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On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*

Vinay V. Deodhar**

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

§ 1. Introduction

Let G be a simply connected semisimple algebraic group over an algebraically closed field Ω. Let $B \supseteq T$ be respectively a Borel subgroup and a maximal torus. Let $W = N(T)/T$ denote the Weyl group of G. One then has the Bruhat decomposition of G into double cosets of B parametrized by W i.e. $G = \bigcup_{w \in W} BwB$.

One also considers the generalized flag manifold G/B which then inherits a disjoint decomposition into cells, $G/B = \bigcup_{y \in W} By \cdot B$. (We use the convention of writing $By \cdot B$ when it is being considered as a subset of G/B and simply ByB when considered as a subset of G itself.)

The cell $By \cdot B$ is known as a Bruhat cell and it is algebraically isomorphic to an affine space Ω^n where n is the length of y with respect to the set of simple reflections determined by B. The closure $X(y)$ of $By \cdot B$ is a projective variety called Schubert variety and it is easy to see that $X(y)$ is a union of Bruhat cells. This gives a partial order $\leqq$ on W called Bruhat ordering. Thus, $x \leqq y$ iff $Bx \cdot B \subseteq X(y)$. (This ordering has been studied extensively; see [$D1$]). For a later use, we introduce the notation $W(y) = \{x \in W \mid x \leqq y\}$.

One also has a 'dual' decomposition of G and G/B obtained by considering the opposite Borel subgroup B^- to B (cf. [Bo], [Ste]). Thus, $G/B = \bigcup_{x \in W} B^- x \cdot B$ (disjoint union). One of the interesting problems now is to describe the intersection pattern of these two decompositions i.e. one is interested in description of $By \cdot B \cap B^- x \cdot B$.

This intersection comes up in several different contexts e.g. [BB], [K–L1], [K–L2]. We describe here one such instance. In order to compute the local cohomology groups of $X(w)$ at point $z \cdot B (z \leqq w)$, one needs a 'good' open

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** Part of the research was done while the author was on leave from Tata Institute, Bombay to Australian National University, Canberra neighbourhood of $z \cdot B$ in $X(w)$. Since the big cell $Bw_0 \cdot B$, w_0 being the element of maximal length in W, is open in G/B, it is easy to see that $zw_0Bw_0 \cdot B \cap X(w)$ is an open neighbourhood of $z \cdot B$ in $X(w)$. A further analysis of this intersection leads one to several intersections of the form $By \cdot B \cap B^- x \cdot B$ for suitable $x, y \in W$.

The first result of this paper is concerned with this problem. We have:

Theorem 1.1. *The Bruhat cell $By \cdot B$ can be decomposed into disjoint (non-empty) subsets $\{D_{\underline{\sigma}}\}_{\underline{\sigma} \in \mathscr{D}}$ ($\mathscr{D}$ is an indexing set which can be described explicitely) such that*

(i) $By \cdot B = \bigcup_{\underline{\sigma} \in \mathscr{D}} D_{\underline{\sigma}}$.

(ii) *For each $\underline{\sigma} \in \mathscr{D}$, there exist unique non-negative integers $m(\underline{\sigma})$ and $n(\underline{\sigma})$ (which can be determined explicitely) such that $D_{\underline{\sigma}} \simeq \Omega^{m(\underline{\sigma})} \times (\Omega^*)^{n(\underline{\sigma})} (\Omega^* = \Omega \setminus \{0\})$.*

(iii) *For each $\underline{\sigma} \in \mathscr{D}$, $\exists x \in W$ such that $D_{\underline{\sigma}} \subseteq B^- x \cdot B$. This element x, which is unique, belongs to $W(y)$ and is denoted by $\pi(\underline{\sigma})$. (Thus one has a map $\pi\colon \mathscr{D} \to$ W (y)).*

Corollary 1.2. $By \cdot B \cap B^- x \cdot B = \bigcup_{\substack{\underline{\sigma} \in \mathscr{D} \\ \pi(\underline{\sigma}) = x}} D_{\underline{\sigma}}$. *(In particular, $By \cdot B \cap B^- x \cdot B \neq \phi$* iff *$x \in W(y)$.)*

As a particular case, one considers a Coxeter element $y = s_1 \cdot \ldots \cdot s_l$ ($l = \text{rank } G$). In this case, $\mathscr{D}$ turns out to be in a bijective correspondence with the power set of S. Moreover, $m(\underline{\sigma}) = 0$ $\forall \underline{\sigma} \in \mathscr{D}$. If $\underline{\sigma} \in \mathscr{D}$ corresponds to a subset $J(\underline{\sigma}) \subseteq S$, then $n(\underline{\sigma}) = l - |J(\underline{\sigma})|$. Further, π is bijective and so $By \cdot B \cap B^- x \cdot B = D_{\underline{\sigma}} \simeq (\Omega^*)^{l - |J(\underline{\sigma})|}$ where $\underline{\sigma} \in \mathscr{D}$ is the unique element such that $\pi(\underline{\sigma}) = x$.

It is possible to extend these results to the case when G is a group associated with Kac-Moody Lie Algebras. These groups have been considered by several mathematicians ([G], [K-P], [M-T], [Sl], [T1], [T2]) and the structure is described in terms of (infinite) root systems associated to Kac-Moody Lie algebras. This description closely resembles that in the finite case. In particular, one has the counterparts of B, G/B and B^-. The proof of Theorem 1.1 goes through with minor changes in this case and hence one gets counterparts of Theorem 1.1 and Corollary 1.2. We skip the actual details. These details have been worked out independently by Z. Haddad in his thesis ([H, § 3]).

The indexing set $\mathscr{D}$ in Theorem 1.1 can be described using a reduced expression $s_1 \cdot \ldots \cdot s_k$ for y ($k = l(y)$). It consists of 'subexpressions' of this expression which satisfy an additional property; the elements of $\mathscr{D}$ are called distinguished subexpressions (cf. Def. 2.3). This notion is in fact defined for any Coxeter group and it turns out to be extremely useful in considering problems associated to the Bruhat ordering. The author had formulated this notion along with the maps π, m and n while considering the Kazhdan-Lusztig polynomials $P_{z,w}(q)$ (cf. [K-L1], [K-L2]). These polynomials are defined recursively in terms of another set $\{R_{x,y}(q)\}$ of polynomials ([K-L1, 2.2a]) and it turns out that the set $\mathscr{D}$ describes $R_{x,y}$'s completely for any Coxeter group. More precisely, we have

Theorem 1.3. *Fix $y \in W$ and a reduced expression $y = s_1 \cdot \ldots \cdot s_k$. Then for any $x \in W(y)$ (i.e. $x \leqq y$),*

$$R_{x,y}(q) = \sum_{\substack{\sigma \in \mathscr{D} \\ \pi(\sigma) = x}} q^{m(\sigma)}(q-1)^{n(\sigma)}.$$

We give a uniform proof which is combinatorial in nature. If W is a Weyl group, then one may deduce Theorem 1.3 from Theorem 1.1 and the following observations of ([K-L1, A3, A4]): If F is a finite field of q elements then $R_{x,y}(q) = |By \cdot B \cap B^- x \cdot B|$.

In an earlier paper ([D3]) the author has described the 'shape' of the polynomial $R_{x,y}$. Theorem 1.3 gives complete information.

Another application of the set $\mathscr{D}$ (or rather a subset $\mathscr{D}_0$ described in Prop. 5.3) is to the so-called L-shellability (cf. [B-W]) of the Bruhat ordering on W and related sets $W^J (J \subseteq S)$. Recently a lot of work has been done on shellable posets and some interesting consequences in algebraic geometry are derived (cf. [Sta]).

