mathscr{E}_1}^{\vee}} \rho \otimes (\phi_! K)_\rho$, *(see* 3.7). *Let* $\bar{Y}^{a_0}$ *be the variety of unipotent elements in* $\bar{Y}$. *There is a unique* $(C, \mathscr{E}^{\cdot}) \in \mathscr{N}$ *with the following property:* $C \subset \bar{Y}$ *and the restriction of* $\phi_! K$ *to* $\bar{Y}^{a_0}$ *is isomorphic to the complex* $IC(\bar{C}, \mathscr{E}^{\cdot})$ *shifted by* $2d_C$ *and extended by* 0 *on* $\bar{Y}^{a_0} - \bar{C}$. *In particular,* $\mathscr{E}^{\cdot} = \mathscr{H}^{2d_C}((\phi_! K)_\rho)|C$. *The map* $\rho \mapsto (C, \mathscr{E}^{\cdot})$ *just defined is a bijection between the set* $\mathscr{A}_{\mathscr{E}_1}^{\vee}$ *and the set*

$$\{(C, \mathscr{E}^{\cdot}) \in \mathscr{N} \mid \Phi(C, \mathscr{E}^{\cdot}) = (L, C_1, \mathscr{E}_1^{\cdot})\}.$$

Part (a) of the Theorem is already contained in 6.4. The rest of this chapter will be devoted to the proof of parts (b) and (c).

6.6. In this section, the notations are those of §4 and §5. For any $a \in A_Y$, let $\bar{Y}^a = \bar{Y} \cap \sigma^{-1}(a)$, $X^a = \varphi^{-1}(\bar{Y}^a) \subset X$, φ^a = restriction of φ to X^a; thus, $\varphi^a\colon X^a \to \bar{Y}^a$. Let $S_1^a = S_1 \cap \sigma^{-1}(a)$, $\bar{S}_1^a = \bar{S}_1 \cap \sigma^{-1}(a)$, $\alpha^a = \alpha \cap \sigma^{-1}(a)$, (for a stratum α of $\bar{S}_1$, see 4.4), $\mathscr{E}_1^a$ = restriction of $\mathscr{E}_1$ to S_1^a, $\bar{\mathscr{E}}_1^a$ = restriction of $\bar{\mathscr{E}}_1$ to $X_{\alpha_0}^a = X^a \cap X_{\alpha_0}$. Let $\hat{S}_1$ be the set of semisimple parts of elements in $\bar{S}_1$, and let $\hat{S}_1^a = \hat{S}_1 \cap \sigma^{-1}(a)$. Note that $\hat{S}_1$ is a smooth, closed subvariety of L, and $\hat{S}_1^a$ is a smooth subvariety of $\hat{S}_1$. Note also that $\bar{S}_1^a$ is the inverse image of $\hat{S}_1^a$ under the natural locally trivial fibration $\bar{S}_1 \to \hat{S}_1$. It follows that $IC(\bar{S}_1^a, \mathscr{E}_1^a)$ is the restriction of $IC(\bar{S}_1, \mathscr{E}_1)$ to $\bar{S}_1^a$. It follows also that the restriction of $K = IC(X, \bar{\mathscr{E}}_1)$ to X^a is $K^a = IC(X^a, \bar{\mathscr{E}}_1^a)$, and that the restriction of $\varphi_! K$ to $\bar{Y}^a$ is $(\varphi^a)_! K^a$. The following result was noticed by Borho-MacPherson [2], in a special case.

(6.6.1) $\varphi_! K|\bar{Y}^a = (\varphi^a)_! K^a$ *is (up to a shift by* $\dim \bar{Y}^a$ *degrees) a pure perverse sheaf on* $\bar{Y}^a$ *(in the sense of* [1]).

Since φ^a is proper, and $K^a = IC(X^a, \bar{\mathscr{E}}_1^a)$, it suffices to prove the following assertion.

(6.6.2) For any $i \geqq 0$, we have $\dim \operatorname{supp} \mathscr{H}^i((\varphi^a)_! K^a) \leqq \dim \bar{Y}^a - i$,

and also the analogous assertion in which K^a is replaced by $K^{*a} = IC(X, \bar{\mathscr{E}}_1^{*a})$; note also that $\bar{Y}^a$ has pure dimension given by 7.2(a). We shall only prove the statement (6.6.2) for K^a; the proof for K^{*a} is the same. The proof is similar to that of (4.5.1). With notations in that proof, we see that it is enough to prove:

(6.6.3) For any $i \geqq 0$ and any stratum α, we have

$$\dim\left\{g \in \bar{Y}^a \mid \dim \varphi^{-1}(g)_a \geqq \frac{i}{2} - \frac{\dim S_1 - \dim \alpha}{2}\right\} \leqq \dim \bar{Y}^a - i.$$

If this is violated for some $i \geqq 0$, it would follow that the set of triples

$$\{(g, xP, x'P) \in \bar{Y}^a \times (G/P) \times (G/P) | x^{-1} g x \in \alpha U_P, x'^{-1} g x' \in \alpha U_P\}$$

has dimension $> \dim \bar{Y}^a - (\dim S_1 - \dim \alpha)$. This set of triples is contained in the set of triples

$$\{(g, xP, x'P) \in G \times (G/P) \times (G/P) | x^{-1} g x \in \alpha^a U_P, x'^{-1} g x' \in \alpha^a U_P\}.$$

But α^a is a union of finitely many L-conjugacy classes in α (each of dimension equal to $\dim(\alpha/\mathscr{Z}_L^0)$), hence, by using twice 1.2(b), we see that the last space of triples has dimension $\leqq 2\nu_G - 2\nu_L + \dim(\alpha/\mathscr{Z}_L^0) = \dim \bar{Y}^a - (\dim S_1 - \dim \alpha)$, (see 7.2(a)). Thus, (6.6.3), and hence also (6.6.1), are proved.

Returning to the setup of 6.5, we now take $a = a_0$; in this case, $\bar{Y}^{a_0}$ is irreducible, (see 7.2(b)). From (6.6.1) and the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber [1] it now follows, just as in [2], that $\varphi_! K | \bar{Y}^{a_0}$ is a direct sum of complexes of the form $K(\mathscr{E}^{\cdot}) = IC(\bar{C}, \mathscr{E}^{\cdot})[2d_C]$ (extended by 0 on $Y^a - C$), where $(C, \mathscr{E}^{\cdot}) \in \mathscr{N}$ are such that $C \subset \bar{Y}^{a_0}$. (Note that $2d_C = \dim Y^a - \dim C$; $[2d_C]$ stands for a shift of a complex by $2d_C$ degrees, as in [1].) We have $\dim(\bar{Y}^{a_0}) = d_0 = 2\nu_G - 2\nu_L + \dim C_1$, (see 7.2(a)).

6.7. **Lemma.** *Given $(C, \mathscr{E}^{\cdot})$ and $(C', \mathscr{E}^{\cdot\prime})$ in $\mathscr{N}$ with $C \subset \bar{Y}^{a_0}$, $C' \subset \bar{Y}^{a_0}$, we have*

$$\dim H_c^{2d_0}(\bar{Y}^{a_0}, K(\mathscr{E}^{\cdot}) \otimes K(\mathscr{E}^{\cdot\prime})) = \begin{cases} 1, & \text{if } C = C' \text{ and } \mathscr{E}^{\cdot *} \approx \mathscr{E}^{\cdot\prime} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let C'' be a unipotent class contained in $\bar{C} \cap \bar{C}$! We first show that

(6.7.1) $$H_c^{2d_0}(C'', K(\mathscr{E}^{\cdot}) \otimes K(\mathscr{E}^{\cdot\prime}) = 0 \quad \text{if} \quad C'' \neq C \quad \text{or} \quad C'' \neq C'.$$

From the hypercohomology spectral sequence, we see that it is enough to prove:

(6.7.2) $$H_c^i(C'', \mathscr{H}^j(K(\mathscr{E}^{\cdot})) \otimes \mathscr{H}^{j'}(K(\mathscr{E}^{\cdot\prime}))) \neq 0 \Rightarrow i + j + j' < 2d_0$$

The hypothesis of (6.7.2) implies: $i \leqq 2\dim(C'')$, $j \leqq \dim(C) - \dim(C'') + 2d_C$, $j' \leqq \dim(C') - \dim(C'') + 2d_{C'}$, and at least one of the last two inequalities is strict, since $C'' \neq C$ or $C'' \neq C'$. It follows that $i + j + j' < \dim(C) + \dim(C') + 2d_C + 2d_{C'} = 2d_0$. Thus, (6.7.2) and hence (6.7.1) are proved. From (6.7.1) it follows immediately that $H_c^{2d_0}(\bigcup C'', K(\mathscr{E}^{\cdot}) \otimes K(\mathscr{E}^{\cdot\prime})) = 0$, where the union is taken over all $C'' \subset \bar{C} \cap \bar{C}'$ such that $C'' \neq C$ or $C'' \neq C'$. Hence, the conclusion of the lemma holds if $C \neq C'$; in the case where $C = C'$, it follows that

$$H_c^{2d_0}(\bar{Y}^{a_0}, K(\mathscr{E}^{\cdot}) \otimes K(\mathscr{E}^{\cdot\prime}) \approx H_c^{2d_0}(C, K(\mathscr{E}^{\cdot}) \otimes K(\mathscr{E}^{\cdot\prime})).$$

The last space is $H_c^{2\dim C}(C, \mathscr{E}^{\cdot} \otimes \mathscr{E}^{\cdot\prime})$ and this is one dimensional if $\mathscr{E}^{\cdot *} \approx \mathscr{E}^{\cdot\prime}$ and is zero otherwise. The lemma is proved.

6.8. For $(C, \mathscr{E}^{\cdot}) \in \mathscr{N}$ such that $C \subset \bar{Y}$, we denote $n_{\mathscr{E}^{\cdot}}$ the multiplicity with which $K(\mathscr{E}^{\cdot})$ appears as a summand of $\varphi_! K | \bar{Y}^{a_0}$, (see 6.6). This is the same as the multiplicity with which $K(\mathscr{E}^{\cdot *})$ appears as a summand of $\varphi_! K^* | \bar{Y}^a$, where $K^* = IC(X, \bar{\mathscr{E}}_1^*)$. From 6.7 it follows immediately that

$$(6.8.1) \qquad \dim H_c^{2d_0}(\bar{Y}^{a_0}, (\varphi_! K | \bar{Y}^{a_0}) \otimes (\varphi_! K^* | \bar{Y}^{a_0})) = \sum_{(C, \mathscr{E}^{\cdot})} n_{\mathscr{E}^{\cdot}}^2.$$

The left hand side of (6.8.1) is just $\dim \tilde{\mathscr{T}}_{a_0}$, (see 5.13) and hence it is equal to $\dim \mathscr{A}_{\mathscr{E}_1}$ (see *loc. cit.*); the right hand side of (6.8.1) is equal to $\dim \operatorname{End}(\varphi_! K | \bar{Y}^{a_0})$. It follows that

$$(6.8.2) \qquad \dim \operatorname{End}(\varphi_! K | \bar{Y}^{a_0}) = \dim \mathscr{A}_{\mathscr{E}_1}.$$

For each $\rho \in \mathscr{A}_{\mathscr{E}_1}^{\vee}$, the restriction of $(\varphi_! K)_\rho$ to $\bar{Y}^{a_0}$ is non-zero, since even the ρ-isotypic part of $\mathscr{T}_{a_0} = H^{2d_0}(\bar{Y}^{a_0}, (\varphi_! K | \bar{Y}^{a_0}) \otimes (\varphi_! K^* | \bar{Y}^{a_0}))$ is non-zero (see 5.13). Since $(\varphi_! K)_\rho | \bar{Y}^{a_0}$ is a direct summand of $\varphi_! K | \bar{Y}^{a_0}$, it follows that it is a direct sum of complexes $K(\mathscr{E}^{\cdot})$, with at least one summand. It follows that the natural map

$$(6.8.3) \qquad \operatorname{End}(\varphi_! K) \to \operatorname{End}(\varphi_! K | \bar{Y}^{a_0})$$

is injective, and due to (4.1.1) and (6.8.2), it is in fact an isomorphism. This implies that the restriction $(\varphi_! K)_\rho | \bar{Y}^{a_0}$ is of the form $K(\mathscr{E}^{\cdot})$, for a well defined $(C, \mathscr{E}^{\cdot}) \in \mathscr{N}$, with $C \subset \bar{Y}$ and this gives a $1-1$ correspondence between $\mathscr{A}_{\mathscr{E}_1}^{\vee}$ and the set of pairs $(C, \mathscr{E}^{\cdot})$ such that $K(\mathscr{E}^{\cdot})$ is a summand of $\varphi_! K | \bar{Y}^{a_0}$.

6.9. *Proof of* 6.5(b), (c). We first show that the map in 6.5(b) is surjective. For this, it is enough to show that, for any $g \in C$, we have $H_c^{2d_C}(\varphi^{-1}(g) - \varphi^{-1}(g)_{\alpha_0}, K) = 0$, and this would follow from the equality $H_c^{2d_C}(\varphi^{-1}(g)_\alpha, K) = 0$, for any stratum $\alpha \neq \alpha_0$ of $\bar{S}_1$, (with notations in 4.5). The hypercohomology spectral sequence reduces us to proving: $H_c^i(\varphi^{-1}(g)_\alpha, \mathscr{H}^j(K)) \neq 0 \Rightarrow i + j < 2d_C$. If the last group is non-zero, then we must have $i \leqq 2 \dim \varphi^{-1}(g)_\alpha \leqq 2d_C - (\dim S_1 - \dim \alpha)$, (by 1.2(b)), and $j < \dim X - \dim X_\alpha = \dim S_1 - \dim \alpha$, (since $\alpha \neq \alpha_0$); it follows that $i + j < 2d_C$, as desired. Thus, the map in 6.5(b) is surjective. If $\mathscr{E}^{\cdot}$ is an irreducible, G-equivariant local system on C, we denote by $m_{\mathscr{E}^{\cdot}}$ its multiplicity in $R^{2d_C} f_!(\bar{\mathscr{E}}_1) | C$ and by $\tilde{m}_{\mathscr{E}^{\cdot}}$ its multiplicity in $\mathscr{H}^{2d_C}(\varphi_! K) | C$. The proof so far shows that $m_{\mathscr{E}^{\cdot}} \geqq \tilde{m}_{\mathscr{E}^{\cdot}}$, It is easy to see that, with notations in 6.8, we have $\tilde{m}_{\mathscr{E}^{\cdot}} = n_{\mathscr{E}^{\cdot}}$; from the results in 6.8, it then follows that

$$(6.9.1) \qquad \sum_{(C, \mathscr{E}^{\cdot})} \tilde{m}_{\mathscr{E}^{\cdot}}^2 = \dim \mathscr{A}_{\mathscr{E}_1}.$$

We shall now prove that

$$(6.9.2) \qquad \sum_{(C, \mathscr{E}^{\cdot})} m_{\mathscr{E}^{\cdot}}^2 = \dim \mathscr{A}_{\mathscr{E}_1}.$$

From the definition of $m_{\mathscr{E}^{\cdot}}$, we see that

$$\dim H_c^{2\dim C}(C, R^{2d_C} f_!(\bar{\mathscr{E}}_1) \otimes R^{2d_C} f_!(\bar{\mathscr{E}}_1^*)) = \sum_{\mathscr{E}^{\cdot}} m_{\mathscr{E}^{\cdot}}^2.$$

(sum over all $\mathscr{E}^{\cdot}$ such that $(C,\mathscr{E}^{\cdot})\in\mathscr{N}$). Let $Z_{\alpha_0,\alpha_0,C}$ and $Z^{a_0}_{\alpha_0,\alpha_0}$ be defined just as in 6.3. From the Leray spectral sequence of the natural map $Z_{\alpha_0,\alpha_0,C}\to C$, all of whose fibres have dimension $\leqq 2d_c$, it follows that

$$H_c^{2d_0}(Z_{\alpha_0,\alpha_0,C},\bar{\mathscr{E}}_1\boxtimes\bar{\mathscr{E}}_1^*)\approx H_c^{2\dim C}(C,R^{2d_c}f_!(\bar{\mathscr{E}}_1)\otimes R^{2d_c}f_!(\bar{\mathscr{E}}_1^*)).$$

If we sum the dimensions of these vector spaces over all $C\subset\bar{Y}$, we find the dimension of $H_c^{2d_0}(Z^{a_0}_{\alpha_0,\alpha_0},\bar{\mathscr{E}}_1\boxtimes\bar{\mathscr{E}}_1^*)$; indeed, the $Z_{\alpha_0,\alpha_0,C}$ form a partition of $Z^{a_0}_{\alpha_0,\alpha_0}$ into locally closed pieces of dimension $\leqq d_0$. Hence, we have

$$\sum_{(C,\mathscr{E}^{\cdot})} m^2_{\mathscr{E}^{\cdot}}=\dim H^{2d_0}(Z^{a_0}_{\alpha_0,\alpha_0},\bar{\mathscr{E}}_1\boxtimes\bar{\mathscr{E}}_1^*)=\dim\mathscr{T}_{a_0}=\dim\mathscr{A}_{\mathscr{E}_1},$$

(see 5.13) and (6.9.2) is proved. From (6.9.1) and (6.9.2) and from the inequality $m_{\mathscr{E}^{\cdot}}\geqq\tilde{m}_{\mathscr{E}^{\cdot}}$, it follows that $m_{\mathscr{E}^{\cdot}}=\tilde{m}_{\mathscr{E}^{\cdot}}$, for all $(C,\mathscr{E}^{\cdot})$. This means that the surjective map in 6.5(b) must be an isomorphism.

From 6.5(a) it follows that $(C,\mathscr{E}^{\cdot})$ is in the set $\Phi^{-1}(L,C_1,\mathscr{E}_1^{\cdot})$ if and only if $m_{\mathscr{E}^{\cdot}}\neq 0$. As we have seen this is equivalent to the condition that $\tilde{m}_{\mathscr{E}^{\cdot}}=n_{\mathscr{E}^{\cdot}}\neq 0$, i.e. to the condition that $K(\mathscr{E}^{\cdot})$ is a direct summand of $\varphi_! K|\bar{Y}^{a_0}$. This completes the proof of Theorem 6.5.

§ 7. Induced unipotent classes

7.1. We again fix $S_1\subset L\subset P$ and $\mathscr{E}_1$ on S_1, with $(S_1,\mathscr{E}_1)$ a cuspidal pair for L. Let $\sigma\colon G\to A$ be the Steinberg map, see 5.1. For each $a\in A_Y$, we define $\bar{Y}^a=\bar{Y}\cap\sigma^{-1}(a)$, $X^a=\phi^{-1}(\bar{Y}^a)\subset X$.

7.2. **Proposition.** (a) *For any $a\in A_Y$, both X^a and $\bar{Y}^a$ have pure dimension $2\nu_G-2\nu_L+\dim(S_1/\mathscr{Z}_L^0)$.*

(b) *If S_1 contains some unipotent element, then A_Y contains a_0 = class of the identity element in G and X^{a_0}, $\bar{Y}^{a_0}$ are irreducible.*

Proof. (a) We have $X^a=\{(g,xP)\in G\times(G/P)|x^{-1}gx\in\bar{S}_1U_P\cap\sigma^{-1}(a)\}$. All fibres of the map $(g,xP)\mapsto xP\colon X^a\to G/P$ are isomorphic to $\bar{S}_1U_P\cap\sigma^{-1}(a)=(\bar{S}_1\cap\sigma^{-1}(a))U_P$, (see 5.1). Let $C_1,\dots,C_r$ be the L-conjugacy classes in $S_1\cap\sigma^{-1}(a)$. Then $\bar{S}_1\cap\sigma^{-1}(a)=\bar{C}_1\cup\dots\cup\bar{C}_r$. We have $\dim C_i=\dim(S_1/\mathscr{Z}_L^0)$ for $i=1,\dots,r$. It follows that X^a has pure dimension as stated. We know that $\phi(X)=\bar{Y}$; it follows that $\phi(X^a)=\bar{Y}^a$, hence $\dim\bar{Y}^a\leqq\dim X^a$. Assume that $\bar{Y}^a$ has some irreducible component D of dimension $\delta<\dim X^a$. Then there exists an open dense set D_0 in D such that $\dim\phi^{-1}(g)=\dim(X^a)-\delta$ for all $g\in D_0$. The fibre product $X^a\times_{\bar{Y}^a}X^a$ contains the fibre product $\phi^{-1}(D_0)\times_{D_0}\phi^{-1}(D_0)$ hence it has dimension $\geqq\dim D_0+2(\dim(X^a)-\delta)=2\dim(X^a)-\delta$. By 5.3(b), (with $L'=L,S_1'=S_1$) the fibre product $X^a\times_{\bar{Y}^a}X^a$ $(=Z^a)$ has dimension $\leqq 2\nu_G-2\nu_L+\dim(S_1/\mathscr{Z}_L^0)=\dim(X^a)$. It follows that $\dim(X^a)\geqq 2\dim(X^a)-\delta$ hence $\delta\geqq\dim(X^a)$. This contradicts $\delta<\dim(X^a)$ and proves (a).

(b) With the assumption in (b), S_1 contains a unique unipotent conjugacy class of L. Let us denote it C_1. We have $\bar{S}_1\cap\sigma^{-1}(a_0)=\bar{C}_1$ hence $X^{a_0}=\{(g,xP)\in G\times(G/P)|x^{-1}gx\in\bar{C}_1\cdot U_P\}$. This is clearly an irreducible variety (since $\bar{C}_1\cdot U_P$ and G/P are irreducible). Since $\bar{Y}^{a_0}=\phi(X^{a_0})$, $\bar{Y}^{a_0}$ must be also irreducible.

The following result is proved in [11]; unlike the proof there, the proof given below will not make use of the rather delicate "property A" of [11] for C_1.

