Designs, Codes and Cryptography, 3, 155–166 (1993)

Some Characterizations of Quasi-Symmetric Designs with a Spread

S.S. SANE
Department of Mathematics, University of Bombay, Bombay 400098, India

M.S. SHRIKHANDE
Mathematics Department, Central Michigan University, Mt. Pleasant, MI 48859

Communicated by R. Mullin
Received November 11, 1991; Revised November 2, 1992.

Abstract. The design PG_2 (4, q) of the points and planes of PG (4, q) forms a quasi-symmetric 2-design with block intersection numbers $x = 1$ and $y = q + 1$. We give some characterizations of quasi-symmetric designs with $x = 1$ which have a spread through a fixed point. For instance, it is proved that if such a design D is also smooth, then $D \cong PG_2$ (4, q).

1. Introduction

A t-(v, k, λ) design D is called quasi-symmetric (q.s.) if any pair of its blocks intersect in x or y points ($x < y$). There has been much recent interest in such designs. For example, for $t = 2$: [1], [3], [9], [10] and in case $t = 3$: [4], [11], [13].

Cameron [5], classified q.s. 3-designs with $x = 0$. In Sane and M.S. Shrikhande [13], q.s. 3-designs with $x > 0$ were considered and the case $x = 1$ was studied. The unique 4-(23, 7, 1) Witt design and its residual the 3-(22, 7, 4) design provide examples of q.s. 3-designs with $x = 1$. It was conjectured in [13], that these are the only possibilities. This conjecture was proved true by Calderbank and Morton [4]. See Sane and Pawale [11] for an alternative short proof. As is well known, the combined efforts of Ito and others (see e.g., references in [15]) prove that up to complementation, the 4-(23, 7, 1) is the only q.s. 4-design. Cameron [6] proved that no q.s. t-designs exist for $t \geq 5$. For these and other results on quasi-symmetric designs, see M.S. Shrikhande and Sane [16].

The classification problem for q.s. 2-designs is open. One of the approaches has been to place some restrictions on the intersection numbers x, y and/or put some extra structural conditions on the design. For instance, in [1]: $x = 0$ and the design has no three mutually disjoint blocks, or in [10]: $x = 0$, $y = 2$ and the design satisfies some additional geometrical conditions. The paper [14] contains some generalizations of [10] to the case $x = 0$, $y \geq 2$. Cameron [7] studies the other extreme considered in [1], namely assume $x = 0$ and the q.s. design D has a spread—i.e., a family of pairwise disjoint blocks whose union is the point set of D.

In this paper our interest is in q.s. 2-(v, k, λ) designs with $x = 1$. If $(x, y) = (1, 2)$, it is well known that D is the residual of a biplane. An infinite family of q.s. designs with $x = 1$, $y = q + 1$ is the design PG_2 (4, q) of points and planes in PG (4, q). In PG_2 (4, q), the set of blocks through any fixed point contains a spread. Also PG_2 (4, q) is smooth—i.e., any three noncollinear points are contained in a constant number ($\rho \geq 0$) of blocks. One of the purposes of this paper is to obtain some Dembowski-Wagner type characterizations of q.s. 2-(v, k, λ) designs with $x = 1$. The well known Dembowski-Wagner results [8] characterize the symmetric design of points and hyperplanes of $PG\,(n, q)$ in terms of smoothness.

In Section 2, it is shown (Proposition 2.6), that if $x = 1$, $y \geq 3$ and the q.s. 2-design has a spread consisting of β blocks through a fixed point, then the integers $\gamma = (\beta - 1)/(y - 1)$ and $m = (k - 1)/(y - 1)$ satisfy the inequality $1 \leq \gamma \leq m - 1$. We refer to designs achieving these bounds as *critical designs*. Theorem 2.8 obtains a classification of q.s. 2-designs with $x = 1$ which have a spread and are critical. Section 3 is devoted to obtaining some characterizations of the q.s. 2-design $PG_2\,(4, q)$. If D is a q.s. 2-design with the same parameters as $PG_2\,(4, q)$, then Theorem 3.6 obtains several conditions equivalent to assuming $D \cong PG_2\,(4, q)$. We then consider q.s. 2-designs D with $x = 1$, $y \geq 3$ and suppose, in addition, that D has a spread and is also smooth. We shown (in Theorem 3.8) that any three noncollinear points are contained in exactly one block and all line sizes are y. The proof depends on Cameron's result [7] on q.s. 2-designs with an intersection number 0 and having a spread. We prove in Theorem 3.9, that if D is a q.s. 2-design with $x = 1$ has a spread and is also smooth, then $D \cong PG_2\,(4, q)$. Theorem 3.10 is another characterization result. Assume D is q.s. 2-(v, k, λ) design with $x = 1$ and suppose every point triple is contained in at least one block. Then Theorem 3.10 gives several Dembowski-Wagner type characterizations for D.