The present paper is arranged as follows: In §2 we give some preliminaries on the Bruhat decomposition and Bruhat ordering. We also give a formal definition of distinguished subexpressions. In §3, we give a proof of Theorem 1.1 and Corollary 1.2. In §4 we examine closely the special case of a Coxeter element. §5 is devoted to a study of some of the properties of distinguished subexpressions which enables us to prove Theorem 1.3. In §6 we consider the application to L-shellability of Bruhat orderings.

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Part of this work was done while the author was a Research Fellow at Australian National University, Canberra and he takes this opportunity to express his gratitude for the hospitality extended to him by the members of that department.

§2. Notation and preliminaries

We first expand the notation used in the introduction. Let Φ be the root system of the pair (G, T). Let Φ^+ be the set of positive roots corresponding to B and let Δ be the set of simple roots in Φ^+. For $\alpha \in \Phi$, parametrize the one-parameter subgroup U_α in such a way that $h \cdot x_\alpha(t) \cdot h^{-1} = x_\alpha(\alpha(h) \cdot t)\ \forall\, t \in \Omega,\ h \in T$. Let U^+ (respectively U^-) be the maximal unipotent subgroup of G corresponding to Φ^+ (respectively $-\Phi^+$). Given an order in Φ^+, any element $u \in U^+$ can be uniquely written as $\prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha)$ with respect to this order.

For $\alpha\in\Phi^+$, let $s_\alpha\in N(T)$ be defined by $s_\alpha=x_\alpha(1)\cdot x_{-\alpha}(-1)\cdot x_\alpha(1)$. One then has the following:

Lemma 2.1. *For $t\in\Omega^*$, $x_\alpha(t^{-1})\cdot s_\alpha\cdot x_\alpha(t)=h\cdot s_\alpha\cdot x_\alpha(-t^{-1})\cdot s_\alpha^{-1}$ for a suitable $h\in T$.*

A proof of this lemma can be easily derived from [Ste, Lemma 19].

By abuse of notation, we write s_α for its image in $W=N(T)/T$.

Fix $y\in W$ and a reduced expression $y=s_1\dots s_k$ ($s_i\in S$, the set of simple reflections corresponding to Δ). For $1\leqq j\leqq k$, let α_j be the simple root corresponding to s_j i.e. $s_j=s_{\alpha_j}$. (Note that α_j's need not be distinct.) For $1\leqq j\leqq k+1$, define $U_j=U^+\cap{}^{s_j\dots s_k}U^-$ and $U^j=U^+\cap{}^{s_k\dots s_j}U^+$. (For a subset A of G and $g\in G$, ${}^gA=gAg^{-1}$). One then has:

Lemma 2.2.

(i) *U_j is the subgroup of U^+ generated by 1-parameter subgroups corresponding to the set of roots $\{\phi\in\Phi^+ | s_k\dots s_j(\phi)\,\varepsilon-\Phi^+\}$. This set consists precisely of the roots $\{\alpha_j, s_j(\alpha_{j+1}),\dots, s_j\dots s_{k-1}(\alpha_k)\}$. Further, any element $u\in U_j$ can be uniquely written as*

$$x_{\alpha_j}(t_j)\cdot x_{s_j(\alpha_{j+1})}(t_{j+1})\dots x_{s_j\dots s_{k-1}(\alpha_k)}(t_k);$$

thus $U_j\simeq\Omega^{k-j+1}$.

(ii) $U_1\supseteq{}^{s_1}U_2\supseteq{}^{s_1s_2}U_3\supseteq\dots\supseteq{}^{s_1\dots s_k}U_{k+1}=\{\mathrm{id}\}$.

(iii) *U^j is the subgroup of U^+ generated by 1-parameter subgroups corresponding to the set of roots $\{\phi\in\Phi^+ | s_j\dots s_k(\phi)\in\Phi^+\}$.*

(iv) $U^1\subseteq U^2\subseteq\dots\subseteq U^{k+1}=U^+$.

(v) *For any $1\leqq j\leqq k+1$, $U^+=U_j\cdot{}^{s_j\dots s_k}U^j$ with a uniqueness of expression.*

(vi) *Any element $\xi\in By\cdot B$ can be uniquely written as $us_1\dots s_k\cdot B$ with $u\in U_1$. Thus $By\cdot B\simeq U_1\simeq\Omega^k$.*

A proof of this lemma is straightforward and can be easily deduced from the structure of the group U^+ as given in ([Ste, Lemma 17]).

We now consider some fundamental properties of the Bruhat ordering. For the rest of this section, (W,S) denotes any Coxeter group. One knows several equivalent ways of defining Bruhat ordering ([D1, Thm. 1.1]) and we choose the following: For $y\in W$ fix a reduced expression $y=s_1\dots s_k$. Then $x\leqq y$ iff there exists a subsequence $1\leqq p_1<\dots<p_t\leqq k$ ($t\geqq 0$) of indices such that $x=s_1\dots\hat{s}_{p_1}\dots\hat{s}_{p_t}\dots s_k$. By abuse of language, we say that the above is a subexpression of $y=s_1\dots s_k$ whose value is x. (The correct way is of course to define the subexpression as a sequence $(p_1,\dots,p_t)$ of increasing integers.)

There is another way to formalize this notion: A subexpression is a sequence $\underline{\sigma}=(\sigma_0,\dots,\sigma_k)$ of elements of W such that (i) $\sigma_0=\mathrm{id}$ and (ii) $\sigma_{j-1}^{-1}\sigma_j\in\{\mathrm{id},s_j\}\ \forall\, 1\leqq j\leqq k$. The correspondence between these two formulations is obvious viz. $\underline{\sigma}=(\sigma_0,\sigma_1,\dots,\sigma_k)$ corresponds to $s_1\dots\hat{s}_{p_1}\dots\hat{s}_{p_t}\dots s_k$ where $\{p_1,\dots,p_t\}=\{j|\sigma_{j-1}^{-1}\sigma_j=\mathrm{id}\}$. Let $\mathscr{S}$ denote the set of all subexpressions viewed as sequences with $k+1$ elements as above. We note that for $\underline{\sigma}\in\mathscr{S}$ and $1\leqq j\leqq k$, one has a trichotomy: $\sigma_{j-1}<\sigma_j$ or $\sigma_{j-1}=\sigma_j$ or $\sigma_{j-1}>\sigma_j$. (This is a consequence of condition (ii) of subexpressions). For $\underline{\sigma}\in\mathscr{S}$, define $I(\underline{\sigma})=\{j|1\leqq j\leqq k$ and $\sigma_{j-1}=\sigma_j\}$ and $n(\underline{\sigma})=|I(\underline{\sigma})|$. Also define $m(\underline{\sigma})=|\{j|1\leqq j\leqq k$ and $\sigma_{j-1}>\sigma_j\}|$. We will use these maps later on. Finally, let $\pi\colon\mathscr{S}\to W$ denote the projection onto the last factor i.e. $\pi(\underline{\sigma})=\sigma_k$. It is clear that $\pi(\mathscr{S})=W(y)=\{x\in W|x\leqq y\}$ (cf. [D1, Theorem 1.1]).

We next come to the key definition of set $\mathscr{D}$:

Definition 2.3. An element $\underline{\sigma}\in\mathscr{S}$ is called a distinguished subexpression if it satisfies the following additional condition:

(iii) $\sigma_j\leqq\sigma_{j-1}s_j \forall 1\leqq j\leqq k$.

Let $\mathscr{D}$ be the set of all distinguished subexpressions. We denote the restrictions of maps π, m, n to $\mathscr{D}$ by the same letters respectively.

Remark 2.4. It is not apriori clear if $\pi(\mathscr{D})=W(y)$. However, it is true and we will prove it in Proposition 5.2 by methods independent of the results of §3 and §4.