7.3. **Corollary.** *Assume that S_1 contains the unipotent L-conjugacy class C_1 and that $\mathscr{E}_1|\mathscr{Z}_L^0$ is constant.*

(a) *Let C be the unique unipotent conjugacy class in G such that $C_1 U_P \cap C$ is dense in $C_1 U_P$. Then C is the unique unipotent class which is open, dense in $\bar{Y}^{a_0}$.*

(b) *G acts transitively on $\phi^{-1}(C)$.*

(c) *P acts transitively on $C_1 U_P \cap C$.*

(d) *Let $g \in C_1 U_P \cap C$ and let $\bar{g} = \pi_P(g) \in L$. The natural map $\gamma\colon Z_P(g)/Z_P^0(g) \to Z_G(g)/Z_G^0(g)$ is injective and the natural map*

$$Z_P(g)/Z_P^0(g) \xrightarrow{\gamma'} Z_L(\bar{g})/Z_L^0(\bar{g})$$

is surjective.

Proof. (a) We have $\overline{C_1 U_P} \subset \bar{C}$. As $\bar{Y}^{a_0}$ is the union of the G-conjugates of $\overline{C_1 U_P}$, it is contained in $\bar{C}$. Since $C \subset \bar{Y}^{a_0}$, we have $\bar{Y}^{a_0} = \bar{C}$. Hence C is open dense in $\bar{Y}^{a_0}$. The uniqueness of such C follows from the irreducibility of $\bar{Y}^{a_0}$, (7.2(b)).

(b) As we have seen in the proof of 7.2, the map $\phi\colon X^{a_0} \to \bar{Y}^{a_0}$ is surjective and $X^{a_0}, \bar{Y}^{a_0}$ are irreducible of the same dimension. It follows that all fibres of $\phi\colon \phi^{-1}(C) \to C$ are finite. Any G-orbit on $\phi^{-1}(C)$ maps onto C since G is transitive on C; hence any G-orbits on $\phi^{-1}(C)$ must have dimension equal to that of $\phi^{-1}(C)$ hence it is dense in $\phi^{-1}(C)$, as $\phi^{-1}(C)$ is irreducible. It follows that any two G-orbits on $\phi^{-1}(C)$ must intersect each other, so that there is only one G-orbit on $\phi^{-1}(C)$ and (b) follows.

(c) Let g, g' be two elements of $C_1 U_P \cap C$. Then $g' = x^{-1} g x$ for some $x \in G$. Since $g' \in C_1 U_P$, it follows that $(g, xP) \in \phi^{-1}(C)$. By (b), (g, xP) must be in the same G-orbit as $(g, P) \in \phi^{-1}(C)$. Hence there exists $y \in G$ such that $y^{-1} g y = g$, $yP = xP$. Then $y = xz$, $z \in P$ and $g = y^{-1} g y = z^{-1} x^{-1} g x z = z^{-1} g' z$. Thus, g, g' are conjugate under $z \in P$ and (c) is proved.

(d) The isotropy group (in G) of g is $Z_G(g)$; it must have the same dimension as the isotropy group (in G) of $(g, P) \in \phi^{-1}(C)$, which is $Z_P(g)$, since the G-orbit of g has the same dimension as the G-orbit of (g, P), (see b). From the equality $\dim Z_P(g) = \dim Z_G(g)$ it follows that $Z_P^0(g) = Z_G^0(g)$, hence γ is injective.

We now prove following [11, 1.5] that γ' is surjective. By (a), $g U_P \cap C$ is dense in $g U_P$ and from (c) it follows that $Z_L(\bar{g}) U_P$ acts transitively (by conjugation) on $g U_P \cap C$. Since $g U_P$ is irreducible it follows that $Z_L^0(\bar{g}) U_P$ must also act transitively on $g U_P \cap C$. Hence for any element $z \in Z_L(\bar{g})$ there exists $z_1 \cdot v \in Z_L^0(\bar{g}) \cdot U_P$ such that $z g z^{-1} = z_1 v g v^{-1} z_1^{-1}$ so that $v^{-1} z_1^{-1} z \in Z_P(g)$. Under the map γ', the coset of $v^{-1} z_1^{-1} z$ is mapped to the coset of $z_1^{-1} z$ which is the same as the coset of z. Thus, γ' is surjective.

7.4. **Corollary.** *With the notations of* 7.3, *we have $d_C = 0$ (d_C as in* 6.5*). The G-equivariant local system $\mathscr{H}^0(\phi_! K)|C$ corresponds to the representation of $Z_G(g)/Z_G^0(g)$ induced by the representation of $Z_P(g)/Z_P^0(g)$ obtained by composing*

γ' and the irreducible representation of $Z_L(\bar{g})/Z_L^0(\bar{g})$ corresponding to the L-equivariant local system $\mathscr{E}_1|C_1$ on C_1.

Proof. The equality $d_C=0$ follows from $\dim C = \dim \bar{Y}^{\alpha_0} = 2\nu_G - 2\nu_L + \dim C_1$ (see 7.2(a)). The map $\phi: \phi^{-1}(C)\to C$ can be identified with the natural map $G/Z_P(g)\to G/Z_G(g)$.

The restriction of K to $\phi^{-1}(C)\approx G/Z_P(g)$ is a G-equivariant local system; it is the same as the restriction of $\bar{\mathscr{E}}_1$ since $\phi^{-1}(C)\subset X_{\alpha_0}$, (see 4.4). It corresponds to a representation of $Z_P(g)/Z_P^0(g)$, hence it is determined by its restriction to $P/Z_P(g)\subset G/Z_P(g)$. That restriction is the inverse image under $P/Z_P(g)\to(L/Z_L(\bar{g})\approx C_1$ of the local system $\mathscr{E}_1|C_1$ on C_1. The Corollary follows from these remarks.

§ 8. Restriction to a parabolic subgroup

8.1. In [21, 4.4], Springer gave a description of the restriction of an irreducible representation of the Weyl group in G to a parabolic subgroup, in geometric terms, involving unipotent elements and Borel subgroups containing them; his proof was subject to certain restrictions on the characteristic of k. In [15, § 1], Shoji proved a closely related result. In [3, 3.1], Borho-MacPherson found another approach to Springer's result, but their formulation is less convenient for applications than Springer's, in that they formulate the answer in terms of an unknown intersection cohomology space. In this chapter, we shall give a proof of Springer's result valid in any characteristic and which applies also to the more general situation considered in § 6.

8.2. Let $P\subset P'$ be two parabolic subgroups of G with Levi subgroups L, L' respectively, such that $L\subset L'$. Let $C_1\subset L$ be a unipotent class and let $\mathscr{E}_1^{\boldsymbol{\cdot}}$ be an L-equivariant local system on C_1 such that $(C_1,\mathscr{E}_1^{\boldsymbol{\cdot}})\in\mathscr{N}_L^{(0)}$. Let $S_1=\mathscr{Z}_L^0 C_1$, and let $\mathscr{E}_1$ be the local system $1\boxtimes\mathscr{E}_1^{\boldsymbol{\cdot}}$ on S_1. Let $\mathscr{A}_{\mathscr{E}_1}$ be the algebra associated in 3.4 to $(L,S_1,\mathscr{E}_1)$ and G. Let $\mathscr{A}'_{\mathscr{E}_1}$ be the algebra defined in the same way but replacing G by L'. Then $\mathscr{A}'_{\mathscr{E}_1}$ is in a natural way a subalgebra of $\mathscr{A}_{\mathscr{E}_1}$. Let $(C',\mathscr{E}^{\boldsymbol{\cdot\prime}})\in\mathscr{N}_{L'}$ be an element in $\Phi^{-1}(L,C_1,\mathscr{E}_1^{\boldsymbol{\cdot}})$, ($\Phi$ defined with respect to L', see 6.4) and let ρ' be the corresponding irreducible representation of $\mathscr{A}'_{\mathscr{E}_1}$, (see 6.5(c)).

Let $(C,\mathscr{E}^{\boldsymbol{\cdot}})\in\mathscr{N}_G$. Define the integer $m_{\mathscr{E}^{\cdot\prime},\mathscr{E}^{\cdot}}$ to be the multiplicity of $\mathscr{E}^{\boldsymbol{\cdot\prime}}$ in the local system $R^{2d_{C,C'}}(f_{C,C'})(\mathscr{E}^{\boldsymbol{\cdot}})$ on C', where $f_{C,C'}:\pi_{P'}^{-1}(C')\cap C\to C'$ is the restriction of $\pi_{P'}: P'\to L'$ and $d_{C,C'}=\frac{1}{2}(\dim(C)-\dim(C'))$.

8.3. **Theorem.** (a) *If $m_{\mathscr{E}^{\cdot\prime},\mathscr{E}^{\cdot}}$ is non-zero, then $(C,\mathscr{E}^{\boldsymbol{\cdot}})\in\Phi^{-1}(L,C_1,\mathscr{E}_1^{\boldsymbol{\cdot}})$, ($\Phi$ defined with respect to G, see* 6.4).

(b) *Assume that $(C,\mathscr{E}^{\boldsymbol{\cdot}})\in\Phi^{-1}(L,C_1,\mathscr{E}_1^{\boldsymbol{\cdot}})$ and let ρ be the corresponding irreducible representation of $\mathscr{A}_{\mathscr{E}_1}$, (see* 6.5(c)). *Let $(\rho':\rho)$ be the multiplicity of ρ' in the restriction of ρ to $\mathscr{A}'_{\mathscr{E}_1}$. Then $m_{\mathscr{E}^{\cdot\prime},\mathscr{E}^{\cdot}}=(\rho':\rho)$.*

8.4. For the proof of 8.3 we shall consider the set $(S_1)_{\text{reg}}$ defined as in 3.1 (with respect to G); we have a commutative diagram

$$\pi\left(\begin{array}{ccc} \tilde{Y} & \xhookrightarrow{j_0} & X \\ \big\downarrow \leftarrow \pi' & & \big\downarrow \phi' \\ \tilde{Y}_1 & \xhookrightarrow{j_1} & X_1 \\ \big\downarrow \leftarrow \pi'' & & \big\downarrow \phi'' \\ Y & \xhookrightarrow{j_2} & Y \end{array}\right)\phi$$

where $Y=\bigcup_{x\in G} x(S_1)_{\mathrm{reg}}x^{-1}$, (see 3.1), $\bar{Y}=\bigcup_{x\in G} x(\bar{S}_1 U_P)x^{-1}$

$$Y'=\bigcup_{y\in L'} y(S_1)_{\mathrm{reg}}y^{-1}, \qquad \bar{Y}'=\bigcup_{y\in L'} y\bar{S}_1(U_P\cap L')y^{-1},$$
$$\tilde{Y}=\{(g,xL)\in G\times(G/L)\,|\,x^{-1}gx\in(S_1)_{\mathrm{reg}}\},$$
$$\tilde{Y}_1=\{(g,xL')\in G\times(G/L')\,|\,x^{-1}gx\in Y'\},$$
$$X=\{(g,xP)\in G\times(G/P)\,|\,x^{-1}gx\in\bar{S}_1 U_P\},$$
$$X_1=\{g,xP')\in G\times(G/P')\,|\,x^{-1}gx\in\bar{Y}'U_{P'}\}.$$

The map ϕ' is $(g,xP)\mapsto(g,xP')$, the map ϕ'' is $(g,xP')\mapsto g$ and $\phi=\phi''\phi'$. The map π' is $(g,xL)\mapsto(g,xL')$, the map π'' is $(g,xL')\mapsto g$, and $\pi=\pi''\pi'$. The map j_0 is $(g,xL)\mapsto(g,xP)$, the map j_1 is $(g,xL')\mapsto(g,xP')$ and the map j_2 is $g\mapsto g$; the maps j_0,j_1,j_2 define imbeddings of $\tilde{Y}$, $\tilde{Y}_1$, Y as open smooth dense subvarieties of X, X_1, $\bar{Y}$ respectively.

The map π: $\tilde{Y}\to Y$ is a finite covering and a principal bundle with group $\mathscr{W}_{S_1}$ $=$stabilizer of S_1 in $N_G(L)/L$, see 3.2. The map π': $\tilde{Y}\to\tilde{Y}_1$ is a finite covering and a principal covering with group $\mathscr{W}'_{S_1}=$stabilizer of S_1 in $N_{L'}(L)/L$.

Let $\tilde{\mathscr{E}}_1$ be the G-equivariant local system on $\tilde{Y}$ defined in 3.2 in terms of $S_1,\mathscr{E}_1$. Then we have

(8.4.1) $$\mathscr{A}'_{\mathscr{E}_1}\cong\mathrm{End}\,\pi'_*(\tilde{\mathscr{E}}_1).$$

This can be seen as follows. In the diagram

$$\begin{array}{l} \hat{Y}_1=\{(g,x)\in G\times G\,|\,x^{-1}gx\in Y'\}\cong\{(g,y')\in G\times Y'\}\xrightarrow{pr_2} Y' \\ \big\downarrow \beta \\ \tilde{Y}_1 \end{array}$$

$(\beta(g,x)=(g,xL'))$, the local system $\beta^*\pi'_*(\tilde{\mathscr{E}}_1)$ on $\hat{Y}_1$ is the inverse image under pr_2^* of an L'-equivariant local system $\mathscr{F}_1$ on Y'; $\mathscr{F}_1$ is the restriction of $\pi'_*(\tilde{\mathscr{E}}_1)$ to Y', identified with a subset of $\tilde{Y}_1$, via $y'\mapsto(y',L')$. By 3.5 applied to L' instead of G, we have $\mathscr{A}'_{\mathscr{E}_1}\cong\mathrm{End}(\mathscr{F}_1)$. As $\tilde{Y}_1$ is the space of L'-orbits on $\hat{Y}_1$ for a free action of L' on $\hat{Y}_1$, we have $\mathrm{End}(\mathscr{F}_1)=\mathrm{End}\,pr_2^*(\mathscr{F}_1)=\mathrm{End}(\pi'_*(\tilde{\mathscr{E}}_1))$ and (8.4.1) follows. (Compare with the proof of 3.5.)

It follows that we have a canonical decomposition

(8.4.2) $$\pi'_*(\tilde{\mathscr{E}}_1) = \bigoplus_{\rho'_1 \in \mathscr{A}'^{\vee}_{\mathscr{E}_1}} (\rho'_1 \otimes \pi'_*(\tilde{\mathscr{E}}_1)_{\rho'_1})$$

where $\pi'_*(\tilde{\mathscr{E}}_1)_{\rho'_1}$ are irreducible, G-equivariant, local systems on $\tilde{Y}_1$.

By 3.5, 3.7, we have $\mathscr{A}_{\mathscr{E}_1} \cong \mathrm{End}(\pi_*(\tilde{\mathscr{E}}_1))$ and a canonical decomposition

$$\pi_*(\tilde{\mathscr{E}}_1) = \bigoplus_{\rho_1 \in \mathscr{A}^{\vee}_{\mathscr{E}_1}} (\rho_1 \otimes \pi_*(\tilde{\mathscr{E}}_1)_{\rho_1}).$$

Since taking direct image of a local system under a finite unramified covering is essentially the same as taking an induced representation of the fundamental group we see that for any $\rho'_1 \in \mathscr{A}'^{\vee}_{\mathscr{E}_1}$ we have

(8.4.3) $$\pi''_*(\pi'_*(\tilde{\mathscr{E}}_1)_{\rho'_1}) = \bigoplus_{\rho_1 \in \mathscr{A}^{\vee}_{\mathscr{E}_1}} (\pi_*(\tilde{\mathscr{E}}_1)_{\rho_1} \otimes \bar{\mathbb{Q}}_l^{(\rho'_1 : \rho_1)}).$$

We now consider the complex of sheaves $K = IC(X, \bar{\mathscr{E}}_1)$ on X, defined in 4.4. (Recall that $\bar{\mathscr{E}}_1$ is a local system on $\{(g, xP) \in G \times (G/P) \mid x^{-1} g x \in S_1 U_P\} \subset X$.) From 4.1 and 4.5 we see that

$$\phi_!(K) = \bigoplus_{\rho_1 \in \mathscr{A}^{\vee}_{\mathscr{E}_1}} (\rho_1 \otimes \phi_!(K)_{\rho_1}) \quad \text{and} \quad \phi_!(K)_{\rho_1} = IC(\bar{Y}, \pi^*(\tilde{\mathscr{E}}_1)_{\rho_1}).$$

From 4.5 applied to L' instead of G, we see that

(8.4.4) $$\phi'_!(K) \mid \bar{Y}' = IC(\bar{Y}', \mathscr{F}_1)$$

and from this it follows that

(8.4.5) $$\phi'_!(K) = IC(X_1, \pi'_*(\tilde{\mathscr{E}}_1)).$$

(Note that X_1 has the same singularities as $\bar{Y}'$, in the following sense: X_1 is the space of P'-orbits for the free P'-action on $\hat{X}_1 = \{(g, x) \in G \times G \mid x^{-1} g x \in \bar{Y}' U_{P'}\}$, $(g, x) \mapsto (g, xp')$ and we have $\hat{X}_1 \approx \bar{Y}' \times U_{P'} \times G$.)

From (8.4.1) it now follows that

$$\mathrm{End}(\phi'_!(K)) \cong \mathscr{A}'_{\mathscr{E}_1} \quad \text{and} \quad \phi'_!(K) = \bigoplus_{\rho'_1 \in \mathscr{A}'^{\vee}_{\mathscr{E}_1}} (\rho'_1 \otimes \phi'_!(K)_{\rho'_1})$$

where

(8.4.6) $$\phi'_!(K)_{\rho'_1} = IC(X_1, \pi'_*(\tilde{\mathscr{E}}_1)_{\rho'_1}).$$

Next, we show that

(8.4.7) $$\phi''_!(\phi'_!(K)_{\rho'_1}) = IC(\bar{Y}, \pi''_*(\pi'_*(\tilde{\mathscr{E}}_1)_{\rho'_1})),$$

for any $\rho'_1 \in \mathscr{A}'^{\vee}_{\mathscr{E}_1}$.

From (8.4.6) we see that the restriction of $\phi''_!(\phi'_!(K)_{\rho'_1})$ to Y is the local system $\pi''_*(\pi'_*(\tilde{\mathscr{E}}_1)_{\rho'})$. Since ϕ'' is proper, (8.4.7) is a consequence of (8.4.6), of the assertion (8.4.8) below and of the analogous assertion with K replaced by $K^* = IC(X, \bar{\mathscr{E}}_1^*)$:

(8.4.8) For any $i>0$, we have $\dim \operatorname{supp} \mathscr{H}^i(\phi''_!(\phi'_!(K)_{\rho'_1}))<\dim(\bar{Y})-i$.

This is checked as follows. We have $\operatorname{supp} \mathscr{H}^i(\phi''_!(\phi'_!(K))\rho'_1) \subset \operatorname{supp} \mathscr{H}^i(\phi''_!(\phi'_!(K)))=\operatorname{supp} \mathscr{H}^i(\phi_!(K))$ hence (8.4.8) is a consequence of (4.5.1). Thus, (8.4.7) is verified. Combining (8.4.7) with (8.4.3) we see that, for any $\rho'_1 \in \mathscr{A}'^{\vee}_{\mathscr{E}_1}$, we have

$$\phi''_!(\phi'_!(K)_{\rho'_1}) \cong \bigoplus_{\rho_1 \in \mathscr{A}^{\vee}_{\mathscr{E}_1}} (\phi_!(K)_{\rho_1} \otimes \bar{\mathbb{Q}}_l^{(\rho'_1:\rho_1)}). \tag{8.4.9}$$

For $\rho_1 \in \mathscr{A}^{\vee}_{\mathscr{E}_1}$, let $(C_{\rho_1}, \mathscr{E}^{\bullet}_{\rho_1})$ be the element of $\Phi^{-1}(L, C_1, \mathscr{E}^{\bullet}_1) \subset \mathscr{N}_G$ corresponding to ρ_1, by 6.5. From 6.5, it follows that $\mathscr{H}^{2d(C_{\rho_1}, C_1)}(\phi_!(K)_{\tilde{\rho}_1})|C_{\rho_1}$ is $\mathscr{E}^{\bullet}_{\rho_1}$ if $\tilde{\rho}_1=\rho_1$ and is 0 if $\tilde{\rho}_1 \neq \rho_1$. (Here $d(C_{\rho_1}, C_1)=(\nu_G-\frac{1}{2}\dim C_{\rho_1})-(\nu_L-\frac{1}{2}\dim C_1)$.) Hence, from (8.4.9), it follows that

(8.4.10) $(\rho'_1:\rho_1)=$ multiplicity of $\mathscr{E}^{\bullet}_{\rho_1}$ in the local system

$\mathscr{H}^{2d(C_{\rho_1}, C_1)}(\phi''_!(\phi'_!(K)_{\rho'_1}))|C_{\rho_1}$.

Now let $(C'_{\rho'_1}, \mathscr{E}^{\bullet}_{\rho'_1})$ be the element of $\Phi^{-1}(L, C_1, \mathscr{E}^{\bullet}_1) \subset \mathscr{N}_{L'}$, ($\Phi$ defined with respect to L') corresponding to ρ'_1, by 6.5. From 6.5 it follows that

$$\operatorname{supp} \phi'_!(K)_{\rho'_1} \subset D=\{(g, xP') \in G \times (G/P') \mid x^{-1}gx \in \bar{C}'_{\rho'_1} U_{P'}\}.$$

Hence

(8.4.11) $(\rho'_1:\rho_1)=$ multiplicity of $\mathscr{E}^{\bullet}_{\rho_1}$ in the local system

$$\mathscr{H}^{2d(C_{\rho_1}, C_1)}((\phi''|D)_!(\phi'_!(K)_{\rho'_1}|D))|C_{\rho_1}.$$

Let $D^{\mathrm{op}}=\{(g, xP') \in G \times (G/P') \mid x^{-1}gx \in C'_{\rho'_1} U_{P'}\}$, (an open subset of D). We now show that

(8.4.12) the natural map

$$\mathscr{H}^{2d(C_{\rho_1}, C_1)}(\phi''|D^{\mathrm{op}})_!(\phi'_!(K)_{\rho'_1}|D^{\mathrm{op}}))|C_{\rho_1}$$
$$\to \mathscr{H}^{2d(C_{\rho_1}, C_1)}((\phi''|D)_!\,\phi'_!(K)_{\rho'_1}|D))|C_{\rho_1}$$

is surjective.

