On the Energy Transfer to Small Disturbances in Fluid Flow (Part I)*

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Summary. The "energy in a small disturbance" in a viscous compressible heat-conductive medium is defined as a positive definite quantity characterizing the mean level of fluctuation in the disturbance which, in the absence of heat transfer at the boundaries and of work done by boundary forces or body forces, and in the absence of heat and material sources, is a monotone non-increasing function of time. For small disturbances a quantity satisfying these requirements is found. When viscosity and heat conductivity are neglected, it reduces to the familiar acoustic energy in the theory of sound. Stability in the mean of such a fluid system can thus be discussed with reference to the growth and decay of the energy in the disturbance.

The effects of body forces, heat and material sources are discussed. RAYLEIGH's criterion for the stability of systems involving heat sources is derived and its limitations shown. Transfer of energy from a steady main stream to a disturbance is then examined, and the particular case of a parallel main stream is worked out in detail. The last analysis will be useful in the discussion of the mechanism of hydrodynamic instability for a viscous compressible heat-conductive flow. In addition to the work done by the REYNOLD's stress, there is another major energy transfer term caused by the transport of entropy spots across layers of fluids of different mean temperature.

Zusammenfassung. Die „Energie in einer kleinen Störung" in einem zähen, kompressiblen und wärmeleitenden Medium wird als eine positiv definierte Größe eingeführt, welche die mittlere Schwankung in der Störung charakterisiert und bei Abwesenheit von Wärme- und Massequellen eine monoton nicht anwachsende Funktion der Zeit ist, falls kein Wärmeübergang an der Oberfläche stattfindet und keine Arbeit von den Oberflächen- und Volumskräften geleistet wird. Für kleine Störungen wird eine solche Funktion angegeben. Sie reduziert sich, wenn Zähigkeit und Wärmeleitung vernachlässigt werden, auf die bekannte akustische Energie in der Theorie des Schalles. Mit Hilfe des Anwachsens und Abnehmens der Störungsenergie kann dann die Stabilität im Mittel diskutiert werden.

Die Einflüsse der Volumskräfte und der Wärme- und Massequellen werden besprochen. Das RAYLEIGHsche Kriterium für die Stabilität eines Systems mit Wärmequellen wird hergeleitet und sein Geltungsbereich gezeigt. Der Energieübergang von der stationären Hauptströmung auf die Störungen wird untersucht und der Sonderfall der Parallelströmung im einzelnen ausgearbeitet. Letztere Untersuchung erscheint nützlich bei der Diskussion des Mechanismus der hydrodynamischen Instabilität in einer zähen, kompressiblen, wärmeleitenden Strömung. Zusätzlich zur Arbeit, die von den REYNOLDSschen Spannungen geleistet wird, gibt es noch ein weiteres wichtiges Energieübertragungsglied, das vom Transport von Entropienestern quer durch Flüssigkeitsschichten mit verschiedener mittlerer Temperatur herrührt.

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Notations

a = velocity of sound of the flow.
$\bar{a}$ = velocity of sound in the main stream.
a_0 = sound speed in an undisturbed uniform medium.
C_p = specific heat at constant pressure.
C_v = specific heat at constant volume.
d_j = defined by Eq. (33 e).
E = energy in a disturbance; see Eqs. (9) and (34).
e_{ij} = rate of strain tensor $= \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.
F_i = component of body force per unit mass in the x_i-direction.
F_x, F_y = component of body force per unit mass in the x-, y-direction, respectively.
g = gravitational acceleration.
h = a non-negative quantity; see Eq. (27).
h_j = defined by Eq. (33 g).
i, j, k = indices: 1, 2, or 3; summation convention being used throughout.
K = coefficient of conductivity.
$\bar{K}, K_0$ = K at $T = \bar{T}$ and $T = T_0$, respectively.
$K' = K - \bar{K}$.
$\bar{K}_1 = \left(\frac{dK}{dT}\right)_{T=\bar{T}}$.
M, m' = rate of mass production per unit volume.
$\vec{n}$ = normal vector at the boundaries.
p = pressure of the flow.
$\bar{p}$ = pressure of the main stream.
p' = pressure in the disturbance.
p_0 = pressure in an undisturbed uniform medium.
Q' = rate of heat release per unit volume.
Q^* = defined by Eq. (43).
R = gas constant.
S = entropy of flow.
$\bar{S}$ = entropy of the main stream.
S' = entropy in the disturbance.
S_0 = entropy in an undisturbed uniform medium.
T = temperature of flow.
$\bar{T}$ = temperature of the main stream.
T' = temperature in the disturbance.
T_0 = temperature in an undisturbed uniform medium.
t = time.
u_i = component of velocity of flow in the x_i-direction.
$\bar{u}_i$ = component of velocity of the main stream in the x_i-direction.
u_i' = component of velocity of the disturbance in the x_i-direction.
u = component of velocity of flow in the x-direction.
$\bar{u}$ = component of velocity of the main stream in the x-direction.

u' = component of velocity of the disturbance in the x-direction.
v' = component of velocity of the disturbance in the y-direction.
x_i & (x, y) = Cartesian coordinates.
$\gamma = C_p/C_v$.
δ_{ij} = KRONECKER delta.
Φ = viscous dissipation function $= \left(\frac{1}{2}\tau_{ij}\,e_{ij}\right)$.
$\bar{\Phi} = \frac{1}{2}\bar{\tau}_{ij}\bar{e}_{ij} \geqslant 0$.
$\Phi' = \frac{1}{2}\tau_{ij}'\,e_{ij}' \geqslant 0$.
ϱ = density of flow.
$\bar{\varrho}$ = density of the main stream.
ϱ' = density of the disturbance.
ϱ_0 = density in the undisturbed uniform medium.
σ_{ij} = stress tensor in the flow field.
$\bar{\sigma}_{ij}$ = stress tensor in the main stream.
σ_{ij}' = stress tensor in the disturbance, Eq. (4).
$d\sigma$ = an element of surface area.
τ_{ij} = viscous stress tensor in the flow field.
$\bar{\tau}_{ij}$ = viscous stress tensor in the main stream, Eq. (30 a).
τ_{ij}' = viscous stress tensor in the disturbance, Eqs. (4) and (33 a).
τ_{ij}^* = defined by Eq. (33 f).
$d\tau$ = a volume element.
μ = coefficient of viscosity.
$\bar{\mu}, \mu_0$ = coefficient of viscosity at $T = \bar{T}$ and $T = T_0$, respectively.
$\bar{\mu}_1 = \left(\frac{d\mu}{dT}\right)_{T=\bar{T}}$.
λ = defined by Eq. (24 a).

Bar "–" signifies that quantities to which it is attached are associated with the main stream, e. g., $\bar{p}$ denotes the main stream pressure.

Prime "′" signifies that quantities to which it is attached are associated with the disturbance, e. g., p' denotes the pressure in the disturbance.

Subscript "$_0$" signifies that quantities to which it is attached are associated with the undisturbed uniform state, e. g., p_0 denotes the undisturbed pressure.

I. Introduction

The "energy in a small disturbance" is a very useful concept. First, it enables us to understand why and how a disturbance is amplified, and provides some indications as to the mechanism of instability in many flow systems. Secondly, it is the foundation upon which the proof of uniqueness of solution of many correctly formulated initial-value problems depends.