Example 2.5. Consider $s, s'\in S$ such that $ss'\neq s's$. Let $y=ss's$ then $l(y)=3$ and the preceding is a reduced expression. The set $\mathscr{S}$ consists of 8 elements of which only one is *not* distinguished viz. $\underline{\sigma}=(\mathrm{id}, s, s, s)$. In general, however, $\mathscr{D}$ turns out to be a considerably smaller set than $\mathscr{S}$.

§3. Proof of Theorem 1.1

Throughout this section, W denotes Weyl group associated to a semisimple simply connected algebraic group G over Ω. We use the notation set in §2. We begin with the following:

Proposition 3.1. *Let $u_1\in U_1$ be fixed. For $0\leqq j\leqq k$, let $\sigma_j\in W$ be the unique element such that $u_1 s_1 \dots s_j\in B^-\sigma_j B$. Then $\underline{\sigma}=(\sigma_0, \sigma_1, \dots, \sigma_k)$ is a distinguished subexpression.*

Proof. We first note that the following holds for G: For $w\in W$ and $s\in S$, $wBs\subseteq B^- wsB\cup B^- wB$. Moreover, $wBs\cap B^- wB\neq\phi\Rightarrow l(ws)\geqq l(w)$. This is a variant of the 'usual' condition $(T3)$ satisfied by the Tits system (G, B, N, S) and it is very easy to prove using Lemma 2.1.

Clearly $\sigma_0=\mathrm{id}$. Consider $1\leqq j\leqq k$. Then $u_1 s_1\dots s_j=u_1 s_1\dots s_{j-1}s_j\in B^-\sigma_{j-1}Bs_j$. Hence from above, $u_1 s_1 \dots s_j\in B^-\sigma_{j-1}s_j B$ or $B^-\sigma_{j-1}B$ with the second possibility holding only if $l(\sigma_{j-1}s_j)\geqq l(\sigma_{j-1})$. It is now clear that either $\sigma_j=\sigma_{j-1}s_j$ or $\sigma_j=\sigma_{j-1}$ with $l(\sigma_{j-1}s_j)\geqq l(\sigma_{j-1})$. In either case, it can be seen that conditions (ii) and (iii) are satisfied for the sequence $(\sigma_0, \sigma_1, \dots, \sigma_k)$. Hence $\underline{\sigma}\in\mathscr{D}$. This completes the proof.

We now have a map $\eta\colon U_1\to\mathscr{D}$ defined in an obvious way. We take a look at the fibers. Note that it is apriori not clear if η is surjective; we will prove it in the course of the proof of Theorem 1.1.

Next we have the following:

Proposition 3.2. *Fix $\underline{\sigma}\in\mathscr{D}$ and an integer j $(1\leqq j\leqq k)$ Define a subset $\Omega(\underline{\sigma}, j)$ of Ω as follows.*

$$\Omega(\underline{\sigma}, j)=\begin{cases}\Omega & \text{if } \sigma_{j-1}>\sigma_j\\ \{0\} & \text{if } \sigma_{j-1}<\sigma_j\\ \Omega^* & \text{if } \sigma_{j-1}=\sigma_j.\end{cases}$$

Then one has:

(i) *There exists an injective morphism* $f_j\colon \Omega(\underline{\sigma}, j)\times U_{j+1}\to U_j$ *such that*

$$\sigma_{j-1} f_j(t, u_{j+1})\, s_j\cdot\ldots\cdot s_k = b_j^-\, \sigma_j u_{j+1} s_{j+1}\cdot\ldots\cdot s_k\cdot v_{j+1} \qquad (*)$$

for suitable $b_j^-\in B^-$ *and* $v_{j+1}\in U^{j+1}$.

(ii) *The image of* f_j *is a locally closed subset of* U_j *and* f_j *is an isomorphism onto its image.*

Proof. Case (a). $\sigma_{j-1}>\sigma_j$.
In this case, $\sigma_j=\sigma_{j-1}s_j$ with $\sigma_j(\alpha_j)\in\Phi^+$ and $\Omega(\underline{\sigma}, j)=\Omega$. Define $f_j\colon \Omega\times U_{j+1}\to U_j$ by $f_j(t, u_{j+1})=x_{\alpha_j}(t)\cdot s_j u_{j+1} s_j^{-1}$. It is easy to verify that (i) and (ii) hold in this case with Image $f_j=U_j$.

Case (b). $\sigma_{j-1}<\sigma_j$.
In this case, $\sigma_j=\sigma_{j-1}s_j$ and $\Omega(\underline{\sigma}, j)=\{0\}$. Define $f_j\colon \{0\}\times U_{j+1}\to U_j$ by $f_j(0, u_{j+1}) =s_j u_{j+1} s_j^{-1}$. Again it is easy to check that (i) and (ii) hold in this case. We note that the image of f_j is the closed set ${}^{s_j}U_{j+1}$.

Case (c). $\sigma_{j-1}=\sigma_j$.
In this case, $\Omega(\underline{\sigma}, j)=\Omega^*$. Also, $l(\sigma_{j-1}s_j)\geqq l(\sigma_{j-1})$ and so $\sigma_{j-1}(\alpha_j)\in\Phi^+$. Given $t\in\Omega^*$ and $u_{j+1}\in U_{j+1}$, Lemma 2.2(v) gives a unique $\tilde{u}_{j+1}\in U_{j+1}$ and $v_{j+1}\in U^{j+1}$ such that

$$x_{\alpha_j}(t)\cdot u_{j+1}=\tilde{u}_{j+1}s_{j+1}\cdot\ldots\cdot s_k\cdot v_{j+1}^{-1}\cdot s_k^{-1}\ldots s_{j+1}^{-1} \qquad (**)$$

Define $f_j\colon \Omega^*\times U_{j+1}\to U_j$ by $f_j(t, u_{j+1})=x_{\alpha_j}(t^{-1})\cdot s_j\cdot\tilde{u}_{j+1}\cdot s_j^{-1}\in U_j$ where $\tilde{u}_{j+1}$ is the element in U_{j+1} given by (*). It is easy to see that f_j is injective. Since $t\rightsquigarrow t^{-1}$ is a morphism from Ω^* to Ω^*, it can be checked that f_j is a morphism too. The image of f_j is the open set $U_j\setminus {}^{s_j}U_{j+1}$. To see this, consider $x_{\alpha_j}(d)\cdot s_j\cdot\tilde{u}_{j+1}\cdot s_j^{-1}$ with $d\neq 0$ and $\tilde{u}_{j+1}\in U_{j+1}$. By (**),

$$x_{\alpha_j}(-d^{-1})\cdot\tilde{u}_{j+1}=u_{j+1}s_{j+1}\ldots s_k v_{j+1}\cdot s_k^{-1}\ldots s_{j+1}^{-1}$$

with $u_{j+1}\in U_{j+1}$. It is easy to check that $f_j(d^{-1}, u_{j+1})=x_{\alpha_j}(d)\cdot s_j\tilde{u}_{j+1}\cdot s_j^{-1}$. This proves (ii). Next,

$$\begin{aligned}\sigma_{j-1} f_j(t, u_{j+1})\cdot s_j\ldots s_k &=\sigma_{j-1}\cdot x_{\alpha_j}(t^{-1})\cdot s_j\tilde{u}_{j+1}\cdot s_j^{-1}\cdot s_j\ldots s_k\\ &=\sigma_{j-1}\cdot x_{\alpha_j}(t^{-1})\cdot s_j\tilde{u}_{j+1}s_{j+1}\ldots s_k.\end{aligned}$$

Using (**), we get

$$\begin{aligned}\sigma_{j-1} f_j(t, u_{j+1})\cdot s_j\ldots s_k &=\sigma_{j-1}x_{\alpha_j}(t^{-1})\cdot s_j\cdot x_{\alpha_j}(t)\,u_{j+1}s_{j+1}\ldots s_k\cdot v_{j+1}\\ &=\sigma_{j-1}hs_j x_{\alpha_j}(-t^{-1})\cdot s_j^{-1}u_{j+1}s_{j+1}\ldots s_k\cdot v_{j+1} \qquad \text{for some } h\in T.\end{aligned}$$

(This follows from Lemma 2.1).

$$=b_j^-\,\sigma_{j-1}u_{j+1}s_{j+1}\ldots s_k\cdot v_{j+1}$$

where $b_j^-=\sigma_{j-1}hs_j x_{\alpha_j}(-t^{-1})s_j^{-1}\sigma_{j-1}^{-1}\in B^-$ since $\sigma_{j-1}(\alpha_j)\in\Phi^+$. Since $\sigma_{j-1}=\sigma_j$ in this case, (*) is satisfied and (i) is proved. This completes the proof of this proposition.

