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On the Convergence of Multi-Grid Methods with Transforming Smoothers

Theory with Applications to the Navier-Stokes Equations*

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Summary. In the present paper we give a convergence theory for multi-grid methods with transforming smoothers as introduced in [31] applied to a general system of partial differential equations. The theory follows Hackbusch's approach for scalar pde and allows a convergence proof for some well-known multi-grid methods for Stokes- and Navier-Stokes equations as DGS by Brandt-Dinar, [5], TILU from [31] and the SIMPLE-methods by Patankar-Spalding, [23].

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1 Introduction

In the present paper we give a general convergence theory for multi-grid methods with transforming smoothing as introduced in [31], applying to linear and quasilinear saddle-point problems as the Stokes and Navier-Stokes equations. The theory allows a convergence proof for some well-known multi-grid methods for those equations such as the so-called distributive Gauß-Seidel by Brandt and Dinar, [5], transforming ILU, introduced in [30, 31], and the SIMPLE-methods originally due to Patankar and Spalding, [23]. Recently the importance of these methods is growing more and more as illustrated by a number of papers on that topic (see [3, 9, 21, 24, 25, 26, 31] and the references there). The results given here are extensions of the ones in [30].

Sections 1.1 and 1.2 contain a short discussion of multi-grid technique and convergence theory as well as a brief description of the incompressible, steady-state Navier-Stokes equations, serving as model problem. After a short outline of r-transforming smoothing in paragraph 2.1, we give a criterion for the convergence of a multi-grid method with r-transforming smoother applied to a general system of partial differential equations. Based on this criterion the smoothing

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property and consequently two-grid convergence for a general block triangular system is discussed in section 2.3, providing advice for the construction of suitable transformations. In sections 3.1 and 3.2 the theory from chapter 2 is applied to the Stokes and Navier-Stokes equations.

1.1 Multi-Grid Technique

1.1.1 Notations and Preliminaries. Multi-grid methods are iterative solvers of optimal efficiency, gained by a skilful combination of two parts, smoothing and coarse-grid correction.

Let the linear boundary-value problem

$$Ku=f \quad \text{in } \Omega \tag{1.1.1a}$$

$$u=u_R \quad \text{on } \partial\Omega \tag{1.1.1b}$$

with a differential operator $K\colon U \to F$ between some Sobolev-spaces be given on a domain $\Omega c \mathbf{R}^d$. Let (1.1.1) be discretized on $l_{\max}$ grids

$$\Omega_l, \quad l=1, \ldots, l_{\max} \tag{1.1.2a}$$

with stepsizes

$$0<h_{l+1}<h_l, \quad l=1, \ldots, l_{\max}-1. \tag{1.1.2b}$$

Let further

$$K_l\colon\ U_l \to F_l \tag{1.1.3}$$

be the discretization of K on Ω_l, U_l and F_l some spaces of grid-functions, being discrete analogues of the Sobolev-spaces U and F. Let the discrete problem

$$K_l u_l = f_l \quad \text{in } \Omega_l, \quad l=1, \ldots, l_{\max} \tag{1.1.4a}$$

$$u_l = u_{R,l} \quad \text{on } \partial\Omega_l, \quad l=1, \ldots, l_{\max} \tag{1.1.4b}$$

be well-posed. Now, classical iterations can be used as smoothers, as they primarily reduce the high frequency components of the error. We denote one step of such an iteration ("smoothing step") by S. Furthermore, let some prolongation and restriction

$$p\colon\ U_{l-1} \to U_l, \quad r\colon\ F_l \to F_{l-1}, \quad l=2, \ldots, l_{\max} \tag{1.1.5}$$

between the spaces of grid-functions be given. Then the iteration matrix of a two-grid method on the grids Ω_l and Ω_{l-1} with ν_1 pre- and ν_2 post-smoothing steps is given by

$$T_{2,l}(\nu_1, \nu_2) = S^{\nu_2}(I_l - p_l (K_{l-1})^{-1} r_l K_l) S^{\nu_1}. \tag{1.1.6}$$

For a detailed introduction into multi-grid technique and algorithms see [13].

