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Inventiones mathematicae

Arakelov's Theorem for Abelian Varieties

G. Faltings

Fachbereich Mathematik, Universität-Gesamthochschule Wuppertal,
Gaußstr. 20, D-5600 Wuppertal 1, Federal Republic of Germany

§ 1. Introduction

We start with an algebraic curve B over an algebraically closed field k of characteristic zero, and let $S \subseteq B$ be a finite set of points. In [1] Arakelov has shown that there are only finitely many families of algebraic curves of genus $g>1$ on B, with good reduction outside S, except for isotrivial families (isotrivial = becomes constant on a finite cover of B).

We want to consider the same question for principally polarized abelian varieties. Here the answer is more complicated:

There is a condition $(*)$ such that the number of abelian varieties fulfilling $(*)$ is finite, while any variety not fulfilling it can be deformed. The condition $(*)$ says essentially that all endomorphisms of the cohomology of the abelian variety are endomorphisms of the abelian variety itself.

The method of proof consists of a combination of Arakelov's methods and Deligne's description of abelian varieties via Hodge-structures. In the next chapter we recall the necessary prerequisites, and after that we prove the theorem in two steps as in [1].

We first derive a boundedness-theorem and then we show that families fulfilling $(*)$ cannot be deformed. From the form of the theorem it seems that it is difficult to take it over to characteristic $p>0$ (see [7]). The author has learned about this subject from L. Szpiro, who kindly printed out to him that the following results are already known:

a) L. Moret-Bailly has proved a boundedness-theorem in any characteristic. The result is contained in his thesis. (It will appear in the proceedings of the "seminar on pencils of abelian varieties, Paris 1981/82", and in the Comptes-Rendues). Unfortunately this theorem is weaker in characteristic zero than ours.

b) L. Szpiro and L. Moret-Bailly have a very good theorem about boundedness for $S=\emptyset$ (any characteristic).

c) They have derived rigidity for relative dimension two. (To appear also in "seminar on pencils of abelian varieties"). In characteristic zero our results cover relative dimension up to three.

The results were found during a stay at the I.H.E.S. I have to thank P. Deligne for some help concerning the example of an abelian variety not fulfilling (*).

The referee has told me that the results of Zucker (Ann. of Math. **109**) should allow to treat more general Hodge-structures on $B-S$. As I am not an expert in this field I leave this to the reader. In any case I thank the referee for his suggestions.

§ 2. Notations

k always denotes an algebraically closed field of characteristic 0, and in the proofs we assume that $k=\mathbb{C}$, which we may do by the Lefschetz-principle. B denotes a connected complete smooth curve over k, and $S \subseteq B$ a finite set of points. We want to consider families

$$p: X \to B-S$$

of abelian varieties of relative dimension g over S. By [4] we know that giving such a family is the same as giving a polarizable Hodge-structure (or variation of Hodge-structure) of degree 1 on $B-S$, that is (except for the polarization) a locally constant sheaf $\mathbf{V}$ on $B-S$, locally isomorphic to $\mathbb{Z}^{2g}$, plus a subbundle

$$\mathscr{W} = \mathscr{W}_X \subseteq \mathbf{V} \otimes_{\mathbb{Z}} \mathcal{O}_B$$

of rank g, such that $\mathscr{W}$ and its complex conjugate span the fibre of $\mathbf{V}$ in each point. We furthermore assume that it has a principal polarization, which means a skew-symmetric form

$$\langle\, ,\rangle : \mathbf{V} \times \mathbf{V} \to \mathbb{Z},$$

which identifies $\mathbf{V}$ with its own dual, vanishes on $\mathscr{W} \times \mathscr{W}$, and which has the additional property that for $w \in \mathscr{W}$ and $w \neq 0$

$$-\frac{1}{2\pi i}\langle w, \bar{w}\rangle > 0.$$

Definition. (See [4], § 4.4.) A family

$$p: X \to B-S$$

satisfies (*) if any anti-symmetric endomorphism A of

$$\mathbf{V} = R^1 p_* \mathbb{Z}$$

defines an endomorphism of X (i.e., is of type $(0,0)$). A is called anti-symmetric if

$$\langle \mathrm{A}u, v\rangle = -\langle u, \mathrm{A}v\rangle.$$

(We may use étale or de Rham cohomology to formulate that over base fields $k \neq \mathbb{C}$.)