We now come to the proof of Theorem 1.1. We take the set $\mathscr{D}$ of distinguished subexpressions (cf. §2) as the indexing set. Recall the map $\eta: U_1 \to \mathscr{D}$ given by Proposition 3.1. For $\underline{\sigma} \in \mathscr{D}$, define $D_{\underline{\sigma}} = \{u_1 y \cdot B \in By \cdot B \mid \eta(u_1) = \underline{\sigma}\}$. (Note that $By \cdot B \simeq U_1$). It is then clear that

(i) $By \cdot B = \bigcup_{\underline{\sigma} \in \mathscr{D}} D_{\underline{\sigma}}$, a disjoint union.

(ii) Let $\underline{\sigma} \in \mathscr{D}$ be fixed. Define subsets $A_j \subseteq U_j$ $(1 \leqq j \leqq k+1)$ by downward induction on j as follows: $A_{k+1} = \mathrm{id}$ and $A_j = f_j(\Omega(\underline{\sigma}, j) \times A_{j+1})$ for $1 \leqq j \leqq k$, f_j as in Proposition 3.2. It is clear that A_j is a locally closed subset of U_j which is isomorphic to a product $\Omega^{m_j(\sigma)} \times (\Omega^*)^{n_j(\sigma)}$ where $m_j(\underline{\sigma}) = |\{p \mid j \leqq p \leqq k$ and $\Omega(\underline{\sigma}, p) = \Omega\}|$ and $n_j(\underline{\sigma}) = |\{p \mid j \leqq p \leqq k$ and $\Omega(\underline{\sigma}, p) = \Omega^*\}|$. Note that $m_1(\underline{\sigma}) = m(\underline{\sigma})$ and $n_1(\underline{\sigma}) = n(\underline{\sigma})$. Thus $A_1 \simeq \Omega^{m(\sigma)} \times (\Omega^*)^{n(\sigma)}$ we claim that $A_1 = \eta^{-1}(\underline{\sigma})$.

Let $u_1 \in A_1$ then by definition, there exist sequences $\{u_j\}_{1 \leqq j \leqq k+1}$ and $\{t_j\}_{1 \leqq j \leqq k}$ such that for each j, (a) $u_j \in A_j$, (b) $t_j \in \Omega(\underline{\sigma}, j)$ and (c) $u_j = f_j(t_j, u_{j+1})$. Using (*) of Proposition 3.2, it is easy to see that $u_1 s_1 \ldots s_j \in B^- \sigma_j \cdot B \,\forall j$ and so by definition $\eta(u_1) = \underline{\sigma}$ or $A_1 \subseteq \eta^{-1}(\underline{\sigma})$.

Conversely, let $u_1 \in \eta^{-1}(\underline{\sigma})$. We prove by induction on j that there exist sequences $\{u_j\}_{1 \leqq j \leqq k+1}$ (starting with the given u_1) and $\{t_j\}_{1 \leqq j \leqq k}$ such that (a) $t_j \in \Omega(\underline{\sigma}, j)$ and (b) $u_j = f_j(t_j, u_{j+1})$. To see this, observe that the image of f_j is U_j (respectively ${}^{s_j}U_{j+1}$, $U_j \setminus {}^{s_j}U_{j+1}$) if $\sigma_{j-1} > \sigma_j$ (respectively $\sigma_{j-1} < \sigma_j$, $\sigma_{j-1} = \sigma_j$). It is easy to see that $u_1 \in \eta^{-1}(\underline{\sigma})$ implies $u_1 \in \mathrm{Image}\, f_1$. Thus $\exists$ unique $t_1 \in \Omega(\underline{\sigma}, 1)$ and $u_2 \in U_2$ such that $u_1 = f_1(t_1, u_2)$ Having defined $u_1, \ldots, u_j$ and $t_1, \ldots, t_{j-1}$, it can be proved that $u_j \in \mathrm{Image}\, f_j$ (or else $u_1 \notin \eta^{-1}(\underline{\sigma})$). The definition of t_j and u_{j+1} is now clear. Since $u_{k+1} \in A_{k+1}$ and $u_j = f_j(t_j, u_{j+1})$, it is clear that $u_j \in A_j \forall_j$ and so $u_1 \in A_1$. Thus $A_1 = \eta^{-1}(\underline{\sigma})$. Now $D_{\underline{\sigma}} = A_1 y \cdot B \simeq A_1 \simeq \Omega^{m(\sigma)} \times (\Omega^*)^{n(\sigma)}$ proving (ii). Note that this also proves that $D_{\underline{\sigma}} \neq \phi \,\forall \underline{\sigma} \in \mathscr{D}$.

(iii) Fix $\underline{\sigma} \in \mathscr{D}$ and let $u_1 y \cdot B \in D_{\underline{\sigma}}$. Then $\eta(u_1) = \underline{\sigma}$. Also, $u_1 s_1 \ldots s_k \in B^- \sigma_k \cdot B$ by definition of map η and so $u_1 y \cdot B \in B^- \sigma_k \cdot B$. Thus $D_{\underline{\sigma}} \subseteq B^- \sigma_k \cdot B$. It is also clear that $\sigma_k \in W(y)$.

This completes the proof of Theorem 1.1.

Proof of Corollary 1.2.

$$By \cdot B \cap B^- x \cdot B = \Big(\bigcup_{\underline{\sigma} \in \mathscr{D}} D_{\underline{\sigma}}\Big) \cap B^- x \cdot B = \bigcup_{\substack{\underline{\sigma} \in \mathscr{D} \\ \pi(\underline{\sigma}) = x}} D_{\underline{\sigma}}$$

by Theorem 1.1 (iii).

Since $\pi(\underline{\sigma}) \in W(y)$, it is clear that $By \cdot B \cap B^- x \cdot B \neq \Phi$ only if $x \in W(y)$. As mentioned earlier, it can be proved independently of this section (Prop. 5.3) that $\pi: \mathscr{D} \to W(y)$ is onto. Hence $x \in W(y) \Rightarrow By \cdot B \cap B^- x \cdot B \neq \phi$.

§4. The special case of a Coxeter element

In this section, we consider the case when y is a Coxeter element of W (or that of a parabolic subgroup of W). Thus y has a reduced expression $y = s_1 \ldots s_k$ with $s_i \neq s_j$ for $i \neq j$. In this case, a lot of simplifications take place and one gets a very simple description of the sets $D_{\underline{\sigma}}$'s that occur in Theorem 1.1.

First of all, every subexpression in $\mathscr{S}$ is a distinguished one i.e. $\mathscr{S}=\mathscr{D}$. Next, the map $\pi:\mathscr{S}\to W(y)$ is bijective and the Bruhat ordering in $W(y)$ is simply given by: $x\leqq z(\leqq y)$ iff $I(\pi^{-1}(x))\supseteq I(\pi^{-1}(z))$. It can be easily seen that $\mathscr{D}\simeq\mathscr{P}(\Delta)$ in this case.) Further, $m(\underline{\sigma})=0$ for all $\underline{\sigma}\in\mathscr{D}$ so that $l(\pi(\underline{\sigma}))=k-n(\underline{\sigma})$.

