Invent. math. 80, 161–179 (1985)

Inventiones mathematicae
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Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map*

Feliks Przytycki

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, PL-00-950 Warsaw, Poland

§ 0. Introduction

Anthony Manning in [M] proved the following

Theorem [Manning]. *Suppose that each critical point c of a polynomial f: $\mathbb{C}\to\mathbb{C}$ satisfies $f^n(c)\not\to\infty$ and that* $\mathrm{cl}(\mathrm{Orb}_f^+(c))\cap J(f)=\emptyset$. *(*$\mathrm{cl}\,\mathrm{Orb}_f^+(c)$ *denotes closure of the orbit of c under forward iterations with f. J(f) denotes the Julia set for f). Then for the maximal measure m (i.e. such that $h_m(f)=h_{\mathrm{top}}(f)=\deg f$) the Lyapunov characteristic exponent $\int\log|f'|\,dm=\chi_m(f)$ is equal to* $\deg f$. *The Hausdorff dimension of m is* 1.

The Condition $\mathrm{cl}\,\mathrm{Orb}_f^+(c)\cap J(f)=\emptyset$ for every critical point c is equivalent to the expanding assumption on $J(f)$ (see [F], [J] or [Br]). It means there exists $n>0$, such that $|(f^n)'(z)|>1$ for all z in $J(f)$. Hausdorff dimension of a probability measure m (or of a set Y) will be denoted $\mathrm{HD}(m)$ (respect. by $\mathrm{HD}(Y)$). It is defined by: $\mathrm{HD}(m)=\inf\{\mathrm{HD}(Y): m(Y)=1\}$. $h_m(f)(h_{\mathrm{top}}(f))$ denotes measure (topological) entropy. The condition $f^n(c)\not\to\infty$ means that the attractive basin of ∞ is a topological disc A_∞. Also $J(f)=\mathrm{Fr}\,A_\infty$.

In the present paper I will prove the equality $\mathrm{HD}(m)=1$ in a more general situation. Consider any open topological disc A in the Riemann sphere S^2 such that $\mathrm{Card}(S^2\setminus A)>1$. Assume there exists a holomorphic map f defined on a neighbourhood of $\mathrm{cl}\,A$ such that $f(A)=A$, $f(\mathrm{Fr}\,A)=\mathrm{Fr}\,A$ and A is attracted to a sink $a\in A$ i.e. $f(a)=a$, $f^n|A\to a$ (This yields $d=\deg(f|A)>1$, degree means number of pre-images of any regular value).

I define and discuss now the measure m which will be under consideration. I learned this is so-called harmonic measure, see the Note 1 at the end of this Introduction.

Let R: $D^2\to A$ be a Riemann map (conformal homeomorphism) from the unit disc D^2 such that $R(0)=a$. Let

$$g(z)=z\cdot\prod_{i=2}^{d}\frac{z-a_i}{1-\bar{a}_i z}$$

* This paper in a preprint form was entitled "On the boundary of an attractive basin of a sink for a rational map on the Riemann sphere"

where $\{a_2, \ldots, a_i\} = R^{-1}(f^{-1}(a)))$. Observe that, due to the maximum principle, $|R^{-1} \circ f \circ R/g| \leqq 1$ and $|g/R^{-1} \circ f \circ R| \leqq 1$ on D^2. Hence, for some real number α, we have $g = \exp(i\alpha) R^{-1} \circ f \circ R$. If $\alpha \neq 0$ replace R by $R \circ \exp(-i\alpha/1-d)$. Then

$$g = R^{-1} \circ f \circ R.$$

The length measure h on $S^1 = \operatorname{Fr} D^2$ is g-invariant (*Proof.* Let φ denote an arbitrary continuous function on S^1 and $\bar{\varphi}$ its harmonic extension to D^2. $\int \varphi \, dl = \bar{\varphi}(0) = \bar{\varphi}(g(0)) = \int \varphi \circ g \, dl$. $\square$). Let $\bar{R} = R$ on D^2 and $\bar{R}$ be defined as the radial limit of R on a set $\mathscr{D}(\bar{R})$ of those points from S^1 where the non-tangential limit for R exists. It is known that $l(\mathscr{D}(\bar{R})) = 1$. Clearly $\bar{R}$ is measurable and in our situation $\mathscr{D}(\bar{R})$ is g-invariant

$$g^{-1}(\mathscr{D}(\bar{R})) = \mathscr{D}(\bar{R}).$$

Put $m = \bar{R}_*(l)$, this is an f-invariant measure.

We consider characteristic exponents $\chi_\mu(F) = \int \log |F'| \, d\mu$ in this paper, for functions F and ergodic measures μ. We always check $\log|F'|$ is μ-integrable; then $\chi_\mu(F) = \lim_{n \to \infty} \frac{1}{n} \log |(F^n)'(z)|$ for μ-almost every z and $\chi_\mu(f)$ is finite. If $\operatorname{supp}(\mu) \subset \mathbb{C}$ it is no matter we consider F' with respect to the standard Riemannian metric on S^2 or the Euclidean one on $\mathbb{C}$. In our cases the same happens even when $\operatorname{supp} \mu \not\subset \mathbb{C}$.

Theorem. *If f, g and m satisfy the above hypotheses then*

$$h_m(f) = h_l(g) = \chi_l(g) = \chi_m(f), \tag{1}$$

$$\operatorname{HD}(m) = h_m(f)/\chi_m(f) = 1, \tag{2}$$

and

$$\lim_{r \to 0} \frac{\log m(B(z,r))}{\log r} = \operatorname{HD}(m) = 1 \quad \text{for } m\text{-almost } z \in \operatorname{Fr} A. \tag{3}$$

Property (3) is a weak differentiability property for $\bar{R}$ and $\bar{R}^{-1}$.

Question. Does (3) hold for any Riemann map R of D^2 to a bounded domain $A \subset \mathbb{C}$ (without assumptions about existence of any holomorphic dynamics on A which extends beyond A)?

We give two proofs that $h_l(g) = h_m(f)$. The first proof is given in §2. We use there coding by the "tree" of pre-images of a point in A and the ideas from [Mi-P]. The second proof given in §4 bases on the following theorem about any Riemann map R (see [Be], [C-L] or [T]):

Theorem [Beurling]. *The logarithmic capacity $\gamma(S^1 \setminus \mathscr{D}(\bar{R})) = 0$. For every $z \in \operatorname{Fr} A$, $\gamma(R^{-1}(\{z\})) = 0$.*

Roughly speaking we prove in §4 that if the sets $\bar{R}^{-1}(\{z\})$ are "thin" in the sense of potential theory then they are "thin" in the sense of ergodic theory.

The both proofs are valid in more general situations. In particular it happens that

$$h_\mu(g) = h_{\bar{R}_*(\mu)}(f)$$

for every probability, g-invariant, ergodic measure μ, provided only $\bar{R}_*(\mu)$ is defined, i.e. provided $\mu(\mathscr{D}\bar{R})=1$.

For example $\mu(\mathscr{D}\bar{R})=1$ if μ is a *Gibbs measure* (equilibrium state) for $g|S^1$ and a Hölder continuous function on S^1 (see [Bo] for the definition). Indeed if $\mu(S^1\setminus\mathscr{D}\bar{R})>0$ then for a compact set $E\subset S^1\setminus\mathscr{D}(\bar{R})$ with $\mu(E)>0$ we would have

$$0=\gamma(E)\geqq \mathrm{HD}(E)\geqq\frac{h_\mu(g)}{\chi_\mu(g)}>0. \qquad \text{(cf. § 4)}$$

The discussion of the both proofs will be continued in § 5.2, 3.

An easy case of our Theorem is when f is expanding on $\mathrm{Fr}A$. Then R extends to $\bar{R}$ continuous on $\mathrm{cl}D^2$, see [Br]. Also A can be considered as bounded in $\mathbb{C}$ because this expanding property implies $S^2\setminus\mathrm{cl}A\neq 0$. Otherwise f would be rational on S^2 and A would contain the whole set $\mathrm{Sing}f$ of critical points $2\cdot\deg f-2=\mathrm{Card}(\mathrm{Sing}f)\geqq\deg(f|A)$ so A could not be simply connected.