Precisely what is the energy in a disturbance is rather difficult to define. We shall venture to give a functional definition after considering a few special examples where such a concept is used and generally accepted. But, to avoid any misunderstandings, it is necessary to define first what is meant by the terms "disturbance" and "main stream" used in this paper, since these terms are known to be used in more than one sense.

In our study any time-independent solution of the basic equations governing the motion of a fluid can be considered as a "main-stream", and any deviation from this solution will be described as a "disturbance". For a given physical set-up, the boundary conditions usually determine one and only one time-independent solution. This solution may be taken as the main stream. Any small deviation from the main stream is a small disturbance. In the course of time this small disturbance may grow or decay. In this paper we shall be interested in small disturbances only.

The definition given above for the main stream automatically rules out the possibility that the main stream may vary with time. Furthermore, for space-wise periodic disturbances, the above definition of disturbance does not guarantee that the fluctuations in it, averaged over one wave length, is necessarily zero.

Let us now consider a few special instances in which the concept of energy in a disturbance has been introduced and generally accepted. In an incompressible medium, the energy in a small disturbance is usually taken to be its kinetic energy. On this basis, REYNOLD shows that the main factor responsible for the growth of a small disturbance in an incompressible flow is the work done by the REYNOLD's stress [1]. Indeed, LIN [2] contributed much in clarifying the mechanism of stability of laminar boundary layer by a study of this energy production term. In acoustics, the energy in a small disturbance is taken as the sum of the kinetic energy of the disturbance plus the energy of condensation. On this basis, it is possible to derive an energy production term for systems involving heat release [3] which is a mathematical description of a criterion first stated by RAYLEIGH [4]. RAYLEIGH's criterion has been used extensively in explaining the cause of instability in many thermal acoustic systems by RAYLEIGH himself [5] and in recent years by many other authors.

In the case of an incompressible medium, the main stream acts as the energy source while in the thermal acoustic systems, the heat sources supply the energy to the disturbance. Two things stand out clearly from the above examples. First, the energy in a disturbance must be a positive definite quantity having the dimension of the energy. Secondly, if there are no energy sources in the system, the energy in the disturbance must be a *monotone non-increasing function of time*. Whether these two properties are sufficient to define uniquely the energy in a disturbance is not obvious. However, they do limit greatly the possible choice as to which quantity may represent the energy in a disturbance.

Part I of this paper deals with the energy in a disturbance in a perfect gas and an incompressible fluid. The search for such a quantity is not only of basic importance in understanding the mechanism of instability of many flow systems where conduction and viscous effects are important, but it also allows one to examine from a single point of view the different physical phenomenon involving the question of stability, whether it be purely hydrodynamical or thermal or gravitational.

II. Energy in Small Disturbances

The state of a flow field is specified when the pressure p, density ϱ, temperature T, and velocity u_i $(i = 1, 2, 3)$, are known as functions of the space and time variables: x_i and t. The six unknowns: p, ϱ, T, u_1, u_2, u_3, are governed by the six equations:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho u_j}{\partial x_j} = 0, \tag{1a}$$

$$\varrho \frac{\partial u_i}{\partial t} + \varrho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad i = 1, 2, 3, \tag{1b}$$

$$\varrho C_v \frac{\partial T}{\partial t} + \varrho u_j C_v \frac{\partial T}{\partial x_j} + p \frac{\partial u_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K \frac{\partial T}{\partial x_j}\right) + \Phi \tag{1c}$$

and the equation of state. In these equations, σ_{ij} denotes the stress tensor, and Φ denotes the viscous dissipation function. They are related to the velocity field through the viscous stress tensor, τ_{ij}, and the rate of strain tensor, e_{ij}. Thus,

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij} \tag{2a}$$

where

$$\tau_{ij} = \mu\, e_{ij} - \frac{1}{3}\mu\, e_{kk}\,\delta_{ij}, \tag{2b}$$

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{2c}$$

and

$$\Phi = \frac{1}{2}\tau_{ij}\, e_{ij} = \frac{\mu}{6}\left[(e_{11} - e_{22})^2 + (e_{22} - e_{33})^2 + (e_{33} - e_{11})^2 + \right.$$
$$\left. + 6\,(e_{12}{}^2 + e_{23}{}^2 + e_{31}{}^2)\right] \geqslant 0. \tag{2d}$$

C_v, μ, and K in the equations denote respectively, the spezific heat at constant volume, coefficient of viscosity, and coefficient of heat conductivity of the medium. They are, in general, functions of the temperature and the pressure. For simplicity, we assume that C_v is a constant[1] and neglect the pressure dependence of the viscosity and conductivity, which is small for a gas under normal conditions. Unless the contrary is explicitly indicated, we shall take our medium as a perfect gas whose equation of state is

$$p = \varrho R T \tag{1d}$$

[1] In actuality, results obtained in this and the following section are not affected by the assumption. It is relevant only in Section IV.

where R is the gas constant. On several occasions, we shall make comparisons between our results and the case when the medium is an incompressible fluid, for which medium the equation of state is $\varrho = a$ constant.

In a uniform homogeneous medium at rest $u_i = 0$, and p, ϱ, T, μ, and K are all constant. The latter will be denoted by p_0, ϱ_0, etc. Suppose that some small disturbances are introduced into the system. The pressure, velocity, density, etc., will then be $p = p_0 + p'$, $u_i = u_i'$, $\varrho = \varrho_0 + \varrho'$, etc., where $\left|\frac{p'}{p_0}\right| \ll 1$, $\left|\frac{u_i'}{a_0}\right| \ll 1$, $\left|\frac{\varrho'}{\varrho_0}\right| \ll 1$, etc., ($a_0 = \sqrt{\frac{\gamma p_0}{\varrho_0}}$ being the sound velocity). Substituting these into the system (1 a)–(1 d) and neglecting all quadratic terms of the small quantities, one obtains the following linearized system:

$$\frac{\partial \varrho'}{\partial t} + \varrho_0 \frac{\partial u_j'}{\partial x_j} = 0, \tag{3a}$$

$$\varrho_0 \frac{\partial u_i'}{\partial t} = \frac{\partial \sigma_{ij}'}{\partial x_j}, \tag{3b}$$

$$\varrho_0 C_v \frac{\partial T'}{\partial t} + p_0 \frac{\partial u_j'}{\partial x_j} = K_0 \frac{\partial^2 T'}{\partial x_j \partial x_j}, \tag{3c}$$

$$\frac{p'}{p_0} = \frac{\varrho'}{\varrho_0} + \frac{T'}{T_0} \tag{3d}$$

where

$$\sigma_{ij}' = -p' \delta_{ij} + \tau_{ij}' = -p' \delta_{ij} + \mu_0 \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) - \frac{2}{3} \mu_0 \frac{\partial u_k'}{\partial x_k} \delta_{ij}. \tag{4}$$