Let $x\in W(y)$. Since π is injective and $m(\pi^{-1}(x))=0$, $By\cdot B\cap\bar{B}x\cdot B=D_{\pi^{-1}(x)}\simeq(\Omega^*)^{k-l(x)}$. Unlike the general case (where the description is complicated), this isomorphism is quite simple to describe and this is done as follows: Identify U_1 with Ω^k as given in Lemma 2.2(i). Then

$$D_{\underline{\sigma}}=\{(a_1,\dots,a_k)\,|\,a_j=0 \text{ for } j\notin I(\underline{\sigma}) \text{ and } a_j\neq 0 \text{ for } j\in I(\underline{\sigma})\}.$$

This can be seen by taking a closer look at the proofs of Theorem 1.1 and Proposition 3.2. Since $m(\underline{\sigma})=0$, case (a) of Proposition 3.2 doesn't occur and hence $a_j=0$ for $j\notin I(\underline{\sigma})$. Next, we look at relation (**) of Proposition 3.2. Since α_j's are distinct, (**) gets simplified and $\tilde{u}_{j+1}=u_{j+1}$. From this it follows that $a_j\neq 0$ for $j\in I(\underline{\sigma})$.

Remark 4.1. Consider the element $\mathrm{id}\in W(y)$. Then $\pi^{-1}(\mathrm{id})=\underline{\sigma}^{(0)}=(\mathrm{id},\mathrm{id},\dots,\mathrm{id})$. Then from above,

$$By\cdot B\cap B^-\mathrm{id}\cdot B=D_{\underline{\sigma}^{(0)}}=\{(a_1,\dots,a_k)\in\Omega^k\,|\,a_j\neq 0\,\forall_j\}.$$

This description is basically the same as given in ([L, Prop. 2.2]).

Remark 4.2. The closure of $D_{\underline{\sigma}}(\underline{\sigma}\in\mathscr{D})$ in the affine space U_1 is very easy to describe. One has:

$$\bar{D}^a_{\underline{\sigma}}=\bigcup_{I(\underline{\sigma})\supset I(\underline{\tau})}D_{\underline{\tau}}=\{(a_1,\dots,a_k)\,|\,a_j=0 \text{ for } j\notin I(\underline{\sigma})\}$$

In other words, for $x\in W(y)$,

$$\bar{D}^a_{\pi^{-1}(x)}=\bigcup_{x\leqq z}D_{\pi^{-1}(z)}=By\cdot B\cap\bigcup_{x\leqq z}B^-z\cdot B$$

Remark 4.3. One can also consider the closure of $D_{\underline{\sigma}}(\underline{\sigma}=\pi^{-1}(x))$ in the *projective* space G/B. It can be shown that

$$\bar{D}^p_{\pi^{-1}(x)}=\bigcup_{x\leqq z'\leqq z\leqq y}(Bz\cdot B\cap B^-z'\cdot B)$$

In particular, for $x=\mathrm{id}$,

$$\bar{D}^p_{\pi^{-1}(\mathrm{id})}=\bigcup_{z'\leqq z\leqq y}(Bz\cdot B\cap B^-z'\cdot B).$$

Since $D_{\pi^{-1}(x)}$ is a torus, one gets a toroidal imbedding (cf. [Ke]).

Remark 4.4. The closures in the general case of a non-Coxeter element are more subtle to describe and we take this up in another paper.

§5. Proof of Theorem 1.3 and structure of $\mathscr{D}$

For this and the next section, (W, S) is any Coxeter group.

For $y \in W$ fix a reduced expression $y = s_1 \cdot \ldots \cdot s_k$. As explained in §2, we then have sets $\mathscr{S}$ and $\mathscr{D}$ of subexpressions and distinguished subexpressions respectively. We also have maps $\pi: \mathscr{S} \to W(y)$ $(= \{x \in W \mid x \leq y\})$, $m: \mathscr{S} \to \mathbb{Z}^+$ and $n: \mathscr{S} \to \mathbb{Z}^+$.

We begin the proof of Theorem 1.3 by a simple lemma:

Lemma 5.1. *For $\underline{\sigma} \in \mathscr{S}$, $l(\pi(\underline{\sigma})) = k - n(\underline{\sigma}) - 2m(\underline{\sigma})$.*

The proof of this lemma is easy to derive from definitions of maps π, m and n.

Next we introduce some subsets of $\mathscr{S}$ and $\mathscr{D}$ as follows: For $x \in W(y)$, let $\mathscr{S}(x) = \pi^{-1}(x)$ and $\mathscr{D}(x) = \mathscr{S}(x) \cap \mathscr{D}$. For $i \in \mathbb{Z}^+$, let $\mathscr{S}_i(x) = \mathscr{S}(x) \cap m^{-1}(i)$ and $\mathscr{D}_i(x) = \mathscr{S}_i(x) \cap \mathscr{D}$.

We want to compare these sets with sets defined analogously for the element ys_k and the reduced expression $s_1 \ldots s_{k-1}$ for it. Let $\mathscr{S}'$, $\mathscr{D}'$, $\mathscr{S}_i'(x)$, $\mathscr{D}_i'(x)$ denote these sets and m', n' denote corresponding maps. We have a natural map $\theta: \mathscr{S} \to \mathscr{S}'$ given by $\theta((\sigma_0, \sigma_1, \ldots, \sigma_k)) = (\sigma_0, \sigma_1, \ldots, \sigma_{k-1})$. It is easy to see that $\theta(\mathscr{D}) \subseteq \mathscr{D}'$. Let $x \in W(y)$. Then we have the following lemma for "comparisons":

Lemma 5.2.

(a) *If $xs_k \leq x$ then* (i) $xs_k \in W(ys_k)$,

(ii) $k \notin I(\underline{\sigma}) \forall \underline{\sigma} \in \mathscr{D}(x)$ *so that* $\theta(\mathscr{D}(x)) \subseteq \mathscr{D}'(xs_k)$,

(iii) $n(\underline{\sigma}) = n'(\theta(\underline{\sigma})) \forall \underline{\sigma} \in \mathscr{D}(x)$ *so that* $\theta(\mathscr{D}_i(x)) \subseteq \mathscr{D}_i'(xs_k) \forall i$ *and*

(iv) $\theta: \mathscr{D}_i(x) \to \mathscr{D}_i'(xs_k)$ *is bijective* $\forall i$.

(b) *If $x \leq xs_k$ but $xs_k \notin W(ys_k)$ then* (i) $x \in W(ys_k)$,

(ii) $k \in I(\underline{\sigma}) \forall \underline{\sigma} \in \mathscr{D}(x)$ *so that* $\theta(\mathscr{D}(x)) \subseteq \mathscr{D}'(x)$,

(iii) $n(\underline{\sigma}) = n'(\theta(\underline{\sigma})) + 1 \forall \underline{\sigma} \in \mathscr{D}(x)$ *so that* $\theta(\mathscr{D}_i(x)) \subseteq \mathscr{D}_i'(x) \forall i$ *and*

(iv) $\theta: \mathscr{D}_i(x) \to \mathscr{D}_i'(x)$ *is bijective* $\forall i$.