To prove the equalities

$$\lim_{r\to 0}\frac{\log mB(z,r)}{\log r}=\frac{h_m(f)}{\chi_m(f)}=\mathrm{HD}(m)$$

is easy for the expanding case and in the general case they follow from the papers [Ma$_1$], [K-S], [Le$_1$], [Y]. I briefly sketch the proofs in Sect. 3.

The equality $h_l(g)=\chi_l(g)$, is known as Pesin formula, see [Pe].

Section 1 will be devoted to $\chi_m(f)=\chi_l(g)$. I use an idea in harmonic functions which I learned from [He].

The special case $g(z)=z^d$ corresponds to Manning's polynomial case. Then $h_m(f)=\chi_m(f)=\log d$. If $f|\mathrm{Fr}A$ is expanding or if f is rational and $\deg f=d$ on S^2 we have $h_{\mathrm{top}}(f|\mathrm{Fr}A)=\log d$ and m is unique maximal measure (for the expanding case see Lemma 2 or [Ja], for the second case $h_{\mathrm{top}}(f)=\log d$ is proved in [G] uniqueness in [Lju], [Ma$_2$]). I do not know whether the inequality $h_{\mathrm{top}}(f|\mathrm{Fr}A)\leqq\log d$ is true in general. (If g is not z^d then m is not a maximal measure since $h_m(f)<\log d$ and we always have $h_{\mathrm{top}}(f|\mathrm{Fr}A)\geqq\log d$ by Lemma 4 or [Mi-P].)

Note 1. The idea of extending the $\mathrm{HD}(m)=1$ result from the polynomial to the general case but for the measure m rather than the maximal one belongs to Manning. On his lecture in Warsaw he suggested to look at $J(f)$ from inside the attractive basin. Also he raised the Question.

After preparing the preprint version I learned from Carleson that the measure m is so-called harmonic measure at a with respect to A_a, see [T]. It has also the Brownian motion characterization as the probability distribution of the first hitting of $\mathrm{Fr}A$ for paths starting from a. For the precise definition, discusion and references for estimating $\mathrm{HD}(m)$ in a general situation (without holomorphic dynamics) see [Ø].

In the preprint [Ca$_2$] which appeared at the same time as mine Carleson proved $\mathrm{HD}(m)\leqq 1$ in the case $\mathrm{Fr}A$ has strong self-similarity properties, but not assuming the existence of f. There is striking similarity between his ideas and Manning's and my way.

My formula $\mathrm{HD}(m)=\frac{\log 2}{\log 2+\frac{1}{2}G(c)}$ for Cantor-like Julia set for z^2+c (Proposition 4) gives an illustration of Carleson's inequality $\mathrm{HD}(m)<1$ (harmonic measure at ∞ and the equilibrium distribution for the logarithmic potential coincide).

Recently the preprint [Le$_2$] having some common points with my paper arrived in Warsaw.

Note 2. The considerations of the present paper are continued in the paper [P].

Very recently my attention was payed to the paper [Mk] which contains a sketch of the proof that $\mathrm{HD}(m)\leqq 1$ in general.

§ 1. $\chi_l(g)=\chi_m(f)$

Proposition 1. *Let $R: D^2\to A\subset\mathbb{C}$ be a Riemann mapping. Let $g(z)=z\prod_{i=2}^{d}\frac{z-a_i}{1-\bar{a}_i z}$, $|a_i|<1$. Put $f=R\circ g\circ R^{-1}: A\circlearrowleft$ and assume that f' extends continuously to $\mathrm{cl}\,A$. (f' understood in the Riemannian metric from S^2.)*

Assume also that f' takes the value 0 on a finite set $\{p_1,\dots,p_s\}\subset\mathrm{Fr}\,A$ and that, for some constant $\alpha>0$, $|f'(z)|>(\rho(z,p_i))^\alpha$ for every i and z from a small neighbourhood of p_i in $\mathrm{cl}\,A$ (ρ is the metric in S^2). Then

$$\int_{S^1}\log|g'|\,dl=\int_{\mathrm{Fr}\,A}\log|f'|\,dm.$$

Proof. Assume first that $\bar{R}$ is continuous, A is bounded in $\mathbb{C}$, and f' has no zeros on $\mathrm{Fr}\,A$ (which corresponds to the expanding case). Clearly g' and $f'\circ R$ have the same zeros in D^2. Denote them by $b_1,\dots,b_{d-1}$ and put $B(z)=\prod_{i=1}^{d-1}\frac{z-b_i}{1-\bar{b}_i z}$. Then

$$\frac{\int_{S^1}\log|g'|\,dl}{\int_{\mathrm{Fr}\,A}\log|f'|\,dm}=\frac{\int_{S^1}\log\left|\frac{g'}{B}\right|dl}{\int_{S^1}\log\left|\frac{f'\circ\bar{R}}{B}\right|dl}=\frac{\log\left|\frac{g'}{B}\right|(0)}{\log\left|\frac{(R'\circ g)\,g'((R^{-1})'\circ R)}{B}\right|(0)}=1.$$

We used the fact that we are integrating harmonic functions since $\frac{g'}{B}$, $\frac{f'\circ\bar{R}}{B}$ have no zeros in $\mathrm{cl}\,D^2$. (Derivative in this Proof is considered in the Euclidean metric in $\mathbb{C}$). In the general case I refer to the following facts [D]:

(a) (see [D] p. 50). Let φ be a holomorphic function on D^2 which is univalent (Schlicht) (i.e. $z_1\neq z_2$ implies $\varphi(z_1)\neq\varphi(z_2)$) but not necessarily bounded. Then $\varphi\in H^p$ for all $p<1/2$. (This means the integrals $\int_{S^1}|\varphi(tz)|^p\,dl(z)$ are bounded for $t<1$ by a common constant.)

(b) (see [D] pp. 17, 22). Let $\varphi\in H^p$ for some $p>0$. Then the nontangential limit $\bar{\varphi}$ exists l-almost everywhere. $\log|\bar{\varphi}|$ is l-integrable and

$$\int_{S^1}\log|\bar{\varphi}|\,dl\geqq\limsup_{t\to 1}\int_{S^1}\log|\bar{\varphi}(tz)|\,dl(z).$$

If also $\frac{1}{\varphi} \in H^p$ then

$$\log|\varphi(tz)| \to \log|\bar{\varphi}(z)|, \quad \text{in the } L^1 \text{ sense as } t \nearrow 1.$$

(To see the importance of the univalency assumption check the example $\varphi(z) = \exp((1+z)/(1-z))$, see [D] p. 22.)

This applied to $\varphi = R$ gives $\bar{R}$ used for the definition of m in the Introduction.

(b) will immediately prove Proposition 1, if we check $\frac{f' \circ R}{B}, \frac{B}{f' \circ R} \in H^p$. Assume A is bounded.

Lemma 1. $\frac{f' \circ R}{B}, \frac{B}{f' \circ R} \in H^p$ *for some* $p < \frac{1}{2}$.

Proof. $\frac{f' \circ R}{B} \in H^\infty$ (is bounded). On the other hand

$$\left|\frac{f' \circ R}{B}(z)\right| \geqq C \left(\prod_{i=1}^{s} |R(z) - p_i|^\alpha\right)$$

for every $z \in \mathrm{cl}\, D^2$, some $C > 0$. But $1/(R(z) - p_i) \in H^\beta$, $\beta < \frac{1}{2}$, as univalent function. So $\frac{B}{f' \circ R} \in H^{\beta/\alpha}$. $\square$

To check that $\int_{\mathrm{Fr}A} \log|f'|\, dm$ does not depend on whether we consider a metric from S^2 or from $\mathbb{C}$ we take some $k: S^2 \circlearrowleft$ such that $k(\mathrm{Fr}A) \subset \mathbb{C}$. For example $k = \frac{1}{z-a}$ for some $a \notin \mathrm{Fr}A$. Now

$$\int_{\mathrm{Fr}k(A)} \log|(k \circ f \circ k^{-1})' \circ k|\, d(k_*(m))$$
$$= \int_{\mathrm{Fr}A} \log|f'|\, dm + \int_{\mathrm{Fr}A} \log|k' \circ f|\, dm - \int_{\mathrm{Fr}A} \log|k'|\, dm,$$

which is $\int_{\mathrm{Fr}A} \log|f'|\, dm$, provided $\log|k'|$ is m-integrable. But

$$\int_{\mathrm{Fr}A} |\log|k'||\, dm = \int_{S^1} |\log|k' \circ \bar{R}||\, dl = \int_{S^1} \left|\log\left|\left(\frac{1}{\bar{R} - a}\right)^2\right|\right| dl$$
$$= -2 \int_{S^1} |\log|\bar{R} - a||\, dl < \infty$$

by the univalency of $R - a$.