For an incompressible fluid, (3 d) must be replaced by $\varrho' = 0$. The system of equations governs the change of the six variables: p', ϱ', T', u_i', and, therefore, the evolution of the disturbance. What is the total energy in such a disturbance? The energy in the disturbance should be a positive definite quantity which, in the absence of heat transfer at the boundaries and of work done by them, must be a monotone non-increasing function of time; the rate of change of the quantity should depend on the coefficients of viscosity, μ_0, and heat condition, K_0, and is zero when $\mu_0 = K_0 = 0$. Such a quantity can indeed be constructed for our system by first multiplying (3 a) with $\frac{a_0^2}{\gamma} \frac{\varrho'}{\varrho_0}$, (3 b) with u_i', (3 c) with $\frac{T'}{T_0}$, and then adding the three equations:

$$\frac{\partial}{\partial t}\left[\frac{1}{2} \frac{a_0^2 \varrho'^2}{\gamma \varrho_0} + \frac{1}{2} \varrho_0 u_i' u_i' + \frac{1}{2} \frac{p_0 C_v T'^2}{T_0}\right]$$

$$= \frac{\partial}{\partial x_j} (\sigma_{ij}' u_i') + \frac{\partial}{\partial x_j} \left(\frac{K_0 T'}{T_0} \frac{\partial T'}{\partial x_j}\right) - \Phi' - \frac{K_0}{T_0} \frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j} \tag{5}$$

where

$$\Phi' = \frac{1}{2} \tau_{ij}' e_{ij}' = \frac{1}{2} \tau_{ij}' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) = \frac{\mu_0}{6} [(e_{11}' - e_{22}')^2 +$$

$$+ (e_{22}' - e_{33}')^2 + (e_{33}' - e_{11}')^2 + 6 (e_{12}'^2 + e_{23}'^2 + e_{31}'^2)] \geqslant 0,$$

and γ is the ratio of specific heat at constant pressure to that at constant volume. In the derivation of (5), use has been made of Eq. (3 d)[2]. For stationary insulated boundaries, we have:

$$u' = 0, \; \frac{\partial T'}{\partial n} = 0 \tag{6}$$

at the solid boundaries, $\vec{n}$ being the normal vector at the boundaries. If the disturbance dies down at infinity, we must have

$$u' \to 0, \; T' \to 0 \tag{7}$$

at infinity. Now if we integrate (5) over the entire region occupied by the medium and make use of the boundary conditions (6) and (7), we find

$$\frac{\partial}{\partial t}\int\left[\frac{1}{2}\frac{a_0^2\,\varrho'^2}{\gamma\,\varrho_0} + \frac{1}{2}\varrho_0\,u_i'\,u_i' + \frac{1}{2}\frac{\varrho_0\,C_v\,T'^2}{T_0}\right]d\tau =$$
$$= -\int \Phi'\,d\tau - \frac{K_0}{T_0}\int\frac{\partial T'}{\partial x_j}\frac{\partial T'}{\partial x_j}\,d\tau \tag{8}$$

which is evidently $\leqslant 0$. Thus, the quantity

$$E = \int\left[\frac{1}{2}\varrho_0\,u_i'\,u_i' + \frac{1}{2}\frac{a_0^2\,p'^2}{\gamma\,\varrho_0} + \frac{1}{2}\frac{\varrho_0\,C_v\,T'^2}{T_0}\right]d\tau \tag{9}$$

possesses all the properties pertaining to the energy in the system and will be *defined* as the total energy in the disturbance. It is, in fact, an energy of the disturbance in the sense that it characterizes the mean level (or intensity) of fluctuation in the disturbance.

If the disturbance is periodic in space so that it does not die out at the infinity in some direction and (7) ceases to be valid, the integral in (8) must be restricted, in that direction, to one single period to ensure its convergence. With the understanding of this change of the range of integration, it is clear that (8) is nevertheless still valid for periodic disturbances since the periodicity condition which replaces (7) in that particular direction still eliminates any contribution to (8) from the terms

$$\frac{\partial}{\partial x_j}(\sigma_{ij}'\,u_i') + \frac{\partial}{\partial x_j}\left(\frac{K_0\,T'}{T_0}\frac{\partial T'}{\partial x_j}\right)$$

of (5).

[2] Eq. (5) is still valid for an incompressible medium for which (3d) must be replaced by the condition $\varrho' = 0$. In fact, in such a case, we have a pair of uncoupled energy equations governing the changes in kinetic and thermal energies:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varrho_0\,u_i'\,u_i'\right] = \frac{\partial}{\partial x_j}(\sigma_{ij}'\,u_i') - \Phi',$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\varrho_0\,C_v\,T'^2}{T_0}\right] = \frac{\partial}{\partial x_j}\left(\frac{K_0\,T'}{T_0}\frac{\partial T'}{\partial x_j}\right) - \frac{K_0}{T_0}\frac{\partial T'}{\partial x_j}\frac{\partial T'}{\partial x_j}.$$

The sum of these two is identical to (5), provided that ϱ' is put equal to zero in the latter.

If we restrict ourselves to phenomena in which conduction effects are negligible (i. e., $K_0 = 0$), the energy equation (3 c) and the continuity equation (3 a) can be combined to give the isentropic relation:

$$\frac{\varrho'}{\varrho_0} = \frac{1}{\gamma - 1} \frac{T'}{T_0} = \frac{1}{\gamma} \frac{p'}{p_0}.$$

(The arbitrary function of integration is zero if we assume further that the disturbance is generated mechanically so that the entropy which is initially uniform maintains its uniformity.) In such case, E reduces to the form:

$$E = \int \left[\frac{1}{2} \varrho_0 u_i' u_i' + \frac{1}{2} \frac{a_0^2 \varrho'^2}{\varrho_0}\right] d\tau, \tag{10}$$

which is immediately recognized as the total acoustic energy in the system. In particular, $\int \frac{1}{2} \varrho_0 u_i' u_i' \, d\tau$ represents the total kinetic energy in the disturbance while $\int \frac{1}{2} \frac{a_0^2 \varrho'^2}{\varrho_0} d\tau$ is the total energy of condensation or the potential energy in the disturbance [6]. If we now have in addition, $\mu_0 = 0$ (non-viscous medium), then

$$E = \text{const.} \tag{11}$$

In a similar fashion for the general case of non-zero K_0 and μ_0, we can call $\frac{1}{2} \varrho_0 u_i' u_i'$ the kinetic energy in the disturbance per unit volume and *define* the quantity

$$\frac{1}{2} \frac{a_0^2 \varrho'^2}{\gamma \varrho_0} + \frac{1}{2} \frac{\varrho_0 C_v T'^2}{T_0}, \tag{12}$$

as the generalized potential energy per unit volume in the disturbance. In the general case, the potential energy will then consist of two parts, namely, that associated with the pressure fluctuations (compression), and that associated with entropy spottiness (resulting from heat exchange). This can be seen more readily if we write $\frac{T'}{T_0}$ and $\frac{\varrho'}{\varrho_0}$ in terms of the pressure fluctuation, p', and the entropy fluctuation, S'. The entropy of a perfect gas is related to the pressure and temperature by

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{\gamma - 1}{\gamma}} e^{\frac{S - S_0}{C_p}} \tag{13a}$$

where S is the entropy. Likewise

$$\frac{\varrho}{\varrho_0} = \left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}} e^{-\frac{S - S_0}{C_p}} \tag{13b}$$

These equations clearly show that density and temperature changes can be produced either by a change in p (compression) or a change in S (heat exchange). For small changes, these equations can be written:

$$\frac{T'}{T_0} = \frac{\gamma - 1}{\gamma} \frac{p'}{p_0} + \frac{S'}{C_p}, \tag{14a}$$

$$\frac{\varrho'}{\varrho_0} = \frac{1}{\gamma} \frac{p'}{p_0} - \frac{S'}{C_p}. \tag{14b}$$