This will imply that

$$(\rho'_1:\rho_1) \leqq x_{\rho'_1, \rho_1} \tag{8.4.13}$$

where

(8.4.14) $x_{\rho'_1, \rho_1}=$ multiplicity of $\mathscr{E}^{\bullet}_{\rho_1}$ in the local system

$$\mathscr{H}^{2d(C_{\rho_1}, C_1)}((\phi''|D^{\mathrm{op}})_!(\phi'_!(K)_{\rho'_1}|D^{\mathrm{op}}))|C_{\rho_1}.$$

To prove (8.4.12) it is clearly enough to prove that

$$\mathscr{H}^{2d(C_{\rho_1}, C_1)}((\phi''|D-D^{\mathrm{op}})_!(\phi'_!(K)_{\rho'_1}|D-D^{\mathrm{op}}))|C_{\rho_1}=0,$$

or, equivalently, that for any $g \in C_{\rho_1}$, we have

$$H_c^{2d(C_{\rho_1}, C_1)}(\phi''^{-1}(g) \cap (D-D^{\mathrm{op}}), \phi'_!(K)_{\rho'_1})=0.$$

Since $D-D^{\mathrm{op}}$ can be partitioned into the locally closed pieces

(8.4.15) $$D_{\tilde{C}'}=\{(g,xP')\in G\times(G/P')\mid x^{-1}gx\in\tilde{C}'U_{P'}\},$$

for the various L'-unipotent classes $\tilde{C}'\subset\bar{C}'_{\rho'_1}-C'_{\rho'_1}$, we see that it is enough to show that for every such $\tilde{C}'$, we have

$$H_c^{2d(C_{\rho_1},C_1)}(\phi''^{-1}(g)\cap D_{\tilde{C}'},\phi'_!(K)_{\rho'_1})=0.$$

The hypercohomology spectral sequence shows that the last equality is a consequence of the following statement:

$$H_c^i(\phi''^{-1}(g)\cap D_{\tilde{C}'},\mathcal{H}^j(\phi'_!(K)_{\rho'_1}))\neq 0\Rightarrow i+j<2d(C_{\rho_1},C_1).$$

The hypothesis of this statement implies $i\leqq 2\dim(\phi''^{-1}(g)\cap D_{\tilde{C}'})\leqq(2\nu_G-\dim C_{\rho_1})-(2\nu_{L'}-\dim\tilde{C}')$, (see 1.2(b)) and $\mathcal{H}^j(\phi'_!(K)_{\rho'_1})|D_{\tilde{C}'}\neq 0$, hence $j<(2\nu_{L'}-\dim\tilde{C}')-(2\nu_L-\dim C_1)$, (by 6.5); it follows that $i+j<2d(C_{\rho_1},C_1)$, as desired.

Thus, (8.4.12) and hence (8.4.13), are proved.

We want to show that

(8.4.16) $$(\rho'_1:\rho_1)=x_{\rho'_1,\rho_1}.$$

In view of (8.4.13) and the equality $\sum_{\rho'_1}\dim(\rho'_1)(\rho'_1:\rho_1)=\dim(\rho_1)$, it is enough to prove the equality

(8.4.17) $$\sum_{\rho'_1}\dim(\rho'_1)x_{\rho'_1,\rho_1}=\dim(\rho_1).$$

We first compute the sum $\tilde{x}_{\tilde{C}'}=\sum\dim(\rho'_1)x_{\rho'_1,\rho_1}$, summation over all $\rho'_1\in\mathcal{A}'^{\vee}_{\mathcal{E}_1}$ such that $C'_{\rho'_1}$ is a fixed L'-unipotent class $\tilde{C}'\subset Y'$. From (8.4.14), and 6.5, we see that, if we set $\delta=(\nu_{L'}-\frac{1}{2}\dim C'_{\rho'_1})-(\nu_L-\frac{1}{2}\dim C_1)$, we have

(8.4.18) $\tilde{x}_{\tilde{C}'}=$ multiplicity of $\mathcal{E}^{\bullet}_{\rho_1}$ in the local system

$$\mathcal{H}^{2d(C_{\rho_1},C_1)-2\delta}((\phi''|D_{\tilde{C}'})_!(\mathcal{H}^{2\delta}(\phi'_!(K))|D_{\tilde{C}'}))|C_{\rho_1}.$$

(Note that the definition of $D_{\tilde{C}'}$ in (8.4.15) makes sense for any unipotent class $\tilde{C}'\subset L'$.) From 6.5(b) it follows that

$$\mathcal{H}^{2\delta}(\phi'_!(K))|D_{\tilde{C}'}\approx R^{2\delta}(\phi'|X_{\alpha_0})_!(\bar{\mathcal{E}}_1)|D_{\tilde{C}'}$$

where $X_{\alpha_0}=\{(g,xP)\in G\times(G/P)\mid x^{-1}gx\in S_1U_P\}$, hence (8.4.18) becomes

(8.4.19) $\tilde{x}_{\tilde{C}'}=$ multiplicity of $\mathcal{E}^{\bullet}_{\rho_1}$ in the local system

$$\mathcal{H}^{2d(C_{\rho_1},C_1)-2\delta}((\phi''|D_{\tilde{C}'})_!(R^{2\delta}(\phi'|X_{\alpha_0})_!(\bar{\mathcal{E}}_1)|D_{\tilde{C}'}))|C_{\rho_1}.$$

By the spectral sequence of the composition $\phi''\circ\phi'$, the last local system is the same as the local system

$$\mathcal{H}^{2d(C_{\rho_1},C_1)}((\phi''\circ\phi'|X_{\alpha_0}^{\tilde{C}'})_!(\bar{\mathcal{E}}_1))|C_{\rho_1},$$

where $X_{\alpha_0}^{\tilde{C}'}=\{(g,xP)\in X_{\alpha_0}\mid\phi'(g,xP)\in D_{\tilde{C}'}\}$.

The $X_{\alpha_0}^{\tilde{C}'}$ form a partition of X_{α_0} into locally closed pieces of dimension $\leqq d(C_{\rho_1}, C_1)$. It follows that $\sum_{\tilde{C}'} \tilde{x}_{\tilde{C}'}$ = multiplicity of $\mathscr{E}^{\bullet}_{\rho_1}$ in the local system

$$\mathscr{H}^{2d(C_{\rho_1}, C_1)}((\phi \mid X_{\alpha_0})_!(\bar{\mathscr{E}}_1)) \mid C_{\rho_1}.$$

By 6.5(b), the last local system is the same as $\mathscr{H}^{2d(C_{\rho_1}, C_1)}(\phi_! K) \mid C_{\rho_1}$, and from 6.5, it follows that this local system contains $\mathscr{E}^{\bullet}_{\rho_1}$ with multiplicity $\dim(\rho_1)$. Thus, we have $\sum_{\tilde{C}'} \tilde{x}_{\tilde{C}'} = \dim(\rho_1)$.

Thus, (8.4.17) and hence (8.4.16) are proved.

8.5. We can now prove Theorem 8.3. With the notations of that theorem, we consider the variety $V = \{(g, xP') \in C \times (G/P') \mid x^{-1} g x \in C' U_{P'}\}$. This is a subvariety of X_1 since C' must be contained in $\bar{Y}'$. The L'-equivariant local system $\mathscr{E}^{\bullet\prime}$ on C' gives rise to a G-equivariant local system $\bar{\mathscr{E}}^{\bullet\prime}$ on V as follows. We pull back $\mathscr{E}^{\bullet\prime}$ to $\hat{V} = \{(g, x) \in C \times G \mid x^{-1} g x \in C' U_{P'}\}$ by the map $(g, x) \mapsto C'$-component of $x^{-1} g x$; we obtain a $G \times L'$-equivariant local system on $\hat{V}$ which must be the inverse image under $\hat{V} \to V$: $(g, x) \mapsto (g, xP')$ of a well-defined G-equivariant local system $\bar{\mathscr{E}}^{\bullet\prime}$ on V.

Let f: $V \to C$ be the projection on the g-coordinate. According to (6.1.1),

(8.5.1) $m_{\mathscr{E}^{\bullet\prime}, \mathscr{E}^{\bullet}}$ is equal to the multiplicity of $\mathscr{E}^{\bullet}$ in the local system $R^{2d} f_! \bar{\mathscr{E}}^{\bullet\prime}$ (on C), where $d = (\nu_G - \frac{1}{2}\dim C) - (\nu_{L'} - \frac{1}{2}\dim C')$. From the definition of ρ' (see 8.2), it follows that $\bar{\mathscr{E}}^{\bullet\prime}$ is the local system $\mathscr{H}^{2\delta}(\phi'_!(K)_{\rho'}) \mid V$, where $\delta = (\nu_{L'} - \frac{1}{2}\dim C') - (\nu_L - \frac{1}{2}\dim C_1)$, hence $m_{\mathscr{E}^{\bullet\prime}, \mathscr{E}^{\bullet}}$ is equal to the multiplicity of $\mathscr{E}^{\bullet}$ in the local system $R^{2d}(\phi'' \mid V)_!(\mathscr{H}^{2\delta}(\phi'_!(K)_{\rho'}) \mid V)$ on C.

With the assumption of 8.3(b) it now follows that $m_{\mathscr{E}^{\bullet\prime}, \mathscr{E}^{\bullet}} = x_{\rho', \rho}$, (see (8.4.14)) and, by (8.4.16), the conclusion of 8.3(b) follows.

With the assumption of 8.3(a), ($m_{\mathscr{E}^{\bullet\prime}, \mathscr{E}^{\bullet}} \neq 0$), it follows that $\mathscr{E}^{\bullet}$ appears with $\neq 0$ multiplicity in the local system $R^{2d}(\phi'' \mid V)_!(R^{2\delta}(\phi' \mid X_{\alpha_0}^{C'})_!(\bar{\mathscr{E}}_1) \mid V)$ on C, or equivalently, in the local system $R^{2d+2\delta}(\phi \mid X_{\alpha_0}^{C'})_!(\bar{\mathscr{E}}_1) \mid C$, (see 8.4). It also follows that $\mathscr{E}^{\bullet}$ appears with non-zero multiplicity in $R^{2d+2\delta}(\phi \mid X_{\alpha_0})_!(\bar{\mathscr{E}}_1) \mid C$. (Indeed, when C' varies, the $X_{\alpha_0}^{C'}$ form a partition of X_{α_0} into locally closed pieces whose intersections with $\phi^{-1}(g)$, $(g \in C)$ have dimension $\leqq d + \delta$.) From 6.4, it now follows that $(C, \mathscr{E}^{\bullet}) \in \Phi^{-1}(L, C_1, \mathscr{E}^{\bullet}_1)$ and 8.3(a) is proved.

§ 9. On the structure of the algebra $\mathscr{A}_{\mathscr{E}_1}^{\bullet}$

9.1. In this chapter we fix $(L, C_1, \mathscr{E}^{\bullet}_1) \in \mathscr{M}_G$ and a parabolic subgroup $P \subsetneqq G$ having L as a Levi subgroup. Let $S_1 = \mathscr{Z}_L^0 C_1$ and let $\mathscr{E}_1$ be the local system $1 \boxtimes \mathscr{E}^{\bullet}_1$ on S_1. Let ϕ: $X \to \bar{Y} = \bar{Y}_{(L, S_1)}$, $\phi_!(K)$ be defined in terms of L, S_1, $\mathscr{E}_1$, P as in 4.2, 4.4 and let $\mathscr{A}_{\mathscr{E}_1}$ be the algebra defined as in 3.4. Let $P_1, P_2, \ldots, P_r$ be the parabolic subgroups of G which are minimal with the property that they contain strictly P. Let $L^1, L^2, \ldots, L^r$ be the corresponding Levi subgroups of $P^1, P^2, \ldots, P^r$, containing L. We shall denote by C the unipotent class in G such that $C \cap C_1 U_P$ is dense in $C_1 U_P$ and by C^i the unipotent class in L^i such that $C^i \cap C_1 U_{P \cap L^i}$ is dense in $C_1 U_{P \cap L^i}$, $(1 \leqq i \leqq r)$. We shall denote by $\hat{C}$ the unipotent class in G which contains C_1 and by $\hat{C}^i$ the unipotent class in L^i which contains C_1.

With these notations we can state

9.2. **Theorem.** a) *$N_G(L)/L$ is a Coxeter group with simple reflections $s, s_2, \dots, s_r$, where s_i is the unique element of order 2 of $N_{L_i}(L)/L$.*

b) $\mathscr{W}_{\mathscr{E}_1} = N_G(L)/L$.

c) *The local system $\mathscr{H}^0(\phi_! K)|C$ is $\neq 0$, irreducible. Hence there is a unique $\rho \in \mathscr{A}_{\mathscr{E}_1}^\vee$ such that $\mathscr{H}^0(\phi_! K)|C = \mathscr{H}^0(\phi_! K)_\rho|C$; moreover,* $\dim(\rho) = 1$.

d) *There is a unique isomorphism of algebras $\mathscr{A}_{\mathscr{E}_1} \xrightarrow{\sim} \bar{\mathbb{Q}}_l[\mathscr{W}_{\mathscr{E}_1}]$ which maps $\mathscr{A}_{\mathscr{E}_1, w}$ onto $\bar{\mathbb{Q}}_l w$ ($\forall w \in \mathscr{W}_{\mathscr{E}_1}$) and is such that it makes ρ(in (c)) correspond to the unit representation of $\mathscr{W}_{\mathscr{E}_1}$.*

For the proof, we shall need the following

9.3. **Lemma.** *Let P be a proper parabolic subgroup of G and let L be a Levi subgroup of P. Let g be a unipotent element in L, and let $cl_G(g), cl_L(g)$ denote the conjugacy class of g in G and L respectively. Then $(\nu_G - \frac{1}{2}\dim cl_G(g)) - (\nu_L - \frac{1}{2}\dim cl_L(g)) = \dim Z_{U_P}(g) > 0$.*

Proof. Fix a maximal torus $T_1 \subset L$ and consider the subgroup $U_P^{(0)}$ of U_P generated by all root subgroups of U_P (with respect to T_1) corresponding to roots α with the following property: the sum of coefficients in α of the simple roots of G which are not roots of L is maximum possible. Then $U_P^{(0)}$ is isomorphic as a group to k^n, for some $n > 0$. It is normalized by L, and the action of L on it corresponds to a linear action of L on k^n. A unipotent linear transformation of k^n ($n > 0$) must have a fixed point set of dimension > 0. It follows that $\dim Z_{U_P^{(0)}}(g) > 0$ hence $\dim Z_{U_P}(g) > 0$. The equality in the lemma follows from (2.9.3), (2.9.4). The lemma is proved.

9.4. *Proof of Theorem* 9.2. We first show that

$$\text{(9.4.1)} \quad \mathscr{H}^{2d_{\hat{C}}}(\phi_! K)|\hat{C} \neq 0, \quad \text{where} \quad d_{\hat{C}} = (\nu_G - \tfrac{1}{2}\dim \hat{C}) - (\nu_L - \tfrac{1}{2}\dim C_1).$$

By 6.5, it is enough to show that $H_c^{2d_{\hat{C}}}(\phi^{-1}(g) \cap X_{\alpha_0}, \bar{\mathscr{E}}_1) \neq 0$ where $g \in C_1$ and $X_{\alpha_0}, \bar{\mathscr{E}}_1$ are as in 4.4. The variety $V = \{xP \in G/P | x \in Z_G^0(g)\}$ is contained in $\phi^{-1}(g) \cap X_{\alpha_0}$ (by $xP \mapsto (g, xP)$), and is non-empty, irreducible of dimension $d_{\hat{C}}$, (see 2.9(b)).

Hence to prove (9.4.1) it is enough to show that the restriction of $\mathscr{E}_1$ to V is constant. Consider the commutative diagram

$$\begin{array}{ccccc} Z_G^0(g) & \xrightarrow{\hat{i}} & \hat{X}_{\alpha_0} = \{(g', x) \in G \times G | x^{-1} g' x \in \mathscr{Z}_L^0\, C_1\, U_P\} & \xrightarrow{\gamma} & C_1 \\ \downarrow j & & \downarrow j' & & \\ V & \xrightarrow{i} & X_{\alpha_0} = \{(g', xP) \in G \times (G/P) | x^{-1} g' x \in \mathscr{Z}_L^0\, C_1\, U_P\}. & & \end{array}$$

Here $\hat{i}(x) = (g, x)$, $i(xP) = (g, xP)$, $j(x) = xP$, $j'(g', x) = (g', xP)$, $\gamma(g', x) = pr_{C_1}(x^{-1} g' x)$. By definition, we have $(j')^* \bar{\mathscr{E}}_1 = \gamma^* \mathscr{E}_1^\bullet$. Since $\gamma \hat{i}$ maps $Z_G^0(g)$ to a point, the local system $\hat{i}^* \gamma^*(\mathscr{E}_1^\bullet)$ on $Z_G^0(g)$ is constant. This is the same as the local system $j^* i^* \bar{\mathscr{E}}_1$. Now V is the orbit space $Z_G^0(g)/Z_G^0(g) \cap P$; by 2.9(c), the group $Z_G^0(g) \cap P$ is connected. Since $j^* i^* \bar{\mathscr{E}}_1$ is constant, it follows that $i^* \bar{\mathscr{E}}_1$ is constant, i.e. the restriction of $\bar{\mathscr{E}}_1$ to V is constant and (9.4.1) is proved. From 9.3 it follows that $d_{\hat{C}}>0$. On the other hand, by 7.4, we have $d_C=(\nu_G-\frac{1}{2}\dim C)-(\nu_L-\frac{1}{2}\dim C_1)=0$. It follows that $C\neq\hat{C}$. By 7.4, we have $\mathscr{H}^0(\phi_! K)|C\neq 0$. This and (9.4.1) imply that the algebra $\mathscr{A}_{\mathscr{E}_1}$ has at least two non-isomorphic representations, (see 6.5 (c)). Hence

(9.4.2) $$|\mathscr{W}_{\mathscr{E}_1}|\geqq 2.$$

Applying this to L^i instead of G, we see that $\mathscr{W}_{\mathscr{E}_1}\cap(N_{L^i}(L)/L)$ has at least two elements. On the other hand, $N_{L_i}(L)/L$ has at most two elements (it acts faithfully on the one dimensional torus $\mathscr{Z}_L^0/\mathscr{Z}_{L^i}^0$). It follows that $N_{L^i}(L)/L$ has order 2 and it is contained in $\mathscr{W}_{\mathscr{E}_1}$, for any i, $i\leqq i\leqq r$. Hence the Coxeter graph of L is stable under the opposition involution of the Coxeter graph of L^i, $(1\leqq i\leqq r)$. This implies (see for example [7, 5.9]) that $N_G(L)/L$ is a Coxeter group with simple reflections $s_1,s_2,\dots,s_r$, where s_i is the non-trivial element of $N_{L^i}(L)/L$. Hence (a) holds. Since $s_i\in\mathscr{W}_{\mathscr{E}_1}$ and the s_i generate $N_G(L)/L$, it follows that $\mathscr{W}_{\mathscr{E}_1}=N_G(L)/L$, hence (b) holds.

We consider the 2-dimensional subalgebra $\mathscr{A}^i_{\mathscr{E}_1}$ of $\mathscr{A}_{\mathscr{E}_1}$ spanned by $\mathscr{A}_{\mathscr{E},1}$ and $\mathscr{A}_{\mathscr{E}_1,s_i}$. This algebra has at most two irreducible representations. Hence the set $\Phi^{-1}(L,C_1,\mathscr{E}_1^{\cdot})\subset\mathscr{N}_{L^i}$, ($\Phi$ defined with respect to L^i) has at most two elements. Note that $C^i\neq\hat{C}^i$. Indeed, from 9.3, applied to L^i instead of G, we see that $(\nu_{L^i}-\frac{1}{2}\dim\hat{C}^i)>(\nu_L-\frac{1}{2}\dim C_1)$. On the other hand, from 7.4 applied to L^i instead of G, we see that $\nu_{L^i}-\frac{1}{2}\dim\hat{C}^i=\nu_L-\frac{1}{2}\dim C_1$. It follows that $\dim\hat{C}^i<\dim C^i$ hence $C^i\neq\hat{C}^i$, as asserted.

Using (9.4.1) for L^i instead of G and $\hat{C}^i$ instead of $\hat{C}$ and 7.4 for L^i instead of G, it now follows that $\Phi^{-1}(L,C_1,\mathscr{E}_1^{\cdot})\subset\mathscr{N}_{L^i}$ has exactly two elements; one is supported by C_i, the other is supported by $\hat{C}^i$; they correspond by 6.5 to irreducible representation ρ^i, $\hat{\rho}^i$ of $\mathscr{A}^i_{\mathscr{E}_1}$.

The local system $\mathscr{H}^0(\phi_! K)|C$ is non-zero, by 7.4. Let $\rho\in\mathscr{A}^\vee_{\mathscr{E}_1}$ be such that $\mathscr{H}^0(\phi_! K)_\rho|C\neq 0$. We now show that the restriction of ρ to the subalgebra $\mathscr{A}^i_{\mathscr{E}_1}$ does not contain the representation $\hat{\rho}^i$ of that subalgebra. By 8.3 and (8.5.1) it is enough to show that $(\nu_G-\frac{1}{2}\dim C)-(\nu_{L^i}-\frac{1}{2}\dim\hat{C}^i)<0$. From 7.4 we see that $\nu_G-\frac{1}{2}\dim C=\nu_L-\frac{1}{2}\dim C_1=\nu_{L^i}-\frac{1}{2}\dim C^i$, and it remains to use the inequality $\dim\hat{C}^i<\dim C^i$ which has been already noted. It follows that the restriction of ρ to $\mathscr{A}^i_{\mathscr{E}_1}$ is a direct sum of copies of the representation ρ^i. Hence, if b_{s_i} is a basis element of $\mathscr{A}_{\mathscr{E}_1,s_i}$, then b_{s_i} acts on ρ as a scalar times the identity. Since the b_{s_i} $(1\leqq i\leqq r)$ generate $\mathscr{A}_{\mathscr{E}_1}$ as an algebra, it follows that any element of $\mathscr{A}_{\mathscr{E}_1}$ acts on ρ as a scalar times the identity. Since ρ is irreducible, it must be one-dimensional, and it is uniquely determined by the property that $\rho|\mathscr{A}^i_{\mathscr{E}_1}=\rho^i$. Hence, (c) is proved.