2. Quasi-Symmetric Designs with $x = 1$ and Spreads

Let D denote a quasi-symmetric (q.s.) 2-design with usual parameters v, b, r, k, λ and block intersection cardinalities x and y ($x < y$). We assume always that D is a proper q.s. design, i.e., both x and y actually occur as block intersection sizes. It is well known (see e.g. [16]) that $y - x$ divides $k - x$, so denote the integer $(k - x)/(y - x)$ by m. As mentioned in the introduction, we shall be interested in the case $x = 1$ only. So from now, D will denote a proper q.s. 2-design with parameters v, b, r, $k = mq + 1$, λ, $x = 1$, $y = q + 1$ ($q \geq 1$).

The following result is well known.

Result 2.1. D is a q.s. 2-design with $x = 1$, $y = 2$ if and only if D is the residual of a biplane (= symmetric design with $\lambda = 2$).

In view of the above result, we assume from now on that $q \geq 2$. The following is a well known infinite family of q.s. 2-designs with $x = 1$.

Example 2.2. Let $PG\,(4, q)$ be the four dimensional projective geometry over a finite field of order q. Let D be the incidence structure whose points and blocks are respectively the points and planes of $PG\,(4, q)$. Then D is a q.s. 2-(v, k, λ) design with

$$v = q^4 + q^3 + q^2 + q + 1,\ k = \lambda = q^2 + q + 1,\ x = 1,\ y = q + 1.$$

Following usual convention (e.g., [2]), we denote the design D formed by the points and planes of $PG(4, q)$ by $PG_2(4, q)$. It is easily seen that in $PG_2(4, q)$, the planes through any fixed point p contain a subcollection S of $q^2 + 1$ planes which partition the points $PG_2(4, q)$ other than p. A collection of such planes is said to form a spread of $PG_2(4, q)$. One of our aims in this paper is to obtain characterizations of the design $PG_2(4, q)$.

The next result is easy to prove and will be used often.

LEMMA 2.3. *Let D be a q.s. 2-(v, k, λ) design with $x = 1$, $y = q + 1$. Fix a block B of D. Then the induced structure D_B consisting of the points of B, and those blocks of D which intersect B in y points, forms a 2-(v^*, k^*, λ^*) design with $v^* = k = mq + 1$, $k^* = y = q + 1$, $\lambda^* = \lambda - 1$. In addition,*

$$r^* = m(\lambda - 1). \tag{1}$$

$$b^* = m(\lambda - 1)(mq + 1)/(q + 1). \tag{2}$$

We observe from (2), that all our q.s. 2-designs have $\lambda \geq 2$. The next lemma is also easily verified.

LEMMA 2.4. *Let D satisfy the hypothesis of Lemma 2.3. Suppose further that D has a spread—i.e., a set $S = \{B_1, B_2, \ldots, B_\beta\}$ of blocks such that $B_i \cap B_j = \{z\}$, for all $1 \leq i \neq j \leq \beta$, where z is a fixed point and $\cup_{i=1}^{\beta} B_i$ = the point set of D. Then,*

$$v = \beta mq + 1. \tag{3}$$

$$r = \lambda\beta. \tag{4}$$

$$b = \lambda\beta(\beta mq + 1)/(mq + 1). \tag{5}$$

LEMMA 2.5. *Under the hypothesis of Lemma 2.4, put $\gamma = (\beta - 1)/q$. Then the following assertions hold:*

(i) γ is an integer with $0 \leq \gamma \leq m$.
(ii) If $\gamma = m$, then D has repeated blocks.
(iii) $\gamma = 0$ is impossible.

Proof. Let C be a block not containing z. Suppose C meets γ_i blocks of the spread in i points. Then we obtain

$$\gamma_1 + \gamma_{q+1} = \beta. \tag{6}$$

$$\gamma_1 + (q + 1)\gamma_{q+1} = k = mq + 1. \tag{7}$$

This implies that

$$\gamma_{q+1} = m - (\beta - 1)/q. \tag{8}$$

Thus $\gamma = (\beta - 1)/q$ is an integer with $0 \leq \gamma \leq m$. If $\gamma = 0$, then $\beta = 1$ and using (3), we obtain $v = k$, a contradiction. If $\gamma = m$, then $\gamma_{q+1} = 0$ and $\gamma_1 = \beta$. This implies that all the blocks meeting $B = B_1$ in at least two points must contain z. Using Lemma 2.3, we get $r^* = b^*$. Then (1) and (2) imply $mq + 1 = q + 1$, i.e., $k = y$, we see that D has repeated blocks. This completes the proof of Lemma 2.5.

Assuming from now on that the q.s. design D has no repeated blocks and has a spread, then Lemma 2.5 gives the following inequality connecting the integers $\gamma = (\beta - 1)/q$ and $m = (k - 1)/(y - 1)$:

$$1 \leq \gamma \leq m - 1. \tag{9}$$

We now come to one of our basic tools.