To finish the proof of Proposition 1 one needs still to prove Lemma 1 for A unbounded. One can assume $\infty \notin \{p_1, \ldots, p_s, f(p_1), \ldots, f(p_s)\}$. Then there are constants $C_1, C_2 > 0$ such that

$$C_1 \left|\frac{1}{z^2}\right| \leqq |f'(z)| \leqq C_2 |f(z)|^2 \quad \text{for all } z$$

in a neighbourhood of $\{\infty\}\cup f^{-1}(\{\infty\})$. Now use the univalency of R and finite-valency of $f\circ\bar{R}=\bar{R}\circ g$. $\square$

Remark. If in Proposition 1, we do not assume anything about zeros of f' on $\operatorname{Fr} A$ then still in view of part of (b) $\log|f'|$ is m-integrable and

$$\chi_l(g)\geqq\chi_m(f).$$

§2. $h_l(g)=h_m(f)$

Proposition 2. *Let R: $D^2\to A$ be a Riemann mapping and, for* $g=z\cdot\prod_{i=2}^{d}\frac{z-a_i}{1-\bar{a}_i z}$, $|a_i|<1$, *assume that $f=R\circ g\circ R^{-1}$ extends to a C^1-function f in a neighbourhood U of* $\operatorname{cl} A$ *(in S^2). Then*

$$h_l(g)=h_m(f).$$

Proof. First prove it in the Manning case where $g=z^d$ and $f|_{\operatorname{Fr} A}$ is expanding (so $\bar{R}$ is continuous). Choose an arbitrary point $z_0\in A\setminus R(\{0\})$. Then the points from $f^{-n}(z_0)$ are (n,ε)-separated for some ε depending on z_0. Indeed suppose that $f(z_1)=f(z_2)$ and z_1 is close to z_2 close to $\operatorname{Fr} A$ (far from $R(0)$). Then z_1 and z_2 can be joined by an interval $\gamma\subset U$ and $\gamma f'(z)d|z|=0$ implies $f'(z)$ is close to 0 for $z\in\gamma$. But this contradicts the assumption $f'|\operatorname{Fr} A$ is nowhere 0.

Equidistribute measures σ_n on the points from $g^{-n}(R^{-1}(z_0))$ (i.e. $\sigma_n(\{z\})=d^{-n}$). Then $\sigma_n\to l$ in weak*-topology, $n\to\infty$, hence by continuity of $\bar{R}$, $R_*(\sigma_n)\to m$. Since the points from $f^{-n}(\{z_0\})$ are (n,ε)-separated and $\operatorname{Card} f^{-n}(\{w\})=d^n$, $h_m(f)\geqq\log d=h_l(g)$ (see Misiurewicz's proof of the variational principle, for example [Mi]). Of course $h_m(f)\leqq h_{\text{top}}(f)\leqq h_{\text{top}}(g)=\log d$ since f is a topological factor of g ($\bar{R}$ is continuous).

In the general case σ_n converge to the measure with the maximal entropy rather than to l. Nevertheless we shall use the sets $f^{-n}(z_0)$ to make a coding of the dynamics f. Let us describe it first in the expanding case.

Lemma 2. *Assume $f|\operatorname{Fr} A$ is expanding. Let R_g be the factor map of the full shift (Σ^d,σ) to (S^1,g) $\left(\Sigma^d=\prod_1^{\infty}\{1,\dots,d\}\right.$, σ is the left $\left.\text{shift}\right)$, according to the Markov partition of S^1 into arcs p^ip^{i+1} where p^i are consecutive preimages of a fixed point $p\in S^1$. Then there exists a continuous factor map R_f of (Σ^d,σ) to $(\operatorname{Fr} A,f)$ such that*

$$\bar{R}\circ R_g=R_f \quad \text{and} \quad \operatorname{Card}(R_f^{-1}(z))\leqq C$$

for a constant number C and every $z\in\operatorname{Fr} A$.

Proof. This lemma is almost Jakobson Theorem, [Ja] Theorem 3: For any rational function $F:S^2\circlearrowleft$ which is expanding on the Julia set $J(F)$, $(J(F),F)$ is an at most C-to-1 factor of the full shift $(\Sigma^{\deg F},\sigma)$.

I will repeat Jakobson's proof to help the reader to understand the more complicated non-expanding case which will follow and to introduce the coding notation.

Describe coding of (S^1, g) (this coding itself was known before [Ja] appeared). Fix arbitrary $w \in D^2 \setminus \bigcup_{k>0} g^k(\mathrm{Sing}\, g)$. For every sequence $\alpha = (i_1, i_2, \ldots) \in \Sigma^d$ define a sequence of points $(w_n(\alpha))_{n=1}^{\infty}$, $w_n \in g^{-n}(w)$, by induction: Order points from $g^{-1}(w)$ into sequence $w^1, \ldots\ldots, w^d$ join w^i with w by a curve $\gamma_i \subset D^2 \setminus \bigcup_{k>0} g^k(\mathrm{Sing}\, g)$. Let $w_0(\alpha) = w$. After defining $w_{n-1}(\alpha)$ consider the curve $\gamma(w_n) = g_{\tau}^{-(n-1)}(\gamma_{i_n})$ where the branch $g_{\tau}^{-(n-1)}$ of $g^{-(n-1)}$ is chosen so that one end of $\gamma(w_n)$ is w_{n-1}. Define w_n as the other end of $\gamma(w_n)$.

We call such sequences of points $(w_n) = (w_n(\alpha))$ *admissible*. Admissible sequences (for g) converge uniformly since $g|S^1$ is expanding. Define $R_g(\alpha) = \lim_{n\to\infty} w_n(\alpha)$.

We define similarly admissible sequences $z_n(\alpha)$ for f in A using the curves $R(\gamma_i)$. $z_n(\alpha)$ converge uniformly since $\bar{R}$ is continuous. We define $R_f(\alpha) = \lim_{n\to\infty} z_n(\alpha)$.

$\bigcup_{i=1}^{d} \bigcup_{n=0}^{\infty} f^{-n}(\gamma_i)$ forms a "tree". This is the graph with vertices $z_n(\alpha)$, edges $\gamma(z_n(\alpha))$ for $\alpha \in \Sigma^d$. Admissible sequences z_n are just consecutive vertices along infinite "branches". The points $z_n(\alpha)$ code cylinders in Σ^d given by the first n coordinates. It is more comfortable to look at our coding as coding by admissible sequences of points $(z_n(\alpha))$, than by sequences of integers α.

Since $\mathrm{Sing} f \cap \mathrm{Fr}\, A = \emptyset$ there exists $\varepsilon > 0$, $k > 0$ such that for every $l \geqq k$

(i) if $v_1, v_2 \in f^{-1}(z_l(\alpha))$, $v_1 \neq v_2$, then $\rho(v_1, v_2) \geqq \varepsilon$,

(ii) $\rho(\lim_{n\to\infty} z_n(\alpha), z_l(\alpha)) < \frac{\varepsilon}{2}$.