Using the one dimensional $\mathscr{A}_{\mathscr{E}_1}$-module ρ, we can define an isomorphism of $\mathscr{A}_{\mathscr{E}_1}$ with the group algebra of $\mathscr{W}_{\mathscr{E}_1}$ as follows. In each summand $\mathscr{A}_{\mathscr{E}_1,w}$, $(w\in\mathscr{W}_{\mathscr{E}_1}$ we choose as basis element b_w the unique element which acts as the identity map on the $\mathscr{A}_{\mathscr{E}_1}$-module ρ. It is clear that $b_w b_{w'}$ must be equal to $b_{ww'}$ so that the basis (b_w) provides the required isomorphism $\mathscr{A}_{\mathscr{E}_1}\xrightarrow{\sim}\bar{\mathbb{Q}}_l[\mathscr{W}_{\mathscr{E}_1}]$. This completes the proof of the theorem.

9.5. **Proposition.** *With the assumptions in* 9.1, *the local system* $\mathscr{H}^{2d\hat{c}}(\phi_! K)|\hat{C}$ *in* (9.4.1) *is* $\neq 0$, *irreducible and the unique* $\hat{\rho}\in\mathscr{A}_{\mathscr{E}_1}^{\vee}$ *such that* $\mathscr{H}^{2d\hat{c}}(\phi_! K)|\hat{C} = \mathscr{H}^{2d\hat{c}}(\phi_! K)_{\hat{\rho}}|\hat{C}$ *corresponds under* 9.2(d) *to the sign representation of* $\mathscr{W}_{\mathscr{E}_1} = N_G(L)/L$.

We first show that the proposition is a consequence of the following statement.

(9.5.1) For each i, $1\leqq i\leqq r$, we have $C^i U_{P^i}\cap C=\emptyset$.

If we assume (9.5.1) then the variety V in 8.5 (defined for $\hat{C}$, C^i, P_i instead of C, C', P') is empty; hence from (8.5.1) and 8.3 it follows that the restriction of $\hat{\rho}$ to $\mathscr{A}^i_{\mathscr{E}_1}$ does not contain ρ^i. (Here $\hat{\rho}$ is any irreducible representation of $\mathscr{A}_{\mathscr{E}_1}$ such that $\mathscr{H}^{2d\hat{c}}(\phi_! K)_{\hat{\rho}}|\hat{C}\neq 0$.) It follows that $\hat{\rho}|\mathscr{A}^i_{\mathscr{E}_1}$ is a direct sum of copies of the representation $\hat{\rho}^i$. If we identify $\mathscr{A}_{\mathscr{E}_1}$ with $\bar{\mathbb{Q}}_l[\mathscr{W}_{\mathscr{E}_1}]$ as in 9.2(d) then $\mathscr{A}^i_{\mathscr{E}_1}$ becomes the subalgebra of $\bar{\mathbb{Q}}_l[\mathscr{W}_{\mathscr{E}_1}]$ spanned by 1 and s_i and $\hat{\rho}^i$ is the one-dimensional representation of that subalgebra on which s_i acts as -1. Hence $s_i = -1$ on $\hat{\rho}$ for any i, $1\leqq i\leqq d$. Since $\hat{\rho}$ is irreducible, it follows that $\hat{\rho}$ must be the sign representation of $\mathscr{W}_{\mathscr{E}_1}$ and, in particular, it is uniquely determined and the proposition is proved (assuming (9.5.1).

We now show that (9.5.1) is a consequence of the following statement:

(9.5.2) Let P' be a parabolic subgroup of G with Levi subgroup L' and let $u\in L'$, $v\in U_{P'}$ be unipotent elements. Then $cl_G(u)\subset\overline{cl_G(u\,v)}$. (Here cl_G denotes conjugacy class in G).

We take $L'=L^i$, $P'=P^i$, $u\in C^i$, $v\in U_{P^i}$ and assume that $u\,v\in\hat{C}$. From (9.5.2) it follows that $\dim cl_G(u)\leqq\dim\hat{C}$ hence $\dim Z_G(u)\geqq\dim Z_G(g)$ where $g\in C_1$. From 9.3, we have $\dim Z_G(u)=\dim Z_{L^i}(u)+2\dim Z_{U_{P^i}}(u)$, and $\dim Z_G(g)=\dim Z_{L^i}(g) +2\dim Z_{U_{P^i}}(g)$. Moreover, from 9.3 (for L^i instead of G) we have $\dim Z_{L^i}(u) =\dim Z_L(g)<\dim Z_{L^i}(g)$. It follows that $\dim Z_{U_{P^i}}(u)>\dim Z_{U_{P^i}}(g)$. The last inequality is impossible for the following reason. Consider the action of L^i on U_{P^i}, by conjugation. The fixed point set of an element in C^i has constant dimension $(=\dim Z_{U_{P^i}}(u))$; hence the fixed point set of an element in the closure $\bar{C}^i$ must have dimension $\geqq\dim Z_{U_{P^i}}(u)$. Since $g\in\bar{C}^i$, we find a contradiction. Thus, we have proved that (9.5.2) implies (9.5.1).

It remains to prove (9.5.2). It is easy to see that the action of $\mathscr{Z}^0_{L'}$ on $U_{P'}$ (by conjugation) has the property that the closure of any $\mathscr{Z}^0_{L'}$-orbit contains the unit element. It follows that u belongs to the closure of the set $\{z\,u\,v\,z^{-1}|z\in\mathscr{Z}^0_{L'}\}$ and (9.5.2) follows. This completes the proof of the proposition.

§ 10. Examples in the classical groups

10.1. We wish to describe the set $\mathscr{M}_G$. (Recall (cf. 6.4) that $\mathscr{M}_G$ consists of all G-conjugacy classes of triples $(L, C_1, \mathscr{E}_1^{\bullet})$ where L is a Levi subgroup of a parabolic subgroup of G, C_1 is a unipotent conjugacy class of L and $\mathscr{E}_1^{\bullet}$ is an irreducible L-equivariant local system on C_1 such that $(C_1, \mathscr{E}_1^{\bullet})\in\mathscr{N}_L^{(0)}$.) Our description will be in the form of a list of elements of $\mathscr{M}_G$ and for each $m\in\mathscr{M}_G$

we will indicate the number of elements in $\Phi^{-1}(m)$, (see 6.4). For this, we can reduce ourselves to the case where G is almost simple and simply connected. Indeed, let $\pi: G\to G/\mathscr{Z}_G^0$ be the natural homomorphism. Then π induces a bijection between the sets $\mathscr{N}_G$ (resp. $\mathscr{N}_G^{(0)}, \mathscr{M}_G$) and the corresponding sets for $G/\mathscr{Z}_G^0$, which is compatible with the map Φ. Thus, we are reduced to the case where G is semisimple. In that case, let $\tilde{\pi}: \tilde{G}\to G$ be the simply connected covering of G. Then $\tilde{\pi}$ induces a bijection between the sets $\mathscr{N}_G$ (resp. $\mathscr{N}_G^{(0)}$, $\mathscr{M}_G$) and the subsets of the corresponding sets for $\tilde{G}$, defined by the condition that the kernel of $\tilde{\pi}$ acts trivially; this is again compatible with Φ. Thus, we are reduced to the case where G is semisimple and simply connected. In that case, there is a natural bijection between the sets $\mathscr{N}_G$ (resp. $\mathscr{N}_G^0, \mathscr{M}_G$) and the product of the corresponding sets for the various almost simple factors of G. Thus, we are reduced to the case where G is almost simple and simply connected.

10.2. We shall use the following method. We assume that the set $\mathscr{N}_L^{(0)}$ has been already determined for all Levi subgroups L of proper parabolic subgroups of G. Hence we can list all elements of $\mathscr{M}_G$ corresponding to $L\neq G$. For each such element $m=(L, C_1, \mathscr{E}_1^{\boldsymbol{\cdot}})$, we know from 9.2 that $\mathscr{W}_{\mathscr{E}_1}=N(L)/L$ and that $\mathscr{A}_{\mathscr{E}_1}\approx \bar{\mathbb{Q}}_l[\mathscr{W}_{\mathscr{E}_1}]$. By 6.5, the set $\Phi^{-1}(m)$ has then exactly as many elements as the set of irreducible representations of $N(L)/L$. Taking the sum over m of the integers $|\Phi^{-1}(m)|$, we get the number of elements in $\mathscr{N}_G-\mathscr{N}_G^{(0)}$. The number of elements in $\mathscr{N}_G$ is equal to the sum over a set of representatives g for the unipotent classes in G of the numbers of irreducible representations of the finite groups $Z_G(g)/Z_G^0(g)$. Hence the number of elements in $\mathscr{N}_G$ can be determined in each case, (see [18, 16, 12–14]). Therefore the numbers of elements in $\mathscr{N}_G^{(0)}$ can be determined as the difference between the number of elements in $\mathscr{N}_G$ and in $\mathscr{N}_G-\mathscr{N}_G^{(0)}$.

10.3. In this section we shall use the method in 10.2 in the case where $G=SL_n(k)$. The unipotent classes in G are completely described by the sizes of the Jordan blocks of their elements hence correspond to partitions of n. (Thus the unit element corresponds to the partition $1+1+\ldots+1$ and a regular unipotent element corresponds to the partition n.) If g is a unipotent element such that the corresponding partition is $\alpha_1+\alpha_2+\ldots+\alpha_t=n$, then $Z_G(g)/Z_G^0(g)$ is a cyclic group of order g.c.d. $(\alpha_1,\alpha_2,\ldots,\alpha_t,n')$ where n' is the part of n prime to the characteristic exponent of k. Hence

$$|\mathscr{N}|=\sum_{\substack{0\leqq\alpha_1\leqq\alpha_2\leqq\ldots\leqq\alpha_t\\ \alpha_1+\alpha_2+\ldots=n}} \text{g.c.d.}\,(\alpha_1,\alpha_2,\ldots,\alpha_t,n').$$

It is easy to see that the last expression can be rewritten as follows:

(10.3.1) $\quad |\mathscr{N}|=\sum\phi(\alpha)\,p(n/\alpha)\quad$ (sum over all divisors α of n')

where ϕ is the Euler function and $p(n/\alpha)$ is the number of partitions of n/α.

We shall prove by induction on n, that:

(10.3.2) *$\mathscr{N}^{(0)}$ (for $SL_n(k)$) consists of the pairs $(C,\mathscr{E})\in\mathscr{N}$ where C is the class of a regular unipotent element g and $\mathscr{E}$ corresponds to an irreducible representation $Z_G(g)/Z_G^0(g)=\mathscr{Z}_G\to\bar{\mathbb{Q}}_l^*$ whose image has exactly n elements.*

We may assume that $n \geqq 1$ and that the statement (10.3.2) is known for $\tilde{n} < n$. Let $(L, C_1, \mathscr{E}_1^{\bullet}) \in \mathscr{M}_G$. We may assume that

$$L = S(GL_{\alpha_1} \times GL_{\alpha_2} \times \dots \times GL_{\alpha_t}) \subset G, \qquad \alpha_1 + \alpha_2 + \dots + \alpha_t = n.$$

Assume also that $L \neq G$, i.e. that all α_i are strictly less than n. By the induction hypothesis, C_1 is the class of a regular unipotent element $g_1 \in L$ and $\mathscr{E}_1^{\bullet}$ corresponds to a representation $\theta: Z_L(g_1)/Z_L^0(g_1) \to \bar{\mathbb{Q}}_l^*$ with the following property: for each $i \in [1, t]$. the restriction of θ to the image of $\mathscr{Z}_{SL_{\alpha_i}(k)} \to Z_L(g_1)/Z_L^0(g_1)$ maps that image onto a subgroup of order α_i of $\bar{\mathbb{Q}}_l^*$. Now $Z_L(g_1)/Z_L^0(g_1)$ is a cyclic group of order g.c.d. $(\alpha_1, \alpha_2, \dots, \alpha_t, n') \leqq \alpha_i$. It follows that $\alpha_1 = \alpha_2 = \dots = \alpha_t = \alpha$ where α is a divisor of n' and that there are exactly $\phi(\alpha)$ possibilities for θ. It is clear that $N(L)/L$ is naturally isomorphic to the symmetric group $\mathfrak{S}_t$. Let $\mathscr{E}_1$ be the local system $1 \boxtimes \mathscr{E}_1^{\bullet}$ on $\mathscr{Z}_L^0 \cdot C_1$.

From 9.2, it follows that $\mathscr{A}_{\mathscr{E}_1} \approx \bar{\mathbb{Q}}_l[\mathfrak{S}_t]$, hence $\Phi^{-1}(L, C_1, \mathscr{E}_1^{\bullet})$ consists of $p(t) = p(n/\alpha)$ elements, (see 6.5).

It follows that

$$|\mathscr{N} - \mathscr{N}^{(0)}| = \sum \phi(\alpha)\, p(n/\alpha),$$

sum over all divisors α of n' such that $\alpha \neq n$. Comparing with (10.3.1) it follows that

$$(10.3.3) \qquad |\mathscr{N}^{(0)}| = \begin{cases} \phi(n), & \text{if } n \text{ is invertible in } k \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\mathscr{N}^{(0)}$ is empty, unless $|\mathscr{Z}_G| = n$. Assume now that $|\mathscr{Z}_G| = n$. Then $n' = n$.

Let us consider again the triple $(L, C_1, \mathscr{E}_1^{\bullet}) \in \mathscr{M}_G$ where L corresponds, as above, to the partition $\alpha_1 + \alpha_2 + \dots + \alpha_t = n$, and $\alpha_1 = \alpha_2 = \dots = \alpha_t = \alpha$ is a divisor of n, $\alpha \neq n$. The group $\mathscr{Z}_G \subset L$ acts on $\mathscr{E}_1^{\bullet}$ by scalar multiplication on each stalk via a character of order α. From the definition of Φ, it follows that, for any $(C, \mathscr{E}) \in \Phi^{-1}(L, C_1, \mathscr{E}_1^{\bullet})$, the group $\mathscr{Z}_G \subset G$ acts on $\mathscr{E}$ by scalar multiplication on each stalk via a character of order α. Hence, if $(C, \mathscr{E})$ is such that C is the class of a regular unipotent element g and $\mathscr{E}$ corresponds to a character of order n of $Z_G(g)/Z_G^0(g) = \mathscr{Z}_G$, then $(C, \mathscr{E})$ cannot be in $\Phi^{-1}(L, C_1, \mathscr{E}_1^{\bullet})$, and therefore $(C, \mathscr{E})$ is in $\mathscr{N}^{(0)}$. Moreover, (10.3.3) shows that all elements of $\mathscr{N}^{(0)}$ are obtained in this way. This completes the proof of (10.3.2). This proof shows also that two elements $(C, \mathscr{E})$, $(C', \mathscr{E}')$ of $\mathscr{N}$ are in the same fibre of Φ: $\mathscr{N} \to \mathscr{M}_G$ if and only if $\mathscr{Z}_G$ acts on $\mathscr{E}$ and on $\mathscr{E}'$ via the same character; thus, $\mathscr{M}_G$ is in $1-1$ correspondence with the set of characters of $\mathscr{Z}_G$.

10.4. In this section, we assume that $G = Sp_{2n}(k)$ and that $\operatorname{char}(k) \neq 2$. Let x_n be the number of elements in $\mathscr{N}$. In this case the unipotent classes in G are in $1-1$ correspondence with the partitions $2n = 1 \cdot i_1 + 2 \cdot i_2 + 3 \cdot i_3 + \dots$ where $i_1, i_2, i_3, \dots \geqq 0$ and $i_1, i_3, i_5, \dots$ are even. (i_a is the number of Jordan blocks of size a of a unipotent element.) The group of components of the centralizer of a unipotent element corresponding to such a partition is an elementary abelian

2-group of order $2^{\#\{a\,\mathrm{even}|i_a>0\}}$. It follows that

$$\begin{aligned}\sum_{n\geq 0} x_n t^n &= \sum_{\substack{i_1,i_2,\ldots\geq 0\\ i_1,i_3,i_5\ldots\,\mathrm{even}}} 2^{\#\{a\,\mathrm{even}|i_a>0\}} t^{(1i_1+2i_2+3i_3+\ldots)/2}\\
&=\Big(\sum_{i'_1\geq 0} t^{i'_1}\Big)\Big(\sum_{i'_3\geq 0} t^{3i'_3}\Big)\Big(\sum_{i'_5\geq 0} t^{5i'_5}\Big)\ldots\times\Big(1+\sum_{i'_2\geq 1} 2t^{i'_2}\Big)\Big(1+\sum_{i'_4\geq 1} 2t^{2i'_4}\Big)\ldots\\
&=(1-t)^{-1}(1-t^3)^{-1}(1-t^5)^{-1}\ldots\times\frac{1+t}{1-t}\cdot\frac{1+t^2}{1-t^2}\ldots\\
&=\frac{\prod_{i\geq 1}(1+t^i)}{\prod_{i\geq 1}(1-t^i)\prod_{i\geq 1}(1-t^{2i-1})}\\
&=\frac{1}{\prod_{i\geq 1}(1-t^i)^2}\cdot\frac{\prod_{i\geq 1}(1-t^{2i})}{\prod_{i\geq 1}(1-t^{2i-1})}\\
&=\frac{1}{\prod_{i\geq 1}(1-t^i)^2}\cdot\sum_{j\geq 0} t^{j(j+1)/2}\end{aligned}$$

the last step being an identity of Gauss [5, Th. 354], (t is an indeterminate). If we define $p_2(j)$ by the identity

$$\prod_{i\geq 1}(1-t^i)^{-2}=\sum_j p_2(j)\,t^j$$

it follows that

(10.4.1) $$x_n=\sum_{j\geq 0} p_2(n-(1+2+\ldots+j)).$$

We shall prove by induction on n that

(10.4.2) *$\mathcal{N}^{(0)}$ (for $Sp_{2n}(k)$) has exactly one element, if $2n=2+4+\ldots+2j$ for some $j\geq 0$ and is empty otherwise.*

We may assume that $n\geq 2$ and that the statement (10.4.2) is known for $\tilde{n}<n$. Let $(L, C_1, \mathscr{E}_1^{\bullet})\in\mathscr{M}_G$ be such that $L\neq G$. Then L must be a product $GL_{\alpha_1}(k)\times GL_{\alpha_2}(k)\times\ldots\times GL_{\alpha_t}(k)\times Sp_{2r}(k)$, $\alpha_1+\alpha_2+\ldots+\alpha_t+r=n$, $r<n$. It follows from (10.3.2) that each α_i must be equal to 1 and from the induction hypothesis that $r=j(j+1)$ for some $j\geq 1$ and that $(C_1, \mathscr{E}_1^{\bullet})$ is uniquely determined by L. The group $N(L)/L$ is easily seen to be a Coxeter group of type B_{n-r}. Hence, if $\mathscr{E}_1$ is the local system $\bar{1}\boxtimes\mathscr{E}_1^{\bullet}$ on $\mathscr{Z}_L^0\cdot C_1$, it follows from 9.2 that $\mathscr{A}_{\mathscr{E}_1}\approx\bar{\mathbb{Q}}_l[\mathscr{W}_{\mathscr{E}_1}]=\bar{\mathbb{Q}}_l(N(L)/L)$ has exactly $p_2(n-r)$ irreducible representations hence $\Phi^{-1}(L, C_1, \mathscr{E}_1^{\bullet})$ consists of $p_2(n-r)$ elements. It follows that

$$|\mathcal{N}-\mathcal{N}^{(0)}|=\sum_{\substack{j\geq 0\\ \frac{1}{2}j(j+1)<n}} p_2(n-(1+2+\ldots+j)).$$

Comparing with (10.4.1), it follows that

$$|\mathcal{N}^{(0)}| = \begin{cases} 1, & \text{if } n = 1+2+\ldots+j, \text{ for some } j \\ 0, & \text{otherwise} \end{cases}$$

and (10.4.2) is proved.

10.5. In this section, we assume that $G = G_n$ is either $Sp_{2n}(k)$ or a simply connected group of type B_n and that $\operatorname{char}(k) = 2$. Note that $|\mathcal{N}_G|$ is equal to the number of unipotent conjugacy classes in the finite group $Sp_{2n}(F_q)$ where q is any power of 2, hence it is also equal to the number of unipotent representations of $Sp_{2n}(F_q)$ (see [10, 9.8]). We find that

$$|\mathcal{N}| = \sum_{j \geqq 0} p_2(n - 2(1+2+\ldots+j)).$$

Just as in 10.4, we see that

$$|\mathcal{N}^{(0)}| = \begin{cases} 1, & \text{if } n = 2(1+2+\ldots+j) \quad \text{for some } j \geqq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{10.5.1}$$

The elements of $\mathcal{M}_G$ are the triples $(L, C_1, \mathcal{E}_1^{\cdot})$ where L is a product

$$\underbrace{GL_1(k) \times \ldots \times GL_1(k)}_{t \text{ factors}} \times G_r, \qquad (n = t + r)$$

and $r = j(j+1)$ for some $j \geqq 0$; $(C_1, \mathcal{E}_1^{\cdot})$ is uniquely determined by L. $N(L)/L$ is a Coxeter group of type B_{n-r}. The number of elements in $\Phi^{-1}(L, C_1, \mathcal{E}_1^{\cdot})$ is $p_2(n-r)$.