1.1.2 Convergence Theory. Our theory is a generalization of Hackbusch's one given in [13]. For a detailed introduction see there.

Let U_l be the space of grid-functions u_l from (1.1.4a) with the norm $\|.\|_{U_l}$ and F_l the space of grid-functions f_l from (1.1.4a) with the norm $\|.\|_{F_l}$. For a scalar differential operator K it is sufficient to choose $\|.\|_{U_l}=\|.\|_{F_l}=\|.\|_0$, where

(1.1.7) $$\|u_l\|_0=(c^2 \sum_{x\in\Omega_l} |u_l(x)|^2)^{1/2}$$

is the Euclidean norm with a suitable scaling factor c, say $c=h$.

The situation becomes more complicated, if K represents a system of partial differential equations

(1.1.8a) $$K=\begin{pmatrix} K_{11} & \dots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{n1} & \dots & K_{nn} \end{pmatrix}$$

with

(1.1.8b) $$K_{ij}=K_{ij}(D)=\sum_{|\alpha|\leq k_{ij}} c_\alpha D^\alpha$$

where α is a multi-index, $D=\partial/\partial x$. Let $m_1, \dots, m_n, m'_1, \dots, m'_n$ be numbers with

(1.1.9a) $$2m=\sum_{i=1}^{n} (m_i+m'_i)$$

so that

(1.1.9b) $$k_{ij}\leq m_i+m'_j$$

(cf. [1]). By virtue of these numbers we introduce the following norms for the discrete spaces:

(1.1.10a) $$\|u_l\|^2_{U_l}=\sum_{i=1}^{n} h_l^{-2m_i}\|u_{l,i}\|^2_0$$

and

(1.1.10b) $$\|f_l\|^2_{F_l}=\sum_{i=1}^{n} h_l^{-2m'_i}\|f_{l,i}\|^2_0$$

where u_i refers to the i-th block of u corresponding to (1.1.8a).

The discrete spaces can be written as

(1.1.10c) $$U_l=X_l^{m_1,\dots,m_n}=X_l^{m_1}x\dots xX_l^{m_n}$$

and

(1.1.10d) $$F_l=Y_l^{-m'_1,\dots,-m'_1}=Y^{-m'_1}x\dots xY_l^{-m'_n}$$

with spaces of grid-functions X_l^μ and Y_l^μ, μ denoting the order of differentiability of the corresponding continuous spaces (cf. [12]).

For a positive-definite and symmetric operator K_l of order $2m$

$$|u_l|_s = \| K_l^{s/2m} u_l \|_0 \tag{1.1.11}$$

defines a norm which is independently of h equivalent to the discrete Sobolev norms for spaces of grid functions as introduced in [15], provided $-m \leqq s \leqq m$ and the region Ω_l satisfies "property C" from [14].

The following theorem providing sufficient conditions for the convergence of the two-grid method $T_{2,l}$ is due to Hackbusch (cf. [13]).

Theorem 1.1.1. *Let S_l satisfy the smoothing property for K_l,* i.e. *there exist $\eta(\nu)$ and $\nu'(h)$ so that*

$$\| K_l S_l^\nu \|_{F \leftarrow U} \leqq \eta(\nu) \quad \forall \nu\colon\ 1 \leqq \nu \leqq \nu'(h_l), \quad l \geqq 2,$$

$$\eta(\nu) \to 0 \quad \textit{for } \nu \to \infty, \quad \nu'(h) = \infty \quad \textit{or} \quad \nu'(h) \to \infty \quad \textit{for } h \to 0, \tag{1.1.12}$$

and let K_l fulfill the approximation-property:

$$\exists C_A \to 0, \quad \textit{independent of } h \textit{ so that}$$

$$\| K_l^{-1} - p(K_{l-1})^{-1} r \|_{U \leftarrow F} C_A, \quad \forall l \geqq 2, \tag{1.1.13}$$

then there exist $\underline{h}$ and $\underline{\nu} \in N$:

$$\| T_{2,l}(\nu, 0) \|_{U \leftarrow U} \leqq C_A \, \eta(\nu) < 1 \tag{1.1.14}$$

holds for ν with $\nu'(h_l) \geqq \nu \geqq \underline{\nu}(h_l)$ and $h_2 \leqq \underline{h}$ and the two-grid method $T_{2,l}$ from (1.1.6) *converges monotonically, independently of h.*

Proof. Follows immediately by $T_{2,l}(\nu, 0) = (K_l^{-1} - p(K_{l-1})^{-1} r)(K_l S_l^\nu)$.

Remark 1.1.2. The norm of $T_{2,l}(\nu_1, \nu_2)$ is estimated similarly (cf. [13]). Under the additional assumptions

$$\| S_l^\nu \|_{U_l \leftarrow U_l} \leqq C_s \quad \forall l \geqq 1, \quad 0 < \nu < \nu'(h_l) \tag{1.1.15}$$

and

$$\underline{C}_p^{-1} \| u_{l-1} \|_{U_{l-1}} \leqq \| p u_{l-1} \|_{U_l} \leqq \bar{C}_p \| u_{l-1} \|_{U_{l-1}} \quad \forall u_{l-1} \in U_{l-1}, \quad l \geqq 1, \tag{1.1.16}$$

which are usually satisfied, the smoothing-property (1.1.12) and the approximation-property (1.1.13) yield the h-independent convergence of the corresponding multi-grid method with W-cycle too.

Proof. See [13], Theorem 7.1.2.

Remark 1.1.3. The smoothing-property allows a stability argument, i.e. it carries over to problems perturbed by lower order terms (cf. [13], crit. 6.2.8).

1.2 Model Problems

We consider the following saddle-point problem:

$$\text{(1.2.1)} \qquad \left.\begin{aligned} A_l u_l + C_l p_l &= f_1 \\ B_l u_l &= f_2 \end{aligned}\right\} \quad \text{in } \Omega_l \subset \mathbf{R}^d$$
$$u_l = u_R \quad \text{on } \partial\Omega_l$$

with

$$\text{(1.2.2a)} \qquad A_l\colon X_{l,1}^{t+\frac{\alpha}{2}} \to Y_{l,1}^{t-\frac{\alpha}{2}}, \quad \text{bounded,}$$

$$\text{(1.2.2b)} \qquad B_l\colon X_{l,1}^{t+\frac{\alpha}{2}} \to Y_{l,2}^{t-\beta+\frac{\alpha}{2}}, \quad \text{bounded,}$$

$$\text{(1.2.2c)} \qquad C_l\colon X_{l,2}^{t+\beta-\frac{\alpha}{2}} \to Y_{l,1}^{t-\frac{\alpha}{2}}, \quad \text{bounded,}$$

for some $t \in \mathbf{R}$ and X_l^μ and Y_l^μ from (1.1.10). Let further A_l be invertible with bounded inverse and by means of A_l^{-1} we define

$$\text{(1.2.3)} \qquad E_l = B_l A_l^{-1} C_l\colon X_{l,2}^{t+\beta-\alpha/2} \to Y_{l,2}^{t-\beta+\alpha/2},$$

and require E_l to be bounded and to have a bounded inverse. Then K_l^{-1} exists and is bounded. For a precise discussion of regularity cf. [12] and [6]. Such a problem is given by the Stokes and linearized incompressible Navier-Stokes equations in primitive variables as described below, or by the mixed formulation of the biharmonic equation (cf. [12]).