The potential energy per unit volume can be written as

$$\frac{1}{2} \frac{a_0^2 \varrho'^2}{\gamma p_0} + \frac{1}{2} \frac{\varrho_0 C_v T'^2}{T_0} = \frac{1}{2} \varrho_0 a_0^2 \left(\frac{p'}{\gamma p_0}\right)^2 + \frac{1}{2} \frac{\gamma - 1}{\gamma} p_0 \left(\frac{S'}{R}\right)^2. \tag{15}$$

The first term of the right hand side of (15) represents the compression effect while the second represents the effect of heat exchange. [Again if $K_0 = 0$, the equations (3 a) and (3 c) can be combined to give $\frac{\partial S'}{\partial t} = 0$, so that if $S' = 0$ initially, $S' = 0$ at all times. The right hand side of Eq. (15) reduces to the single term $\frac{1}{2} \varrho_0 a_0^2 \left(\frac{p'}{\gamma p_0}\right)^2$ which is equal to $\frac{1}{2} \frac{a_0^2 \varrho'^2}{\varrho_0}$, the acoustic energy of condensation.] Thus, the total energy, E, in the disturbance can also be written as

$$E = \int \left[\frac{1}{2} \varrho_0 u_i' u_i' + \frac{1}{2} \varrho_0 a_0^2 \left(\frac{p'}{\gamma p_0}\right)^2 + \frac{1}{2} \frac{\gamma - 1}{\gamma} p_0 \left(\frac{S'}{R}\right)^2\right] d\tau. \tag{16}$$

The fact that entropy spottiness, S', should also be considered as a form of energy in the disturbance may at first seem to be a little puzzling, especially because entropy is normally taken as a measure of the unavailable energy. The important thing to be recognized is that we are speaking here of the *energy in a disturbance*; and as such, changes in entropy distribution of a gas will always induce a change in the fluid motion and hence, a change in the kinetic energy in the disturbance. Perhaps the simplest example illustrating this effect is the following one. Consider an infinite medium initially (say $t = 0$) at rest (i. e., $u_i' = 0$). We assume that pressure is uniform initially (i. e., $p' = 0$ at $t = 0$) and that the entropy distribution is not uniform at the initial instant (i. e., $S' \neq 0$ at $t = 0$). Hence, initially the first two terms in (16) are zero while the last term is positive. For $t > 0$, heat conduction will cause a diffusion of S' and hence, a change in density distribution. This change in density, ϱ', causes a relative motion between different parts of fluid (by the continuity equation) so that $\int \frac{1}{2} \varrho_0 u_i' u_i' \, d\tau$ is now greater than zero. In this example we find a partial conversion of the potential energy associated with the entropy spottiness into the kinetic energy of the disturbance. This example also shows that with the absence of external heat addition such conversion in a perfect gas is possible in a first order theory only when the coefficient of heat conductivity is different from zero. For, if $K_0 = 0$, Eqs. (3 a) and (3 c) can be combined to give $\frac{\partial S'}{\partial t} = 0$ so that

$$\frac{\partial}{\partial t} \int \frac{1}{2} \frac{\gamma - 1}{\gamma} p_0 \left(\frac{S'}{R}\right)^2 d\tau = 0,$$

in which case we have simply

$$\frac{\partial}{\partial t}\int\left[\frac{1}{2}\varrho_0 u_i' u_i' + \frac{1}{2}\varrho_0 a_0^2\left(\frac{p'}{\gamma p_0}\right)^2\right]d\tau = -\int \Phi'\, d\tau. \tag{17}$$

Similarly, there is no conversion from the entropy fluctuation into the kinetic energy of the disturbance in an incompressible medium, for there the velocity field is independent of the temperature field. Since the heat conduction effect in a gaseous medium is not very marked, the conversion of the potential energy associated with S' into kinetic energy and vice versa will generally not be significant except in high frequency phenomena, or for large amplitude disturbances. The conversion may also be of some importance near solid boundaries where the flow velocity is small.

Just like the viscous dissipation, Φ', which tends to even out any velocity fluctuations, the non-negative integral

$$\int \frac{K_0}{T_0}\frac{\partial T'}{\partial x_j}\frac{\partial T'}{\partial x_j}\,d\tau,$$

tends to wipe out any entropy spottiness. As such, it may be called the thermal dissipation.

Finally, the energy in disturbances defined above has also a geometrical interpretation. The state of a fluid at any instant is characterized by its velocity and two thermodynamic variables. Consequently, the state of fluid is completely specified by an ordered set of five functions (of the three space variables) giving the values of the two thermodynamic variables and three velocity components. It can thus be conveniently represented by a point in a five-dimensional function space. If we take a Cartesian coordinate system in this space with the origin at the point representing our undisturbed state, and the five axes giving the values of u_1', u_2', u_3', p', S' as functions of the space variables x_1, x_2, x_3, then, at any instant, the disturbed state of the fluid will be represented by a point in the neighborhood of the origin of our coordinate system. With proper scaling along each axis, the distance between the point and the origin can be made equal to the energy in the disturbance. Eq. (8) then says that the point representing the disturbed state always tends toward the origin in the absence of external energy supply. From this point of view, it is not surprising that the total energy in the disturbance should actually include a term S'. [In defining the disturbed state of the field by a point in the function space, we could have chosen u_1', u_2', u_3', ϱ', T' as the coordinate axes. By virtue of relations (14) this system is related to the system u_1', u_2', u_3', p', S' by a rotation about the velocity axes.]

III. Addition of Energy to Small Disturbances by Body Force, Heat and Material Sources

The first step toward generalizing the foregoing analysis for studying energy transfer from a non-uniform main stream to small disturbances is to consider the effects of the externally imposed body forces, heat and material sources on the change of total energy of the disturbance in a uniform medium. These effects appear through additional terms in the linearized equations of continuity, momentum, and energy as follows[3]:

$$\frac{\partial \varrho'}{\partial t} + \varrho_0 \frac{\partial u_j'}{\partial x_j} = m', \tag{18a}$$

$$\varrho_0 \frac{\partial u_i'}{\partial t} = \frac{\partial \sigma_{ij}'}{\partial x_j} + \varrho_0 F_i', \tag{18b}$$

$$\varrho_0 C_v \frac{\partial T'}{\partial t} + p_0 \frac{\partial u_j'}{\partial x_j} = K_0 \frac{\partial^2 T'}{\partial x_j \partial x_j} + Q' + m' R T_0, \tag{18c}$$

$$\frac{p'}{p_0} = \frac{\varrho'}{\varrho_0} + \frac{T'}{T_0} \tag{18d}$$

where m' and Q' are respectively the rate of production of mass and heat per unit volume, F_i' is the body force per unit mass, and m', Q', F_i' are so small that $\left|\frac{\varrho'}{\varrho_0}\right|, \left|\frac{p'}{p_0}\right|, \left|\frac{T'}{T_0}\right|, \left|\frac{u_i'}{a_0}\right| \ll 1$. Multiply (18 a) with $\frac{a_0^2}{\gamma} \frac{\varrho'}{\varrho_0}$, (18 b) with u_i, and (18 c) with $\frac{T'}{T_0}$ as before, and add them together, and then integrate the result over the whole flow field. We find, after making use of the boundary conditions (6) and (7), the following result:

$$\frac{\partial E}{\partial t} = \frac{1}{\varrho_0} \int p' m' \, d\tau + \frac{1}{T_0} \int T' Q' \, d\tau + \varrho_0 \int F_i' u_i' \, d\tau - \\ - \frac{K_0}{T_0} \int \frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j} d\tau - \int \Phi' \, d\tau \tag{19a}$$

where the range of integration extends over the entire domain occupied by the fluid. Again, if the disturbance is periodic in certain directions, (19 a) is still valid provided that the integral is taken over one single period in those directions (cf., p. 221).