(c) *If $x \leq xs_k$ and $xs_k \in W(ys_k)$ then define subsets $\mathscr{A}(x)$ and $\mathscr{B}(x)$ of $\mathscr{D}(x)$ by: $\mathscr{A}(x) = \{\underline{\sigma} \in \mathscr{D}(x) \mid k \notin I(\underline{\sigma})\}$ and $\mathscr{B}(x) = \mathscr{D}(x) \setminus \mathscr{A}(x)$. Then one has:* (i) $n(\underline{\sigma}) = n'(\theta(\underline{\sigma}))$ $\forall \underline{\sigma} \in \mathscr{A}(x)$ *so that* $\theta(\mathscr{A}(x) \cap \mathscr{D}_i(x)) \subseteq \mathscr{D}_{i-1}'(xs_k) \forall i$, (ii) $\theta: \mathscr{A}(x) \cap \mathscr{D}_i(x) \to \mathscr{D}_{i-1}'(xs_k)$ *is bijective* $\forall i$, (iii) $n(\underline{\sigma}) = n'(\theta(\underline{\sigma})) + 1 \forall \underline{\sigma} \in \mathscr{B}(x)$ *so that* $\theta(\mathscr{B}(x) \cap \mathscr{D}_i(x)) \subseteq \mathscr{D}_i'(x) \forall i$ *and* (iv) $\theta: \mathscr{B}(x) \cap \mathscr{D}_i(x) \to \mathscr{D}_i'(x)$ *is bijective* $\forall i$.

The proof of this lemma is not difficult; we prove parts c(iii) and c(iv) as an illustration.

Proof of c(iii) and c(iv)

Since $\mathscr{B}(x) = \mathscr{D}(x) \setminus \mathscr{A}(x)$, $k \in I(\underline{\sigma}) \forall \underline{\sigma} \in \mathscr{B}(x)$, i.e. $\sigma_{k-1} = \sigma_k = x \forall \underline{\sigma} \in \mathscr{B}(x)$. Hence $\theta(\underline{\sigma}) \in \mathscr{D}'(x)$. Also, $n(\underline{\sigma}) = n'(\theta(\underline{\sigma})) + 1$ as is clear and so $m(\underline{\sigma}) = m'(\theta(\underline{\sigma}))$ by Lemma 5.1. Thus $\theta(\mathscr{B}(x) \cap \mathscr{D}_i(x)) \subseteq \mathscr{D}_i'(x) \forall i$. Also, θ is clearly injective on $\mathscr{B}(x) \cap \mathscr{D}_i(x)$. Finally, given $\underline{\sigma}' \in \mathscr{D}_i'(x)$, let $\underline{\sigma} = (\underline{\sigma}', x)$ then $\underline{\sigma} \in \mathscr{B}(x)$ since $\sigma_{k-1} s_k \geq \sigma_{k-1}$ and $k \in I(\underline{\sigma})$. Also $\underline{\sigma} \in \mathscr{D}_i(x)$ as is clear. So θ is surjective on $\mathscr{B}(x) \cap \mathscr{D}_i(x)$. This proves c(iii) and c(iv).

Now we can give

Proof of Theorem 1.3

Using notation introduced by us, we have to prove that

$$R_{x,y}(q)=\sum_{\sigma\in\mathscr{D}(x)}(q-1)^{n(\sigma)}\cdot q^{m(\sigma)}.$$

We proceed by induction on $l(y)$.

If $l(y)=0$, $y=x=\mathrm{id}$ and the above formula is trivially true. So let $k=l(y)\geqq 1$.

We have the following recursive relations ([K-L1, § 2]).

Case (a). If $xs_k\leqq x$ then $R_{x,y}(q)=R_{xs_k,ys_k}(q)$.

Case (b). If $x\leqq xs_k$ but $xs_k\notin W(ys_k)$ then $R_{x,y}(q)=(q-1)\cdot R_{x,ys_k}(q)$.

Case (c). If $x\leqq xs_k$ and $xs_k\in W(ys_k)$ then $R_{x,y}(q)=(q-1)R_{x,ys_k}(q)+q\cdot R_{xs_k,ys_k}(q)$.

Clearly these cases correspond respectively to the three parts of Lemma 5.2 which enables us to establish the induction step. We give details of case (c) as an illustration. By induction,

$$\begin{aligned}R_{x,ys_k}(q)&=\sum_{\sigma'\in\mathscr{D}'(x)}(q-1)^{n'(\sigma')}\cdot q^{m'(\sigma')}\\&=\sum_i\Big(\sum_{\sigma'\in\mathscr{D}_i'(x)}(q-1)^{n'(\sigma')}\Big)q^i\\&=\sum_i\Big(\sum_{\sigma\in\mathscr{B}(x)\cap\mathscr{D}_i(x)}(q-1)^{n(\sigma)-1}\Big)q^i\\&=\sum_{\sigma\in\mathscr{B}(x)}(q-1)^{n(\sigma)-1}\cdot q^{m(\sigma)}.\end{aligned}$$

Similarly, $R_{xs_k,ys_k}(q)=\sum_{\sigma\in\mathscr{A}(x)}(q-1)^{n(\sigma)}\cdot q^{m(\sigma)-1}$. Hence it is clear that $R_{x,y}(q)=\sum_{\sigma\in\mathscr{D}(x)}(q-1)^{n(\sigma)}\cdot q^{m(\sigma)}$. This establishes the induction step in this case. The proof of Theorem 1.3 is now complete.

Rewriting the expression on right hand side of Theorem 1.3, we get:

$$R_{x,y}(q)=|\mathscr{D}_0(x)|\cdot(q-1)^{k-l(x)}+|\mathscr{D}_1(x)|\cdot(q-1)^{k-l(x)-2}\cdot q+\dots.$$

However, this gives no information about vanishing of terms i.e. when $|\mathscr{D}_i(x)|=0$ or $\mathscr{D}_i(x)=\phi$. Our next proposition brings out this aspect and we get a sharp description about the "shape" of $R_{x,y}(q)$ which was stated by the author in [D3, Rem. 2.2]:

Proposition 5.3. *Let $x\in W(y)$. Then one has:*

(i) $\mathscr{S}_i(x)\neq\phi\Rightarrow 2i\leqq k-l(x)$.

(ii) $\mathscr{S}_i(x)\neq\phi\Rightarrow\mathscr{D}_i(x)\neq\phi$.

(iii) $\mathscr{S}_i(x)\neq\phi$ *and* $i\neq 0\Rightarrow\mathscr{S}_{i-1}(x)\neq\phi$.

(iv) $\mathscr{D}_0(x)$ *consists of exactly one element.*

(v) *If* $\mathscr{D}_0=\bigcup_{x\in W(y)}\mathscr{D}_0(x)$ *then* $\pi:\mathscr{D}_0\to W(y)$ *is a bijection.*

Proof. (i) Follows from Lemma 5.1.