The incompressible Navier-Stokes equations describing the motion of a viscous, incompressible fluid with the Reynolds-number Re, the velocity $u = (u_1, \ldots, u_d)^T$, the pressure p, under the outer force $f = (f_1, \ldots, f_d)^T$, inside a region $\Omega \subset \mathbf{R}^d$ are given by

$$\left.\begin{aligned} &\text{(1.2.4a)} \qquad -\Delta u + \mathrm{Re}(u \cdot \nabla) u + \nabla p = f \\ &\text{(1.2.4b)} \qquad \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \Omega$$

and the usual Dirichlet boundary conditions for fixed walls

$$\text{(1.2.4c)} \qquad u = u_R \quad \text{on } \partial\Omega.$$

For strongly viscous flows the Reynolds-number is very low. Thus we can neglect the nonlinear convective term $\mathrm{Re}(u \cdot \nabla) u$, and get the linear Stokes equations

$$\text{(1.2.5a)} \qquad -\Delta u + \nabla p = f \quad \text{in } \Omega,$$
$$\text{(1.2.5b)} \qquad \operatorname{div} u = 0$$
$$\text{(1.2.5c)} \qquad u = u_R \quad \text{on } \partial\Omega.$$

To get a discrete problem of form (1.2.1), we have to apply a suitable discretization process yielding stable discrete operators. That is, we replace the continuous Sobolev-spaces, on which the operators from (1.2.4) and (1.2.5) are defined, by discrete spaces. This can be done by finite difference techniques as well as by finite element ones. Especially we have to take care that Brezzi's condition (cf. [6, 12]) is fulfilled, in order to achieve stability of the discretisation. Examples of such discretizations can be found in [2, 11, 16, 18, 27].

Of course, linearizing (1.2.4) in the usual way also yields a problem of form (1.2.1). These two problems are the main applications for our theory.

2 General Theory

2.1 *Transforming Smoothers*

If K_l from (1.1.3) is regular, but indefinite, the construction of appropriate smoothers is not obvious. To split K_l in the sense of Varga, [28], we construct nonsingular matrices $\bar{K}_l$ and $\bar{\bar{K}}_l$, called *r-* or *l-transformation* respectively, so that a splitting of

$$\bar{\bar{K}}_l K_l \bar{K}_l = M - N \tag{2.1.1}$$

is reasonable. As $\bar{K}_l$ and $\bar{\bar{K}}_l$ are nonsingular, (2.1.1) gives rise to a splitting of K_l

$$K_l = \bar{\bar{K}}_l^{-1} M \bar{K}_l^{-1} - \bar{\bar{K}}_l^{-1} N \bar{K}_l^{-1}. \tag{2.1.2}$$

The corresponding iteration, called "*transforming iteration*", is given by:

Let an arbitrary starting guess $u_l^{(0)}$ be given. Then the $i+1^{\text{st}}$ iterate is calculated from the i^{th} one via

$$u_l^{(i+1)} = u_l^{(i)} - \bar{K}_l M^{-1} \bar{\bar{K}}_l (K_l u_l^{(i)} - f_l). \tag{2.1.3}$$

If $\bar{\bar{K}}_l = I$, we speak of *r-transforming* iteration. R-transforming iterations are a generalization of the squared methods by Kaczmarz, [19], and Cimmino, [7], and of the distributive relaxations by Brandt and Dinar, [4, 5]. They are of special interest to construct smoothers for indefinite systems, as the Navier-Stokes equations (cf. [31]) and the shallow-water equations (cf. [22]). So, widely used iteration-schemes, such as the SIMPLE-family, e.g. SIMPLE (see [23]), SIMPLER, SIMPLEST, PCS, PISO, etc. (see [21, 24]), and distributive relaxations like DGS (see [5]) and PGA (see [9, 10]) are classified in [30] and [31] as r-transforming iterations, thus providing a platform for theoretical investigations about the properties of those methods.