We show that for every $x \in \mathrm{Fr}\, A$, $\mathrm{Card}\, R_f^{-1}(x) \leqq d^k$. Suppose the contrary. Then there exists $s > 0$ such that $\mathrm{Card}(\{z_s(\alpha) : \alpha \in R_f^{-1}(x)\}) > d^k$. Then there exist $\alpha, \beta \in R_f^{-1}(x)$ such that $z_s(\alpha) \neq z_s(\beta)$ and $f^{s-k}(z_s(\alpha)) = f^{s-k}(z_s(\beta))$. So there exists l, $s > l \geqq k$ such that $f^{s-l}(z_s(\alpha)) = f^{s-l}(z_s(\beta))$ and $f^{s-l-1}(z_s(\alpha)) \neq f^{s-l-1}(z_s(\beta))$. This gives, due to (i), (ii) $R_f(\sigma^{s-l-1}(\alpha)) \neq R_f(\sigma^{s-l-1})(\beta))$ so $R_f(\alpha) \neq R_f(\beta)$, contradiction. □

Now we can finish proving Proposition 2 for the expanding case. We have

$$h_l(g) - h_m(f) = h_l(g|f)$$

see [A-Ro]. The definition of the relative metric entropy $h_l(g|f)$ is here:

$$h_l(g|f) = \sup\{h_l(g|f, \xi) : \xi \text{ is a finite partition of } S^1\},$$

$$h_l(g|f, \xi) = \lim_{n\to\infty} n^{-1} H_l\left(\bigvee_{i=0}^{n-1} g^{-i}(\xi) \middle| \bar{R}^{-1}(\varepsilon)\right) = H_l(\xi | \xi^- \vee \bar{R}^{-1}(\varepsilon))$$

where ε is partition of $\mathrm{Fr}\, A$ into points.

$h_l(g|f) = 0$ since the partition $\bar{R}^{-1}(\varepsilon)$ consists of finite sets (of cardinality at most C), by Lemma 2.

Now we pass to proving Proposition 2 in the general (maybe nonexpanding) case:

Lemma 3. *Let R_g: $\Sigma^d \to S^1$ be as in Lemma 2. Put $\nu = (R_g^{-1}) * (l)$. Then there exists a measurable factor map*

$$R_f: (\Sigma^d, \nu, \sigma) \to (\operatorname{Fr} A, m, f)$$

such that $R_f = \bar{R} \circ R_g$ ν-a.e. and

$$h_\nu(\sigma) - h_m(f) = h_\nu(\sigma | f) = 0.$$

To prove this Lemma 3, I need two more lemmas:

Lemma 4. *Let $A \subset M$ be a compact subset of a Riemannian manifold M, with positive Riemannian measure m, i.e. $m(A) > 0$. Let f: $U \to M$ be a C^1-mapping on U, which is some neighbourhood of A. We say $x \in A$ is (f, A, ε)-singular value if there exist $y, z \in A \cap f^{-1}(x)$, $y \neq z$ and $\rho(y, z) < \varepsilon$.*

Then for every ζ, $0 < \zeta < 1$ there exists $\varepsilon = \varepsilon(f, \zeta)$ such that for m-almost every $x \in A$ if $(x_i)_{i=0}^n$ is a piece of an f-trajectory, i.e. $f(x_i) = x_{i+1}$, such that $x_n = x$, $x_i \in A$ and n is sufficiently large, then

$$\operatorname{Card}\{i\colon x_i \text{ is } (f(A, \varepsilon)\text{-singular value}\} < \zeta \cdot n.$$

Proof. See [Mi-P]. □

Lemma 5. *Let f be an analytic mapping defined on a neighbourhood of* $\operatorname{cl} A$, *where A is an open connected domain in S^2, $f(A) = A$, $f(\operatorname{Fr} A) = \operatorname{Fr} A$,* $\operatorname{Card}(S^2 \setminus A) \geqq 3$, *such that $f | \operatorname{Fr} A$ is expanding.*

Then all admissible sequences (z_n) (i.e. *built as in the proof of Lemma* 2) *converge exponentially and uniformly nontangentially to the respective points in* $\operatorname{Fr} A$, i.e.

$$\operatorname{dist}(z_n, \operatorname{Fr} A) / \operatorname{dist}(z_n, \lim_{n \to \infty} z_n) > \text{Const} > 0. \tag{1}$$

Proof. For a small neighbourhood V of $\operatorname{Fr} A$ $f(A \setminus V) \subset A \setminus V$ so f cannot be isometry in the hyperbolic metric on A, so it shortens distances (see [Hi] Th. 15.1.3). So A is attractive domain of a sink $a = \lim_{n \to \infty} f^n | A$.

Exponential convergence of z_n follows from the expanding property. Now for some $\varepsilon > 0$ choose N such that for every admissible sequence $\{z_n\}$ $B(z_N, \varepsilon) \ni \lim_{n \to \infty} z_n$. Since f is expanding all the branches f_ν^{-k} on $B(z_N, \varepsilon)$ (with $f_\nu^{-k}(z_N) \in A$) have uniformly bounded distorsions (independently of the choise of $\{z_n\}_{n > N}$). So the ratios in (1) change only by uniformly bounded factors. □

Now we can give:

Proof of Lemma 3. Since every admissible sequence w_n (for g in D^2) converges nontangentially (by Lemma 5) and since R has nontangential limit $\bar{R}$ l-almost everywhere we conclude that for ν-almost every $\alpha \in \Sigma^d$ the sequence $z_n(\alpha) = R(w_n(\alpha))$ converges (see the notation in Proof of Lemma 2). We define

$$R_f(\alpha) = \lim z_n(\alpha).$$

Now we need to prove that

$$h_\nu(\sigma | f) = 0.$$

Observe that for every η, $\varepsilon>0$ there exist a set $E_1\subset\Sigma^d$ with $\nu(\Sigma^d\setminus E_1)<\eta$ and an integer N_1 such that if $n>N_1$ then for every $\alpha\in E_1$

$$\rho(z_n(\alpha), R_f(\alpha))<\varepsilon.$$

Then by the Birkhoff-Khinchin ergodic theorem, for every integer $k>0$ there exist $E_2\subset\Sigma^d$ with $\nu(\Sigma^d\setminus E_2)<\sqrt{\eta}$ and an integer N_2 such that if $n\cdot k\geqq N_2$ and $\alpha\in E_2$ then:

$$n^{-1}\sum_{i=0}^{n-1}\chi_{E_1}(\sigma^{ik}(\alpha))>1-2\sqrt{\eta}\quad\text{where } \chi_{E_1}(x)=\begin{cases}1 & \text{for } x\in E_1\\ 0 & \text{for } x\notin E_1\end{cases}.$$

(Remark that we do not use ergodicity of σ^k.)

Take for future an arbitrary $k>0$, $0<\zeta<1$ and $\varepsilon=\frac{1}{4}\varepsilon(f^k, \mathrm{cl}\,A, \zeta)$ according to Lemma 4. Assume also $N_1\geqq k$.

Let ξ be the partition of $\mathrm{Fr}\,A$ into points and put $\vartheta=R_f^{-1}(\xi)$. Let $\mathscr{A}^n$ be the partition of Σ^d, into d^n cylinders given by the first n coordinates. Denote by Π_n: $\Sigma^d\to\{1,\dots,d\}^n$ the projection to these coordinates and define, for every $x\in\mathrm{Fr}\,A$

$$S_{n,x}=\Pi_n(E_2\cap R_f^{-1}(x)).$$

For $a\geqq 0$ put $i(a)=-a\cdot\log a$.

We have

$$\begin{aligned}
h_\nu(\sigma|f)&=\lim_{n\to\infty} n^{-1}H_\nu(\mathscr{A}^n/\vartheta)\leqq\lim_{n\to\infty} n^{-1}H_\nu(\mathscr{A}^n\vee\{E_2,\Sigma^d\setminus E_2\}/\vartheta)\\
&=\lim_{n\to\infty} n^{-1}\int_{\Sigma^d}\Big(\sum_{a\in\mathscr{A}^n} i(\nu(a\cap E_2/\vartheta(x)))+\sum_{a\in\mathscr{A}^n} i(\nu(a\setminus E_2/\vartheta(x)))\Big)\,d\nu(x)\\
&\leqq\limsup_{n\to\infty}\int_{\mathrm{Fr}\,A} n^{-1}\log\mathrm{Card}(S_{n,x})\,dm(x)\\
&\quad+\lim_{n\to\infty} n^{-1}\int_{\Sigma^d}\big(i(\nu((\Sigma^d\setminus E_2)/\vartheta(x)))+\nu((\Sigma^d\setminus E_2)/\vartheta(x))\cdot n\cdot\log d\big)\,d\nu(x)\\
&\leqq\int_{\mathrm{Fr}\,A}\limsup_{n\to\infty} n^{-1}\log\mathrm{Card}(S_{n,x})\,dm(x)+\nu(\Sigma^d\setminus E_2)\cdot\log d.
\end{aligned}$$

So we need to estimate $\mathrm{Card}(S_{n,x})$. Call z^1, $z^2\in A$ $(n,\varepsilon,\lambda,k)$-separated for positive integers n, k, for $\varepsilon>0$, $0\leqq\lambda\leqq 1$ if

$$\mathrm{Card}\{i\colon 0\leqq i<n, \rho(f^{ik}(z^1), f^{ik}(z^2))\geqq\varepsilon\}\geqq\lambda n.$$

Observe that for any two elements α, β of $E_2\cap R_f^{-1}(x)$, if $n-N_1\geqq N_2$, then the points $z_n(\alpha)$, $z_n(\beta)$ are not $\left(1+\left[\frac{n-N_1}{k}\right], 2\varepsilon, 4\sqrt{\eta}, k\right)$-separated.