(ii) For $\underline{\sigma}\in\mathcal{S}_i(x)$, recall that the set $I(\underline{\sigma})=\{j|\sigma_{j-1}=\sigma_j\}$ has $n(\underline{\sigma})=k-l(x)-2i$ distinct integers. We arrange them in the increasing order and consider these $(k-l(x)-2i)$-tuples along with the lexicographic order on them induced from the set of *all* (not necessarily increasing) $(k-l(x)-2i)$-tuples of positive integers. Choose $\underline{\sigma}\in\mathcal{S}_i(x)$ such that the corresponding $(k-l(x)-2i)$-tuple is the smallest. We claim that $\underline{\sigma}\in\mathcal{D}_i(x)$ in that case. If not, $\exists j\in I(\underline{\sigma})$ such that $\sigma_{j-1}\geqq\sigma_{j-1}s_j$ (cf. §2, Def. 2.3). By the strongest version of exchange condition for (W,S) ([D2, Prop. 3.1(iii)]), $\exists r$ such that (a) $1\leqq r\leqq j-1$, (b) $r\notin I(\underline{\sigma})$ and (c) $\sigma_{r-1}s_r\sigma_{r-1}^{-1}=\sigma_{j-1}s_j\sigma_{j-1}^{-1}=t$, say. We also note that $\sigma_r=\sigma_{r-1}s_r$ since $r\notin I(\underline{\sigma})$ and $\sigma_j=\sigma_{j-1}$ since $j\in I(\underline{\sigma})$. Consider now a sequence $\underline{\tau}=\{\tau_0,\ldots,\tau_k\}$ given by: $\tau_p=t\sigma_p$ for $r\leqq p\leqq j-1$ and $\tau_p=\sigma_p$ otherwise. It is easy to check that $\underline{\tau}\in\mathcal{S}$. Also $\tau_k=\sigma_k=x$ so that $\pi(\underline{\tau})=x$. Next, $I(\underline{\tau})=I(\underline{\sigma})\cup\{r\}\setminus\{j\}$ so that $n(\underline{\tau})=n(\underline{\sigma})$ and so $\underline{\tau}\in\mathcal{S}_i(x)$. Since $r\leqq j-1$, it is clear that the $(k-l(x)-2i)$-tuple corresponding to $\underline{\tau}$ is smaller than the one for $\underline{\sigma}$ which contradicts the choice of $\underline{\sigma}$. Hence $\underline{\sigma}\in\mathcal{D}_i(x)$ as claimed. Thus $\mathcal{D}_i(x)\neq\phi$.

(iii) can be proved using the strongest version of exchange condition in a way similar to one in (ii).

(iv) Since $x\in W(y)$ and π is onto, $\mathcal{S}(x)\neq\phi$. Thus $\mathcal{S}_i(x)\neq\phi$ for some $i\geqq 0$. Using (iii) repeatedly, $\mathcal{S}_0(x)\neq\phi$ and hence $\mathcal{D}_0(x)\neq\phi$ by (ii). We have now to show that $\underline{\sigma},\underline{\tau}\in\mathcal{D}_0(x)\Rightarrow\underline{\sigma}=\underline{\tau}$. We achieve this by showing $I(\underline{\sigma})=I(\underline{\tau})$. Let, if possible, $I(\underline{\sigma})\neq I(\underline{\tau})$. Consider the maximum integer j belonging to the symmetric difference of these two sets. Thus, for $j+1\leqq p\leqq k$, $p\in I(\underline{\sigma})$ iff $p\in I(\underline{\tau})$. It is then clear that $\sigma_p=\tau_p$ for all $j+1\leqq p\leqq k$. (This can be proved by using downward induction on p and the fact that $\sigma_p^{-1}\sigma_{p+1}=\tau_p^{-1}\tau_{p+1}$ $\forall j+1\leqq p\leqq k-1$.) Now let $j\in I(\underline{\sigma})$ but $j\notin I(\underline{\tau})$ (for the sake of definiteness). Then $\sigma_j=\sigma_{j+1}=\tau_{j+1}=\tau_j s_{j+1}$. Since $m(\underline{\tau})=0$, $\tau_{j+1}>\tau_j$. But this means that $\sigma_j>\sigma_j s_{j+1}$ which is a contradiction since $\underline{\sigma}\in\mathcal{D}$. Thus $I(\underline{\sigma})=I(\underline{\tau})$ and $\underline{\sigma}=\underline{\tau}$. This proves (iv). (v) follows from (iv).

This completes the proof of Proposition 5.3.

Remark 5.4. With reference to the proof of (ii) above, it can be proved that $\underline{\sigma}\in\mathcal{S}_i(x)$ with $\sum_{p\in I(\sigma)} p$ minimal must also belong to $\mathcal{D}_i(x)$. However, these two minimality conditions for elements of $\mathcal{S}_i(x)$ are independent of each other.

Remark 5.5. One may sharpen the statement (i) above as follows:

(i)′ $\mathcal{S}_i(x)\neq\phi\Rightarrow 2i\leqq k-l(x)-2$ unless (a) $x=y$ or (b) $x^{-1}y\in\bigcup_{w\in W} wSw^{-1}$. In case (a) $i=0$ and $\mathcal{S}_0(x)$ is a singleton set. In case (b) $k-l(x)-1$ is even and $\mathcal{S}_{\frac{k-l(x)-1}{2}}(x)$ is a singleton set. We thus recover Lemma 2.1 of [D3].

Remark 5.6. If one considers the natural order on $\mathcal{S}$ given by $\underline{\sigma}\leqq\underline{\tau}$ iff $\sigma_i\leqq\tau_i$ $\forall 1\leqq i\leqq k$ then it can be proved that for any $x_1\leqq x_2\in W(y)$ and any $\underline{\tau}\in\mathcal{S}(x_2)$, $\underline{\mu}\leqq\underline{\tau}$ where $\underline{\mu}$ is the unique element in $\mathcal{D}_0(x_1)$. In particular, $\pi\colon\mathcal{D}_0\to W(y)$ is an order-preserving bijection. The above order on $\mathcal{D}$ is related to the closure problem mentioned in Remark 4.4.

§6. *L*-Shellability of Bruhat Ordering

As mentioned in the introduction, the set $\mathcal{D}_0 = \bigcup_{x \in W(y)} \mathcal{D}_0(x)$ enables us to prove L-shellability of Bruhat ordering. We refer the reader to [B–W] for the definition of this concept. Here we prove a proposition regarding $\mathcal{D}_0$ which is the crux of the matter. This even proves the "relative version" viz. any interval $[x, y]$ in W^J for any $J \subseteq S$ is L-shellable (see below for definitions).

We first need some notation. For $x \in W(y)$, consider a chain $C: y = y_0 \geqq y_1 \geqq \ldots \geqq y_r = x$ of elements in $W(y)$ such that $l(y_i) = k - i$. It is then easy to see that there exists a sequence $(p_1, \ldots, p_r)$ of distinct integers and elements $\underline{\sigma}^{(i)} \in \mathcal{S}_0(y_i)$ such that $I(\underline{\sigma}^{(i)}) = \{p_1, \ldots, p_i\}$. We denote this sequence of integers by $\phi(C)$. Note that this sequence need not be increasing. Now consider all chains between y and x and corresponding sequences of r integers equipped with the lexicographic order. We then have:

Proposition 6.1

(i) *$\exists$ a unique chain $\tilde{C}: y = \tilde{y}_0 \geqq \tilde{y}_1 \geqq \ldots \geqq \tilde{y}_r = x$ for which $\phi(\tilde{C}) = (\tilde{p}_1, \ldots, \tilde{p}_r)$ is increasing. Moreover, the corresponding elements $\underline{\tilde{\sigma}}^{(i)}$ belong to $\mathcal{D}_0(\tilde{y}_i)$ respectively.*

(ii) *For any chain C from y to x, $\phi(\tilde{C}) \leqq \phi(C)$.*

(iii) *If $J \subseteq S$ is such that x and y belong to W^J, the set of minimal coset representatives of W/W^J* (cf. [D1, §3]), *then $\tilde{y}_i \in W^J \, \forall i$.*