Thus, we see that there are three types of terms which may be responsible for an increase of the energy and, therefore, the fluctuation level in the disturbance. These terms can be interpreted as the energy sources (although when their values in a particular problem turn out to be negative, they really represent energy sinks). Since, in small disturbance theory, pressure and entropy fluctuations are two independent modes of fluctuations [7], it is desirable to replace $\frac{T'}{T_0}$ in the third integral on the right hand side of (19 a), by (14 a). We then have the alternate form:

$$\frac{\partial E}{\partial t} = \frac{1}{\varrho_0} \int p' \left(m' + \frac{Q'}{C_p T_0}\right) d\tau + \varrho_0 \int F_i' u_i' \, d\tau + \frac{1}{C_p} \int S' Q' \, d\tau - \\ - \frac{K_0}{T_0} \int \frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j} d\tau - \int \Phi' \, d\tau. \tag{19b}$$

[3] Here we assume that the fluid injected at any point has always a velocity, pressure, temperature and density equal to the local velocity u_i, pressure p, temperature T, and density ϱ of the flow field. Consequently, if the rate of mass injected is M, there will be a momentum addition associated with it of amount $M u_i$, and an energy of amount $M\left(C_p T + \frac{1}{2} u_i u_i\right)$.

Before we apply the above result to the discussion of a few special cases, a few remarks are in order. First of all, there are two classes of phenomena to which the equation (19 a) or (19 b) can be readily applied: resonance and instability. When all the driving functions: m', F_i', Q', are externally imposed, i. e., their amplitudes, frequencies, etc., are given, energy can be continuously fed into the disturbance, increasing its mean level of fluctuation, by proper phasing of the driving functions with the various flow variables. Thus, for example, if m' is in phase with p', or F_i' with u_i', or Q' with T', energy will be continuously fed into the disturbance. Such a phenomena is described as resonance. The amplitude of the disturbance will eventually be limited by the dissipation and, in some cases, non-linear behavior of the system. In a similar manner, a 180-degree out of phasing of these variables will result in a severe attenuation of the disturbance.

In physical systems where the driving functions themselves are functions of the flow variables, and are zero when there is no disturbance in the system, any chance disturbance in the system may cause a change of the energy level in the disturbance which may either reinforce or attenuate the disturbance. When the disturbance reinforces itself through the action of the driving terms, the system is said to be unstable with respect to that type of disturbance. If the disturbance is attenuated, the system is said to be stable with respect to that disturbance. The flow system is said to be stable when it is stable with respect to all types of disturbances. Interpreted geometrically, the point representing the disturbed state in the five-dimensional function space will return (in the course of time) to the origin of the coordinate system if the flow system is stable. Otherwise, there will be at least one disturbed state whose representing point in the vector function space will wander away from the origin in the course of time.

The simplest application of the energy relation, Eq. (19 a), is perhaps the familiar phenomenon of thermal instability under a gravitational field. Here $m' = 0$ and $Q' = 0$; the energy source responsible for this instability is then $\varrho_0 \int F_i'\, u_i'\, d\tau$. If we take the x_3-axis in the direction of gravitation, then $F_1' = 0$, $F_2' = 0$, and $F_3' = \frac{\varrho'}{\varrho_0} g$, so that the energy production term is:

$$\varrho_0 \int F_i'\, u_i'\, d\tau = g \int \varrho'\, u_3'\, d\tau.$$

Hence, a heavier (than average) lump of fluid ($\varrho' > 0$), when it is made to move against the direction of gravity ($u_3' < 0$), extracts energy from the initiating disturbance. This extraction of energy from the disturbance will continue until the disturbance dies out. On the other hand, if the disturbance is such as to impart to the heavier lump of fluid a motion in the direction of gravity, energy will be fed into the disturbance reinforcing it and causing further motion of the lump in the direction of gravity. The same explanation applies to cases where the gravitational field is replaced by an acceleration field (TAYLOR stability [8]).

Another case of some interest in combustion phenomena is the stability of systems containing heat sources. Let us first examine the case where conduction effects are negligible. When K_0 is put equal to zero, Eq. (19 b) can be decomposed into two equations:

$$\frac{\partial}{\partial t}\int\left[\frac{1}{2}\varrho_0\, u_i'\, u_i' + \frac{1}{2}\varrho_0\, a_0^2\left(\frac{p'}{\gamma\, p_0}\right)^2\right]d\tau =$$

$$= \frac{1}{\varrho_0}\int p'\left(m' + \frac{Q'}{C_p\, T_0}\right)d\tau + \varrho_0\int F_i'\, u_i'\, d\tau - \int \Phi'\, d\tau, \qquad (21\,\text{a})$$

$$\frac{\partial}{\partial t}\int\left[\frac{1}{2}\,\frac{\gamma-1}{\gamma}\,p_0\left(\frac{S'}{R}\right)^2\right]d\tau = \frac{1}{C_p}\int S'\, Q'\, d\tau \qquad (21\,\text{b})$$

the first of which shows how mechanical energy in the disturbance (i. e., the sum of kinetic energy and potential energy associated with compression) changes with time while the second describes the change in level of entropy fluctuation in the disturbance. That such a decomposition is possible follows from the fact that if we eliminate the divergence $\frac{\partial u_j'}{\partial x_j}$ from (18 a) and (18 c), we obtain the anticipated result:

$$\varrho_0\, T_0\frac{\partial S'}{\partial t} = Q'. \qquad (22)$$

Eq. (21 b) then follows immediately from this if we interchange the order of integration and differentiation in that expression. Eq. (21 a) is then derived by subtracting (21 b) from (19 b).

The possibility of decomposing (19 b) into (21 a) and (21 b) when $K_0 = 0$ indicates that in such cases there will be no transfer between (or conversion of) the potential energy associated with the entropy fluctuation on the one hand, and the kinetic energy and energy of condensation on the other, even in presence of mass, momentum, and heat sources [see remarks in connection with Eq. (15)]. This situation is similar to the well-known state of affairs in the first order theory of an incompressible medium where there is no coupling between the velocity fluctuation and temperature fluctuation (see footnotes on p. 221). If we now neglect the effects of body forces ($F_i' = 0$) and material sources ($m' = 0$), Eq. (21) becomes:

$$\frac{\partial}{\partial t}\int\left[\frac{1}{2}\varrho_0\, u_i'\, u_i' + \frac{1}{2}\varrho_0\, a_0^2\left(\frac{p'}{\gamma\, p_0}\right)^2\right]d\tau = \frac{\gamma-1}{\gamma}\,\frac{1}{p_0}\int p'\, Q'\, d\tau - \int\Phi'\, d\tau, \qquad (23\,\text{a})$$

$$\frac{\partial}{\partial t}\int\left[\frac{1}{2}\,\frac{\gamma-1}{\gamma}\,p_0\left(\frac{S'}{R}\right)^2\right]d\tau = \frac{1}{C_p}\int S'\, Q'\, d\tau. \qquad (23\,\text{b})$$

Eq. (23 a) is, in fact, the generalized Rayleigh's criterion, taking into account, in addition, the viscous losses. For, in the absence of viscosity ($\Phi' = 0$), Eq. (22 a) states that the mechanical energy in the disturbance will grow in time when the fluctuating heat release rate has a component in phase with the pressure fluctuation—Rayleigh's criterion (see [3] and [4]). However, the additional Eq. (23 b) shows that Rayleigh's criterion in its original form may in some special instances be misleading.