Otherwise there existed l, $0\leqq l\leqq\left[\frac{n-N_1}{k}\right]$ such, that $\rho(f^{kl}(z_n(\alpha)), f^{kl}(z_n(\beta)))\geqq 2\varepsilon$ and $\sigma^{kl}(\alpha)$, $\sigma^{kl}(\beta)\in E_1$. But then $\rho(\lim_{i\to\infty} z_i(\sigma^{kl}(\alpha)), \lim_{i\to\infty} z_i(\sigma^{kl}(\beta)))\geqq\rho(f^{kl}z_n(\alpha), f^{kl}z_n(\beta))-2\varepsilon+\text{small positive number}>0$. Hence

$$R_f(\sigma^{kl}(\alpha))\neq R_f(\sigma^{kl}(\beta)),$$

hence $R_f(\alpha)\neq R_f(\beta)$, contradiction.

Fix arbitrary $a\in\Pi_{N_1}(\Sigma^d)$ and consider

$$S_{n,x}(a)=S_{n,x}\cap\Pi_n((\Pi_{N_1}\circ f^{n-N_1})^{-1}(a)).$$

Fix now $c\in S_{n,x}(a)$. Take arbitrary $D\subset\left\{0,1,\dots,\left[\frac{n-N_1}{k}\right]\right\}$ with $\operatorname{Card}D=\left[4\sqrt{\eta}\left(\left[\frac{n-N_1}{k}\right]+1\right)\right]$ and consider

$$S_{n,x}(a,D)=\{b\in S_{n,x}(a)\colon \rho(f^{kl}(z_n(b))),f^{kl}(z_n(c)))\geqq 2\varepsilon \text{ implies } l\in D\}.$$

Consider the "tree" $\mathcal{T}$ of points: $G_0=\{z_n(b)\colon b\in S_{n,x}(a,D)\}$ $G_l=f^{kl}(G_0)$ for $l=0,\dots,\left[\frac{n-N_1}{k}\right]+1$. ($\mathcal{T}$ is a "tree" in a different sense that this in the proof of Lemma 2.) We say $e\in G_l$ branches if $\operatorname{Card}(f^{-k}(e)\cap G_{l-1})\geqq 2$.

We have for every $z\in G_0$

$$\operatorname{Card}\{l\colon f^{kl}(z)\text{ branches}\}\leqq\operatorname{Card}D+\zeta\cdot\left[\frac{n}{k}\right].$$

This follows from Lemma 4, namely that branchings in $\mathcal{T}$, except of indices in $D+1$, can occur only at $(f^k,\operatorname{Fr}A,4\varepsilon)$-singular values and just $4\varepsilon=\varepsilon(f^k,\operatorname{cl}A,\zeta)$.

So by consideration like in [Mi-P]

$$\operatorname{Card}G_0=\operatorname{Card}S_{n,x}(a,D)\leqq d^{4\sqrt{\eta}n+\zeta n}$$

so

$$\operatorname{Card}(S_{n,x})\leqq d^{N_1}\cdot 2^{\frac{n}{k}}\cdot d^{n(4\sqrt{\eta}+\zeta)}.$$

$$\limsup_{n\to\infty} n^{-1}\log\operatorname{Card}(S_{n,x})\leqq\frac{1}{k}\log 2+(4\sqrt{\eta}+\zeta)\log d. \tag{2}$$

This is what we need since k can be arbitrarily large and ν, ζ arbitrarily small. □

Remark 1. (2) implies that Capacity $(R_f^{-1}(x)\cap E_2)\leqq 4\sqrt{\eta}+\zeta+\frac{1}{k}$ if we consider in Σ^d the metric $\rho((i_s),(j_s))=\sum_{s=1}^{\infty}d^{-s}|i_s-j_s|$. Capacity of a set X means here:

$$\lim_{\varepsilon\to 0}\left(\frac{\log(\inf\{\operatorname{Card}U_\varepsilon : U_\varepsilon \text{ is a covering of } X \text{ by balls with radius } \varepsilon\})}{-\log\varepsilon}\right.$$

This implies that there exists a sequence $E_2(n)\subset\Sigma^d$ with $\nu(E_2(n))\nearrow 1$ such that for every $x\in\operatorname{Fr}A$ Capacity $(R_f^{-1}(x)\cap E_2(n))=0$. So there exists $E\subset\Sigma^d$ with $\nu(E)=1$ such that for every $x\in\operatorname{Fr}A$, $\operatorname{HD}(R_f^{-1}(x)\cap E)=0$.

(HD is Hausdorff dimension; we use the fact that HD$\leqq$Capacity and that $\operatorname{HD}\bigcup_{n=1}^{\infty}X_n=\sup_n\operatorname{HD}(X_n)$).

Since R_g is Hölder continuous this implies the following:

There exists $E\subset S^1$ with $l(E)=1$ such that for every $x\in\operatorname{Fr}A$, $\operatorname{HD}(\bar{R}^{-1}(x)\cap E)=0$.

This fact is much weaker than Beurling's estimate $\gamma(\bar{R}^{-1}(x))=0$ but our proof emphasises the C^1 property. See § 5.3 for further discussion.

Remark 2. $\bar{R}_*$ is continuous at equilibrium states in the following sense:

Let φ be a Hölder continuous function on S^1, μ_φ be the equilibrium state for (g, φ). Extend φ to a Hölder continuous function on D^2. For any $w_0 \in \operatorname{cl} D^2$ such that $w_0 \notin \bigcup_{i=1}^{\infty} g^i(\operatorname{Sing} g)$ let $\mu_{\varphi, w_0, n}$ be a probability measure distributed on $g^{-n}(w_0)$ with weight on $w \in g^{-n}(w_0)$, $\exp \sum_{i=0}^{n-1} \varphi(g^i(w))$. From [Bo] it follows that $\mu_{\varphi, w_0, n}$ weakly* converges to $h\mu_\varphi$ for $n \to \infty$ and that there exists a constant $C>0$ such that for every w_0 close to S^1, $\alpha \in \Sigma^d$ and $n>0$

$$C^{-1} \leqq \frac{\mu_{\varphi, w_0, n}(w_n(\alpha))}{\mu_\varphi(R_g \Pi_n^{-1} \Pi_n(\alpha))} \leqq C. \tag{3}$$

(Recall that the functions w_n: $\Sigma^d \to \operatorname{cl} D^2$, z_n: $\Sigma^d \to A$, for fixed $w_0 \in \operatorname{cl} D^2$, $z_0 \in A$, were defined in the proof of Lemma 2, Π_n: $\Sigma^d \to \{1, \ldots, d\}^n$ is projection to the first n coordinates, h is a positive Hölder continuous function on S^1.)