In [4], Proposition 4.4.11, there are given conditions which force X to fulfill $(*)$.

We furthermore need some information about moduli-spaces. We fix a number $n \geqq 3$ and denote by $\mathscr{A} = \mathscr{A}_{n,g}$ the quotient of Siegel's upper half-plane $\mathbb{H}_g$ under the congruence-subgroup group Γ_n of level n of $Sp(2g, \mathbb{Z})$. It is a fine moduli-space for principally polarized abelian varieties with a level-n-structure. As before we have on it a principally polarized Hodge-structure $\mathbf{V}$, together with a subbundle $\mathscr{W}$ of $\mathbf{V} \otimes_{\mathbb{Z}} \mathcal{O}$. $\mathscr{W}$ can also be described as the vector-bundle associated with the natural g-dimensional representation of $U(g, \mathbb{C})$, the maximal compact subgroup of $Sp(2g, \mathbb{R})$. We denote its maximal exterior power by ω.

$$\omega = \Lambda^g \mathscr{W}.$$

We know [3] that $\mathscr{A}$ can be imbedded as an open subvariety into a compact variety $\mathscr{A}^*$, such that ω extends to an ample line-bundle on $\mathscr{A}^*$. There further exists another compactification $\bar{\mathscr{A}}$ of $\mathscr{A}$ which dominates $\mathscr{A}^*$, such that $\bar{\mathscr{A}}$ is nonsingular and that

$$D_\infty = \bar{\mathscr{A}} - \mathscr{A}$$

is a divisor with normal crossings (see [2]). We furthermore know that $\mathscr{W}$ extends as a bundle to $\bar{\mathscr{A}}$, such that the hermitian metric defined on it by the polarization of $\mathbf{V}$ has only logarithmic singularities at D_∞ (see [5])*. It is also wellknown that the extension of the second symmetric power is isomorphic to the sheaf of differential forms with logarithmic poles at D_∞:

$$S^2(\mathscr{W}) \cong \Omega^1_{\bar{\mathscr{A}}}[D_\infty].$$

The latter fact can be seen as follows:

There is such an isomorphism over $\mathscr{A}$, and we only have to show that $\Omega^1_{\bar{\mathscr{A}}}[D_\infty]$ is Mumford's extension of $\Omega^1_{\mathscr{A}}$. But this follows from the fact that locally the compactification looks like an embedding

$$(\mathbb{C}^x)^a \times \mathbb{C}^b \subseteq \mathbb{C}^a \times \mathbb{C}^b,$$

and that the extended bundle is generated by sections invariant under $(\mathbb{C}^x)^a$. (See the construction in [5], p. 257).

If we have an abelian variety

$$p: X \to B - S$$

as in the beginning, and if we assume that X has a level-n-structure, we obtain a mapping

$$\phi: B \to \bar{\mathscr{A}}$$

with

$$\phi^{-1}(D_\infty) \subseteq S.$$

The pullback $\phi^*(\mathscr{W})$ on B is an extension of our previous bundle $\mathscr{W}$ (which was defined on $B - S$). It can be described as follows:

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* This means that for a matrix $H = (h_{pq})$ giving the metric the functions $|h_{pq}|$, $|\det(H)|^{-1}$ grow at most like powers of $\log|1/f|$, f a local equation for D_∞

It is known that X extends to a stable family

$$\bar{p}: \bar{X} \to B$$

a semiabelian varieties over B. If

$$s: B \to \bar{X}$$

denotes its zero-section, then

$$\mathscr{W} = s^*(\Omega^1_{\bar{X}/B}).$$

We can show this as follows:

We may enlarge s, replace B by a covering B', or replace X by a larger abelian variety in which it is a factor. We may also change the polarization (since two of them are comparable), and finally we may assume that X is the Jacobian of a family of curves Y over $B-S$, which extends to a semi-stable family $\bar{Y}$ over B. But then some easy and rather explicit calculations show that the canonical metric on $s^*(\Omega^1_{\bar{X}/B})$ has a only logarithmic singularities, and we are done.