10.6. In this section, we assume that $G = SO_N(k)$, $(N \geqq 1)$, and that $\operatorname{char}(k) \neq 2$. In this case, the unipotent classes in G correspond to partitions $N = 1i_1 + 2i_2 + 3i_3 + \ldots$ where $i_1, i_2, i_3 \ldots \geqq 0$ and $i_2, i_4, i_6, \ldots$ are even. (i_a is the number of Jordan cells of size a of a unipotent element.) This correspondence is $1-1$ except that there are two unipotent classes (said to be degenerate) corresponding to any partition such that $i_1 = i_3 = i_5 = \ldots = 0$. The group of components of the centralizer of a non-degenerate unipotent element corresponding to the partition $N = 1i_1 + 2i_2 + 3i_3 + \ldots$ is an elementary abelian 2-group of order $2^{\#\{a \text{ odd}, i_a > 0\} - 1}$; the group of components of the centralizer of a degenerate unipotent element is trivial. Let x_N be the number of elements in $\mathcal{N}$ of form $(C, \mathcal{E})$ with C non-degenerate and let x_N'' be the number of elements in $\mathcal{N}$ of form $(C, \mathcal{E})$ with C degenerate. If t is an indeterminate, we have

$$\begin{aligned}
1 + \sum_{n \geqq 1} \left(2x_N' + \frac{x_N''}{2}\right) t^N &= \sum_{\substack{i_1, i_2, \ldots \geqq 0 \\ i_2, i_4, \ldots \text{ even}}} 2^{\#\{a \text{ odd}, i_a > 0\}} t^{1i_1 + 2i_2 + 3i_3 + \ldots} \\
&= (1 + \sum_{i_1 \geqq 1} 2t^{i_1})(1 + \sum_{i_3 \geqq 1} 2t^{3i_3}) \ldots \times (\sum_{i_2' \geqq 0} t^{4i_2'})(\sum_{i_4' \geqq 0} t^{8i_4'})(\sum_{i_6' \geqq 0} t^{12i_6'}) \ldots \\
&= \frac{1+t}{1-t} \frac{1+t^3}{1-t^3} \ldots \times (1-t^4)^{-1}(1-t^8)^{-1} \ldots \\
&= \prod_{i=1}^{\infty} (1-t^{2i})^{-2} \cdot \prod_{i=1}^{\infty} (1+t^{2i-1})^2 \prod_{i=1}^{\infty} (1-t^{2i}) \\
&= (\sum_{j \geqq 0} p_2(j) t^{2j}) \sum_{m=-\infty}^{\infty} t^{m^2}
\end{aligned}$$

the last step being an identity of Jacobi [5, 19.9(i)]. It follows that

$$2x'_N + \frac{x''_N}{2} = p_2\left(\frac{N}{2}\right) + 2\sum_{m>0} p_2\left(\frac{N}{2} - \frac{m^2}{2}\right), \tag{10.6.1}$$

(we agree to set $p_2(j) = p(j) = 0$ if j is not an integer $\geqq 0$). Note that

$$x''_N = 2p\left(\frac{N}{4}\right), \tag{10.6.2}$$

hence

$$x_N = x'_N + x''_N = \frac{1}{2}\left(p_2\left(\frac{N}{2}\right) - p\left(\frac{N}{4}\right)\right) + 2p\left(\frac{N}{4}\right) + \sum_{m>0} p_2\left(\frac{N}{2} - \frac{m^2}{2}\right). \tag{10.6.3}$$

We shall prove by induction on N that

(10.6.4) *If $N \geqq 3$ then $\mathcal{N}^{(0)}$ for $SO_N(k)$ has exactly one element if N is a square, and is empty otherwise.*

For $N \leqq 4$ this follows from (10.3.2). We now assume that $N \geqq 5$ and that the statement (10.6.4) is known for $\tilde{N} < N$. Let $(L, C_1, \mathcal{E}_1^\bullet) \in \mathcal{M}_G$ be such that $L \neq G$. Then as in the proof of (10.4.2), L must be a product $\underbrace{GL_1(k) \times \ldots \times GL_1(k)}_{t \text{ factors}} \times SO_{N-2t}$, where $N - 2t$ is a square $\geqq 4$ and $(C_1, \mathcal{E}_1^\bullet)$ is uniquely determined by L, or else L is a maximal torus and again $(C_1, \mathcal{E}_1^\bullet)$ is uniquely determined by L. In the first case, $N(L)/L$ is a Coxeter group of type B_t; from 9.2, we see that the number of elements in $\Phi^{-1}(L, C_1, \mathcal{E}_1^\bullet)$ is $p_2(t)$. In the second case (L a maximal torus), $N(L)/L$ is a Coxeter group of type $B_{(N-1)/2}$ if N is odd and of type $D_{N/2}$ if N is even; we see again that $\Phi^{-1}(L, C_1, \mathcal{E}_1^\bullet)$ has $p_2\left(\frac{N-1}{2}\right)$ elements if N is odd and $\frac{1}{2}\left(p_2\left(\frac{N}{2}\right) - p\left(\frac{N}{4}\right)\right) + 2p\left(\frac{N}{4}\right)$ elements if N is even. (The last expression is the number of irreducible representations of a Coxeter group of type $D_{N/2}$.)

Thus, we get an explicit formula for $|\mathcal{N} - \mathcal{N}^{(0)}|$, which when combined with (10.6.3), yields (10.6.4).

10.7. In this section, we assume that $G = SO_{2n}(k)$, ($n \geqq 2$) and that $\operatorname{char}(k) = 2$. (Analogous results will hold for the simply connected group of type D_n over k, since it maps bijectively onto G.)

In this case, $|\mathcal{N}|$ can be computed as in 10.5 and it is given by

$$|\mathcal{N}| = \frac{1}{2}\left(p_2(n) - p\left(\frac{n}{2}\right)\right) + 2p\left(\frac{n}{2}\right) + \sum_{\substack{m \geqq 2 \\ \text{even}}} p_2(n - m^2).$$

Just as in 10.4, we see that

$$|\mathcal{N}^{(0)}| = \begin{cases} 1, & \text{if } n \text{ is an even square,} \\ 0, & \text{otherwise.} \end{cases}$$

The elements of $\mathscr{M}_G$ are the triples $(L, C_1, \mathscr{E}_1^{\bullet})$ where L is a product

$$\underbrace{GL_1(k) \times \ldots \times GL_1(k)}_{t \text{ factors}} \times SO_{2m^2}, \quad (m \geqq 2,\ m \text{ even},\ m^2+t=n),$$

or a maximal torus; $(C_1, \mathscr{E}_1^{\bullet})$ is uniquely determined by L. $N(L)/L$ is a Coxeter group of type B_t (if L is not a torus) and D_n (if L is a torus). The number of elements in the corresponding fibre $\Phi^{-1}(L, C_1, \mathscr{E}_1^{\bullet})$ is the number of irreducible representations of the Coxeter group $N(L)/L$.

§ 11. Some combinatorics

11.1. We now introduce some combinatorial objects which will be used to parametrize the set $\mathscr{N}_G$ in the case where G is a symplectic or special orthogonal group and $\mathrm{char}(k) \neq 2$. The following discussion was suggested by the discussion of symbols in [6, § 3].

11.2. The unipotent classes in $G = Sp_{2n}(k)$, are in $1-1$ correspondence with the set

$$X_{2n} = \{\text{partitions } 2n = 1 \cdot i_1 + 2 \cdot i_2 + 3 \cdot i_3 + \ldots \text{ with } i_1, i_2, i_3, \ldots \geqq 0,\ i_1, i_3, i_5, \ldots \text{ even}\};$$

i_a is the number of Jordan cells of size a of a unipotent element.

If u is a unipotent element of G, the group $Z_G(u)/Z_G^0(u)$ (and hence also its dual $\mathrm{Hom}(Z_G(u)/Z_G^0(u), \bar{\mathbb{Q}}_l)$) is identified in [18, I 2.9] with the F_2-vector space with basis indexed by the set $\Delta_\lambda = \{a \text{ even} \mid i_a \geqq 1\}$ where λ is the partition attached to u. It follows that we have a natural bijection:

$$(11.2.1) \qquad \mathscr{N}_G \leftrightarrow \coprod_{\lambda \in X_{2n}} F_2[\Delta_\lambda].$$

Here $F_2[\Delta_\lambda]$ means: F_2 vector space with basis indexed by Δ_λ.

11.3. The unipotent classes in $G = SO_N(k)$, $\mathrm{char}(k) \neq 2$, are in $1-1$ correspondence (as in 11.2) with the set

$$X'_N = \{\text{partitions } N = 1 \cdot i_1 + 2 \cdot i_2 + 3 \cdot i_3 + \ldots \text{ with } i_1, i_2, i_3, \ldots \geqq 0,\ i_2, i_4, i_6, \ldots \text{ even}\},$$

except that to any partition in X'_N such that $i_1 = i_3 = i_5 = \ldots = 0$, there correspond two (degenerate) unipotent classes in $SO_N(k)$.

If u is a non-degenerate unipotent element of G, the group $Z_G(u)/Z_G^0(u)$ is identified in [18, I 2.9] with the subspace of $F_2[\Delta_\lambda]$, $(\Delta_\lambda = \{a \text{ odd} \mid i_a \geqq 1\})$ defined by the single equation: the sum of coordinates equals 0. Here λ is the partition attached to u. Hence $\mathrm{Hom}(Z_G(u)/Z_G^0(u), \bar{\mathbb{Q}}_l^*)$ may be canonically identified with the quotient $\widetilde{F_2[\Delta_\lambda]}$ of $F_2[\Delta_\lambda]$ by the line spanned by the sum of all vectors in the canonical basis.

If u is degenerate, then $\mathrm{Hom}(Z_G(u)/Z_G^0(u), \bar{\mathbb{Q}}_l^*)$ has a single element. It may be identified with the 0-dimensional F_2-vector space $F_2[\Delta_\lambda]$ where λ is the partition in X_N corresponding to u. (Such a partition is said to be degenerate; the partition corresponding to a non-degenerate unipotent class is said to be non-degenerate). Thus, we have a natural bijection

$$\mathcal{N}_G \leftrightarrow \coprod_{\substack{\lambda \in X'_N \\ \lambda \text{ non-deg.}}} \widetilde{F_2[\Delta_\lambda]} \coprod \Big(\coprod_{\substack{\lambda \in X'_N \\ \lambda \text{ deg.}}} (F_2[\Delta_\lambda] \coprod F_2[\Delta_\lambda]) \Big). \tag{11.3.1}$$

(Note that there are two degenerate unipotent classes corresponding to a given degenerate partition.)

11.4. Let $\tilde{\Psi}_N$, (N even, $\geqq 2$), be the set of all *ordered pairs* (A,B) where A is a finite subset of $\{0,1,2,\dots\}$, B is a finite subset of $\{1,2,3,\dots\}$, which are subject to the following three requirements

(11.4.1) If i is any integer then $\{i,i+1\}$ is not contained in A nor in B.

(11.4.2) $|A|+|B|$ is odd, ($|A|$ = number of elements in A).

(11.4.3) $\sum_{a\in A} a + \sum_{b\in B} b = \frac{1}{2}N + \frac{1}{2}(|A|+|B|)(|A|+|B|-1)$.

Let Ψ_N be the set of equivalence classes on $\tilde{\Psi}_N$ for the equivalence relation generated by

$$(A,B) \sim (\{0\} \cup (A+2), \{1\} \cup (B+2)). \tag{11.4.4}$$

We shall denote the equivalence class of the pair (A,B) under (11.4.4) again by (A,B).

Let $\tilde{\Psi}'_N$, (N integer, $\geqq 3$) be the set of all *unordered pairs* (A,B) where A,B are finite subsets of $\{0,1,2,\dots\}$ which are subject to the requirement (11.4.1) and to the requirement

$$\sum_{a\in A} a + \sum_{b\in B} b = \tfrac{1}{2}N + \tfrac{1}{2}((|A|+|B|-1)^2 - 1) \tag{11.4.5}$$

(which implies that $|A|+|B| \equiv N \pmod 2$).

Let Ψ'_N be the set of equivalence classes on $\tilde{\Psi}'_N$ for the equivalence relation generated by

$$(A,B) \sim (\{0\} \cup (A+2), \{0\} \cup (B+2)). \tag{11.4.6}$$

We shall denote the equivalence class of (A,B) under (11.4.6) again by (A,B).

The sets Ψ_N, Ψ'_N are finite. An element of Ψ'_N of form (A,A) is said to be degenerate; the other elements are said to be non-degenerate.

11.5. Two elements of Ψ_N (or Ψ'_N) are said to be *similar* if they can be represented in the form (A,B), (A',B') with $A\cup B = A'\cup B'$, $A\cap B = A'\cap B'$. In each similarity class in Ψ_N (or in Ψ'_N) there is a unique element which can be represented by (A,B) with $A=\{a_1<a_2<\dots<a_{m'}\}$, $B=\{b_1<b_2<\dots<b_m\}$ such that the following holds: $m'=m+1$ (for Ψ_N, any N, and for Ψ'_N, N odd), $m'=m$ (for Ψ'_N, N even), $a_1 \leqq b_1 \leqq a_2 \leqq b_2 \leqq \dots \leqq a_m \leqq b_m$, and moreover $b_m \leqq a_{m+1}$ (for Ψ_N any N, and for Ψ'_N, N odd).

Such an element is said to be distinguished.

The set of elements in a fixed similarity class may be organized as an F_2-vector space, as follows. Let (A,B) be distinguished in our class, (see above); we assume that $A \neq B$, so that $C=(A\cup B)-(A\cap B)$ is non-empty.

A non-empty subset I of C is said to be an interval if it is of the form $\{i, i+1, i+2, \ldots, j\}$ with $i-1\notin C, j+1\notin C$ and, if moreover (in the case of Ψ_N), we have $i\neq 0$. Let $\mathscr{I}$ be the set of intervals of C. It is non-empty. We denote by I_0 the set of elements of C which do not belong to any interval. Then I_0 is empty or of the form $\{0,1,2,\ldots,h\}$; the latter possibility can only arise for Ψ_N. For each subset $\alpha\subset\mathscr{I}$, set $\alpha'=\mathscr{I}-\alpha$ and we consider

$$A_\alpha=(\bigcup_{I\in\alpha}(I\cap A))\cup(\bigcup_{I\in\alpha'}(I\cap B))\cup(I_0\cap A)\cup(A\cap B)$$

$$B_\alpha=(\bigcup_{I\in\alpha}(I\cap B))\cap(\bigcup_{I\in\alpha'}I\cap A)\cup(I_0\cap B)\cup(A\cap B).$$

Then (A_α, B_α) is similar to (A,B) and the map $\alpha\to(A_\alpha,B_\alpha)$ is a bijection between the set $\mathscr{P}(\mathscr{I})$ of subsets of $\mathscr{I}$ (resp. $\mathscr{P}(\mathscr{I})$ modulo the equivalence relation $\alpha\sim\alpha'$) and the set of elements in the similarity class of (A,B), in the case of Ψ_N (resp. the case of Ψ'_N). Now $\mathscr{P}(\mathscr{I})$ is an F_2-vector space with respect to symmetric difference; it has as canonical basis the one-element subsets of $\mathscr{I}$. Hence $\mathscr{P}(\mathscr{I})=F_2[\mathscr{I}]$ and $\mathscr{P}(\mathscr{I})/(\alpha\sim\alpha')=\widetilde{F_2[\mathscr{I}]}\overset{\text{def}}{=}F_2[\mathscr{I}]$/line spanned by the sum of the standard basis elements. Thus, we have a bijection between the elements in the similarity class of (A,B) and the F_2-vector space $F_2[\mathscr{I}]$ (in case of Ψ_N) and $\widetilde{F_2[\mathscr{I}]}$ (in case of Ψ'_N).

11.6. Start with a partition $\lambda=(1i_1+2i_2+3i_3+\ldots)$ in X_N, N even. We associate to λ an element in Ψ_N as follows. Let $2m$ be an even integer, $2m\geqq i_1+i_2+\ldots$, and let $z_1\leqq z_2\leqq\ldots\leqq z_{2m}$ be the sequence containing the number j exactly i_j times $(\forall j\geqq 1)$ and the number 0 exactly $2m-(i_1+i_2+\ldots)$ times. Let $z'_1<z'_2<\ldots<z'_{2m}$ be the sequence defined by $z'_i=z_i+(i-1)$. This sequence contains m even numbers $2y_1<2y_2<\ldots<2y_m$ and m odd numbers $2y'_1+1<2y'_2+1<\ldots<2y'_m+1$. (This is seen easily by induction on N.)

Again, by induction on N, we see that

$$0\leqq y_1+1\leqq y'_1+2\leqq y_2+2\leqq y'_2+3\leqq\ldots\leqq y_m+m\leqq y'_m+(m+1).$$

Hence, if we set

$$A=\{0, y'_1+2, y'_2+3, \ldots, y'_m+(m+1)\}, \qquad B=\{y_1+1, y_2+2, \ldots, y_m+m\},$$

then (A,B) is a distinguished element of Ψ_N. Its similarity class remains unchanged when m is increased by 1, hence it depends only on λ, not on m. This process can be reversed so that it gives a bijection between X_N and the set of similarity classes in Ψ_N. Let λ, m, A, B be as above. Let $C=(A\cap B)-(A\cap B)$. We have a bijection between the set Δ_λ (see 11.2) and the set $\mathscr{I}_\lambda$ of intervals of C, defined as follows. If we arrange the intervals in $\mathscr{I}_\lambda$ in increasing order $I_1, I_2, \ldots, I_f$ (any element in I_1 is smaller than any element in I_2, etc.) and if we arrange the elements of $\Delta_\lambda=\{a \text{ even} \mid i_a\geqq 1\}$ in increasing order

$a_1 < a_2 < \ldots < a_{f'}$, then $f = f'$ and we make I_h correspond to a_h, $(1 \leqq h \leqq f)$; we have $|I_h| = i_{a_h}$. (This is also checked by induction on N).

Hence we get an isomorphism $F_2[\Delta_\lambda] \cong F_2[\mathscr{I}_\lambda]$; the last space has been identified in 11.5 with the set of elements in the similarity class on Ψ_N corresponding to λ. Hence, we obtain a bijection

$$\Psi_N \leftrightarrow \coprod_{\lambda \in X_N} F_2[\Delta_\lambda].$$

Composing it with the bijection (11.2.1) we obtain a bijection

$$\mathcal{N}_G \leftrightarrow \Psi_N, \quad \text{where} \quad G = Sp_N(k), \quad \operatorname{char} k \neq 2 \tag{11.6.1}$$

In the following table we describe, as an example, the correspondence between similarity classes in Ψ_6 and elements of X_6.

Elements of Ψ_6.	*Elements of* X_6.
$(\{3\},\emptyset),(\emptyset,\{3\})$	6
$(\{0,4\},\{2\}),(\{0,2\},\{4\}),(\{0\},\{2,4\}),(\{0,2,4\},\emptyset)$	$2+4$
$(\{1,4\},\{1\}),(\{1\},\{1,4\})$	$1+1+4$
$(\{0,3\},\{3\})$	$3+3$
$(\{1,3\},\{2\}),(\{2\},\{1,3\})$	$2+2+2$
$(\{0,2,5\},\{2,4\}),(\{0,2,4\},\{2,5\})$	$1+1+2+2$
$(\{1,3,5\},\{1,3\}),(\{1,3\},\{1,3,5\})$	$1+1+1+1+2$
$(\{0,2,4,6\},\{2,4,6\})$	$1+1+1+1+1+1$

11.7. We now start with a partition $\lambda = (1 i_1 + 2 i_2 + \ldots)$ in X'_N. We associate to λ an element in Ψ'_N as follows. Let M be an integer, $M \geqq i_1 + i_2 + i_3 + \ldots, M \equiv N \pmod 2$, and let $z_1 \leqq z_2 \leqq \ldots \leqq z_M$ be the sequence containing the number j exactly i_j times $(j \geqq 1)$ and the number 0 exactly $M - (i_1 + i_2 + \ldots)$ times. Let $z'_1 < z'_2 < \ldots < z'_M$ be the sequence defined by $z'_i = z_i + (i-1)$. This sequence contains $[M/2]$ even numbers $2y_1 < 2y_2 < \ldots < 2y_{[M/2]}$ and $[(M+1)/2]$ odd number $2y'_1 + 1 < 2y'_2 + 1 < \ldots < 2y'_{[(M+1)/2]} + 1$. Here $[x]$ denotes the largest integer $\leqq x$. This is seen by induction on N; in the same way we see that

$$y'_1 \leqq y_1 \leqq y'_2 + 1 \leqq y_2 + 1 \leqq \ldots \leqq y'_{[M/2]} + [M/2] - 1 \leqq y_{[M/2]} + [M/2] - 1,$$

and $y_{\frac{1}{2}(M-1)} + \frac{1}{2}(M-1) - 1 \leqq y'_{\frac{1}{2}(M+1)} + \frac{1}{2}(M+1) - 1$ if M is odd).

Hence, if we set

$$A = \{y'_1, y'_2 + 1, \ldots, y'_{[(M+1)/2]} + [(M+1)/2] - 1\},$$
$$B = \{y_1, y_2 + 1, \ldots, y_{[M/2]} + [M/2] - 1\}$$

then (A, B) is a distinguished element of Ψ'_N. Its similarity class remains unchanged when M is increased by 2, hence it depends only on λ, not on M. This process can be reversed so that it gives a bijection between X'_N and the set of similarity classes in Ψ'_N.

Let λ, M, A, B be as above. Assume that $i_1, i_3, i_5 \ldots$ are not all zero. Then $A \neq B$, i.e. $C = (A \cup B) - (A \cap B)$ is non-empty. We have a bijection between Δ_λ (see

11.3) and the set $\mathscr{I}_\lambda$ of intervals of C, defined in essentially the same way as in 11.6. Hence we get an isomorphism $F_2[\Delta_\lambda] \cong F_2[\mathscr{I}_\lambda]$ carrying the line spanned by the sum of the basis elements in $F_2[\Delta_\lambda]$ to the analogous line in $F_2[\mathscr{I}_\lambda]$. Taking quotients by these lines, we have an induced isomorphism $\widetilde{F_2[\Delta_\lambda]} \cong \widetilde{F_2[\mathscr{I}_\lambda]}$. The last space has been identified in 11.5 with the set of elements in the similarity class in Ψ_N' corresponding to λ. Hence we obtain a bijection

$$\text{(11.7.1)} \quad \{\text{non degenerate elements in } \Psi_N'\} \leftrightarrow \coprod_\lambda \widetilde{F_2[\Delta_\lambda]}$$

(union over all nondegenerate partitions λ in X_N'.)