PROPOSITION 2.6. *Let D be a q.s. 2-(v, k, λ) design with $x = 1$ and $y = q + 1$. Suppose D has a spread consisting of β blocks. Then the integers m, q, λ and γ satisfy the equation*

$$\gamma(m - \gamma)/(m - 1) = (\lambda - 1)(mq + 1)/\lambda q(q + 1). \tag{10}$$

Proof. Let $S = \{B_1, B_2, \ldots, B_\beta\}$ be a spread of D, where $B_i \cap B_j = \{z\}$, a fixed point, for $1 \leq i \neq j \leq \beta$. Count the number of pairs (B, C) of blocks, where $B = B_i \in S$, $z \notin C$ and $|B \cap C| = q + 1$ in two ways. For a fixed B, considering the design D_B of Lemma 2.3, this number is $b^* - r^*$, while for a fixed C, this number is $\gamma_{q+1} = m - \gamma$. Hence we get the equality

$$(b^* - r^*)\beta = (m - \gamma)(b - r). \tag{11}$$

But, using (1) and (2), this gives

$$(b^* - r^*) = m(\lambda - 1)(m - 1)q/(q + 1). \tag{12}$$

Next (4) and (5) imply

$$(b - r) = \lambda mq^2(\gamma q + 1)\gamma/(mq + 1). \tag{13}$$

Now use (11), (12) and (13) to get the desired equation (10).

Remark 2.7. We already know that $1 \leq \gamma \leq m - 1$. For a fixed m, the L.H.S. of (10) is minimized at the two end points $\gamma = 1$ and $\gamma = m - 1$, and in each case the value of the L.H.S. of (10) is 1. We refer to such designs D as *critical designs*.

We are now ready to prove:

THEOREM 2.8. *Let D be a q.s. 2-(v, k, λ) design with $x = 1$ and $y = q + 1$. Suppose D has a spread and is critical. Then one of the following holds:*

(1) *D has parameters of* PG_2 (4, q),
(2) *D has parameters* $v = (q + 1)^2 q + 1$, $k = \lambda = q^2 + q + 1$,
(3) *D has parameters* $v = q(q + 2)(q^2 + q + 1) + 1$, $k = (q + 1)^2$, $\lambda = q + 1$.

Proof. Let m, β and γ be defined as earlier. Then as observed above, the L.H.S. of (10) is at least one. Hence $(mq + 1) > q(q + 1)$, which gives

$$m \geq q + 1 \tag{14}$$

Now since D is critical, $\lambda/(\lambda - 1) = (mq + 1)/(q^2 + q) \geq (q^2 + q + 1)/(q^2 + q)$. If $\gamma = 1$, then (8) gives $\gamma_{q+1} = m - 1$. Using (6), then $\beta = q + 1 \geq \gamma_{q+1} = m - 1$, giving $m \leq q + 2$. Thus we obtain

$$q + 1 \leq m \leq q + 2. \tag{15}$$

If $m = q + 2$, then (10) gives $\lambda/(\lambda - 1) = ((q + 2)q + 1)/(q^2 + q) = (q + 1)/q$. This yields $\lambda = q + 1$. Using (4), we get $r = \lambda\beta = (q + 1)^2$, and $k = mq + 1 = (q + 2)q + 1$. This implies D is symmetric, which is a contradiction. Thus $m = q + 1$. Then $\lambda/(\lambda - 1) = (q^2 + q + 1)/(q^2 + q)$ gives $\lambda = q^2 + q + 1$. Then using (3) and (4) gives $v = (q + 1)^2 q + 1$, $k = q^2 + q + 1$, $r = (q^2 + q + 1)(q + 1)$ and $b = ((q + 1)^2 q + 1)(q + 1)$.

Observe also that in this case (using Lemma 2.3), $v^* = q^2 + q + 1$, $k^* = q + 1$, $\lambda^* = q^2 + q$, $r^* = (q + 1)^2 q$ and $r - r^* = q + 1 = \beta$.

Next assume $\gamma = m - 1$. Then (10) gives $\lambda/(\lambda - 1) = (mq + 1)/(q^2 + q)$, which implies $\lambda = (mq + 1)/((mq + 1) - (q^2 + q))$. From (14), $m \geq q + 1$ with equality iff $\lambda = q^2 + q + 1$. It is easily checked that if $m = q + 1$, then $k = \lambda = q^2 + q + 1$, $v = mq\beta + 1 = q^4 + q^3 + q^2 + q + 1$, which are the parameters of PG_2 (4, q).

Assume therefore $m \geq q + 2$. Suppose first $\lambda \leq q$. Now $\lambda/(\lambda - 1) = (mq + 1)/(q^2 + q)$, gives $\lambda(q^2 + q) = (\lambda - 1)(mq + 1)$. This implies that q divides $\lambda - 1$. But since $\lambda \leq q$ was assumed, we must have $\lambda = 1$, a contradiction. Thus $\lambda \geq q + 1$. Then $(mq + 1) \geq (q + 1)(mq + 1) - (q^2 + q)(q + 1)$, which implies $(q + 1)^2 \geq mq + 1 \geq (q + 2) q + 1$. Therefore, if $\lambda \neq mq + 1$, then $m \leq q + 2 \leq m$, yielding $m = q + 2$. In this case $\lambda/(\lambda - 1) = (mq + 1)/(q^2 + q) = ((q + 2)q + 1)/(q^2 + q) = (q + 1)/q$, giving $\lambda = q + 1$ and $k = mq + 1 = (q + 1)^2$. Then (3) with $\beta = q^2 + q + 1$ yields $v = q(q + 2)(q^2 + q + 1) + 1$. This completes the proof of Theorem 2.8.