Consider, for example, a system with distributed heat sources releasing heat at a rate proportional to the local entropy fluctuation, i. e.,

$$Q' = \lambda S'. \tag{24a}$$

If $\lambda > 0$, and $S' \neq 0$, Eq. (23 b) shows that

$$\frac{\partial}{\partial t} \int \left[\frac{1}{2} \frac{\gamma - 1}{\gamma} \left(\frac{S'}{R} \right)^2 \right] d\tau > 0. \tag{24b}$$

Hence, any small accidental entropy fluctuation introduced into the system will induce a heat release which will reinforce the entropy fluctuation and cause further increase in heat release. The system is therefore unstable[4] with respect to small entropy fluctuations even if we assume that initially there is *no pressure disturbance* in the system at all. In this instance, RAYLEIGH's criterion does not give any indication as to the stability of the system, although the pressure and velocity fluctuations do ultimately increase beyond all bounds and the integral $\int_0^\infty \int p' \, Q' \, d\tau \, dt$ diverges. The seemingly paradoxical situation can perhaps be best clarified by re-writing the governing differential equations (18) with the help of Eqs. (14) and (22) as follows:

$$\frac{1}{\gamma p_0} \frac{\partial p'}{\partial t} + \frac{\partial u_j'}{\partial x_j} = \frac{Q'}{\varrho_0 C_p T_0}, \tag{25a}$$

$$\varrho_0 \frac{\partial u_i'}{\partial t} = - \frac{\partial p}{\partial x_j} + \mu_0 \frac{\partial^2 u_i'}{\partial x_j \, \partial x_j} + \frac{1}{3} \mu_0 \frac{\partial^2 u_j'}{\partial x_i \, \partial x_j}, \tag{25b}$$

$$\frac{\partial S'}{\partial t} = \frac{Q'}{\varrho_0 T_0}. \tag{25c}$$

The first two equations form a pair showing how p' and u_i' vary with heat addition: Q'. The last equation shows how S' varies with Q'. When Q' is related to S' by (24 a), S' will increase exponentially with time. So will p' and u_i' increase exponentially with time—not because Q' is in any way related to p', but rather because Q', which is proportional to S' and which acts as the externally applied forcing function in (25 a), increases exponentially with time.

When $K_0 = 0$ but $\mu_0 \neq 0$, RAYLEIGH's criterion must be modified slightly as follows: The condition that a thermal system is unstable to small disturbance is that the inequality

$$\int p' \, Q' \, d\tau > \frac{\gamma}{\gamma - 1} p_0 \int \Phi' \, d\tau \tag{26}$$

is satisfied. Likewise, a mode of oscillation is a neutral mode if the two are equal.

When $K_0 \neq 0$, the decomposition of (19 b) into (21 a) and (21 b) is no longer permissible. We see that what is essential for the growth of

[4] This can also be seen if we substitute (24a) into (22) and integrate. Thus, we find $S' = S_0' \exp. \left[\frac{\lambda}{\varrho_0 T_0} t \right]$ where S_0' is the entropy distribution at $t = 0$.

disturbance under thermal action is that T' and Q' are positively correlated (i. e., $\int T' Q' \, d\tau > 0$) instead of $\int p' Q' \, d\tau > 0$. In fact, $\int T' Q' \, d\tau$ must be larger than the sum of the viscous and thermal dissipation,

$$T_0 \int \Phi' \, d\tau + K_0 \int \frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j} \, d\tau,$$

before the energy in the disturbance begins to increase with time. This is the generalization of RAYLEIGH's criterion for a thermal system.

In many systems (especially in combustion problems), the insulated boundary condition in (6) is not even approximately satisfied. A more realistic condition is that

$$\frac{\partial T'}{\partial n} + h \, T' = 0 \tag{27}$$

at the boundaries, where $h \geqslant 0$. The value of h depends on the radiation and conduction heat losses at the boundaries [9]. Using this boundary condition instead of the second part of (6), we find

$$\frac{\partial E}{\partial t} = \frac{1}{T_0} \int T' Q' \, d\tau - \frac{h K_0}{T_0} \int T'^2 \, d\sigma - \frac{K_0}{T_0} \int \frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j} \, d\tau - \int \Phi' \, d\tau \tag{28}$$

instead of (19 a). (Here, we assume $m' = 0$ and $F_i' = 0$.) Thus, in general, for the system to be unstable there must be one mode of fluctuations for which the integral of the product of the fluctuating heat release and the temperature fluctuation is greater than the heat lost through the boundaries [the surface integral of Eq. (28)], and the viscous and thermal dissipations [the two volume integrals of Eq. (28)].

One final application of (19 a) is concerned with the amplification of a disturbance through mass production. Such cases are of importance in rocket engineering where large quantities of gaseous products are evolved through combustion of liquid or solid fuels [10]. Eq. (19 a) shows that the most favorable condition for feeding energy into the disturbance occurs when the rate of mass production is in phase with the pressure fluctuation.

IV. Transfer of Energy From a Steady Main Stream

Let there be a steady flow whose pressure, density, temperature, and velocity fields are described by $\bar{p}(x_i)$, $\bar{\varrho}(x_i)$, $\overline{T}(x_i)$, and $\bar{u}_j(x_i)$. Then

$$\frac{\partial \bar{\varrho} \, \bar{u}_j}{\partial x_j} = 0, \tag{29a}$$

$$\frac{\partial \bar{\varrho} \, \bar{u}_j \, \bar{u}_i}{\partial x_j} = - \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{ij}}{\partial x_j}, \tag{29b}$$

$$\frac{\partial}{\partial x_j} (\bar{\varrho} \, \bar{u}_j \, C_v \, \overline{T}) + \bar{p} \, \frac{\partial u_j}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\bar{K} \frac{\partial \overline{T}}{\partial x_j} \right) + \bar{\Phi}, \tag{29c}$$

$$\bar{p} = \bar{\varrho} \, R \, \overline{T} \tag{29d}$$

where $\bar{\tau}_{ij}$ is the viscous stress tensor,

$$\bar{\tau}_{ij} = \bar{\mu}\left[\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\,\frac{\partial \bar{u}_K}{\partial x_K}\right], \tag{30a}$$

and $\bar{\Phi}$ is the viscous dissipation function,

$$\bar{\Phi} = \bar{\tau}_{ij}\,\frac{\partial \bar{u}_i}{\partial x_j}. \tag{30b}$$