Our bijection Ψ_N' (mod similarity) $\leftrightarrow X_N'$ gives rise to a bijection

$$\text{(11.7.2)} \quad \{\text{degenerate elements in } \Psi_N'\} \rightarrow \coprod_\lambda \widetilde{F_2[\Delta_\lambda]}$$

(union over all degenerate partitions λ in X_N'.) Note that a degenerate element in Ψ_N' is alone in its similarity class and $\widetilde{F_2[\Delta_\lambda]}$ has a single element for λ degenerate.

Combining (11.7.1), (11.7.2) and (11.3.1) we obtain a map

$$\text{(11.7.3)} \qquad \mathscr{N}_G \rightarrow \Psi_N' \qquad (G = SO_N(k), \operatorname{char}(k) \neq 2)$$

which is bijective over the set of non-degenerate elements of Ψ_N', and is such that for each degenerate element in Ψ_N', its fibre has 2 elements (corresponding to degenerate unipotent classes with a constant local system.)

In the following tables, we describe as an example, the correspondence between similarity classes in Ψ_N' and elements in X_N' for $N=7$ and 8.

Elements of Ψ_7'	*Elements of* X_7'
$(\{3\}, \emptyset)$	7
$(\{0,4\}, \{1\}), (\{1,4\}, \{0\})$	1+1+5
$(\{1,3\}, \{1\})$	2+2+3
$(\{0,3\}, \{2\}), (\{0,2\}, \{3\})$	1+3+3
$(\{0,2,5\}, \{1,3\}), (\{1,3,5\}, \{0,2\})$	1+1+1+1+3
$(\{0,2,4\}, \{1,4\})$	1+1+1+2+2
$(\{0,2,4,6\}, \{1,3,5\})$	1+1+1+1+1+1+1

Elements of Ψ_8'	*Elements of* X_8'
$(\{0\}, \{4\}), (\{0,4\}, \emptyset)$	1+7
$(\{1\}, \{3\}), (\{1,3\}, \emptyset)$	3+5
$(\{0,2\}, \{1,5\}), (\{0,2,5\}, \{1\})$	1+1+1+5
$(\{2\}, \{2\})$	4+4
$(\{0,3\}, \{1,4\}), (\{0,4\}, \{1,3\})$	1+1+3+3
$(\{0,2\}, \{2,4\}), (\{0,2,4\}, \{2\})$	1+2+2+3
$(\{0,2,4\}, \{1,3,6\}), (\{0,2,4,6\}, \{1,3\})$	1+1+1+1+1+3
$(\{1,3\}, \{1,3\})$	2+2+2+2
$(\{0,2,5\}, \{1,3,5\})$	1+1+1+1+2+2
$(\{0,2,4,6\}, \{1,3,5,7\})$	1+1+1+1+1+1+1+1

§ 12. A generalization of a result of T. Shoji for symplectic groups

12.1. In this chapter (except in 14.5) we assume that $G = Sp(V)$, where V is a $2n$-dimensional vector space with a non-singular symplectic form (,) over k of odd characteristic. The set $\mathcal{N}_G$ has a natural partition given by the fibres of Φ, (see 6.5, 10.4):

$$\mathcal{N}_G = \coprod_{j \geqq 0} \mathcal{N}_G^{(n - \frac{1}{2} j(j+1))} \tag{12.1.1}$$

where $\mathcal{N}_G^{(i)}$ is the fibre of Φ over $(L, C_1, \mathcal{E}_1^{\cdot}) \in \mathcal{M}_G$ with $L = L_i$ of type $\underbrace{GL_1 \times \ldots \times GL_1}_{i \text{ factors}} \times Sp_{2n-2i}$. Moreover, from 6.5(c) and 9.2(d) we get a canonical bijection

$$\mathcal{N}_G^{(i)} \leftrightarrow (N(L_i)/L_i)^{\vee} \tag{12.1.2}$$

(where $^{\vee}$ denotes the set of isomorphism classes of irreducible representations of a finite group).

Let W_n be the group of permutations of the set $\{1, 2, \ldots, n, n', \ldots, 2', 1'\}$ which commute with the involution $j \to j'$, $j' \to j$. Then we can identify

$$N(L_i)/L_i \cong W_i \tag{12.1.3}$$

as follows. Consider a basis $e_1, \ldots, e_n, e'_n, \ldots, e'_1$ of V such that $(e_i, e'_i) = 1$ $(e'_i, e_i) = -1$ and all other scalar products equal to zero. We assume that L_i is the set of $g \in G$ which map each of the vectors $e_1, \ldots, e_i, e'_i, \ldots, e'_1$ into a scalar multiple of itself. Then each element of $N(L_i)/L_i$ defines a permutation of the set of lines $\langle e_1 \rangle, \ldots, \langle e_i \rangle, \langle e'_i \rangle, \ldots, \langle e'_1 \rangle$ and this gives the isomorphism (12.1.3). Combining (12.1.1), (12.1.2), (12.1.3) we get a bijection

$$\mathcal{N}_G \leftrightarrow \coprod_{j \geqq 0} (W_{n - \frac{1}{2} j(j+1)})^{\vee}. \tag{12.1.4}$$

12.2. The purpose of this chapter is to give an explicit (combinatorial) description of the bijection (12.1.4).

To do this, we shall define an explicit bijection

$$\Psi_{2n} \leftrightarrow \coprod (W_{n - \frac{1}{2} j(j+1)})^{\vee}, \tag{12.2.1}$$

union over all $j \geqq 0$ such that $\frac{1}{2} j(j+1) \leqq n$). We define the defect of $(A, B) \in \Psi_{2n}$ to be $d = |A| - |B|$; it is an odd integer. We have a partition

$$\Psi_{2n} = \coprod_{d \text{ odd}} \Psi_{2n, d} \tag{12.2.2}$$

where $\Psi_{2n, d}$ is the set of elements of defect d in Ψ_{2n}.

We have a canonical bijection

$$\Psi_{2n', 1} \xrightarrow{\sim} \Psi_{2n' + d(d-1), d}, \ (d \text{ odd}),$$

defined by

$$(A, B) \mapsto (\{0, 2, 4, \ldots, 2d-4\} \cup (A + 2d - 2), B), \qquad \text{if } d \geqq 1$$

and by

$$(A,B)\mapsto(A,\{1,3,5,\dots,1-2d\}\cup(B+2-2d)),\qquad \text{if } d\leqq -1.$$

This, together with (12.2.2) give us a bijection

(12.2.3) $$\Psi_{2n}\leftrightarrow \coprod_{d\,\mathrm{odd}} \Psi_{2n-d(d-1),1} = \coprod_{j\geqq 0} \Psi_{2n-j(j+1),1}.$$

(The last equality is given by the change of variable $j=d-1$ if $d\geqq 1$, $j=-d$ if $d\leqq -1$.). We make here the convention that $\Psi_{2n'}$ is empty if $n'<0$.

The bijection (12.2.1) is the composition of (12.2.3) with the bijections

$$\Psi_{2n-j(j+1),1}\leftrightarrow(W_{n-\frac{1}{2}j(j+1)})^\vee,\qquad (j\geqq 0).$$

which are a special case of the bijection

(12.2.4) $$\Psi_{2n,1}\leftrightarrow \mathrm{W}_n^\vee$$

defined as follows. We parametrize the elements of W_n by ordered pairs of partitions $0\leqq\alpha_1\leqq\alpha_2\leqq\dots\leqq\alpha_{m'}$, $0\leqq\beta_1\leqq\beta_2\dots\leqq\beta_{m''}$ with $\sum\alpha_i+\sum\beta_j=n$. (Such a correspondence is described, for example, in [6, 2.7(i)]; in particular, the unit representation corresponds to the pair of partition $n,\emptyset$. The sign representation corresponds to $\emptyset$, $1+1+\dots+1$. The one dimensional representation of W_n defined by $\chi(w)=(-1)^{l'(w)}$, where $l'(w)=$number of $i\in\{1,2,\dots,n\}$ such that $w(i)\in\{n',\dots,2',1'\}$, corresponds to the pair of partitions $\emptyset, n$.) Here m',m'' can be chosen arbitrarily large, by adding zeros to the parts of our partitions. We shall choose them so that $m'=m''+1$. We set

$$A=\{\alpha_1<\alpha_2+2<\alpha_3+4<\dots<\alpha_{m'}+2m'-2\},$$
$$B=\{\beta_1+1<\beta_2+3<\beta_3+5<\dots<\beta_{m''}+2m''-1\}.$$

Then $(A,B)\in\Psi_{2n,1}$ and this gives the required bijection (12.2.4).

Composing the bijections (12.1.4) and (12.2.1) we get a bijection

(12.2.5) $$\mathcal{N}_G\leftrightarrow\Psi_{2n}.$$

We can now state the main result of this chapter.

12.3. **Theorem.** *The bijections* (12.2.5) *and* (11.6.1), (*with* $N=2n$) *coincide.*

Our bijections restrict to bijections $\mathcal{N}_G^{(n)}\leftrightarrow\Psi_{2n,1}$ (here $\mathcal{N}_G^{(n)}$ is as in 12.1).

The fact that these restrictions coincide is essentially equivalent to the main result of T. Shoji in [15]. Note however, that the combinatorics used by Shoji is more complicated (in the author's opinion) than the one used here.

The proof is based on Theorem 8.3 and on the following observation of Shoji: if $n\geqq 3$, an irreducible representation of W_n is completely determined by its restriction to W_{n-1}. (We regard here W_{n-1} as the subgroup of W_n which stabilizes $1\in\{1,\dots,n,n'\dots,1'\}$.) Assume now that $n\geqq 1$. Consider the parabolic subgroup $P'\subset G$ which is the stabilizer of a line D in V, and let L' be a Levi subgroup of P'. Then $L'\cong GL_1(k)\times Sp_{2n-2}(k)$. We may assume the theorem proved for $Sp_{2n-2}(k)$. For each $(C,\mathscr{E}^\cdot)\in\mathcal{N}_G$, $(C',\mathscr{E}'^\cdot)\in\mathcal{N}_L$, the multiplicity $m_{\mathscr{E}'^\cdot,\mathscr{E}^\cdot}$ (see 8.2) can be determined explicity in a geometrical way (see 12.5), by a computation which is almost the same as a computation contained in

Spaltenstein's work [18, II]. (This multiplicity is 0 or 1.) Using this information, 8.3(a) and the induction hypothesis, we can check that any element of $\mathcal{N}_G$ which is mapped by (11.6.1) to an element of $\Psi_{2n,d}$ ($d(d-1)\leqq 2n$), must be also mapped by (12.2.5) to an element of $\Psi_{2n,d}$. We can also check immediately that the inverse images of $\Psi_{2n,d}$ under (11.6.1) and (12.2.5) have the same number of elements. Hence these inverse images coincide, (if $d(d-1)<2n$). They must also coincide for $d(d-1)=2n$ since d must satisfy $d(d-1)\leqq 2n$. Thus, the fibres $\mathcal{N}_G^{(i)}$ of Φ: $\mathcal{N}_G \to \mathcal{M}_G$ are identified in a combinatorial way. From the knowledge of multiplicities $m_{\mathcal{E}'',\mathcal{E}'}$ and from 8.3(b) we see that the irreducible representation of W_i, ($i\geqq 1$), corresponding to any element of $\mathcal{N}_G^{(i)}$ are known after restriction to W_{i-1}. Hence they are known as representations of W_i, provided that $i\geqq 3$. When $i=2$, this method doesn't quite identify the correspondence between $\mathcal{N}_G^{(i)}$ and the representations of W_i: there are two one-dimensional representations of W_2 which restrict to the unit representation of W_1 and there are two one-dimensional representations of W_2 which restrict to the sign representation of W_1. However, from 9.5, we know explicitly which elements of $\mathcal{N}_G^{(2)}$ correspond to the unit and sign representation of W_2 and this provides the missing information. When $i=1$, we use again 9.5 which tells us which elements of $\mathcal{N}_G^{(1)}$ correspond to the unit and sign representation of W_1. (These are the only irreducible representations of W_1.) For $i=0$, the correspondence between $\mathcal{N}_G^{(0)}$ (if non-empty) and $W_0^\vee$ is obvious since both sets have just one element. This completes our sketch of proof of Theorem 12.3.

12.4. **Corollary.** (a) *Two elements of $\mathcal{N}_G$ are in the same fibre of Φ (see 6.5) if and only if the corresponding elements of Ψ_{2n} (under (12.2.5)) have the same defect.*

(b) *$\mathcal{N}_G^{(0)}$ is non-empty if and only if $n=\frac{1}{2}d(d-1)$ for some odd (possibly negative) integer d, (see (10.4.2)). If $n=\frac{1}{2}d(d-1)$, (d odd) the unique element of $\mathcal{N}_G^0$ corresponds under (12.2.5) to the element $(\{0,2,4,\dots,2d-2\},\emptyset)\in\Psi_{2n,d}$, if $d\geqq 1$, and to the element $(\emptyset,\{1,3,5,\dots,1-2d\})\in\Psi_{2n,d}$ if $d\leqq -1$; hence this element of $\mathcal{N}_G^0$ is of form $(C,\mathcal{E}^\cdot)$ where $g\in C$ has Jordan blocks of sizes given by the partition $2n=2+4+6+\dots$.*

(c) *Two elements $(C,\mathcal{E}^\cdot)$, $(\tilde{C},\tilde{\mathcal{E}}^\cdot)$ of $\mathcal{N}_G$ satisfy $C=\tilde{C}$ if and only if the corresponding elements of Ψ_{2n} (under (12.2.5)) are similar.*

12.5. Let V be an N-dimensional vector space ($N\geqq 2$) over k (of characteristic $\neq 2$) with a given non-singular bilinear form (,) such that there exist $\varepsilon\in\{1,-1\}$ with $(v,v')=\varepsilon(v',v)$ for all $v,v'\in V$.

Let e,e' be two isotropic vectors in V such that $(e,e')=1$ and let $\bar{V}$ be the subspace of vectors in V orthogonal to e,e'.

The results in this section are concerned with a comparison between unipotent classes in the isometry group $Is(V)$ of V and the corresponding group $Is(\bar{V})$ for $\bar{V}$. It will be however more convenient to formulate them in terms of the set $Nil(V)$ of nilpotent elements in the Lie algebra of $Is(V)$, that is, in terms of nilpotent maps v: $V\to V$ such that $(vv,v')+(v,vv')=0$ for all v,v'. (Since $\mathrm{char}(k)\neq 2$, one can pass freely between unipotent and nilpotent elements.)

The results in this section are essentially variations on results of Spaltenstein in [18, II.6]. They are needed in the proof of Theorem 12.3. Spaltenstein's results have also played a key role in Shoji's proofs in [15].

Let $\bar{v}\in Nil(\bar{V})$. We denote by $cl(\bar{v})$ its conjugacy class under $Is(\bar{V})$. We have (see [22, 2.28, 2.25]):

(12.5.1) $$\dim cl(\bar{v})=\tfrac{1}{2}(N^2-\varepsilon N)-\tfrac{1}{2}\sum_{i\geqq 1}(r_i+r_{i+1}+\dots)^2+\frac{\varepsilon}{2}\sum_{\substack{i\\ \text{odd}}} r_i$$

where r_i is the number of Jordan cells of size i of $\bar{v}$: $\bar{V}\to\bar{V}$. Note that r_i is even if $i\equiv\frac{1}{2}(\varepsilon-1)\,(\bmod 2)$.

Let $X_{\bar{v}}$ be the variety of all elements v in $Nil(V)$ such that $v(e')=0$ and $v(v)-\bar{v}(v)\in k\cdot e'$ for all $v\in\bar{V}$. (In other words, v induces $\bar{v}$ on $\langle e'\rangle^{\perp}/\langle e'\rangle=\bar{V}$.)

We identify

(12.5.2) $$X_{\bar{v}}\cong\begin{cases}\bar{V}\times k, & \text{if } \varepsilon=-1\\ \bar{V}\times\{0\}, & \text{if } \varepsilon=1\end{cases}$$

by $v\leftrightarrow(x,c)\in\bar{V}\times k$, $v(v+\lambda e)=\bar{v}(v)+\lambda x+(\lambda c-(x,v))e'$ $(v\in\bar{V}, \lambda\in k)$.

We have a partition

$$X_{\bar{v}}=X^0_{\bar{v},1}\coprod X^1_{\bar{v},1}\coprod X^0_{\bar{v},2}\coprod X^1_{\bar{v},2}\coprod X^0_{\bar{v},3}\coprod X^1_{\bar{v},3}\coprod\dots$$

where

$$X^0_{\bar{v},i}=\{v\in X_{\bar{v}}\mid e'\in v^{i-1}(V), e'\notin v^i(V), (e', v^{-(i-1)}e')=0\}$$

$$X^1_{\bar{v},i}=\{v\in X_{\bar{v}}\mid e'\in v^{i-1}(V), e'\notin v^i(V), (e', v^{-(i-1)}e')\neq 0\}.$$

We define a sequence of vector spaces $\bar{V}_0, \bar{V}_1, \bar{V}_2, \dots$ by

$$\bar{V}_0=\bar{v}\bar{V},\ \bar{V}_1=(\bar{v}\bar{V}+\ker\bar{v})/\bar{v}\bar{V},\ \bar{V}_2=(\bar{v}\bar{V}+\ker\bar{v}^2)/(\bar{v}\bar{V}+\ker\bar{v}),\ \text{etc.}$$

Then $\dim\bar{V}_i=r_i$, and $\bar{V}_i$ has a natural quadratic form Q_i defined by

$$Q_i(\bar{v}a+b)=(b,\bar{v}^{i-1}b),\quad (a\in\bar{V}, b\in\ker\bar{v}^i),\quad (i\geqq 1)\quad Q_0(\bar{v}a)=(\bar{v}a,a).$$

If $\varepsilon=-1$, then $Q_1=Q_3=\dots=0$ and $Q_0, Q_2, Q_4, \dots$ are non-singular. If $\varepsilon=1$, then $Q_0=Q_2=Q_4=\dots=0$ and $Q_1, Q_3, Q_5, \dots$ are non-singular.

In terms of the (x,c) coordinates (12.5.2), $X^0_{\bar{v},i}$ and $X^1_{\bar{v},i}$ can be described as follows.

$$X^0_{\bar{v},1}=\{(x,c)\mid x\in\bar{v}\bar{V}, Q_0(x)=-c\}\cong k^{N-\sum_{j\geqq 1} r_j}$$

$$X^1_{\bar{v},1}=\emptyset$$

$$X^0_{\bar{v},i}=\{(x,c)\mid x\in\bar{v}\bar{V}+\ker\bar{v}^{i-1}, \text{ image } \bar{x} \text{ of } x \text{ in } \bar{V}_{i-1} \text{ satisfies } \bar{x}\neq 0,\ Q_{i-1}(\bar{x})=0\}$$

(12.5.3) $$\cong k^{N-\sum_{j\geqq i-1} r_j+(1-\varepsilon)/2}\times(B-0)$$

where B is the quadric $Q_{i-1}=0$ in $\bar{V}_{i-1}$, $(i\geqq 2)$

$$X^1_{\bar{v},2}=\{(x,c)\mid x\in\bar{v}\bar{V}, Q_0(x)\neq -c\}\cong\begin{cases}k^{N-\sum_{j\geqq 1} r_j}\times k^*, & \text{if } \varepsilon=-1\\ \emptyset & \text{if } \varepsilon=1\end{cases}$$

$$X^1_{\bar{v},i}=\{(x,c)\mid x\in\bar{v}\bar{V}+\ker\bar{v}^{i-2}, \text{ image } \bar{x} \text{ of } x \text{ in } \bar{V}_{i-2} \text{ satisfies } Q_{i-2}(\bar{x})\neq 0\}$$

$$=k^{N-\sum_{j\geqq i-2} r_j+(1-\varepsilon)/2}\times(k^{r_{i-2}}-B)$$

where B is the quadric $Q_{i-2}=0$ in $\bar{V}_{i-2}$, $(i\geqq 3)$.

The sizes of Jordan cells of an element $v \in X_{\bar{v}}$ are constant when v runs through one of the pieces $X^0_{\bar{v},i}$, $X^1_{\bar{v},i}$. Let r'_i be the number of Jordan cells of size i of an element v in one of these pieces. Then r'_i are given as follows.

$v \in X^0_{\bar{v},i}$: $r'_i = r_i + 2$, $r'_{i-1} = r_{i-1} - 2$, $r'_j = r_j$, $j \neq i, i-1$, $(i \geqq 2)$.

$v \in X^0_{\bar{v},1}$: $r'_1 = r_1 + 2$, $r'_j = r_j$, $(j \neq 1)$.

$v \in X^1_{\bar{v},i}$: $r'_i = r_i + 1$, $r'_{i-2} = r_{i-2} - 1$, $r'_j = r_j$, $(j \neq i, i-2)$, $(i \geqq 3)$.

$v \in X^1_{\bar{v},2}$: $r'_2 = r_2 + 1$, $r'_j = r_j$, $(j \neq 2)$.

From these formulas, and from (12.5.1), (12.5.3), we see that, for $v \in X^0_{\bar{v}}$, we have

$$\tfrac{1}{2}(\dim cl(v) - \dim cl(\bar{v})) = \begin{cases} \dim(X^0_{\bar{v},i}), & \text{if } v \in X^0_{\bar{v},i} \\ \dim(X^1_{\bar{v},i}) + \tfrac{1}{2} r_{i-1}, & \text{if } v \in X^1_{\bar{v},i} \end{cases}$$

where $cl(v)$ is the conjugacy class of v under $Is(V)$.

For the proof of Theorem 12.3 we must compute the multiplicities $m_{\mathscr{E}^{\cdot}, \mathscr{E}^{\cdot\cdot}}$ (see 8.2) where $\mathscr{E}^{\cdot}$ (resp. $\mathscr{E}^{\cdot\cdot}$) is a 1-dimensional local system on $cl(v)$ (resp. $cl(\bar{v})$) and $v \in X_{\bar{v}}$. This computation is done using the explicit description of the pieces $X^0_{\bar{v},i}$, $X^1_{\bar{v},i}$ given in (12.5.3). We may disregard the pieces $X^1_{\bar{v},i}$ for which $\frac{1}{2}(\dim cl(v) - \dim cl(\bar{v})) > \dim X^1_{\bar{v},i}$, $(v \in X^1_{\bar{v},i})$, i.e. we must consider only the pieces $X^1_{\bar{v},i}$ for which $r_{i-1} = 0$ and the pieces $X^0_{\bar{v},i}$ for all i. We omit further details.