The referee has pointed out to me that in the following chapter we need the equality of Mumford's extension of ω to $\bar{\mathscr{A}}$ and of the pullback of the ample line-bundle ω on $\mathscr{A}^*$. This can be derived from the calculations in [2], Chap. IV. §1, by rewriting them for differential forms with logarithmic singularities (instead of ordinary differential forms): We get that for $l \gg 0$ the global sections of $\omega^{\otimes l}$ on $\mathscr{A}^*$ generate $\omega^{\otimes l}$ on $\bar{\mathscr{A}}$.

§ 3. Boundedness

We want to prove the following

Theorem 1. *Let $\mathscr{L}$ be a line-bundle on $\bar{\mathscr{A}}$. If (B, S) is a curve plus a finite set, there exists a constant c (depending on $\mathscr{L}$, B and S), such that*

$$\deg(\phi^*(\mathscr{L})) \leqq c$$

for any morphism

$$\phi: B \to \bar{\mathscr{A}}$$

with

$$\phi^{-1}(D_\infty) \subseteq S.$$

(ϕ corresponds to an abelian variety over $B-S$ with a level-n-structure).

Proof. A moment's thought shows us that it is sufficient to prove the theorem for $\mathscr{L} = \omega = \Lambda^g \mathscr{W}$ (since $\phi(B) \nsubseteq D_\infty$). We shall prove that for any stable abelian variety X over $B-S$

$$\deg(\mathscr{W}_{\bar{X}}) = \deg(\Lambda^g \mathscr{W}_{\bar{X}})$$

is bounded, where $\bar{X}$ is the Néron-model of X over B, and $\mathscr{W}_{\bar{X}}$ is the extension of $\mathscr{W}_X$ described at the end of the previous chapter.

We proceed as follows:

By the Riemann-Roch theorem it is sufficient to bound the dimension of

$$\Gamma(B, \mathscr{W}_{\bar{X}} \otimes \Omega_B^1[S])$$

($\Omega_B^1[S]$ = differentials with simple poles in S). Any global section of $\mathscr{W}_{\bar{X}} \otimes \Omega_B^1[S]$ defines a holomorphic 2-form on X and therefore a class in

$$H^2(X, \mathbb{C}).$$

We note that the dimension of $H^2(X, \mathbb{C})$ is bounded (use $E_2^{p,q} = H^p(B-S, R^q p_* \mathbb{C}) \Rightarrow H^{p+q}(X, \mathbb{C})$), and so we are done if we prove that the mapping

$$\Gamma(B, \mathscr{W}_{\bar{X}} \otimes \Omega_B^1[S]) \to H^2(X, \mathbb{C})$$

is injective.

By Deligne's theorem ([4], Cor. 3.2.13) this would be true if our 2-forms would have only logarithmic poles at infinity, for some compactification of X. We avoid the construction of such a compactification if we note that for the proof of our claim we may enlarge S, replace B by a finite covering, or replace X by a bigger abelian variety in which it is a factor. We thus reduce to the case that X is the Jacobian of a semi-stable curve Y over $B-S$, with a regular semi-stable model $\bar{Y}$ over B. We furthermore may assume that the mapping from $\bar{Y}$ to B has a section. This gives us a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & B-S, & \end{array}$$

and it is sufficient to show that the mapping

$$\Gamma(B, \mathscr{W}_{\bar{X}} \otimes \Omega_B^1[S]) \to H^2(Y, \mathbb{C})$$

is an injection.

Now for semi-stable curves explicit calculations are rather easy, and they show that the elements of $\Gamma(B, \mathscr{W}_{\bar{X}} \otimes \Omega_B^1[S])$ are precisely the global 2-forms on $\bar{Y}$ with logarithmic poles in the preimage of S. By Deligne's theorem we are ready.

Remark. The proof gives the bound (which is not the best):

$$\deg(\mathscr{W}) \leqq g(3 \cdot \operatorname{genus}(B) + \operatorname{order}(S) + 1).$$

Corollary. *The scheme*

$$\mathbf{Hom}((B, S), (\bar{\mathscr{A}}, D_\infty))$$

which classifies the morphisms

$$\phi: B \to \bar{\mathscr{A}}$$

with

$$\phi^{-1}(D_\infty) \subseteq S$$

is of finite type over k.