Remark 2.9. Tonchev [17] has shown that there are exactly five isomorphism classes of quasi-symmetric 2-(31, 7, 7) designs. They all have rank 16 over $GF(2)$ and one of these designs is formed by the points and planes in $PG(4, 2)$.

The following table lists the possible parameters of designs D other than those of PG_2 (4, q) for $q = 2, 3$, in Theorem 2.8.

Table 1.

q	v	b	r	k	λ	x	y	Existence of D
2	19	57	21	7	7	1	3	Does not exist [18]
3	49	196	52	13	13	1	4	?
2	57	133	21	9	3	1	3	?
3	196	637	52	16	4	1	4	?

3. Some Characterizations of the q.s. Design PG_2 (4, q)

In this section, we give some characterizations of the q.s. design PG_2 (4, q). These results (Theorems 3.6, 3.8, 3.9, 3.10) are in the same spirit as the well-known Dembowski-Wagner Theorems ([8]) which characterize the symmetric design of the points and hyperplanes of PG (n, q). We begin with the following.

LEMMA 3.1. *Let D be a q.s.* 2-(v, k, λ) *with block intersection numbers* $x = 1$, $y = q + 1 (q \geq 2)$. *Let* $k = mq + 1$ *and* $v = m\beta q + 1$. *Suppose* (p_1, p_2) *is a fixed point-pair of D. Let* a_i *denote the number of points p distinct from* p_1, p_2 *such that the triple* (p_1, p_2, p) *is contained in exactly i blocks of D* ($i = 0, 1, \ldots, \lambda$).
Then

$$\Sigma (i - 1)(i - \lambda)a_i = \lambda[(\lambda - 1)(q - 1) + (m\beta q - 1) - \lambda(mq - 1)]. \tag{16}$$

Proof. Standard two-way counting produces the following equations:

$$\Sigma a_i = v - 2 = m\beta q - 1. \tag{17}$$

$$\Sigma i a_i = \lambda(k - 2) = \lambda(mq - 1). \tag{18}$$

$$\Sigma i(i - 1)a_i = \lambda(\lambda - 1)(q - 1). \tag{19}$$

from which the relation (16) follows.

COROLLARY 3.2. *Let D be a q.s.* 2-(v, k, λ) *satisfy the hypothesis of Lemma 3.1. Suppose further every point triple of D occurs in at least one block. Then* $m\beta - 1 \leq \lambda(m - 1)$. *Furthermore equality holds if and only if every point-triple is contained in* 1 *or* λ *blocks.*

Proof. By the above lemma, $(\lambda - 1)(q - 1) + (m\beta q - 1) \leq \lambda(mq - 1)$, which implies $m\beta - 1 \leq \lambda(m - 1)$. Equality holds iff $a_i(i - 1)(i - \lambda) = 0$, for all i. This implies $a_i = 0$, for $i \neq 1$ and $i \neq \lambda$. This shows that $m\beta - 1 = \lambda(m - 1)$ iff every point-triple occurs in 1 or λ blocks.

We now give some definitions and concepts needed. Details not given here may be found in [2, p. 573].

DEFINITION 3.3. *A projective incidence space is an incidence structure* $S = (P, \mathcal{L}, \epsilon)$ *satisfying the following Veblen-Young Axioms:*

(A1) *Any two points* p, q *are on exactly one line* $\overline{pq}$.
(A2) *If* p, q, r, s *are four distinct points and if* $\overline{pq}$ *and* $\overline{rs}$ *intersect, the* $\overline{pr}$ *and* $\overline{qs}$ *intersect.*
(A3) *Every line has at least three points.*
(A4) *There are two disjoint lines.*

An incidence structure is called *cohesive* if any two distinct points are on a line (= block). If p, q are two distinct points of a design D, then the line $\overline{pq}$ is the intersection of all blocks of D through p and q. Three points p, q, r are collinear if they are on a line, otherwise $\{p, q, r\}$ are said to form a triangle. A plane $\overline{pqr}$ is the intersection of all blocks of D containing the triangle $\{p, q, r\}$. A cohesive incidence structure with triangles is called *smooth* if every triangle is contained in the same number ρ of blocks ($\rho \geq 0$). Denote the set of lines and planes of an incidence structure S by L and P.

LEMMA 3.4. *Let* D *be a* 2-(v, k, λ) *design. Suppose* D *is smooth. Then for any line* ℓ, $|\ell| = (\lambda k - \rho v)/(\lambda - \rho)$, *where* ρ *denotes the number of blocks containing any triangle.*

Proof. [2, page 578].