Proposition 3. *For every* $z_0 \in A$ $\left(z_0 \notin \bigcup_{i=1}^{\infty} f^i(\operatorname{Sing} f \mid A)\right)$

$$R_*(\mu_{\varphi, R^{-1}(z_0), n}) \to \bar{R}_*(h\mu_\varphi).$$

Proof. Let F be an arbitrary continuous function on $\operatorname{cl} A$. For every $\sigma>0$ there exists $\Sigma \subset \Sigma^d$, a compact set, such that $(R_g^{-1})_*(\mu_\varphi)(\Sigma) \geqq 1-\delta$, R_g is $1-1$ on Σ, the functions z_n converge uniformly to R_f on Σ and $R_f|\Sigma$ is continuous. Then $F \circ \bar{R}$ (defined on D^2 and μ_φ almost everywhere on S^1) is continuous on the compact set

$$W = \bigcup_{n=1}^{\infty} w_n(\Sigma) \cup R_g(\Sigma).$$

Decompose $F \circ \bar{R} = F_1 + F_2$ where F_1 is continuous on $\operatorname{cl} D^2$, $F_1 | W = F \circ \bar{R} | W$ and $|F_2|$ is bounded above by $2 \cdot \sup |F|$. Then

$$\begin{aligned} \int_{\operatorname{Fr} A} F\, d(\bar{R}_*(h\mu_\varphi)) &= \int_{S^1} F \circ \bar{R}\, d h\mu_\varphi = \int_{S^1} F_1\, d h\mu_\varphi + \int_{S^1} F_2\, d h\mu_\varphi \\ &= \lim_{n \to \infty} \int_{S^1} F_1\, d\mu_{\varphi, R^{-1}(z_0), n} + \int_{S^1} F_2\, d h\mu_\varphi. \end{aligned}$$

We have $|\int_{S^1} F_2\, d\mu_\varphi| \leqq \delta \sup |F_2|$, so we only need to estimate

$$\limsup_{n \to \infty} \left| \int_{D^2} F_2\, d\mu_{\varphi, R^{-1}(z_0), n} \right|.$$

Let E_n = denote $\{\alpha \in \Sigma^d : \Pi_n(\alpha) \notin \Pi_n(\Sigma)\}$. Observe that $\operatorname{supp} F_2 \subset w_n(E_n)$ and $(R_g^{-1}) * (\mu_\varphi)(E_n) \leqq \delta$. So, in view of (3),

$$\left| \int_{D^2} F_2\, d\mu_{\varphi, R^{-1}(z_0), n} \right| \leqq \delta \cdot C \cdot \sup |F_2|. \quad \square$$

§ 3. $\mathrm{HD}(m) = h_m(f)/\chi_m(f)$

We give a sketch of proof of the so-called volume lemma, i.e. that $\lim_{r\to 0} \frac{\log m(B(x,r))}{\log r}$ exists for almost every $x \in \mathrm{Fr}\,A$ and equals $h_m(f)/\chi_m(f)$. (On the other hand it equals $\mathrm{HD}(m)$ by Proposition 2.1 in [Y], use the Besicovitch Covering Theorem in the proof there, Theorem 1.1 in [Gu].)

We will be brief since this is a general fact about 1-dimensional dynamics and nothing new in view of [Le_1] and [Y]. Besides I was informed that Mañé can prove this (for the maximal measure on $J(f)$) before I started to work on the present paper. See also [Le_2].

First recall that we have checked:

$$\int_{\mathrm{Fr}\,A} |\log |f'|| \, dm < \infty. \tag{1}$$

This implies the existence of Lyapunov characteristic exponents almost everywhere, in particular that along the forward trajectory of almost every point $1/|f'|$ grows subexponentially.

Since $\mathrm{dist}(z, \mathrm{Sing} f) > \mathrm{Const} \cdot |f'(z)|$, (1) implies:

$$\int_{\mathrm{Fr}\,A} |\log \mathrm{dist}(z, \mathrm{Sing} f)| \, dm < \infty$$

(the derivative and distances are in S^2).

This implies that the forward trajectory of almost every point approaches $\mathrm{Sing} f$ (the set of critical points) subexponentially.

To have

$$\limsup_{r\to 0} \frac{\log m(B(x,r))}{\log r} \leqq h_m(f)/\chi_m(f)$$

one can repeat Mañé's arguments [Ma_1]: Consider a function $\rho = \lambda^{n(x)}$ for λ small $(0 < \lambda < 1)$ and $n(x)$ the first return time under forward iterations with f, to a "good" set S (this defines ρ on S, put $\rho \equiv 1$ outside S). Since $\log \rho$ is integrable there exists a countable, finite entropy partition ξ such that $B(x, \rho(x)) \supset \xi(x)$ for almost every x. Here from the Shannon-McMillan-Breiman Theorem, we have

$$m\left(\bigcap_{n=0}^{N} f^{-n} B(f^n(x), \rho(f^n(x)))\right) \geqq m\left(\bigvee_{n=0}^{N} f^{-n}(\xi)(x)\right) \geqq \exp - N(h_m(f) + \sigma)$$

for σ arbitrarily small, N large.

The sets $\bigcap_{n=0}^{N} f^{-n} B(f^n(x), \rho(f^n(x)))$ "approximate" all small balls with the origin x because of the bounded distortion property, $\left(\text{Because of (1) one can adequately estimate } \left|\frac{f'(w_1)}{f'(w_2)} - 1\right|\right)$. The radii are about $\exp - N\chi_m(f)$.

The inequality

$$\liminf_{r\to 0}\frac{\log m(B(x,r))}{\log r}\geqq h_m(f)/\chi_m(f)$$

is easier to prove:

Consider a finite partition of $\mathrm{cl}A$ with boundaries to which almost every point approaches at most subexponentially and consider the sets $\bigcap_{n=0}^{N} f^{-n}B(f^n(x), C\lambda^n)$ with $0<\lambda<1$, $\lambda\approx 1$. □

§ 4. Relative entropy and Hausdorff dimension of fibres

I will show in this section how to deduce $h_l(g)=h_m(f)$ form Beurling's Theorem.

Denote by ε the partition of a space into points (it will always be clear of what space). Let $l(\cdot/\bar{R}^{-1}(\varepsilon))$ denote conditional measures on the measurable partition $\bar{R}^{-1}(\varepsilon)$. $l(\cdot/\bar{R}^{-1}(\varepsilon)(x))$ denotes the measure on that set from the partition which contains x.

We prove that for almost every $x\in S^1$:

$$0\overset{(\mathrm{i})}{=}\mathrm{HD}(l(\cdot/\bar{R}^{-1}(\varepsilon)(x))\overset{(\mathrm{ii})}{\geqq}\liminf_{r\to 0}\frac{\log l(B(x,r)/\bar{R}^{-1}(\varepsilon)(x))}{\log r}\overset{(\mathrm{iii})}{\geqq}\frac{h_l(g|f)}{\chi_l(g)}.$$

This yields $h_l(g|f)=h_l(g)-h_m(f)=0$.

(i) follows from Beurling's Theorem since for any compact set $E\subset\bar{R}^{-1}(\varepsilon)(x)$, $0=\gamma(E)$, which implies (see [T] Theorem III, 19.) that $\mathrm{HD}(E)=0$ and we can choose sets E such that $l(E/\bar{R}^{-1}(\varepsilon)(x))\to 1$.

(ii) follows from [Y] Proposition 2.1.

So we shall concentrate on (iii). Let ξ be a finite partition of S^1 into arcs and $r=\exp -n(\chi_l(g)+\sigma)$ for small $\sigma>0$. Then

$$B(x,r)\subset\xi^{(n)}(x)=\bigvee_{i=0}^{n-1} g^{-i}(\xi)(x),\qquad \text{for } n=n(x,\xi) \tag{1}$$

sufficiently large.

So

$$\frac{\log l(B(x,r)/\bar{R}^{-1}(\varepsilon)(x))}{\log r}\geqq -\frac{1}{n(\chi_l(g)+\sigma)}\log l(\xi^{(n)}(x)/\bar{R}^{-1}(\varepsilon)(x)).$$

The proof of (iii) will be complete if we prove that

$$\begin{aligned} h_l(g/\bar{R}^{-1}(\varepsilon))-\delta &\leqq\liminf_{n\to\infty}(-\log l((\xi^{(n)}(x)/\bar{R}^{-1}(\varepsilon)(x)) \\ &\leqq\limsup_{n\to\infty}(-\log l(\xi^{(n)}(x)/\bar{R}^{-1}(\varepsilon)(x))\leqq h_l(g/\bar{R}^{-1}(\varepsilon)) \end{aligned}\tag{2}$$

where $\delta\to 0$ when $\xi\nearrow\varepsilon$. This is a weak version of the *Shannon-McMillan-Breiman Theorem* (for ergodic transformation) *relative* to $\bar{R}^{-1}(\varepsilon)$. This would be clearly true if $g^{-1}(\bar{R}^{-1}(\varepsilon))=\bar{R}^{-1}(\varepsilon)$. The proof is then the same as in the classical, non-relative case, see for example [Pa].