§ 4. Deformations

Let

$$\phi: B \to \bar{\mathscr{A}}$$

be a morphism as before, with

$$\phi^{-1}(D_\infty) \subseteq S.$$

We want to determine the tangent-space at ϕ of

$$\mathbf{Hom}((B, S), (\bar{\mathscr{A}}, D_\infty)).$$

By the usual calculations it is isomorphic to

$$\mathrm{Hom}_B(\phi^*(\Omega^1_{\bar{\mathscr{A}}}[D_\infty]), \mathscr{O}_B) = \mathrm{Hom}_B(S^2(\mathscr{W}_X), \mathscr{O}_B).$$

We know by general arguments that the canonical metric on $\mathscr{W}_X$ has positive curvature. It is known that this tends to prevent the existence of linear forms on such a bundle. There are some inequalities which have to become equalities, and we leave it to the reader to verify the following

Lemma. *Suppose $\mathscr{E}$ is a bundle on B which has a hermitian metric with logarithmic singularities and non-negative curvature. If*

$$\phi: \mathscr{E} \to \mathscr{O}_B$$

is a nontrivial morphism let

$$\mathscr{F} = \mathrm{Kern}(\phi).$$

Then $\mathscr{E}$ is isomorphic to the orthogonal direct sum

$$\mathscr{E} \cong \mathscr{F} \oplus \mathscr{O}_B,$$

and ϕ corresponds to the projection onto the second direct summand.

Retourning to our problem we see that any nonzero element s of

$$\mathrm{Hom}_B(\mathbf{S}^2(\mathscr{W}_X), \mathscr{O}_B)$$

gives rise to a section

$$t \in \Gamma(B, \mathbf{S}^2(\mathscr{W}_X))$$

of constant norm 1, and such that $\langle h, \bar{t}\rangle$ is holomorphic for any local section h of $\mathbf{S}^2 \mathscr{W}_X$.

Here $\langle\ ,\ \rangle$ denotes the scalarproduct on

$$\mathbf{S}^2(\mathbf{V}) \otimes_{\mathbf{Z}} \mathscr{O}_B \quad (\mathbf{V} = R^1 p_* \mathbf{Z})$$

derived from the Hodge-structure on $\mathbf{S}^2(\mathbf{V})$.

We denote by

$$\nabla: \mathbf{S}^2(\mathbf{V}) \otimes_{\mathbf{Z}} \mathscr{O}_B \to \mathbf{S}^2(\mathbf{V}) \otimes_{\mathbf{Z}} \Omega^1_B$$

the connection (defined on $B - S$).

Then

$$\langle \nabla(t), \nabla(\bar{t})\rangle = \partial\bar{\partial}\langle t, \bar{t}\rangle = 0$$

and for h a local section of $\mathbf{S}^2(\mathscr{W})$ we have

$$\langle \nabla(t), \bar{h}\rangle = \partial\langle t, \bar{h}\rangle = \overline{\bar{\partial}\langle h, \bar{t}\rangle} = 0.$$

By the fundamental properties of ∇ we known that $\nabla(t)$ has components of bidegree $(2,0)$ and $(1,1)$. The second equality says that it is of bidegree $(1,1)$, and by the first identity it vanishes. Thus t is parallel.

Finally some easy calculations show that there exists a (parallel) endomorphism A of $\mathbf{V}\otimes_{\mathbb{Z}}\mathbb{C}$, such that

i) A is pure of type $(-1,1)$
ii) A is anti-symmetric for $\langle\ ,\ \rangle$,

such that

$$s(h_1\otimes h_2) = \langle h_1, A(h_2)\rangle$$

for local sections h_1, h_2 of $\mathscr{W}_{\bar{X}}$.

Conversely it is clear that any such A define an element s (using for example the results of Schmid [6] to treat the behaviour at S). We have thus nearly proved the

Theorem 2.

$$\mathbf{Hom}((B,S),(\mathscr{A}, D_\infty))$$

is smooth.

Its tangent-space at a point corresponding to an abelian variety X over $B-S$ is isomorphic to the space of $\langle\ ,\ \rangle$-anti-symmetric endomorphisms of

$$\mathbf{V}\otimes_{\mathbb{Z}}\mathbb{C} = R^1\, p(\mathbb{C})$$

of pure type $(-1,1)$.

Proof. It remains to show that any A as above defines deformations. A as well as its complex conjugate $\bar{A}$ are ∇-parallel, so that the eigenvalues of $\bar{A}A$ are constant on B. They are easily seen to be (positive) real numbers, and we shall define a deformation corresponding to A provided these eigenvalues are smaller than $+1$.