LEMMA 3.5. *Let* D *be a smooth* 2-(v, k, λ) *design, and* p *a fixed point of* D. *Let* D' *be a tactical configuration whose points are the lines through* p, *and whose blocks are the blocks of* D *through* p. *Then* D' *is a 2-design with parameters* $(v', b', r', k', \lambda')$ *given by* $v' = (v - 1)/q'$, $k' = (k - 1)/q'$, $\lambda' = \rho$, $r' = \lambda$, $b' = r$, *where* $q' + 1$ *is the (constant) line size of* D, $q' \geq 1$.

Proof. [2, page 579].

We are now in a position to prove one of our characterization theorems.

THEOREM 3.6. *Let* D *be a q.s.* 2-(v, k, λ) *design with* $v = q^4 + q^3 + q^2 + q + 1$, $k = q^2 + q + 1$, $\lambda = q^2 + q + 1$, $x = 1$, $y = q + 1$.
Then the following are equivalent.

(i) $D \cong PG_2(4, q)$.
(ii) *Every plane meets every block.*
(iii) *Every line has* $q + 1$ *points.*
(iv) *The planes of* D *are precisely the blocks of* D.
(v) *Any three points are contained in at least one block.*
(vi) *There are constants* μ_1, μ_2 $(\mu_1 \neq \mu_2)$ *such that any three points are contained in* μ_1 *or* μ_2 *blocks.*
(vii) D *is smooth.*

Proof. It is clear that (i) implies all the rest. Suppose (iii) holds and ℓ is any line of D. Since $y = q + 1$, the λ blocks containing ℓ are mutually disjoint outside ℓ and since $v - (q + 1) = v - |\ell| = \lambda(k - |\ell|)$, it follows that any three points are contained in 1 or λ blocks, where the first possibility occurs iff the three points form a triangle. Hence D is smooth (with $\rho = 1$). Also then every block is a plane and vice versa. This implies that for any block B, the induced design D_B is a projective plane of order q. Using this the Veblen-Young axioms of Definition 3.3 can be verified. Hence (iii) implies (i).

Let (vii) hold. Then every three collinear points are contained in λ blocks and every three noncollinear points are contained in a fixed number ρ of blocks, $\rho \geq 1$. Corollary 3.2 implies $\rho = 1$. Using Lemma 3.4, we obtain $|\ell| = q + 1$, for any line ℓ. Thus (vii) implies (iii). Essentially the same argument shows that (v) implies (vii) and hence (iii). Let (vi) hold and suppose $\mu_1 < \mu_2$. If $1 \leq \mu_1$, then Corollary 3.2 forces $1 = \mu_1 < \mu_2 = \lambda$ and hence (v) is proved. Otherwise $\mu_1 = 0 < \mu_2 = \mu$ (say). Using Lemma 3.1, we get $a_o + a_\mu = v - 2$, $\mu a_\mu = (k - 2)\lambda$, and $\mu(\mu - 1)a_\mu = \lambda(\lambda - 1)(q - 1)$, where a_i = number of points p with the property that the triple (p_1, p_2, p_3) is contained in i blocks and (p_1, p_2) is a fixed pair. This implies $(\mu - 1)(k - 2) = (\lambda - 1)(q - 1)$, giving a non-integral solution for μ. This contradiction proves $\mu \geq 1$ and hence (v) is proved. Let now (iv) hold. Then $\rho = 1$ and Lemma 3.4 shows that $|\ell| = q + 1$, for all the lines ℓ. Thus (iii) holds. Next let (ii) hold. Let B be any block. Since every line contained in B has size $\leq q + 1$, B contains a triangle (p_1, p_2, p_3). Let P be the plane containing this triangle. Clearly then $P \subseteq B$. For every $p \in B$, $p \neq p_i$ $(i = 1, 2, 3)$, there is at least one block on p meeting B in p alone. Since (ii) holds, this implies that $p \in P$, i.e., $P = B$. Thus (ii) implies (iv). Conversely assume that (iv) holds. Since any plane P is generated by three noncollinear points say, (p_1, p_2, s). Let B be any fixed block containing (p_1, p_2). Since B is a plane and since the line $\overline{p_1p_2}$ has size $\leq q + 1$ and is contained in B, it follows that for all $p \notin \overline{p_1p_2}$, (p_1, p_2, p) is contained in B alone (otherwise the plane containing (p_1, p_2, p) will have size $\leq q + 1$, a contradiction). So every block containing (p_1, p_2) other than B must intersect B in $\overline{p_1p_2}$. Hence the line $\overline{p_1p_2}$ has size $q + 1$ and hence (iii) holds.

This completes the proof of Theorem 3.6.

We shall need the following result of P.J. Cameron [7] about q.s. 2-designs, with $x = 0$ and having a spread.

THEOREM 3.7. *Let D be a q.s. 2-(v, k, λ) design with $x = 0$ and $y \geq 1$. Let D have a spread (i.e., a family of disjoint blocks which partition the points of D). Then one of the following holds:*

(i) $\lambda = 1$ *(equivalently $y = 1$).*
(ii) *D is an affine design (i.e., $v = m^2y$, $k = my$, $\lambda = (my - 1)/(m - 1)$ and any two blocks disjoint from a common block are disjoint from each other) (equivalently D has a parallelism such that every parallel class partitions the point set of D).*
(iii) $v = y(2y + 1)(2y + 3)$, $k = y(2y + 1)$, $\lambda = y(2y - 1)$, *where $y \geq 2$.*

The next result concerns q.s. 2-designs with $x = 1$.