In practical applications they are mostly applied as perturbed transforming smoothers, leaving out some inconvenient terms in the product system. I.e. for splitting (2.1.1) a simplified system $\bar{\bar{K}}_l K_l \bar{K}_l$ is used, while the transformations are replaced by $\tilde{\bar{\bar{K}}}_l'$ and $\tilde{\bar{K}}_l$ resp., resulting in the iteration:

Let an arbitrary starting guess $u_l^{(0)}$ be given. Then the $i+1^{\text{st}}$ iterate is calculated from the i^{th} one via

$$u_l^{(i+1)} = u_l^{(i)} - \tilde{\bar{K}}_l M^{-1} \tilde{\bar{\bar{K}}}_l' (K_l u_l^{(i)} - f_l). \tag{2.1.4}$$

Remark 2.1.1. The r-transformation is a mapping

$$\bar{K}_l:\ \bar{U}_l \to U_l \tag{2.1.5}$$

with

(2.1.6) $$\bar{U}_l = X_l^{\bar{m}_1} \times \ldots \times X_l^{\bar{m}_n}.$$

The operator of the transforming method (2.1.3) is given by

(2.1.7a) $$S_{t,l} = \bar{K}_l S_l \bar{K}_l^{-1}$$

with the "product-iteration" operator

(2.1.7b) $$S_l = M^{-1} N$$

which acts directly on the product system. The one of the perturbed r-transforming method reads

(2.1.8) $$\tilde{S}_{t,l} = I_l - \tilde{K}_l M^{-1} K_l.$$

Proof. Follows immediately from (2.1.3) and (2.1.4). qed

Concerning the convergence of perturbed methods, we quote from [31].

Remark 2.1.2. If iteration (2.1.4) converges, its fixed point is the solution of the original system (1.1.4). This is not possible with the so-called product iteration (2.1.7b) which is applied directly to the product system.

Proof. Immediate, as the defect in (2.1.4) is taken w.r.t. the original equation. qed

In the present paper we concentrate on the analysis of r-transforming smoothers.

2.2 *The Convergence of a Two-Grid Method Using R-Transforming Smoothers*

Let K_l: $U_l \to F_l$ be the discretization of a system of partial differential equations as given in (1.1.8a) and denote the discrete problem by

(2.2.1) $$K_l x_l = b_l, \qquad l = 1, \ldots, l_{\max}.$$

Let further the r-transformation $\bar{K}_l$: $\bar{U}_l \to U_l$ also be a stable discretization of a system of partial differential equations of form (1.1.8a). Using the coefficients m_i, $\bar{m}_i$, m'_i from (1.1.9) and (2.1.6) we define

(2.2.2a) $$\bar{H}_l = \operatorname{blockdiag}\{h_l^{\bar{m}_1} I_l, \ldots, h_l^{\bar{m}_n} I_l\},$$

(2.2.2b) $$H_l = \operatorname{blockdiag}\{h_l^{m_1} I_l, \ldots, h_l^{m_n} I_l\},$$

and

(2.2.2c) $$H'_l = \operatorname{blockdiag}\{h_l^{m'_1} I_l, \ldots, h_l^{m'_n} I_l\}.$$

By virtue of (2.2.2a–c) we introduce the norms

(2.2.3a) $$|||u_l|||_{U_l} = |\bar{H}_l^{-1} \bar{K}_l^{-1} u_l|_0$$

(2.2.3b) $$\| f_l \|_{F_l} = |H'_l f_l|_0$$

(2.2.3c) $$\| u_l \|_{\bar{U}_l} = |\bar{H}_l^{-1} u_l|_0$$

on the corresponding spaces.