§ 13. A generalization of a result of T. Shoji for special orthogonal groups

13.1. In this chapter, we assume that $G = SO(V)$ where V is an N-dimensional vector space with a non-singular bilinear form (,) over k of odd characteristic, $(N \geqq 3)$. The set $\mathscr{N}_G$ has a natural partition given by the fibres of Φ (see 6.5, 10.6):

$$\mathscr{N}_G = \coprod_{\substack{j \geqq 0 \\ j \equiv N(2)}} \mathscr{N}_G^{((N-j^2)/2)} \tag{13.1.1}$$

where $\mathscr{N}_G^{(i)}$ is the fibre of Φ over $(L, C_1, \mathscr{E}_1^{\cdot}) \in \mathscr{M}_G$ with $L = L_i$ of type $\underbrace{GL_1 \times \ldots \times GL_1}_{i \text{ factors}} \times SO_{N-2i}$ if $N \geqq 2i$ and $L = L_i =$ maximal torus if $N = 2i$.

Moreover, from 6.5 (c) and 9.2 (d) we get a canonical bijection

$$\mathscr{N}_N^{(i)} \leftrightarrow (N(L_i)/L_i)^\vee. \tag{13.1.2}$$

If $N > 2i$, we can identify, just as in (12.1.3),

$$N(L_i)/L_i \cong W_i. \tag{13.1.3}$$

If $N = 2i \geqq 2$, we can identify in the same way $N(L_i)/L_i$ with the Weyl group W_G of G. This, together with (13.1.1), (13.1.2), (13.1.3) gives rise to a bijection

$$\mathscr{N}_G \leftrightarrow W_G^\vee \coprod \Big(\coprod_{\substack{j > 1 \\ j \equiv N(2) \\ j^2 \leqq N}} (W_{(N-j^2)/2})^\vee \Big) \tag{13.1.4}$$

13.2. In this chapter we shall give an explicit (combinatorial) description of the bijection (13.1.4). To do this we shall define an explicit map

(13.2.1) $$W_G^\vee \coprod \Big(\coprod_{\substack{j>1 \\ j\equiv N(2) \\ j^2 \leqq N}} (W_{(N-j^2)/2})^\vee \Big) \to \Psi'_N.$$

We define the defect of $(A,B)\in\Psi'_N$ to be the absolute value of $|A|-|B|$. It is an integer $\geqq 0$, of the same parity as N. We have a partition

(13.2.2) $$\Psi'_N = \coprod_{\substack{d\geqq 0 \\ d\equiv N(2)}} \Psi'_{N,d}$$

where $\Psi'_{N,d}$ is the set of elements of defect d in Ψ'_N. We have a canonical bijection

$$\Psi'_{N',1} \xrightarrow{\sim} \Psi'_{N'+d^2-1,d}, \qquad (d\geqq 1),$$

defined by

$$(A,B)\mapsto(\{0,2,4,\ldots,2d-4\}\cup(A+2d-2),B),\ |A|>|B|.$$

This together with (13.2.2) gives us a bijection

(13.2.3) $$\begin{cases} \Psi'_N \leftrightarrow \coprod_{\substack{d\geqq 1\\ \text{odd}}} \Psi'_{N-d^2+1,1}, & (N \text{ odd}) \\ \Psi'_N \leftrightarrow \Psi'_{N,0} \coprod \Big(\coprod_{\substack{d\geqq 2\\ \text{even}}} \Psi'_{N-d^2+1,1}\Big), & (N \text{ even}). \end{cases}$$

We make here the convention that $\Psi'_{N'}$ is empty if $N'<0$.

The map (13.2.1) is the composition of (13.2.3) with certain maps

(13.2.4) $$(W_{(N-d^2)/2})^\vee \simeq \Psi'_{N-d^2+1,1},\ (d\geqq 1)$$

(13.2.5) $$W_G^\vee \to \Psi'_{N,0},\ (N \text{ even}).$$

(If N is odd, $W_G = W_{(N-1/2)}$).

The maps (13.2.4) are a special case of the bijection

(13.2.6) $$W_n^\vee \leftrightarrow \Psi'_{2n+1,1}$$

defined as follows. We associate to an element of $W_n^\vee$ the ordered pair of partitions $0\leqq\alpha_1\leqq\alpha_2\leqq\ldots\leqq\alpha_{m'}$, $0\leqq\beta_1\leqq\beta_2\leqq\ldots\leqq\beta_{m''}$ with $\sum\alpha_i+\sum\beta_j=n$, (as in 12.2). We choose $m'=m''+1$. We set

(13.2.7) $$\begin{cases} A=\{\alpha_1<\alpha_2+2<\alpha_3+4<\ldots<\alpha_{m'}+2m'-2\} \\ B=\{\beta_1<\beta_2+2<\beta_3+4<\ldots<\beta_{m''}+2m''-2\}. \end{cases}$$

Then $(A,B)\in\Psi'_{2n+1,1}$ and this gives the required bijection (13.2.6).

The map (13.2.5) is defined as follows. (Recall that we now have N even.) We associate to an element of $W_G^\vee$ the unordered pair of partitions $0\leqq\alpha_1\leqq\ldots\leqq\alpha_{m'}$, $0\leqq\beta_1\leqq\beta_2\leqq\ldots\leqq\beta_{m''}$ with $N/2=\sum\alpha_i+\sum\beta_j$, as in [6, 2.7(ii)]. We choose $m'=m''$. We define A,B by (13.2.7). Then $(A,B)\in\Psi'_{2n,0}$ and this gives the required map (13.2.5).

Thus, the map (13.2.1) is defined, Composing (13.1.4) and (13.2.1) we get a map

(13.2.8) $$\mathcal{N}_G \to \Psi'_N.$$

The following results is a generalization of Shoji's main result in [15] for SO_N in the same way as 12.3 generalized Shoji's main result in [15] for Sp_{2n}.

13.3. **Theorem.** *The maps* (13.2.8) *and* (11.7.3) *coincide.*

The proof is analogous to that of 12.3.

13.4. **Corollary.** (a) *Two elements of* $\mathcal{N}_G$ *are in the same fibre of* Φ (*see* 6.5) *if and only if their images in* Ψ_N *under* (13.2.8) *have the same defect.*

(b) $\mathcal{N}_G^{(0)}$ *is non-empty if and only if* N *is a square,* (*see* (10.6.4)). *If* $N=d^2$, *the unique element of* $\mathcal{N}_G^{(0)}$ *is mapped by* (13.2.8) *to the element* $(\{0,2,4,\dots,2d-2\},\emptyset)\in\Psi'_{N,d}$; *hence this element of* $\mathcal{N}_G^{(0)}$ *is of form* $(C,\mathscr{E}^{\cdot})$ *where* $g\in C$ *has Jordan blocks of sizes given by the partition* $N=1+3+5\dots$.

(c) *Two elements* $(C,\mathscr{E}^{\cdot}),(\tilde{C},\tilde{\mathscr{E}}^{\cdot})\in\mathcal{N}_G$ *with* C, $\tilde{C}$ *non-degenerate satisfy* $C=\tilde{C}$ *if and only if the corresponding elements in* Ψ'_N (*under* (13.2.8)) *are similar.*

§ 14. Examples in the spin-groups

14.1. Let V be a vector space of dimension $N\geqq 3$ over k (of characteristic $\neq 2$) with a given non-singular symmetric bilinear form (,). Let $C(V)$ be the corresponding Clifford algebra; it is provided with an imbedding $V\subset C(V)$ and the product $v\cdot v'$ of two elements in V satisfies $v\cdot v'+v'\cdot v=2(v,v')$. Let $C^+(V)$ be the subalgebra of $C(V)$ spanned by products of an even number of elements in V. The spin-group $\mathrm{Spin}(V)$ is the subgroup of the group of units of $C^+(V)$ consisting of all products $v_1 v_2\dots v_a$ (a even) where the $v_i\in V$ satisfy $(v_i,v_i)=1$. This is a closed subgroup of the group of units of $C^+(V)$. If $x\in\mathrm{Spin}(V)$, then $v\to xvx^{-1}$ leaves V invariant and defines an element $\beta(x)$ of $SO(V)$; if $x=v_1 v_2\dots v_a$ (as above) then $\beta(x)=\beta(v_1)\beta(v_2)\dots\beta(v_a)$ where $\beta(v_i)(v)=-v+2(v,v_i)v_i$ is (-1) times the reflection with respect to v_i. Thus we have a homomorphism β: $\mathrm{Spin}(V)\to SO(V)$; it is the simply connected covering of $SO(V)$. If N is odd, the centre of $\mathrm{Spin}(V)$ has order 2; it is generated by (-1) times the unit element of $C(V)$. If N is even, the centre of $\mathrm{Spin}(V)$ has order 4; it is generated by $\varepsilon=(-1)$ times the unit element of $C(V)$ and by $\omega=v_1 v_2\dots v_N$ where $v_1,\dots,v_N$ is an orthogonal basis such that $(v_i,v_i)=1$ for all i. We have $\omega^2=\varepsilon^{N/2}$ hence the centre of $\mathrm{Spin}\,V$ is cyclic of order 4 if $N\equiv 2 \pmod 4$ and it is a product of two cyclic groups of order 2 if $N\equiv 0 \pmod 4$. In any case, the kernel of β consists of $1,\varepsilon$.

We shall also denote $\mathrm{Spin}(V)$ as G and $SO(V)$ as $SO_N(k)$ or as $\bar{G}$.

14.2. We want to describe the sets $\mathcal{N}$, $\mathcal{N}^{(0)}$, $\mathcal{M}$ for G. Each of these sets can be partitioned into pieces corresponding to the one-dimensional representations $\mathscr{Z}_G\to\bar{\mathbb{Q}}_l^*$. This decomposition into pieces is compatible with the map Φ. Let us denote $\mathcal{N}_\chi$, $\mathcal{N}_\chi^0$, $\mathcal{M}_\chi$ the pieces corresponding to χ: $\mathscr{Z}_G\to\mathbb{Q}_l^*$. (For $(C,\mathscr{E})\in\mathcal{N}_\chi$ or $\mathcal{N}_\chi^{(0)}$, $\mathscr{Z}_G$ acts on $\mathscr{E}$ according to χ; for $(L,C_1,\mathscr{E}_1^{\cdot})\in\mathcal{M}_\chi$, $\mathscr{Z}_G$ acts on $\mathscr{E}_1^{\cdot}$ according to χ.) The pieces corresponding to the χ which are trivial on the kernel of $\beta: G \to \bar{G}$, are essentially the same as the analogous pieces for the special orthogonal group $\bar{G}$, (see 10.6).

We shall therefore concentrate on the pieces corresponding to the χ which are non-trivial on the kernel of β: $G \to \bar{G}$, i.e. such that $\chi(\varepsilon) = -1$.

14.3. The unipotent classes in G and $\bar{G}$ are in $1-1$ correspondence by β. Let $g \in G$ be a unipotent element in G, and let $\bar{g} = \beta(g) \in \bar{G}$. Let i_a be the number of Jordan cells of $\bar{g}: V \to V$ of size a, so that $N = 1 \cdot i_1 + 2 \cdot i_2 + 3 \cdot i_3 + \dots$. Then $i_2, i_4, i_6, \dots$ are even. It is well known that $Z_G(\bar{g})$ is a semidirect product of a unipotent group and a reductive group isomorphic to $S(O(i_1) \times O(i_3) \times O(i_5) \times \dots) \times (Sp(i_2) \times Sp(i_4) \times Sp(i_6) \times \dots)$. (We write $O(r)$ for the orthogonal group $O_r(k)$, $Sp(2r)$ for the symplectic group $Sp_{2r}(k)$.) It is clear that $Z_G(g) \subset \beta^{-1}(Z_G(\bar{g}))$. If $g' \in \beta^{-1}(Z_G(\bar{g}))$ then $g' g g'^{-1} = g$ or εg. Now εg is not unipotent and $g' g g'^{-1}$ is, hence $g' g g'^{-1} = g$. Hence $Z_G(g) = \beta^{-1}(Z_G(\bar{g}))$. There are two possibilities: (a) $\varepsilon \in Z_G^0(g)$ and then $Z_G^0(g) = \beta^{-1}(Z_G^0(\bar{g}))$ and $Z_G(g)/Z_G^0(g) \xrightarrow{\sim} Z_G(\bar{g})/Z_G^0(\bar{g})$, or (b) $\varepsilon \notin Z_G^0(g)$ and then $\beta^{-1}(Z_G^0(\bar{g}))$ has two connected components, and $Z_G(g)/Z_G^0(g) \to Z_G(\bar{g})/Z_G^0(\bar{g})$ is a central extension with a kernel of order 2.

Assume first that $i_a \geqq 2$ for some odd a. Then there exist two $\bar{g}$-stable non-singular subspaces V', V'' of V both of dimension a, which are orthogonal to each other and an isometry $\gamma: V' \xrightarrow{\sim} V''$ commuting with the restriction of $\bar{g}$ to V', V''. Let $v'_1, \dots, v'_a$ be an orthogonal basis of V' such that $(v'_i, v'_i) = 1$ for all i, and let $v''_1, \dots, v''_a$ be the orthogonal basis of V'' defined by $v''_i = \gamma(v'_i)$. It is clear that for any $\lambda, \mu \in k$ such that $\lambda^2 + \mu^2 = 1$, the vectors $\lambda v'_1 + \mu v''_1, \lambda v'_2 + \mu v''_2, \dots, \lambda v'_a + \mu v''_a$ form an orthogonal basis for the subspace $V_{\lambda,\mu}$ they generate and that subspace is $\bar{g}$-stable and non-singular. Consider the element

$$g_{\lambda,\mu} = (v'_1 v'_2 \dots v'_a)(\lambda v'_1 + \mu v''_1) \dots (\lambda v'_a + \mu v''_a)(v'_1 v'_2 \dots v'_a)(\lambda v'_1 + \mu v''_1) \dots (\lambda v'_a + \mu v''_a) \in \mathrm{Spin}(V).$$

The image $\bar{g}_{\lambda,\mu}$ of $g_{\lambda,\mu}$ in $SO(V)$ commutes with $\bar{g}$. Indeed, the product of the reflections with respect to $v'_1, v'_2, \dots, v'_a$ is equal to -1 on V' and $+1$ on $(V')^\perp$ (hence commutes with $\bar{g}$) and similarly, the product of the reflections with respect to $\lambda v'_1 + \mu v''_1, \dots, \lambda v'_a + \mu v''_a$ is equal to -1 on $V_{\lambda,\mu}$ and $+1$ on $(V_{\lambda,\mu})^\perp$ (hence again commutes with $\bar{g}$). Hence $g_{\lambda,\mu} \in \beta^{-1} Z_G(\bar{g}) = Z_G(g)$. The map $\{(\lambda, \mu) \in k \times k \mid \lambda^2 + \mu^2 = 1\} \to Z_G(g)$ defined by $\lambda, \mu \to g_{\lambda,\mu}$ must have as image an irreducible subset of $Z_G(g)$. Since $g_{1,0} = 1$, it follows that $g_{\lambda,\mu} \in Z_G^0(g)$ for all λ, μ. We have $g_{0,1} = -1$ (since a is odd). It follows that $-1 \in Z_G^0(g)$ hence we are in case (a) above.

Next, assume that $i_a \leqq 1$ for all odd a. Then the identity component of $S(O(i_1) \times O(i_3) \times O(i_5) \times \dots) \times (Sp(i_2) \times Sp(i_4) \times \dots)$ is simply connected. It follows that $\beta^{-1}(Z_G^0(\bar{g}))$ is disconnected and hence we are in case (b) above. In this case, we can describe the group $Z_G(g)/Z_G^0(g)$, as follows. Let $I = \{a \text{ odd} \mid i_a = 1\}$. We can write $V = (\bigoplus_{a \in I} V_a) \oplus \tilde{V}$, (orthogonal direct sum of $\bar{g}$-stable subspaces, with $\dim V_a = a$ for $a \in I$). Assume that I is non empty.

For each $a \in I$ we consider an orthogonal basis $v_1^a, v_2^a, \dots, v_a^a$ of V_a such that $(v_i^a, v_i^a) = 1$ for all a, i. Let $x_a = v_1^a v_2^a \dots v_a^a \in C(V)$, and let $\hat{\Gamma}$ be the subgroup of the group of units of $C(V)$ generated by the x_a $(a \in I)$. (The generators x_a satisfy the relations $x_a^2 = \varepsilon^{a(a-1)/2}$, $x_a x_{a'} = \varepsilon x_{a'} x_a$.) Let Γ be the subgroup of $\hat{\Gamma}$ consisting of elements which are products of an even number of generators x_a. Then $\Gamma \subset \mathrm{Spin}(V) = G$; it is easy to see that $\Gamma \subset Z_G(g)$ and that the natural map $\Gamma \to Z_G(g)/Z_G^0(g)$ is an isomorphism. The central extension $Z_G(g)/Z_G^0(g) \to Z_{\bar{G}}(\bar{g})/Z_{\bar{G}}^0(\bar{g})$ can then be described as $\Gamma \to \Gamma/\{1, \varepsilon\}$. Consider the group algebra of Γ modulo the ideal generated by $\varepsilon + 1$. It is clear that this algebra is just the $+$part of the Clifford algebra of the quadratic form $\sum_{a \in I} (-1)^{a(a-1)/2} X_a^2$ in the $|I|$ variables X_a. By [N. Bourbaki, Algèbre, IX § 9 n° 4], this algebra is simple if $|I|$ is odd and it has exactly two simple components of equal dimension if $|I|$ is even. It follows that, if $|I|$ is odd, Γ has exactly one irreducible representation on which ε acts as -1 (its dimension is $2^{(|I|-1)/2}$); if $|I|$ is even and >0, then Γ has exactly two irreducible representations on which ε acts as -1 (they both have dimension $2^{(|I|-2)/2}$).

If I is empty, then N is divisible by 4 and it is clear that $Z_G(g)/Z_G^0(g)$ is of order 2 hence this group has a unique irreducible representation on which ε acts as -1. In this case, $\beta(g)$ is a degenerate unipotent element in $\bar{G} = SO(V)$. Let g' be a unipotent element in G such that $\beta(g')$ is degenerate, with the same kind of Jordan cells as $\beta(g)$, but not conjugate to $\beta(g)$. Then g, g' can be distinguished as follows. The image of $\omega \in \mathscr{Z}_G$ in one of the groups $Z_G(g)/Z_G^0(g)$, $Z_G(g')/Z_G^0(g')$ is trivial and its image in the other is non-trivial. This follows from the fact that there is an outer automophism of G taking the class of g to the class of g' and interchanging ω and $\varepsilon\omega$.

The previous arguments give the following result.

14.4. **Proposition.** *Let $G = \mathrm{Spin}(V)$, $\dim V = N \geqq 3$ and let $\chi: \mathscr{Z}_G \to \bar{\mathbb{Q}}_l^*$ be a character such that $\chi(\varepsilon) = -1$. Then there is a $1-1$ correspondence between the set $\mathscr{N}_\chi$ (see 14.2) and the set of partitions $N = 1i_1 + 2i_2 + 3i_3 + \dots$ such that $i_1, i_2, i_3, \dots \geqq 0$, $i_2, i_4, i_6, \dots$ even, $i_1, i_3, i_5, \dots \in \{0, 1\}$. The correspondence is obtained by attaching to $(C, \mathscr{E}) \in \mathscr{N}_\chi$ the partition for which i_a is the number of Jordan cells of size a of $\beta(g)$, $(g \in C)$. If $N = li_1 + 2i_2 + 3i_3 + \dots$ corresponds in this way to $(C, \mathscr{E}) \in \mathscr{N}_\chi$, then $\mathscr{E}$ is a local system of dimension 2^r on C, where r is defined as follows. Let $I = \{a \text{ odd} \mid i_a = 1\}$. Then*

$$r = \begin{cases} (|I|-1)/2 & \text{if } |I| \text{ is odd} \\ (|I|)-2)/2 & \text{if } |I| \text{ is even}, >0 \\ 0 & \text{if } |I| = 0. \end{cases}$$

In particular, for a given unipotent class C in G, there is at most one G-equivariant irreducible local system $\mathscr{E}$ on C such that $(C, \mathscr{E}) \in \mathscr{N}_\chi$.

14.5. **Corollary.**

$$* \mathscr{N}_\chi] = \sum_{t \in \mathbb{Z}} p_2\left(\frac{N-(2t^2-t)}{4}\right).$$

(As $p_2(j)$ is taken to be zero unless j is an integer $\geqq 0$, we may as well restrict the sum to those t for which $N \equiv (2t^2 - t)(\mathrm{mod}\, 4)$ i.e. $N \equiv t (\mathrm{mod}\, 4)$.)

Proof. For each integer t, we consider the set $\mathscr{P}_{N,t}$ consisting of all partitions of N such that

(a) each odd part appears at most once
(b) each even part appears an even number of times
(c) the number of parts equal to 1 (mod 4) minus the number of parts equal to 3 (mod 4) is equal to t.

It is clear that $\mathscr{P}_{N,t}$ is empty unless $t \equiv N \pmod 4$. From 14.5, we have $|\mathscr{N}_\chi| = \sum_t |\mathscr{P}_{N,t}|$. It is therefore enough to prove the following result

$$(14.5.1) \qquad |\mathscr{P}_{N,t}| = p_2\left(\frac{N-(2t^2-t)}{4}\right);$$

we now show that this formula is equivalent to Jacobi's triple product formula [5, Th. 352].