THEOREM 3.8. *Let D be a q.s. 2-design with parameters v, $k = mq + 1$, λ and block intersection numbers $x = 1$, $y = q + 1$ ($q \geq 2$). Suppose D has a spread and is also smooth with any three noncollinear points contained in ρ blocks. Then, the following assertions hold:*

(i) $\rho = 1$.
(ii) *Every line of D has size $q + 1$.*
(iii) *For any block B of D, the induced design D_B is a Steiner System with $v^* = mq + 1$, $k^* = q + 1$, $\lambda^* = 1$.*
(iv) *Every block is a plane and every plane is a block.*

Proof. Since D is smooth, we obtain from Lemma 3.5, the 2-design D' with parameters $v' = (v - 1)/q'$, $b' = r$, $r' = \lambda$, $k' = (k - 1)/q'$, $\lambda' = \rho$, where $q' + 1$ is the constant line size of D. Further since D has a spread of β blocks, D' also has a spread with β blocks. Also D' is q.s. with $x' = 0$, $y' = q/q'$. Hence, Cameron's Theorem 2.7 is applicable to D'.

First assume $y' \geq 2$. Then by Theorem 3.7, D' is either an affine design or D' has parameters of a special type. Suppose D' is an affine design. Then the parameters of D' are $v' = (v - 1)/q' = m'^2y'$, $k' = (k - 1)/q' = m'y'$, $\lambda' = \rho = (m'y' - 1)/(m' - 1)$. This implies that $m = m' = \beta$. Since D has a spread of β blocks and $\beta = m$, this means that $\gamma = (\beta - 1)/q = (m - 1)/q$. Since γ is an integer this gives $q + 1 \leq m$. Next (10) and $\gamma = (m - 1)/q$ yield $\lambda = q(mq + 1)/(m - 1)$. Now since $q' + 1$ is the (constant) line size of D, using Lemma 3.4, we obtain $\rho = \lambda(k - q' - 1)/(v - q' - 1)$. Now using $v = 1 + m^2q$, $k = 1 + mq$, $\lambda = q(mq + 1)/(m - 1)$, $\rho = (mq - q')/q'(m - 1)$, we get the relation $(mq^2 + q + 1) = m^2q/q'$. Since $y' = q/q'$ is an integer this means that m divides $q + 1$, which yields $m \leq q + 1$. This now implies $m = q + 1$. Using $(mq^2 + q + 1) = m^2q/q'$ again with $m = q + 1$, this gives $q' = q(q + 1)/(q^2 + 1) = 1 + (q - 1)/(q^2 + 1)$. Since $q \geq 2$, this means that q' is not an integer, a contradiction.

This means that if $y' \geq 2$ is asumed, then by Cameron's Theorem 2.7, the parameters of D' must be $v' = y'(2y' + 1)(2y' + 3)$, $k' = y'(2y' + 1)$, $\lambda' = y'(2y' - 1)$. Then $y' = q/q'$, $m' = k'/y' = ((k - 1)/q')/(q/q') = m$, gives $m = 2y' + 1$. Also $v'/k' = ((v - 1)/q')/((k - 1)/q') = \beta$, then gives $\beta = 2y' + 3$. Since D' is a q.s. 2-(v', k', λ') design with $x' = 0$, y', then using the well-known relation $(r' - 1)(y' - 1) = (k' - 1)(\lambda' - 1)$, gives $r' = (2y' + 1)(k' - 1) + 1 = \lambda$. Now, $y' = q/q'$ implies $q = y'q'$ and by (14), $2y' + 1 = m \geq q + 1$. So $q \leq 2y'$, which then gives $y'q' \leq 2y'$. Hence $q' \leq 2$. Suppose $q' = 2$. Then $q = 2y'$ (which by Lemma 2.5) divides $\beta - 1 = 2y' + 2$. This implies $2y'$ divides 2, which means $y' = 1$, again a contradiction. Thus $q' = 1$, and hence $q = y'$ which divides $2y' + 2$. This implies q divides 2. Since $y' \neq 1$, $q = 2 = y'$. This implies that the parameters of D' are $v' = 70$, $k' = 10$, $\lambda' = 6$. Hence the parameters of D are found to be $v = 71$, $k = 11$, $\lambda = 46$. Also $q = 2$, $\beta = 2y' + 3 = 7$ gives $\gamma = (\beta - 1)/q = 3$. But then $m = 5$, $q = 2$, $\lambda = 46$, $\gamma = 3$ do not satisfy (10) in Lemma 2.6. Thus $y' = 1$ and D' is a Steiner system. Hence $\lambda' = 1$, $\rho = \lambda' = 1$ and $q/q' = y' = 1$. Thus $q = q'$. Hence every line of D through a fixed point p has size $q' + 1 = q + 1$. Since $\rho = 1$, D is smooth with every three noncollinear points contained in exactly one block. Since a line through a fixed point p has size $q + 1$ and all lines of D are of size $q + 1$ (by Lemma 3.4), we get $1 + q = [\lambda(mq + 1) - (mq\beta + 1)]/(\lambda - 1)$. This implies that $\lambda - 1 = m(\lambda - \beta)$. But then by Corollary 3.2, every point triple occurs in 1 or λ blocks. This essentially completes the proof of Theorem 3.8.