Let us suppose that for one reason or another, this flow field is slightly disturbed. The deviation of the flow field from that given above will be indicated by a prime in the superscript, e. g., p' stands for the pressure fluctuations, etc. Hence, the pressure, density, temperature, and velocity of the flow at any point and any instant will be

$$p = \bar{p}(x_i) + p'(x_i, t), \tag{31a}$$

$$\varrho = \bar{\varrho}(x_i) + \varrho'(x_i, t), \tag{31b}$$

$$T = \bar{T}(x_i) + T'(x_i, t), \tag{31c}$$

$$u_j = \bar{u}_j(x_i) + u_j'(x_i, t) \tag{31d}$$

where $\left|\frac{p'}{\bar{p}}\right|, \left|\frac{\varrho'}{\bar{\varrho}}\right|, \left|\frac{T'}{\bar{T}}\right|, \left|\frac{u_j'}{\bar{a}}\right| \ll 1$, $\bar{a}(x_i)$ being the velocity of sound in the undisturbed flow. Substituting these into (1) and (2), and neglecting all quadratic and higher products of small quantities, we find

$$\frac{\partial \varrho'}{\partial t} + \bar{\varrho}\,\frac{\partial u_j'}{\partial x_j} = m', \tag{32a}$$

$$\bar{\varrho}\,\frac{\partial u_i'}{\partial t} = -\frac{\partial p'}{\partial x_i} + \frac{\partial \tau_{ij}'}{\partial x_j} + F_i', \tag{32b}$$

$$\bar{\varrho}\,C_v\,\frac{\partial T'}{\partial t} + \bar{p}\,\frac{\partial u_j'}{\partial x_j} = \bar{T}\,\frac{\partial}{\partial x_j}\left[\frac{\bar{K}}{\bar{T}}\,\frac{\partial T'}{\partial x_j}\right] + Q' + m'\,R\,\bar{T}, \tag{32c}$$

$$\frac{p'}{\bar{p}} = \frac{\varrho'}{\bar{\varrho}} + \frac{T'}{\bar{T}} \tag{32d}$$

where τ_{ij}' is defined by

$$\tau_{ij}' = \bar{\mu}\left[\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\,\frac{\partial u_K'}{\partial x_K}\right] \tag{33a}$$

and

$$m' = \bar{\varrho}\,\frac{\partial u_j'}{\partial x_j} + \frac{\partial d_j}{\partial x_j}, \tag{33b}$$

$$F_i' = -\bar{u}_i\,\frac{\partial \varrho'}{\partial t} + \frac{\partial \tau_{ij}^*}{\partial x_j}, \tag{33c}$$

$$Q' = -C_p\,\bar{T}\,\frac{\partial d_j}{\partial x_j} - \bar{p}\,\frac{\partial u_j'}{\partial x_j} - p'\,\frac{\partial \bar{u}_j}{\partial x_j} + 2\,\tau_{ij}'\,\frac{\partial \bar{u}_i}{\partial x_j} + \\ + \frac{\mu'}{\bar{\mu}}\,\bar{\Phi} + \frac{\bar{K}}{\bar{T}}\,\frac{\partial \bar{T}}{\partial x_j}\,\frac{\partial T'}{\partial x_j} + \frac{\partial h_j}{\partial x_j}, \tag{33d}$$

$$d_j = - \varrho' \bar{u}_j - \bar{\varrho} u_j', \tag{33e}$$

$$\tau_{ij}^* = - \varrho' \bar{u}_i \bar{u}_j - \bar{\varrho} u_i' \bar{u}_j - \bar{\varrho} u_j' \bar{u}_i + \frac{\mu'}{\bar{\mu}} \bar{\tau}_{ij}, \tag{33f}$$

$$h_j = - \varrho' \bar{u}_j C_v \bar{T} - \bar{\varrho} u_j' C_v \bar{T} - \bar{\varrho} \bar{u}_j C_v T' + K' \frac{\partial \bar{T}}{\partial x_j}. \tag{33g}$$

Now $\mu = \mu(T)$ and $K = K(T)$ so that if we introduce the notations $\bar{\mu}_1 = \left(\frac{\partial \mu}{\partial T}\right)_{T=\bar{T}}$ and $\bar{K}_1 = \left(\frac{dK}{dT}\right)_{T=\bar{T}}$, the changes in μ and K will be given by

$$\mu' = \bar{\mu}_1 T', \tag{33h}$$

$$K' = \bar{K}_1 T'. \tag{33i}$$

Let us multiply Eq. (32 a) with $\frac{\bar{a}^2 \varrho'}{\gamma \bar{\varrho}}$, Eq. (32 b) with u_i', and Eq. (32 c) with $\frac{T'}{\bar{T}}$, and then add them together. Again, if E is defined by

$$E = \int \left[\frac{1}{2} \bar{\varrho} u_i' u_i' + \frac{1}{2} \frac{\bar{a}^2 \varrho'^2}{\gamma \bar{\varrho}} + \frac{1}{2} \frac{\bar{\varrho} C_v T'^2}{\bar{T}}\right] d\tau \tag{34}$$

[cf. Eq. (9)], we find

$$\frac{\partial E}{\partial t} = - \int p' u_j' n_j \, d\sigma + \int \tau_{ij}' u_i' n_j \, d\sigma + \int \frac{\bar{K} T'}{\bar{T}} \frac{\partial T'}{\partial x_j} n_j \, d\sigma +$$
$$+ \int \frac{p' m'}{\bar{\varrho}} d\tau + \int F_i' u_i' \, d\tau + \int \frac{Q' T'}{\bar{T}} d\tau -$$
$$- \int \frac{\bar{K}}{\bar{T}} \frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j} d\tau - \int \tau_{ij}' \frac{\partial u_i'}{\partial x_j} d\tau. \tag{35}$$

First of all, consider disturbances which vanish at infinity. Here, Eq. (6) applies. Assuming the solid boundaries to be stationary, u_i' at the boundaries will be zero. If the heat loss from the boundaries satisfies the relation

$$\frac{\partial T'}{\partial n} + h T' = 0 \tag{36}$$

where h is a non-negative function of the space variables defined along the boundaries, and $\vec{n}$ is the normal vector at the boundaries, Eq. (35) becomes:

$$\frac{\partial E}{\partial t} = \int \frac{p' m'}{\bar{\varrho}} d\tau + \int F_i' u_i' \, d\tau + \int \frac{Q' T'}{\bar{T}} d\tau -$$
$$- \int \frac{h \bar{K}}{\bar{T}} T'^2 \, d\tau - \int \frac{\bar{K}}{\bar{T}} \frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j} d\tau - \int \Phi' \, d\tau \tag{37}$$

where Φ' is the dissipation function,

$$\Phi' = \tau_{ij}' \frac{\partial u_i'}{\partial x_j}, \tag{38}$$

and the integration extends over the entire space occupied by the medium. If the disturbance is periodic in certain directions, Eq. (37) turns out to be still valid provided that the range of integration is limited to one wave length in those directions. It is clear that for such a system to be unstable, the first three integrals on the right hand side of Eq. (37) must have a sum greater than that of the last three integrals.