Here we have only $g^{-1}(\bar{R}^{-1}(\varepsilon))\leqq\bar{R}^{-1}(\varepsilon)$.

The crucial computation in the proof of the Shannon-McMillan-Breiman Theorem looks then as follows:

$$\frac{1}{n}I\left(\bigvee_{i=0}^{n-1}g^{-i}(\xi)/\bar{R}^{-1}(\varepsilon)\right)=\frac{1}{n}\sum_{i=0}^{n-1}I\left(g^{-i}(\xi)\Big/\left(\bigvee_{j=i+1}^{n-1}g^{-j}(\xi)\vee\bar{R}^{-1}(\varepsilon)\right)\right)$$
$$\leqq\frac{1}{n}\sum_{i=0}^{n-1}I\left(g^{-i}(\xi)\Big/\left(\bigvee_{j=i+1}^{n-1}g^{-j}(\xi)\vee g^{-i}(\bar{R}^{-1}(\varepsilon))\right)\right)=\frac{1}{n}\sum_{i=0}^{n-1}I_{n-i}\circ g^{i}$$
$$\xrightarrow[l\text{-a.e.}]{n\to\infty}\lim\frac{1}{n}\sum_{i=0}^{n-1}(I(\xi/\xi^{-}\vee\bar{R}^{-1}(\varepsilon))\circ g^{i})=h_l(g\,|\,f,\xi).$$

Here $I(\eta/\zeta)(x)=-\log l(\eta(x)/\zeta(x))$ is the information function and

$$I_j=I\left(\xi\Big/\bigvee_{s=1}^{j-1}g^{-s}(\xi)\vee\bar{R}^{-1}(\varepsilon)\right).$$

This proves the Shannon-McMillan-Breiman Theorem in one direction, namely $\limsup -\frac{1}{n}\log l(\xi^n(x)/\bar{R}^{-1}(\varepsilon)(x))\leqq h_l(g\,|\,f,\xi)$ and in particular the right hand inequality in (2).

However to prove (iii) we need an estimate in the other direction. This can be done by lifting to the inverse limit (natural extension):

Set the notation $\tilde{f},\tilde{g},\tilde{R},\tilde{l}$ for the corresponding lifts of $f,g,\bar{R},l$. Let Π_0 be projection to the 0-coordinate. Then

$$\liminf_{n\to\infty}-\frac{1}{n}\,l(\xi^{(n)}(x)/\bar{R}^{-1}(\varepsilon)(x))$$
$$=\liminf_{n\to\infty}-\frac{1}{n}\,\tilde{l}(\Pi_0^{-1}(\xi^{(n)}(x))/\Pi_0^{-1}(\bar{R}^{-1}(\varepsilon)(x)))$$
$$\geqq\liminf_{n\to\infty}-\frac{1}{n}\,\tilde{l}(\Pi_0^{-1}(\xi^{(n)}(x))/\tilde{R}^{-1}(\varepsilon)(\tilde{x}))=H_{\tilde{l}}(\tilde{g}\,|\,\tilde{f},\Pi_0^{-1}(\xi))$$
$$\geqq h_{\tilde{l}}(\tilde{g}\,|\,\tilde{f})-\delta=h_{\tilde{l}}(\tilde{g})-h_{\tilde{l}}(\tilde{f})-\delta=h_l(g)-h_l(f)-\delta=h_l(g\,|\,f)-\delta.$$

($\tilde{x}\in\Pi_0^{-1}(x)$ and the computation is for a.e. $x,\tilde{x}$).

We used the equality $\tilde{f}^{-1}(\tilde{R}^{-1}(\varepsilon))=\tilde{R}^{-1}(\varepsilon)$ and the fact that if $\xi\nearrow\varepsilon$ than $\tilde{g}^n(\Pi_0^{-1}(\xi))\nearrow\varepsilon$. □

Remark. The *relative volume lemma* holds:

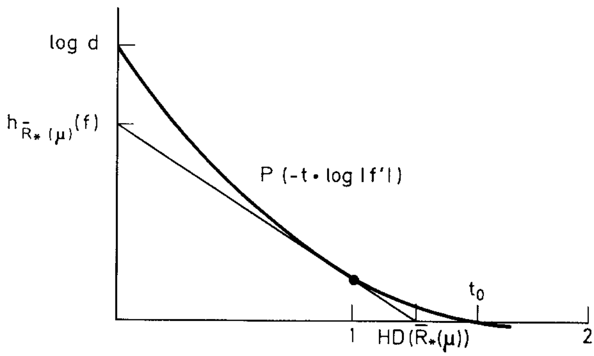
$$l\text{-a.e.}\ \lim_{r\to 0}\frac{\log l(B(x,r)/\bar{R}^{-1}(\varepsilon))}{\log r}=\frac{h_l(g\,|\,f)}{\chi_l(g)}. \tag{3}$$

We have proved this except for the opposite inclusion in (1), for $r=\exp-n(\chi_l(g)-\sigma)$. Here that inclusion trivially holds since g is (uniformly) expanding. However using Mañé's technique, see § 3) one can prove (3) assuming only that the dynamics of g is 1-dimensional (real or complex), $\chi_l(g)$ exists and is positive, as in § 3, l can be an arbitrary g-invariant probability measure.

§ 5. Final remarks

1. The normalized length measure l on S^1 is the equilibrium Gibbs state for the mapping g and the function $-\log|g'|$. Let μ be the Gibbs state for g and $-\log(f' \circ R)$ (in the case $\bar{R}$ is continuous, $f|\mathrm{Fr} A$ is expanding). Then $\bar{R}_*(\mu)$ is the Gibbs state for f and $-\log|f'|$ on $\mathrm{Fr} A$, since $h_{\bar{R}_*(\mu)}(f)=h_\mu(g)$ and $P_f(-\log|f'|)=P_g(-\log|f' \circ R|)$ (the latter equality holds since any (n, ε)-separated set for g in S^1 used in the definition of pressure can be replaced by a set S where $g^n(x)=g^n(y)$ for every $x, y \in S$ and then $\bar{R}(S)$ must be (n, ε)-separated for f in $\mathrm{Fr} A$, cf. begin of the Proof of Proposition 2).

It is known (see [R]) that the pressure for f, $P_f(-t \cdot \log|f'|)$, is equal to 0 for $t=t_0=\mathrm{HD}(\mathrm{Fr} A)$. So if $t_0>1$ we have $\mathrm{HD}(R_*(\mu))>1$, see the McCluskey-Manning picture [McC-M]:



Meanwhile for $m=\bar{R}_*(l)$ $\mathrm{HD}(m)=1$. So $\bar{R}_*(\mu)$ and m are very different. This is striking since $-\log|g'|$ and $-\log|f' \circ \bar{R}|$ are homologous on the open D^2.

2. The condition $\mu(\mathscr{D}\bar{R})=1$ is satisfied by a probability measure μ if it has finite energy i.e.

$$\iint_{S^1 \times S^1} \log \frac{1}{|x-y|} \, d\mu(x) \, d\mu(y) < \infty .$$

This follows from $\gamma(S^1 \setminus \mathscr{D}\bar{R})=0$ (Beurling Theorem) and the definition of logarithmic capacity γ. See [T], Theorem III 7.

How nondegenerate is the measure $\bar{R}_*(\mu)$ then? Has it finite energy? Is it true that

$$\chi_{\bar{R}_*(\mu)}(f)>0 \quad \text{(in particular} > -\infty)? \tag{1}$$

(The importance of (1) has been explained in § 3.

Under the assumptions of Proposition 3 about f the considerations of §2 prove in fact $h_\nu(g)=h_{\bar{R}_*(\nu)}(f)$ for arbitrary probability, g-invariant measure ν such that $\bar{R}_*(\nu)$ is well defined. So if we assume additionally that μ is ergodic and $h_\mu(g)>0$ (or equivalently $\mathrm{HD}(\mu)>0$), then $h_{\bar{R}_*(\mu)}(f)>0$ and, by [Ma$_2$] Lemma II.3, the inequality (1) is true (and $\mathrm{HD}(\bar{R}_*(\mu))>0$), cf. also [P].