We are now ready to prove another characterization result.

THEOREM 3.9. *Let D be a q.s. 2-design with parameters* v, $k = mq + 1$, λ *and block intersections* $x = 1$, $y = q + 1$ $(q \geq 2.)$. *Assume D has a spread and is smooth. Then* $D \cong PG_2(4, q)$.

Proof. By Theorem 3.6, it suffices to prove that D has the same parameters as those of $PG_2(4, q)$. By Lemma 3.4 and Theorem 3.8, we have

$$q + 1 = (\lambda(mq + 1) - \rho(mq\beta + 1))/\lambda - \rho), \text{ where } \rho = 1.$$

This implies the relation

$$\lambda - 1 = m(\lambda - \beta). \tag{20}$$

This gives

$$(m - 1)(\lambda - 1) = m(\beta - 1). \tag{21}$$

So $m - 1$ divides $\beta - 1 = \gamma q$. Now use (10) of Lemma 2.6 to obtain,

$$\gamma q(m - \gamma)/(m - 1) = (\lambda - 1)(mq + 1)/\lambda(q + 1). \tag{22}$$

Since $m - 1$ divides γq, this means that the L.H.S. of (22) is an integer.
Consequently so is the R.H.S. This implies that λ divides $(\lambda - 1)(mq + 1)$, which means λ divides $mq + 1$. The relation (20) gives

$$\lambda - 1 = mu, \text{ where } u = \lambda - \beta. \tag{23}$$

Thus $\lambda = mu + 1$ and λ divides $mq + 1$. This implies that

$$t(mu + 1) = mq + 1, \text{ for some } t \geq 1. \tag{24}$$

This means m divides $t - 1$, so either $t = 1$ or $t \geq m + 1$. Suppose $t \geq m + 1$. Then $(m + 1)(mu + 1) \leq mq + 1$, which implies $mu \leq q$. Consequently $m \leq q$. Again by (10), $\gamma(m - \gamma) \geq m - 1$, and $1 \leq \gamma \leq m - 1$. Then (22) implies $(\lambda - 1)(mq + 1)/(\lambda q(q + 1)) \geq 1$. This yields $mq + 1 \geq q^2 + q$. Now $m \leq q$ gives $q^2 + q \leq q^2 + 1$, which means $q \leq 1$. This yields $q = 1$ and $m = 1$, and hence $k = mq + 1 = 2$, a contradiction. Hence $t = 1$ is the only possibility and from (24), $u = q$. Then (23) gives $\lambda = mq + 1 = k$, and (21) shows that $\lambda = \beta + q = mq + 1$, giving $\beta = (m - 1)q + 1$. Then $\gamma = (\beta - 1)/q = m - 1$ gives $\gamma(m - \gamma)/(m - 1) = 1$. Using (10), this implies $(\lambda - 1)(mq + 1) = \lambda(q^2 + q)$. This gives $m = q + 1$. Hence $k = mq + 1 = q^2 + q + 1$, $\beta = (m - 1)q + 1 = q^2 + 1$, $v = m\beta q + 1 = q^4 + q^3 + q^2 + q + 1$ and $\lambda = q^2 + q + 1$. Thus D has parameters of $PG_2(4, q)$. This completes the proof of Theorem 3.9.

Our final result is the following.

THEOREM 3.10. *Let D be q.s. 2-(v, k, λ) design with $k = mq + 1$ and having $x = 1$, $y = q + 1$ $(q \geq 2)$. Let D have a spread consisting of β blocks through a fixed point. Suppose further that every point triple of D is contained in at least one block.*

Then the following are equivalent.

(i) $D \cong PG_2(4, q)$.
(ii) *D is smooth.*
(iii) $\beta = \lambda - (\lambda - 1)/m$.
(iv) *Every line of D has size $q + 1$.*
(v) *Every three points are contained in μ_1 or μ_2 blocks, where $\mu_1 \neq \mu_2$ are constants.*
(vi) *The blocks of D are planes and conversely.*
(vii) *Every plane meets every block.*

Proof. The equivalence of (i) and (ii) is Theorem 2.9. Also (i) implies the rest of the statements. Let (iv) hold. Let α, β, γ be three distinct noncollinear point of D and ℓ the line through α, β. Let B be a block containing α, β, γ. Suppose $E \neq B$ also contains α, β, γ. Then $|E \cap B| = q + 1 = |\ell|$, and $\ell \subseteq B \cap E$ implies $B \cap E = \ell$. This implies $\gamma \in B \cap E = \ell$, a contradiction. So any three noncollinear points are contained in a constant number ρ (=1) of blocks of D. This shows that D is smooth, giving (ii). Statement (iv) also shows that any three points are in 1 or λ blocks.