If these conditions are fulfilled we define a new Hodge-structure supported by $\mathbf{V}$, by choosing

$$\exp(A)(\mathscr{W}) = (\mathrm{Id}+A)(\mathscr{W}) \subseteq \mathbf{V}\otimes_{\mathbb{Z}}\mathcal{O}_B$$

as space of type $(1,0)$:

We only have to show that for h a local nonzero section of $\mathscr{W}$

$$\langle(\mathrm{Id}+A)(h), \overline{(\mathrm{Id}+A)(h)}\rangle = \langle(\mathrm{Id}-\bar{A}A)(h), \bar{h}\rangle$$

is positive.

This follows from our assumptions because the selfadjoint transformation

$$\mathrm{Id}-\bar{A}A$$

of $\mathscr{W}$ has positive eigenvalues.

We thus have defined a deformation of the Hodge-structure (and therefore of X) parametrized by a neighbourhood of the identity in our space of A's. This proves theorem 2.

Corollary. *There exist only finitely many families of principally polarized abelian varieties of dimension g over B, with good reduction outside S and satisfying* (∗). *If there is one such variety which does not fulfill* (∗), *there are infinitely many.*

Proof. We know that there is a finite Galois-covering B' of B such that any such family has a level n-structure after base-change to B'. We let S' be the preimage of S in B', and π the Galois-group of the covering B'/B. For any representation

$$\rho: \pi \to Sp(2g, \mathbb{Z}/n\mathbb{Z}) \subseteq \mathrm{Aut}(\bar{\mathscr{A}})$$

we obtain an action of π on

$$\mathbf{Hom}((B', S'), (\bar{\mathscr{A}}, D_\infty)),$$

and the set of abelian varieties over B injects into the disjoint union of the fixed-point sets of π, corresponding to all representations ρ.

From Theorem 2 it can be derived that these fixed-point-sets are manifolds at the points given by families

$$p: X \to B - S$$

over $B-S$, and that their tangent-spaces there are the symmetric endomorphisms (over B) of

$$R^1 p_*(\mathbb{C})$$

of type $(-1, 1)$.

They vanish precisely if X fulfills (∗).

Therefore the points corresponding to such X ly in the zero-dimensional components of the fixed-point-sets. As this are varieties of finite type over k (Theorem 1) their number is finite.

Conversely if X does not fulfill (∗), we have constructed deformations of X in the proof of Theorem 2.

Remark. The proof gives that only finitely many pairs $(\mathbf{V}, \langle\ ,\ \rangle)$ occur as cohomology of principally polarized abelian varieties over $B-S$. By an hyperplane section argument the same is true for any normal algebraic variety U.

§ 5. An Example

We want to show that there exist indeed nontrivial examples of abelian varieties which do not satisfy (∗) (constant ones of course fall into this category). We proceed by following the arguments in [4], §4.4. We restrict ourselves to the simplest case, but the method can be generalized.

Let K be a real quadratic number-field, D a quaternion-algebra over K which splits at one of the infinite places of K but not at the other. D has a canonical involution denoted by

$$a \mapsto a^*,$$

such that for $a \in D$

$$N(a) = a\,a^*$$

and

$$T(a) = a + a^*$$

are in K.

We denote by $V_{\mathbb{Q}}$ the rational vectorspace (of dimension 8)

$$V_{\mathbb{Q}} = D.$$

D acts by multiplication on the left and on the right on V. For any element s of D with

$$s^* = -s$$

we obtain a new involution on D, denoted by

$$a^t = s\,a^*\,s^{-1}.$$

We further obtain a skew-symmetric form on $V_{\mathbb{Q}}$ by

$$\langle a, b\rangle = \mathrm{tr}_{K/\mathbb{Q}}(T(a\,s\,b^*)).$$

Then $\langle a c, b\rangle = \langle a, b\,c^t\rangle$, and $\langle\ ,\ \rangle$ is respected by the algebraic group G with

$$G(\mathbb{Q}) = \{a \in D \mid N(a) = 1\},$$

which acts by left-multiplication.