Let Z_1, Z_2 be two indeterminates. We have

(14.5.2)
$$\sum_{\substack{N \geqq 0 \\ t \in \mathbf{Z}}} |\mathscr{P}_{N,t}| Z_1^N Z_2^t = \sum_{\substack{\delta_1, \delta_3, \ldots \in \{0,1\} \\ j_2, j_4, \ldots \geqq 0}} Z_1^{(\delta_1 + 3\delta_3 + 5\delta_5 + \ldots) + 2(2j_2 + 4j_4 + \ldots)} Z_2^{\delta_1 - \delta_3 + \delta_5 - \delta_7 + \ldots}$$
$$= (1-Z_1^4)^{-1}(1-Z_1^8)^{-1}(1-Z_1^{12})^{-1} \ldots (1+Z_1Z_2)(1+Z_1^3Z_2^{-1}) \cdot (1+Z_1^5Z_2)(1+Z_1^7Z_2^{-1}) \ldots.$$

On the other hand

$$\sum_{\substack{N \geqq 0 \\ t \in \mathbf{Z}}} p_2\left(\frac{N-(2t^2-t)}{4}\right) Z_1^N Z_2^t = \sum_{m,t} p_2(m) Z_1^{4m+(2t^2-t)} Z_2^t$$
$$= \Big(\sum_m p_2(m) Z_1^{4m}\Big) \sum_{t \in \mathbf{Z}} (Z_1^2)^{t^2} \left(\frac{Z_1}{Z_2}\right)^t$$
$$= \prod_{i \geqq 1} (1-Z_1^{4i})^{-2} \prod_{i \geqq 1} (1-Z_1^{4i}) \prod_{i \geqq 1} \left(1+Z_1^{4i-2}\frac{Z_2}{Z_1}\right) \prod_{i \geqq 1} \left(1+Z_1^{4i-2}\frac{Z_1}{Z_2}\right)$$

(by Jacobi's formula); this is clearly equal to (14.5.2). Thus, (14.5.1) and hence the Corollary are proved.

We shall prove the following result

14.6. **Proposition.** *If* χ *is as in* 14.4, *we have*

$$|\mathscr{N}_\chi^{(0)}| = \begin{cases} 1, & \text{if } N = j(j+1)/2 \quad \text{for some} \quad j \geqq 2 \\ 0, & \text{otherwise.} \end{cases}$$

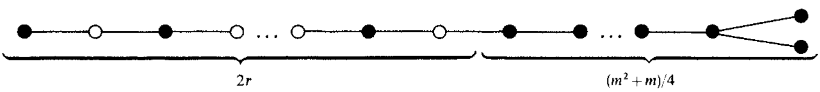
The proof will follow along the same lines as the proofs in § 10.

The desired result is true for $N=3$ or 4, by the known results for $SL_2(k)$, $SL_2(k) \times SL_2(k)$, see (10.3.2). We assume now that $N \geqq 5$ and that the result is already proved for $\tilde{N} < N$. Using the known results for smaller groups we classify the triples $(L, C_1, \mathscr{E}_1^\bullet) \in \mathscr{M}_\chi$ with $L \neq G$, up to conjugacy. ($\mathscr{M}_\chi$ is as in 14.2.) We see that there are the following possibilities.

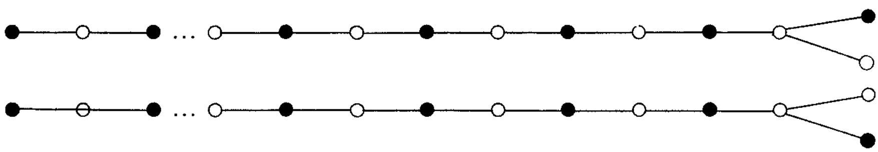
(a) L corresponds to a diagram

$$\underbrace{A_1 \times \ldots \times A_1}_{r \text{ factors}} \times D_{\frac{m^2+m}{4}} \subset D_n \quad \left(\text{where } N=2n,\ n=2r+\frac{m^2+m}{4},\ m \geqq 3,\ r \geqq 1\right)$$

represented by black dots:



(b) L corresponds to exactly one of the following two diagrams $\underbrace{A_1 \times A_1 \times \ldots \times A_1}_{r \text{ factors}} \subset D_n$ (where $N=2n$, $n=2r$) represented by black dots:



(The diagram which actually appears depends on χ.)

(c) L corresponds to a diagram $\underbrace{A_1 \times \ldots \times A_1}_{r \text{ factors}} \times B_{\frac{m^2+m-2}{4}} \subset B_n$ $\left(\text{where } N=2n+1,\ n=2r+\frac{m^2+m-2}{4},\ m \geqq 2,\ r \geqq 1\right)$ represented by black dots:

(d) L corresponds to a diagram $\underbrace{A_1 \times \ldots \times A_1}_{r \text{ factors}} \subset B_n$ (where $N=2n+1$, $n=2r$) represented by black dots:

In each case, $(C_1, \mathscr{E}_1^{\boldsymbol{\cdot}})$ is uniquely determined by L and $N(L)/L$ is a Coxeter group of type B_r.

If $\mathscr{E}_1$ is the local system $1 \boxtimes \mathscr{E}_1^{\boldsymbol{\cdot}}$ on $\mathscr{Z}_L^0 \cdot C_1$, we have $\mathscr{A}_{\mathscr{E}_1} \approx \bar{\mathbb{Q}}_l[\mathscr{W}_{\mathscr{E}_1}] = \bar{\mathbb{Q}}_l[N(L)/L]$, (by 9.2), so that $\Phi^{-1}(L, C_1, \mathscr{E}_1^{\boldsymbol{\cdot}})$ consists of $p_2(r)$ elements. It follows that

$$|\mathscr{N}_\chi - \mathscr{N}_\chi^{(0)}| = \sum_{\substack{m \geqq 0 \\ (m^2+m)/ < N}} p_2\left(\frac{N-(m^2+m)/2}{4}\right).$$

On the other hand, Corollary 14.5 can be rewritten as

$$|\mathscr{N}_\chi| = \sum_{m \geqq 0} p_2\left(\frac{N-(m^2+m)/2}{4}\right)$$

and the desired formula for $|\mathscr{N}_\chi^{(0)}|$ follows.

It is likely that, if $(C, \mathscr{E}) \in \mathscr{N}_\chi^{(0)}$ and $g \in C$, then the sizes of the Jordan cells $\beta(g) \in SO(V)$ give the partition $N = 1+5+9+13+\ldots$ or $N = 3+7+11+15+\ldots$

§ 15. Examples in the exceptional groups

15.1. In this section, we assume that G is simply connected, almost simple of type E_6. From [12], we see that

$$(15.1.1) \qquad |\mathcal{N}| = \begin{cases} 39, & \text{if } \operatorname{char}(k) \neq 2, 3 \\ 27, & \text{if } \operatorname{char}(k) = 3 \\ 44, & \text{if } \operatorname{char}(k) = 2. \end{cases}$$

We now classify the triples $(L, C_1, \mathscr{E}_1^{\bullet}) \in \mathscr{M}_G$ with $L \neq G$ (up to conjugacy) using the already known results for smaller groups. We see that there are the following possibilities:

a) L = maximal torus; then $(C_1, \mathscr{E}_1^{\bullet})$ is uniquely determined by L.

b) L corresponds to the diagram of type $A_2 \times A_2 \subset E_6$ (when $\operatorname{char}(k) \neq 3$); then $L/\mathscr{Z}_L^0 \approx SL_3(k) \times SL_3(k)$ modulo a cyclic group of order 3 imbedded diagonally in the centre. Hence, there are two possibilities for $(C_1, \mathscr{E}_1^{\bullet})$ (see (10.3.2)); the group $\mathscr{Z}_G$ (of order 3) acts on one of them by one of its non trivial characters and on the other by its other non-trivial character.

c) L corresponds to the diagram of type $D_4 \subset E_6$ (when $\operatorname{char}(k) = 2$); then $(C_1, \mathscr{E}_1^{\bullet})$ is uniquely determined (see 10.7).

In each case, $N(L)/L$ is a Coxeter group (of type E_6 in (a), of type G_2 in (b), of type A_2 in (c)).

Hence we can compute the number of elements in the fiber of Φ over any of the elements in a), b), c). (It is the number of irreducible representations of a Coxeter group of type E_6, G_2 or A_2). Hence we can determine $|\mathcal{N} - \mathcal{N}^{(0)}|$. We find:

$$|\mathcal{N} - \mathcal{N}^{(0)}| = \begin{cases} 25+6+6=37, & \text{if } \operatorname{char}(k) \neq 2, 3 \\ 25, & \text{if } \operatorname{char}(k) = 3 \\ 25+6+6+3=40, & \text{if } \operatorname{char}(k) = 2. \end{cases}$$

Comparing with (15.1.1), it follows that

$$(15.1.2) \qquad |\mathcal{N}^{(0)}| = \begin{cases} 2, & \text{if } \operatorname{char} k \neq 2 \\ 4, & \text{if } \operatorname{char} k = 2. \end{cases}$$

In the case where $\operatorname{char}(k) \neq 2, 3$, we denote by $(C, \mathscr{E})$, $(C', \mathscr{E}')$ the two elements of $\mathcal{N}^{(0)}$. The centre $\mathscr{Z}_G$ (of order 3) acts non trivially on the two local systems $\mathscr{E}, \mathscr{E}'$. This follows from the fact that $\mathcal{N}$ (for the adjoint group G_{ad}) has 25 elements, hence by arguments similar to those above, $\mathcal{N}^{(0)}$ (for G_{ad}) is empty. By 2.5, $(C, \mathscr{E}^*)$ is an element of $\mathcal{N}^{(0)}$. ($\mathscr{E}^*$ is the dual of $\mathscr{E}$; it is necessarily distinct from $\mathscr{E}$, since a non-trivial character of $\mathscr{Z}_G$ cannot be equal to its inverse.) Thus, we have $C' = C$, $\mathscr{E}' = \mathscr{E}^*$ and $\mathscr{E}, \mathscr{E}'$ are non-constant. One can show that C is the unique unipotent class of dimension 66.

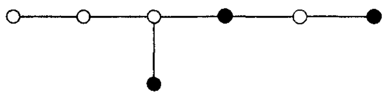
15.2. In this section, we assume that G is simply connected, almost simple of type E_7. From [13], we see that

(15.2.1) $$\mathcal{N} = \begin{cases} 86, & \text{if } \operatorname{char}(k) \neq 2, 3 \\ 92, & \text{if } \operatorname{char}(k) = 3 \\ 72, & \text{if } \operatorname{char}(k) = 2. \end{cases}$$

We now classify the triples $(L, C_1, \mathcal{E}_1^{\cdot}) \in \mathcal{M}_G$, with $L \neq G$, (up to conjugacy) using the already known results for smaller groups. We find the following possibilities:

(a) $L =$ maximal torus; then $(C_1, \mathcal{E}_1^{\cdot})$ is uniquely determined by L.

(b) L corresponds to the following diagram of type $A_1 \times A_1 \times A_1 \subset E_7$, represented by black dots:



(when $\operatorname{char}(k) \neq 2$). Then $L/\mathcal{Z}_L^0 \approx SL_2(k) \times SL_2(k) \times SL_2(k)$ modulo the unique central subgroup of order 4 which is invariant by all permutations of the three factors. In this case $(C_1, \mathcal{E}_1^{\cdot})$ is uniquely determined by L, (see (10.3.2)).

(c) L corresponds to the diagram $E_6 \subset E_7$, (when $\operatorname{char}(k) = 3$). In this case, there are two possibilities for $(C_1, \mathcal{E}_1^{\cdot})$, (see (15.1.2)).

(d) L corresponds to the diagram $D_4 \subset E_7$, (when $\operatorname{char}(k) = 2$). In this case, $(C_1, \mathcal{E}_1^{\cdot})$ is uniquely determined by L, (see 10.7).

In each case, $N(L)/L$ is a Coxeter group (of type E_7 in (a), of type F_4 in (b), of type A_1 in (c), of type B_3 in (d)).

Hence we can compute the number of elements in the fibre of Φ over any of the elements in (a), (b), (c), (d). (It is the number of irreducible representations of a Coxeter group of type E_7, F_4, A_1 or B_3.) Hence we can determine $|\mathcal{N} - \mathcal{N}^{(0)}|$. We find

$$|\mathcal{N} - \mathcal{N}^{(0)}| = \begin{cases} 60 + 25 = 85, & \text{if } \operatorname{char}(k) \neq 2, 3 \\ 60 + 25 + 2 + 2 = 89, & \text{if } \operatorname{char}(k) = 3 \\ 60 + 10 = 70, & \text{if } \operatorname{char}(k) = 2. \end{cases}$$

Comparing with (15.2.1) it follows that

(15.2.2) $$|\mathcal{N}^{(0)}| = \begin{cases} 1, \text{ if } \operatorname{char}(k) \neq 2, 3 \\ 3, \text{ if } \operatorname{char}(k) = 3 \\ 2, \text{ if } \operatorname{char}(k) = 2. \end{cases}$$

By entirely similar arguments, we see that, if $\operatorname{char}(k) \neq 2$, 3, then $\mathcal{N}^{(0)}$ (for the adjoint group G_{ad}) is empty. It follows that

(15.2.3) *If* $\operatorname{char}(k) \neq 2$, 3, *then the unique element* $(C, \mathcal{E}) \in \mathcal{N}^{(0)}$ *is such that* $\mathcal{Z}_G$ *(of order* 2) *acts non-trivially on* $\mathcal{E}$.

One can show that, if $\operatorname{char}(k) \neq 2, 3$, C in (15.2.3) is the unique unipotent class in G of dimension 112, satisfying $|Z_G(g)/Z_G^0(g)| = 12$ for $g \in C$.

15.3. In this section, we assume that G is simple of type E_8. From [13], we see that

$$(15.3.1)\qquad |\mathcal{N}|=\begin{cases}113, & \text{if } \operatorname{char}(k)\neq 2,3,5\\ 117, & \text{if } \operatorname{char}(k)=5\\ 127, & \text{if } \operatorname{char}(k)=3\\ 146, & \text{if } \operatorname{char}(k)=2.\end{cases}$$

We now classify the triples $(L, C_1, \mathscr{E}_1^{\bullet})\in\mathcal{M}_G$ with $L\neq G$ (up to conjugacy) using the already known results for smaller groups. We find the following possibilities:

(a) L = maximal torus; then $(C_1, \mathscr{E}_1^{\bullet})$ is uniquely determined by L.

(*b*) L corresponds to the diagram $E_6\subset E_8$, (when $\operatorname{char}(k)=3$). In this case, there are two possibilities for $(C_1, \mathscr{E}_1^{\bullet})$, (see (15.1.2)).

(c) L corresponds to the diagram $E_7\subset E_8$, (when $\operatorname{char}(k)=2$). In this case, there are two possibilities for $(C_1, \mathscr{E}_1^{\bullet})$, (see (15.2.2)).

(d) L corresponds to the diagram $D_4\subset E_8$, (when $\operatorname{char}(k)=2$). In this case, $(C_1, \mathscr{E}_1^{\bullet})$ is uniquely determined by L.

In each case, $N(L)/L$ is a Coxeter group (of type E_8 in case (a), of type G_2 in case (b), of type A_1 in case (c), of type F_4 in case (d)).

Hence we can compute the number of elements in the fibre of Φ over any of the elements in (a), (b), (c), (d). (It is the number of irreducible representations of a Coxeter group of type E_8, G_2, A_1 or F_4.) Hence we can determine $|\mathcal{N}-\mathcal{N}^{(0)}|$. We find

$$|\mathcal{N}-\mathcal{N}^{(0)}|=\begin{cases}112, & \text{if } \operatorname{char}(k)\neq 2,3\\ 112+6+6=124, & \text{if } \operatorname{char}(k)=3\\ 112+25+2+2=141, & \text{if } \operatorname{char}(k)=2.\end{cases}$$

Comparing with (15.3.1), it follows

$$|\mathcal{N}^{(0)}|=\begin{cases}1, & \text{if } \operatorname{char}(k)\neq 2,3,5\\ 5, & \text{if } \operatorname{char}(k)=5 \text{ or } 2\\ 3, & \text{if } \operatorname{char}(k)=3.\end{cases}$$

From the results in [19] it follows that, when $\operatorname{char}(k)\neq 2,3,5$, the unique element of $\mathcal{N}^{(0)}$ is $(C, \mathscr{E})$, where C is the unique unipotent class such that for $g\in G$, we have $Z_G(g)/Z_G^0(g)\approx\mathfrak{S}_5$, and $\mathscr{E}$ corresponds to the sign character of $\mathfrak{S}_5$.

15.4. In this section, we assume that G is simple of type F_4. From [14, 16], we see that

$$(15.4.1)\qquad |\mathcal{N}|=\begin{cases}26, & \text{if } \operatorname{char}(k)\neq 2,3\\ 28, & \text{if } \operatorname{char}(k)=3\\ 35, & \text{if } \operatorname{char}(k)=2.\end{cases}$$

The triples $(L, C_1, \mathscr{E}_1^{\bullet})\in\mathcal{M}_G$ with $L\neq G$ can be classified as follows (up to conjugacy).

(a) L may be a maximal torus; then $(C_1, \mathscr{E}_1^{\cdot})$ is uniquely determined by L.

(b) L may correspond to the diagram $B_2 \subset F_4$, (when $\mathrm{char}(k)=2$); then $(C_1, \mathscr{E}_1^{\cdot})$ is uniquely determined by L.

$N(L)/L$ is a Coxeter group of type F_4 in case (a) and of type B_2 in case (b); the fibres of Φ over one of the elements (a), (b) of $\mathscr{M}_G$ have a number of elements equal to the number of irreducible representations of the corresponding Coxeter group. From this, we get

$$|\mathscr{N}-\mathscr{N}^{(0)}|=\begin{cases}25, & \text{if } \mathrm{char}(k)\neq 2\\ 30, & \text{if } \mathrm{char}(k)=2\end{cases}$$

and hence

$$|\mathscr{N}^{(0)}|=\begin{cases}1, & \text{if } \mathrm{char}(k)\neq 2,3\\ 3, & \text{if } \mathrm{char}(k)=3\\ 5, & \text{if } \mathrm{char}(k)=2.\end{cases}$$

From [16] it follows that, if $\mathrm{char}(k)\neq 2,3$, the unique element of $\mathscr{N}^{(0)}$ is $(C, \mathscr{E})$ where C is the unique unipotent class such that for $g \in C$, we have $Z_G(g)/Z_G^0(g) \approx \mathfrak{S}_4$ and $\mathscr{E}$ corresponds to the sign character of $\mathfrak{S}_4$.

15.5. In this section, we assume that G is simple of type G_2. It is known that

$$|\mathscr{N}|=\begin{cases}7, & \text{if } \mathrm{char}(k)\neq 2,3\\ 9, & \text{if } \mathrm{char}(k)=3\\ 8, & \text{if } \mathrm{char}(k)=2.\end{cases}$$

There is a unique triple $(L, C_1, \mathscr{E}_1^{\cdot}) \in \mathscr{M}_G$ with $L \neq G$ (up to conjugacy): L is a maximal torus and $(C_1, \mathscr{E}_1^{\cdot})$ is uniquely determined by L. It follows that

$|\mathscr{N}-\mathscr{N}^{(0)}|=6=$ number of irreducible representations of the Weyl group.

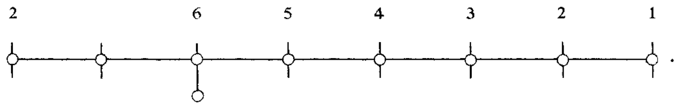
Hence,

$$|\mathscr{N}^{(0)}|=\begin{cases}1, & \text{if } \mathrm{char}(k)\neq 2,3\\ 3, & \text{if } \mathrm{char}(k)=3\\ 2, & \text{if } \mathrm{char}(k)=2.\end{cases} \tag{15.5.1}$$

The elements of $\mathscr{N}^{(0)}$ can be described as follows. Let C be the subregular unipotent class in G. There is a unique one-dimensional non-constant G-equivariant local system $\mathscr{E}$ on C and we have $(C, \mathscr{E}) \in \mathscr{N}^{(0)}$. Let C_0 be the regular unipotent class in G. If $\mathrm{char}(k)=3$, then $(C_0, \mathscr{E}) \in \mathscr{N}^{(0)}$ where $\mathscr{E}$ is any of the two one-dimensional non-constant G-equivariant local systems on C_0. If $\mathrm{char}(k)=2$, then $(C_0, \mathscr{E}) \in \mathscr{N}^{(0)}$ where $\mathscr{E}$ is the unique one-dimensional non-constant G-equivariant local system on C_0.

15.6. Using the results in § 10, 14, 15 and (2.10.1) we obtain a classification of all cuspidal pairs $(S, \mathscr{E})$ of any simply connected almost simple group G. We shall make this explicit in the case where G is of type E_8. In this case there are exactly 13 cuspidal pairs for G (in any characteristic).

In the case where $\dim(k)\neq 2,3,5$, they can be described in a concise way by the following diagram:

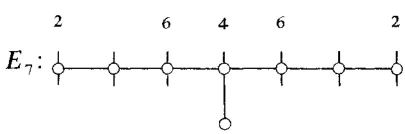


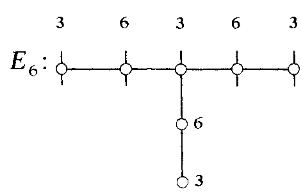
This is interpreted as follows. Each vertex of the diagram corresponds, as it is well known, to an isolated semisimple conjugacy class. Only those vertices which are marked by a number will play a role. Consider one of the marked vertices and let i be the corresponding mark ($1 \leqq i \leqq 6$). Let s be a semisimple element in the corresponding conjugacy class. Then the centre of $Z_G(s)$ is a cyclic group of order i. Let θ be any one dimensional faithful representation of the centre of $Z_G(s)$. Given s, θ, there is a unique cuspidal pair $(S, \mathscr{E})$ for G with the following properties:

(a) $\mathscr{E}$ is one dimensional
(b) there is an element $g \in S$ with semisimple part s
(c) the centre of $Z_G(s)$ acts on the stalk of $\mathscr{E}$ at g by the character θ.

The $13 = \phi(1) + 2\phi(2) + \phi(3) + \phi(4) + \phi(5) + \phi(6)$ cuspidal pairs thus obtained exhaust all cuspidal pairs of G. (ϕ is the Euler function.)

The classification of the cuspidal pairs of the simply connected groups of type E_7, E_6 for $\operatorname{char} k \neq 2, 3$ can be described in an entirely similar way, in terms of digrams as above:





Thus we have 8 (resp. 14) cuspidal pairs for G of type E_7 (resp. E_6).

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