3. Considerations from §4 prove Proposition 3, (that $h_\mu(g)-h_{\bar{R}_*(\mu)}(f)=h_\mu(g|f)=0$ for any probability, g-invariant, ergodic μ such that $\mu(\mathscr{D}\bar{R})=1$) with the assumption that f has a C^1 extension weakened to the assumption:

$$g^{-1}(\bar{R}^{-1}(\varepsilon))\leqq \bar{R}^{-1}(\varepsilon).$$

This latter assumption is even weaker than the assumption that f extends continuously to $\mathrm{cl}\,A$.

On the other hand this part of the proof of Proposition 2 given in §2 which shows that the sets $\bar{R}^{-1}(x)$, $R_\gamma^{-1}(x)$ are thin, does not make use of the analiticity of g, R and f. Another observation is that in the presence of dynamic it is natural instead of the existence of a radial limit to consider the existence of a limit along admissible sequences (definition at the beginning of §2). So the following question arises:

Question. Let $R\colon D^2\to A$ be a C^1 diffeomorphism whose real and imaginary parts have finite Dirichlet integrals.

Let g be a C^1 expanding map defined on a neighbourhood of S^1 in $\mathrm{cl}\,D^2$, preserving S^1, with differential preserving the bundle η normal to S^1, such that, for every $x\in S^1$, the expansion along $\eta(x)$ is not stronger than along $TS^1(x)$.

Assume that $f=R\circ g\circ R^{-1}$ extends C^1 to a neighbourhood of $\mathrm{Fr}\,A$. Is it true then, that

(i) There exists a limit R^a along admissible trajectories on a set $\mathscr{D}R^a$ with $\gamma(S^1\backslash\mathscr{D}R^a)=0$?

(ii) The sets on which R^a is constant have logarithmic capacity 0?

(iii) For every finite g-invariant measure μ on S^1 with finite energy

$$h_{R^a_*(\mu)}(f)=h_\mu(g)?$$

I put finiteness of Dirichlet integrals into the assumptions to have (i) by analogy to the existence then of the radial limit $\bar{R}$ on $\mathscr{D}\bar{R}$ with $\gamma(S^1\backslash\mathscr{D}\bar{R})=0$, see [Ca$_1$], Section 5, Th. 3 (Caution: existence of the radial limit does imply existence of the nontangential limit in general, see [Ca$_1$], p. 62). The assumptions about the strength of expansion g yield admissible sequences converging to S^1 non-tangentially.

(i) implies (iii) by considerations from §2.

4. There is an overlap between Manning's proof of his Theorem (see §0) and Brolin's work [Br] §§ 15, 16. I understand the situation as follows:

Let f be a polynomial $z^d+\alpha_{d-1}z^{d-1}+\dots$, A be the basin of ∞ and ν be an arbitrary, probability, f-invariant measure on $\mathrm{Fr}\,A$. Then, if $c_1,\dots,c_{d-1}$ are the critical points of f in $\mathbb{C}$, we have

$$\chi_\nu(f) = \int_{\operatorname{Fr} A} \log|f'|\,d\nu = \int_{\operatorname{Fr} A} \log d|z-c_1|\dots|z-c_{d-1}|\,d\nu$$
$$= \log d + \sum_{i=1}^{d-1} \int_{\operatorname{Fr} A} \log|z-c_i|\,d\nu.$$

Then conclusion is:

$$\chi_\nu(f) = \log d - \sum_{i=1}^{d-1} u_\nu(c_i), \tag{1}$$

where $u_\nu(v) = \int \log \frac{1}{|z-v|}\,d\nu$ denotes the logarithmic potential for ν at v.

Equidistribute now the measure σ_n on the points from $f^{-n}(z_0)$, $n=1, \dots$ (for arbitrarily fixed $z_0 \in A \setminus \{\infty\}$). Then $\sigma_n \to m_{\max}$, which is the unique measure with the maximal entropy $h_{m_{\max}}(f) = \log d$ (see Proposition 3 in the case A is simply connected and [Lju] in the general case). $m_{\max}$ is also the equilibrium distribution on $\operatorname{Fr} A$ in the sense of potential theory (for the logarithmic potential, see [Br]).

The Brolin-Manning computation:

$$-u\sigma_n(c) = \int \log|z-c|\,du\sigma_n = (\log(\prod_{z \in f^{-u}(z_0)} |z-c|))\frac{1}{d^n}$$
$$= \frac{1}{d^n} \log|f^n(c) - z_0| \xrightarrow[n\to\infty]{} 0, \quad \text{for every } c \notin A, \tag{2}$$

shows that, in the case $c_i \notin A$ for every $i = 1, \dots, d-1$, we have $u_{m_{\max}}(c_i) = 0$, hence by (1).

$$\chi_{m_{\max}}(f) = \log d.$$

Problem. Understand the connections between Lyapunov characteristic exponents and potential theory for rational mappings.

5. We continue our discussion from § 5.4 for the polynomial $f_c(z) = z^2 + c$. Consider the Mandelbrot set $M = \{c \in \mathbb{C} : f_c^n(0) \not\to \infty\}$, when $n \to \infty$, see [Md]. The set $S^2 \setminus M$ is topologically an open disc containing ∞ (see [Do-Hb] or [Do]). For every $c \in \mathbb{C}$ denote by A_c the basin of attraction of ∞ for f_c. m_c denotes the maximal entropy measure on $\operatorname{Fr} A_c = J(f_c)$ (see § 5.4).

Proposition 4. (i) $G(c) = -u_{m_c}(c) = -2u_{m_c}(0) = 2\chi_{m_c} - 2\log 2$ *is the Green's function of* $S^2 \setminus M$ *with its pole at* ∞,

(ii) $\gamma(\operatorname{Fr} M) = 1$,

(iii) $\operatorname{HD}(m_c) = \dfrac{\log 2}{\log 2 + \frac{1}{2} G(c)}$.

Sketch of proof. To have (i) in view of the Brolin-Manning computation, § 5.4, it is enough to check that $\lim_{n\to\infty} 2^{-n} \log|f_c^n(c)|$ is the Green's function. This is straightforward and in fact $G(c) = \log|c| + o(1)$ if $c \to \infty$. So the Robin constant is 0, hence $\gamma(\operatorname{Fr} M)$ (= the mapping radius of $\operatorname{Fr} M$, see [T] p. 84) = 1. (All this was in fact announced by Douady and Hubbard in [Do-Hu] and [Do].

($\lim_{n\to\infty}(f_c^n(c))^{2^{-n}}$): $S^2 \setminus M \to D^2$ is their Riemann mapping). (iii) follows from $\mathrm{HD}(m_c) = \frac{h_{m_c}(f_c)}{\chi_{m_c}(f_c)}$. □

(Recall the similar properties for each individual c: $\gamma(J(f_c)) = 1$, see [Br] or (2); $-u_{m_c}$ is the Green's function of A_c with its pole at ∞, see [T] p. 82.)

Acknowledgement. I wish to thank Anthony Manning for coming to Warsaw in December 1983 and discussions stimulating doing this paper.

I want to thank Lennart Carleson and the staff of the Mittag-Leffler Institute for invitation and care during my stay in March 1984 there and Lennart Carleson and John Garnett for discussions which influenced the final version of the paper.

Finally I want to thank Mrs Anna Goździk from IM PAN for excellent typing from a hardly readable manuscript.

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Oblatum 25-V-1984

Note added in proof

Question on page 2 in the case of Jordan domains has been recently answered in positive by N.G. Makarov in the preprint: On the distorsion of boundary sets under conformal mappings. Makarov proved that m is always absolutely continuous with respect to the Hausdorff measure H_{φ_C} with $\varphi_C(t) = t \cdot \exp\left(C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right)$ with a universal constant $C>0$. He constructed also an example with m singular to H_{φ_c} for some $c>0$.

Under the assumptions of my Theorem, if $\mathrm{Fr}\,A$ is a Jordan curve and $f|\mathrm{Fr}\,A$ is expanding it turns out that either $\mathrm{Fr}\,A$ is an analytic curve or m is singular to H_{φ_c} for some $c=c(f)>0$. A proof will appear in a paper done by M. Urbański, A. Zdunik and myself.