One important special case which deserves special mention is that where the main stream is a two-dimensional parallel shear flow specified by:

$$\bar{p} = p_0, \text{ a constant}, \tag{39a}$$

$$\bar{T} = \bar{T}(y), \tag{39b}$$

$$\bar{\varrho} = p_0/R\,\bar{T}(y), \tag{39c}$$

$$\bar{u}_1 = \bar{u}(y), \tag{39d}$$

$$\bar{u}_2 = 0 \tag{39e}$$

where the x_1-axis is in the direction of the flow and is denoted as x-axis while the x_2-axis is replaced by the y-axis. One must remark that such a flow is an exact solution of the NAVIER-STOKES equations only under very special temperature and velocity profile (i. e., special types of $\bar{T}(y)$ and $\bar{u}(y)$). However, if we confine ourselves to a sufficiently restricted region in the direction of flow, the actual main stream can perhaps be approximated as such without introducing too great an error. In such a case,

$$m' = -\bar{u}\frac{\partial \varrho'}{\partial x} - v'\frac{d\bar{\varrho}}{dy}, \tag{40a}$$

$$F_x' = -\bar{\varrho}\,\bar{u}\frac{\partial u'}{\partial x} - \bar{\varrho}\,v'\frac{d\bar{u}}{dy} + \frac{\partial}{\partial y}\left(\mu'\frac{d\bar{u}}{dy}\right), \tag{40b}$$

$$F_y' = -\bar{\varrho}\,\bar{u}\frac{\partial v'}{\partial x} + \frac{\partial}{\partial x}\left(\mu'\frac{d\bar{u}}{dy}\right), \tag{40c}$$

$$Q' = -\bar{p}\,\bar{u}\frac{\partial}{\partial x}\left(\frac{S'}{R}\right) - \bar{p}\,v'\frac{d}{dy}\left(\frac{\bar{S}}{R}\right) + 2\bar{\mu}\left(\frac{d\bar{u}}{dy}\right)\left(\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial x}\right) + $$
$$+ \frac{\bar{K}}{\bar{T}}\frac{d\bar{T}}{dy}\frac{\partial T'}{\partial y} + \mu'\left(\frac{d\bar{u}}{dy}\right)^2 + \frac{\partial}{\partial y}\left(K'\frac{d\bar{T}}{dy}\right). \tag{40d}$$

Hence,

$$\frac{p'\,m'}{\bar{\varrho}} + F_i'\,u_i' + \frac{Q'\,T'}{\bar{T}} =$$

$$= -\frac{\partial}{\partial x}\left\{\bar{p}\,\bar{u}\left[\frac{1}{2}\left(\frac{\varrho'}{\bar{\varrho}}\right)^2 + \frac{1}{2}\frac{1}{\gamma-1}\left(\frac{T'}{\bar{T}}\right)^2\right] + \bar{\varrho}\,\bar{u}\left[\frac{1}{2}u'^2 + \frac{1}{2}v'^2 + \frac{1}{2}w'^2\right]\right\} -$$

$$-\bar{\varrho}\,u'\,v'\frac{d\bar{u}}{dy} - \bar{\varrho}\,v'\,S'\frac{d\bar{T}}{dy} + \left(2\bar{\mu}\frac{T'}{\bar{T}} - \mu'\right)\frac{d\bar{u}}{dy}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) +$$

$$+\frac{T'}{\overline{T}}\left[\frac{\overline{K}}{\overline{T}}\frac{d\overline{T}}{dy}\frac{\partial T'}{\partial y}+\mu'\left(\frac{d\bar{u}}{dy}\right)^2+\frac{\partial}{\partial y}\left(K'\frac{d\overline{T}}{dy}\right)\right]+$$

$$+\left[\frac{\partial}{\partial x}\left(v'\mu'\frac{d\bar{u}}{dy}\right)+\frac{\partial}{\partial y}\left(u'\mu'\frac{d\bar{u}}{dy}\right)\right]. \tag{41}$$

Consider disturbances which are periodic in x and die down at infinity in all other directions (if such directions exist). If we make use of the condition that all solid boundaries are stationary and integrate Eq. (41) over one wave length in the x-direction and all y, we find

$$\int\frac{p'\,m'}{\bar{\varrho}}\,d\tau+\int F_i'\,u_i'\,d\tau+\int\frac{Q'\,T'}{\overline{T}}\,d\tau=$$

$$=-\int\bar{\varrho}\,u'\,v'\frac{d\bar{u}}{dy}\,d\tau-\int\bar{\varrho}\,v'\,S'\frac{d\overline{T}}{dy}\,d\tau+\int\frac{T'}{\overline{T}}\,Q^*\,d\tau \tag{42}$$

where

$$Q^*=(2\,\bar{\mu}-\bar{\mu}_1\,\overline{T})\frac{d\bar{u}}{dy}\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)+\frac{\overline{K}}{\overline{T}}\frac{d\overline{T}}{dy}\frac{\partial T'}{\partial y}+$$

$$\bar{\mu}_1\,T'\left(\frac{d\bar{u}}{dy}\right)^2+\frac{\partial}{\partial y}\left(\overline{K}_1\,T'\frac{d\overline{T}}{dy}\right). \tag{43}$$

Q^* has the unit of energy per unit volume. The first two terms in the expression for Q^* are similar in nature; they represent the energy transfer to the disturbance due to the interaction of the transport phenomena between the main stream and fluctuations. The last two terms in Eq. (43) represent additional energy transfer to the disturbance resulting from changes in the coefficients of viscosity and heat conduction with the temperature. Eq. (37) can now be written as:

$$\frac{\partial E}{\partial t}=-\int\bar{\varrho}\,u'\,v'\frac{d\bar{u}}{dy}\,d\tau-\int\bar{\varrho}\,v'\,S'\frac{d\overline{T}}{dy}\,d\tau+\int\frac{T'}{\overline{T}}\,Q^*\,d\tau-$$

$$-\int\frac{h\,\overline{K}\,T'^2}{\overline{T}}\,d\sigma-\int\frac{\overline{K}}{\overline{T}}\frac{\partial T'}{\partial x_j}\frac{\partial T'}{\partial x_j}\,d\tau-\int\Phi'\,d\tau. \tag{44}$$

Perhaps the most interesting feature of the equation is the energy production term:

$$-\int\bar{\varrho}\,u'\,v'\frac{d\bar{u}}{dy}\,d\tau-\int\bar{\varrho}\,v'\,S'\frac{d\overline{T}}{dy}\,d\tau. \tag{45}$$

The first term in Eq. (45) is well-known as the work done by the REYNOLD's stress. The second term results from the energy transfer due to the main stream temperature gradient. Note that it is the transport of entropy spottiness across the temperature shear layers that produces this energy transfer. For an incompressible fluid, the first term is solely responsible for the changes in the kinetic energy in the disturbance while the second term which assumes the reduced form: $\varrho_0\,C_v\int v'\,T'\frac{1}{\overline{T}}\frac{d\overline{T}}{dy}\,d\tau$, is the principle term which produces changes in the temperature fluctuation level in the disturbance. Consequently, we expect that at low MACH numbers and small heat transfer rates at the boundaries, the intensity of the velocity fluctuations are determined principally by the energy production term, $\int \bar{\varrho}\, u'\, v' \frac{d\bar{u}}{dy}\, d\tau$, while the intensity of the temperature fluctuation is mainly controlled by the energy production term, $\int \bar{\varrho}\, v'\, S' \frac{d\bar{T}}{dy}\, d\tau$. Finally, the production term $\int \frac{T'}{\bar{T}} Q^*\, d\tau$ is in most cases, an order of magnitude smaller than $\int \bar{\varrho}\, u'\, v' \frac{d\bar{u}}{dy}\, d\tau$ and $\int \bar{\varrho}\, v'\, S' \frac{d\bar{T}}{dy}\, d\tau$. It assumes an important role only in small scale disturbances.

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