By assumption

$$D_{\mathbb{R}} = D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbf{Q} \times M_2(\mathbb{R}) \qquad (\mathbf{Q} = \text{quaternions}),$$

$$G(\mathbb{R}) \cong SU(2, \mathbb{C}) \times SL(2, \mathbb{R})$$

and

$$V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^2 \oplus (\mathbb{R}^2 \otimes \mathbb{R}^2).$$

The decomposition of $V_{\mathbb{R}}$ is written in such a way that $SU(2, \mathbb{C})$ acts on the first summand by its natural representation, $SL(2, \mathbb{R})$ on the first factor of the tensor-product, and $M_2(\mathbb{R})$ by right-multiplication on the second factor.

The bilinear form $\langle\ ,\ \rangle$ decomposes accordingly:

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_2 \otimes \langle\ ,\ \rangle_3.$$

Here $\langle\ ,\ \rangle_1$ (for suitable coordinates) is a real multiple of the skew-form

$$(u_1, u_2) \times (v_1, v_2) \mapsto \mathrm{Im}\,(\bar{u}_1 v_1 + \bar{u}_2 v_2)$$

on $\mathbb{C}^2$, $\langle\ ,\ \rangle_2$ the skew form

$$(u_1, u_2) \times (v_1, v_2) \mapsto u_1 v_2 - u_2 v_1$$

on $\mathbb{R}^2$, and $\langle\ ,\ \rangle_3$ a symmetric form on $\mathbb{R}^2$. We now choose s in such a way that $\langle\ ,\ \rangle_1$ is a positive multiple of the standard form listed above, and that $\langle\ ,\ \rangle_3$ is positive definite.

After extending scalars to the complex-numbers we obtain

$$V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = (\mathbb{C} \oplus \bar{\mathbb{C}}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{R}^2)$$

where the first direct sum has been split such that

$$\bar{\mathbb{C}}^2 = \operatorname{Kern}(\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C} \mapsto \mathbb{C}^2).$$

For $0 \neq w \in \mathbb{C}^2$ we have $-\frac{1}{2\pi i}\langle w, \bar{w}\rangle > 0$, and thus we see that for a suitable one-dimensional complex subspace $L \subseteq \mathbb{C}^2$ we can define a polarized Hodge-structure on V by

$$V_{\mathbb{C}}^{1,0} = (\mathbb{C}^2 \oplus 0) \oplus (L \otimes \mathbb{R}^2).$$

The stabilizer in $G(\mathbb{R})$ of such a Hodge-structure is a maximal compact subgroup

$$M \cong SU(2, \mathbb{C}) \times U(1, \mathbb{C}),$$

and $G(\mathbb{R})$ acts transitively on these Hodge-structures which are parametrized by the upper half-plane

$$\mathbb{H} \cong G(\mathbb{R})/M.$$

Note that any $a \in D$ defines an endomorphism of $V_{\mathbb{Q}}$, but that not all such endomorphisms respect the Hodge-structure.

Otherwise the subspaces

$$\mathbb{C}^2 \oplus 0$$

and

$$0 \oplus \bar{\mathbb{C}}^2$$

of

$$\mathbb{C}^2 \oplus \bar{\mathbb{C}}^2$$

would be fixed by $SU(2, \mathbb{C})$ and $\mathbf{Q} \otimes_{\mathbb{R}} \mathbb{C}$, which cannot happen by a simple dimension-count. Unfortunately the antisymmetric a's still respect the Hodge-structure.

We thus take the direct sum of two copies of $V_{\mathbb{Q}}$: Here the algebra

$$A = M_2(D)$$

acts, with involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

A is generated by its antisymmetric elements, so some of them will not respect the Hodge-structure.

If we choose a lattice

$$V_{\mathbb{Z}} \subseteq V_{\mathbb{Q}}$$

such that $\langle\ ,\ \rangle$ takes integral values on $V_{\mathbb{Z}}$ and has determinant 1, we obtain a family of 8-dimensional abelian varieties over $\mathbb{H}$ which does not satisfy (*). If $\Gamma \subset G(\mathbb{Q})$ is a torsion-free arithmetic subgroup which stabilizes $V_{\mathbb{Z}}$ we can take quotients and we obtain such a family over the compact Riemann surface

$$B = \Gamma \backslash \mathbb{H} = \Gamma \backslash G(\mathbb{R})/M.$$

As Γ is Zariski-dense in G it can be easily seen that this Hodge-structure does not have a fixed part on any finite covering of B.

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