Theorem 2.2.1. *Let K_l and $\bar{K}_l$ of form* (1.1.8a) *be stable. Suppose for $l \geqq 2$ and $1 \leqq i, j, k \leqq n$*

(2.2.4) $$|(\bar{K}_l^{-1})_{ik}|_{0 \leftarrow m_k - \bar{m}_i} = C,$$

(2.2.5) $$|(K_l^{-1} - p K_{l-1}^{-1} r)_{kj}|_{t \leftarrow 0} = C h_l^{m'_j + m_k - t},$$

with $t = m_k - \bar{m}_\kappa$, for all $\kappa = 1, \ldots, n$ for which $(\bar{K}_l^{-1})_{\kappa k} \neq 0$,

and

(2.2.6) $$\| K_l \bar{K}_l S_l^\nu \|_{F_l \leftarrow \bar{U}_l} \leqq C \eta(\nu), \quad \text{for } 0 < \nu < \nu'(h), \quad l \geqq 2,$$

with $\eta(\nu)$ and $\nu'(h)$ according to (1.1.12). *Then the two-grid method converges and*

(2.2.7) $$|||T_{2,l}(\nu, 0)|||_{U_l \leftarrow U_l} \leqq C \eta(\nu)$$

holds.

Proof. With $S_{t,l} = \bar{K}_l S_l \bar{K}_l^{-1}$ being the operator of the r-transforming smoother, we have

$$|||T_{2,l}(\nu, 0)|||_{U_l \leftarrow U_l} \leqq |\bar{H}_l^{-1} \bar{K}_l^{-1} (K_l^{-1} - p K_{l-1}^{-1} r) H_l'^{-1}|_{0 \leftarrow 0} |H'_l K_l S_{t,l}^\nu \bar{K}_l \bar{H}_l|_{0 \leftarrow 0}.$$

By virtue of (2.2.4) and (2.2.5)

$$\begin{aligned} &|(\bar{K}_l^{-1} (K_l^{-1} - p K_{l-1}^{-1} r))_{ij}|_{0 \leftarrow 0} \\ &\leqq \sum_{\mu=1}^{n} |(\bar{K}_l^{-1})_{i\mu}|_{0 \leftarrow m_\mu - \bar{m}_i} \\ &\quad \cdot |(K_l^{-1} - p K_{l-1}^{-1} r)_{\mu j}|_{m_\mu - \bar{m}_i \leftarrow 0} \\ &\leqq C h_l^{\bar{m}_i + m'_j} \end{aligned}$$

holds, yielding $|\bar{H}_l^{-1} \bar{K}_l^{-1} (K_l^{-1} - p K_{l-1}^{-1} r) H_l'^{-1}|_{0 \leftarrow 0} \leqq C$.
The second factor is estimated by

$$|H'_l K_l S_{t,l}^\nu \bar{K}_l \bar{H}_l|_{0 \leftarrow 0} = |H'_l K_l \bar{K}_l S_l^\nu \bar{H}_l|_{0 \leftarrow 0} \leqq C \eta(\nu). \qquad \text{qed}$$

Theorem 2.2.1 is a generalization of Hackbusch's splitting approach, theorem 1.1.1. Instead of a smoothing property of the r-transforming smoother itself, we require only smoothing property (2.2.6) of the product iteration. (2.2.4) is the discrete regularity as investigated by Hackbusch in [14, 15]. (2.2.5) is the approximation property in weak norms. To satisfy (2.2.5), the interpolations and the consistency of the discretization have to meet the order of the product system, which is usually higher than the one needed by the original system. This problem can be avoided by

Remark 2.2.2. Let the coarse-grid matrix K_{l-1} be computed by

(2.2.8) $$K_{l-1}=rK_l\,p,$$

the so-called Galerkin-ansatz. Then only the order of p and $r=p^*$ have to fit for the product-system. No additional consistency assumption is needed.