Let (iii) hold. Then by Corollary 3.2, D is smooth, i.e., (ii) holds. Let D satisfy (v). Suppose $\mu_1 < \mu_2$. Then we must have $1 \leq \mu_1 < \mu_2$. Let α, β, γ be any three collinear points. Then it is easily seen that α, β, γ are contained in exactly λ blocks. This implies that $\mu_2 = \lambda$. Let now α, β, γ be any three noncollinear points. Suppose, if possible, that they are contained in $\mu_2 = \lambda$ blocks. Then γ is contained in all the λ blocks through α, β. This implies that γ is on the line through α, β. This contradiction shows that any three noncollinear points are contained in a constant number ρ $(=\mu_1)$ of blocks. This shows that D is smooth—i.e., (ii) holds.

Let (vi) hold. If (p_1, p_2, p_3) is a triangle contained in two different blocks B, C (say), then the plane $\langle p_1p_2p_3 \rangle$ is contained in $B \cap C$, a contradiction. Thus D is smooth (with $\rho = 1$) and hence (ii) holds.

Finally let (vii) hold. Suppose a triangle (p_1, p_2, p_3) is contained in two different blocks B and C. Then we can find a block E so that $|E \cap B| = 1$ and $|E \cap B \cap C| = 0$. This contradicts (vii). This implies that there is a unique block through (p_1, p_2, p_3). Hence D is smooth (with $\rho = 1$) and (ii) holds. This shows the equivalence of the conditions (i)–(vii) and completes the proof of Theorem 3.10.

Acknowledgment

The first author gratefully acknowledges the hospitality of the Mathematics Department, Central Michigan University while doing this work. The second author acknowledges support of a CMU Summer Fellowship Award #42137.

References

1. Baartmans, A. and Shrikhande, M.S. 1982. Designs with no three mutually disjoint blocks. *Discrete Math.* 40: 129-139.
2. Beth, T., Jungnickel, D. and Lenz, H. 1986. *Design Theory*, Mannheim: B.I. Wissenschaftverlag, 1985, and Cambridge: Cambridge University Press.
3. Calderbank, A.R. 1986. The application of invariant theory to the existence of quasi-symmetric designs. *J. Comb. Theory (A)* 44: 94-109.
4. Calderbank, A.R. and Morton, P. 1990. Quasi-symmetric designs and elliptic curves. *SIAM J. Discrete Math.* 3: 178-196.
5. Cameron, P.J. 1973. Extending symmetric designs. *J. Comb. Theory (A)* 14: 215-220.
6. Cameron, P.J. 1973. Near regularity conditions for designs. *Geom. Dedicata* 2: 213-223.
7. Cameron, P.J. 1992. Quasi-symmetric designs possessing a spread. In: *Combinatorics '88* (ed. A. Barlotti et al.) Mediterranean Press, Rome, Italy, pp. 231-236.
8. Dembowski, P. and Wagner, A. 1960. Some characterizations of finite projective spaces. *Arch. Math.* 11: 465-469.
9. Jungnickel, D. and Tonchev, V.D. 1992. On symmetric and quasi-symmetric designs with the symmetric difference property and their codes. *J. Comb. Theory (A)* 59: 40-50.
10. Mavron, V.C. and Shrikhande, M.S. 1989. Designs with intersection numbers 0 and 2. *Arch. Math.* 52: 407-412.
11. Pawale, R.M. and Sane, S.S. 1991. A short proof of a conjecture on quasi-symmetric 3-designs. *Discrete Math.* 96: 71-74.
12. Sane, S.S. and Shrikhande, M.S. 1986. Finiteness questions in quasi-symmetric designs. *J. Comb. Theory (A)* 42: 252-258.
13. Sane, S.S. and Shrikhande, M.S. 1987. Quasi-symmetric 2, 3, 4-designs. *Combinatorica* 7: 291-301.
14. Sane, S.S. and Shrikhande, M.S. 1992. Quasi-symmetric designs and biplanes of characteristic three. *J. Comb. Theory (A)* 60: 104-116.
15. Shrikhande, M.S. 1990. Designs, intersection numbers and codes, in: IMA Volume in Mathematics and its Applications, Vol. 21, Coding Theory and Design Theory, Part II: Design Theory, 304-318.
16. Shrikhande, M.S. and Sane, S.S. 1991. *Quasi-Symmetric Designs*, London Math. Soc. Lecture Note Series 164, Cambridge: Cambridge Univ. Press.
17. Tonchev, V.D. 1986. Quasi-symmetric 2-(31, 7, 7) designs and a revision of Hamada's conjecture. *J. Comb. Theory (A)* 42: 104-110.
18. Wilbrink, H.A. and Brouwer, A.E. 1978. A (57, 14, 1, 4) strongly regular graph does not exist. *Math. Centrum Preprint*, ZW 121/78.