Proof. (i) Let $\underline{\mu} = (\mu_0, \mu_1, \ldots, \mu_k)$ be the unique element of $\mathcal{D}_0(x)$ (cf. Prop. 5.3(iv)). Let $I(\underline{\mu}) = \{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_r\}$ with $\tilde{p}_1 < \tilde{p}_2 < \ldots < \tilde{p}_r$. Define elements $\tilde{y}_i$ $\sigma \leqq i \leqq r$ as follows: $\tilde{y}_0 = y$ and $\tilde{y}_i = \mu_{\tilde{p}_i} \cdot s_{\tilde{p}_i} \cdot \mu_{\tilde{p}_i}^{-1} \cdot \tilde{y}_{i-1}$ $\forall 1 \leqq i \leqq r$. Since $\mu_{\tilde{p}_i} = \mu_{\tilde{p}_i - 1}$ $\forall i$, it is easy to see that $\tilde{y}_r = x$. Also $l(\tilde{y}_i) < l(\tilde{y}_{i-1})$ since $l(\mu_{\tilde{p}_i} \cdot s_{\tilde{p}_i}) \geqq l(\mu_{\tilde{p}_i})$ and $l(\mu_{\tilde{p}_i}^{-1} \tilde{y}_{i-1}) \geqq l(s_{\tilde{p}_i} \mu_{\tilde{p}_i}^{-1} \tilde{y}_{i-1})$. Thus $r = l(\tilde{y}_0) - l(\tilde{y}_r) = \sum_{i=1}^{r} (l(\tilde{y}_{i-1}) - l(\tilde{y}_i)) \geqq r$. Hence $l(\tilde{y}_{i-1}) - l(\tilde{y}_i) = 1 \, \forall 1 \leqq i \leqq r$ and so $l(\tilde{y}_i) = k - i$. Thus $\tilde{C}: y = \tilde{y}_0 \geqq \tilde{y}_1 \geqq \ldots \geqq \tilde{y}_r = x$ is indeed a chain. Also, $\phi(C) = (\tilde{p}_1, \ldots, \tilde{p}_r)$ and hence is an increasing sequence. It can be proved that the elements $\underline{\tilde{\sigma}}^{(i)}$ corresponding to $\tilde{C}$ belong to $\mathcal{D}_0(\tilde{y}_i)$. Note that $\underline{\tilde{\sigma}}^{(r)} = \underline{\mu}$.

Now given any chain $C: y = y_0 \geqq y_1 \geqq \ldots \geqq y_r = x$ such that $\phi(C) = (p_1, \ldots, p_r)$ is increasing, it can be checked that corresponding elements $\underline{\sigma}^{(i)} \in \mathcal{D}_0(y_i)$. (The property of increasing ensures that.) Thus $\underline{\sigma}^{(r)} \in \mathcal{D}_0(x)$ and so $\underline{\sigma}^{(r)} = \underline{\mu}$. Hence $\{p_1, \ldots, p_r\} = I(\underline{\sigma}^{(r)}) = I(\underline{\mu}) = \{\tilde{p}_1, \ldots, \tilde{p}_r\}$. Since both these sequences are increasing, $p_i = \tilde{p}_i \forall i$. So $C = \tilde{C}$. This proves (i).

(ii) Let, if possible, $\exists$ chain C such that $\phi(C) < \phi(\tilde{C})$. Consider $\underline{\sigma}^{(r)} \in \mathcal{S}_0(x)$ corresponding to C. Then the r-tuple corresponding to $\underline{\sigma}^{(r)}$ (cf. proof of Prop. 5.3(ii)) is a rearrangement of $\phi(C)$ in the increasing order and hence is smaller than it ($\phi(C)$) in the lexicographic order. Therefore the r-tuple corresponding to $\underline{\sigma}^{(r)} <$ that corresponding to $\underline{\mu}$ (which is $(\tilde{p}_1, \ldots, \tilde{p}_r)$). Now as seen in proof of Proposition 5.3(ii), $\exists \underline{\tau} \in \mathcal{D}_0(x)$ such that the r-tuple corresponding to $\underline{\tau} \leqq r$-tuple corresponding to $\underline{\sigma}^{(r)}$. But $\underline{\tau} = \underline{\mu}$ as $\mathcal{D}_0(x)$ is a singleton set. This gives a contradiction. This proves (ii).

(iii) Let $J \subseteq S$ such that $x, y \in W^J$. We use the following fact regarding elements of W^J ([D1, Lemma 3.2]): If $w \leqq w' \in W$ are such that (a) $w \in W^J$ and (b) $l(w') = l(w) + 1$ then either $w' \in W^J$ or $w' = ws$ for some $s \in J$.

Consider the chain $\tilde{C}$ constructed in (i). We prove by downward induction that $\tilde{y}_i \in W^J \forall i$. For $i = r$, $\tilde{y}_r = x \in W^J$. Next, assume $\tilde{y}_i \in W^J$. Then the pair $(\tilde{y}_i, \tilde{y}_{i-1})$ satisfies conditions (a) and (b) mentioned above. Hence either $\tilde{y}_{i-1} \in W^J$ or $\tilde{y}_i^{-1} \tilde{y}_{i-1} \in J$. Let, if possible, the second possibility hold. It can be checked that $\tilde{y}_i^{-1} \tilde{y}_{i-1} = s_k \cdot \ldots \cdot s_{\tilde{p}_i+1} s_{\tilde{p}_i} s_{p_i+1} \cdot \ldots \cdot s_k$. Hence $y \cdot (\tilde{y}_i^{-1} \cdot \tilde{y}_{i-1}) = s_1 \ldots \hat{s}_{\tilde{p}_i} \ldots s_k$ which is a contradiction since $l(ys) \geqq l(y) \forall s \in J$ whereas $l(s_1 \ldots \hat{s}_{\tilde{p}_i} \ldots s_k) < l(y)$. This proves that $\tilde{y}_{i-1} \in W^J$. This proves (iii) by induction.

References

[BB] Bialynicki-Birula, A.: Some theorems on actions of algebraic groups. Annals of Math. **98**, 480–497 (1973)

[B–W] Björner, A., Wachs, M.: Bruhat order of Coxeter groups and shellability. Advances in Math. **43**, 87–100 (1983)

[Bo] Bourbaki, N.: Groupes et algebres de Lie, Chapitre 4, 5 et 6. Paris: Hermann 1968

[D1] Deodhar, V.: Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function. Invent. Math. **39**, 187–198 (1977)

[D2] Deodhar, V.: On the root system of a Coxeter group. Commu. in Alg. **10**(6), 611–630 (1982)

[D3] Deodhar, V.: On the Kazhdan-Lusztig conjectures. Proc. of Koni. Neder. Aka. van Weten., Amsterdam, Series A, **85**, 1–17 (1982)

[G] Garland, H.: The arithmetic theory of loop groups. Publ. Math. I.H.E.S. **52**, 5–136 (1980)

[H] Haddad, Z.: Infinite dimensional flag varieties. Thesis, M.I.T., 1984

[K–P] Kac, V., Peterson, D.: Infinite flag varieties and conjugacy theorems. Proc. Nat. Acad. Sci. USA **80**, 1778–1782 (1983)

[K–L1] Kazhdan, D., Lusztig, G.: Representation of coxeter groups and Hecke algebras. Invent. Math. **53**, 165–184 (1979)

[K–L2] Kazhdan, D., Lusztig, G.: Schubert varieties and Poincaré duality. Proc. Symp. Pure Math. of A.M.S. **36**, 185–203 (1980)

[Ke] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.: Toroidal Embedding I. Lectures Notes in Math., vol. 339. Berlin-Heidelberg-New York: Springer 1973

[L] Lusztig, G.: Coxeter orbits and eigenspaces of Frobenius. Invent. Math. **38**, 101–159 (1976)

[M–T] Moody, R., Teo, K.: Tits' systems with crystallographic Weyl groups. J. of Alg. **21**, 178–190 (1972)

[S1] Slodowy, P.: A character approach to Looijenga's invariant theory for generalized root systems. Preprint

[Sta] Stanley, R.: Interactions between commutative algebra and combinatorics. Report No. 4, Univ. of Stockholm (1982)

[Ste] Steinberg, R.: Lectures on Chevalley groups. Mimeo. notes, Yale Univ. (1967)

[T1] Tits, J.: Resumé de Cours, Annuaire du Collège de France 1980–81, Paris

[T2] Tits, J.: Définition par générateurs et relations de groupes avec BN-pairs. C.R. Acad. Sc. Paris **293**, 317–322 (1981)

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