Proof. see [13], Chapter 6.3. qed

Concerning perturbed r-transforming smoothers as mentioned above, we give the following results

Lemma 2.2.3. *Consider the perturbed r-transforming smoother from* (2.1.8). *Let*

(2.2.9 a) $$P_l=K_l\bar{K}_l-K_l\tilde{K}_l,$$

(2.2.9 b) $$S_l'=I-M_l^{-1}K_l\bar{K}_l,$$

(2.2.9 c) $$S_l''=M_l^{-1}P_l,$$

(2.2.9 d) $$S_l=S_l'+S_l'',$$

and

(2.2.9 e) $$\tilde{K}_l\colon\ \bar{U}_l\to U_l.$$

Let the simplified problem $K_l\bar{K}_l$ satisfy the smoothing property in the norm induced by (2.2.3 b, c)

(2.2.10) $$\|K_l\bar{K}_l\,S_l'^{\nu}\|_{F_l\leftarrow\bar{U}_l}\leqq C_G\,\eta'(\nu)$$

for $1\leqq\nu\leqq\nu'(h_l)$, with $\eta'(\nu)$ and $\nu'(h_l)$ according to (1.1.12). *Let further*

(2.2.11) $$\|S_l'\|_{\bar{U}_l\leftarrow\bar{U}_l}\leqq C_l,\qquad l\geqq 2,$$

and

(2.2.12) $$\lim_{l\to\infty}\|P_l\|_{F_l\leftarrow\bar{U}_l}=0,$$

(2.2.13) $$\lim_{l\to\infty}\|S_l''\|_{\bar{U}_l\leftarrow\bar{U}_l}=0,$$

provided $h_l\to 0$ as $l\to\infty$. If additionally

(2.2.14) $$\|\bar{K}_l^{-1}\|_{U_l\leftarrow\bar{U}_l}\leqq C,\qquad l\geqq 2,$$

holds, then the smoothing property for the perturbed product system

(2.2.15) $$\|K_l\tilde{K}_l\,S_l^{\nu}\|_{F_l\leftarrow\bar{U}_l}\leqq C_G\,\eta(\nu),$$
$$\text{for }0<h_l\leqq h_0(\varepsilon)\quad\text{and}\quad\eta(\nu)=(1+\varepsilon)\,\eta'(\nu),$$
$$\text{for }1\leqq\nu\leqq\nu'(h),\qquad\varepsilon>0,\qquad\text{arbitrary},$$

is satisfied.

Proof. According to the definition of $\tilde{S}_{t,l}$ in (2.1.8) we have

$$\begin{aligned}\tilde{S}_{t,l} &= I - \tilde{K}_l M_l^{-1} K_l = \tilde{K}_l (I - M_l^{-1} K_l \tilde{K}_l) \tilde{K}_l^{-1} \\ &= \tilde{K}_l (I - M_l^{-1} K_l \bar{K}_l + M_l^{-1} P_l) \tilde{K}_l^{-1} \\ &= \tilde{K}_l (S_l' + S_l'') \tilde{K}_l^{-1}\end{aligned}$$

and

$$\| K_l \tilde{K}_l S_l^{\nu} \|_{F_l \leftarrow \bar{U}_l} = \| (K_l \bar{K}_l - P_l)(S_l' + S_l'')^{\nu} \|_{F_l \leftarrow \bar{U}_l} < C_G \eta(\nu)$$

follows from [13], criterion 6.2.8. qed

Theorem 2.2.4. *Let K_l and $\tilde{K}_l$: $\bar{U}_l \to U_l$ be of form* (1.1.8a). *Further let for $l \geqq 2$ and $1 \leqq i, j \leqq n$*

(2.2.16) $$|(\tilde{K}_l^{-1})_{ij}|_{0 \leftarrow m_j - \bar{m}_i} \leqq C.$$

In addition, let (2.2.5) *and* (2.2.15) *hold. Then the two-grid method converges and*

(2.2.17) $$|||T_{2,l}(\nu, 0)|||_{U_l \leftarrow U_l} \leqq C \eta(\nu)$$
with $\eta(\nu)$ from (2.2.15),

where

(2.2.18) $$|||u_l|||_{U_l} = \| \bar{H}_l^{-1} \tilde{K}_l^{-1} u_l \|.$$

Proof. Similar to the one of Theorem 2.2.1. qed

Theorem 2.2.4 and Lemma 2.2.3 apply to perturbations of lower order (see [13, 30]).

2.3 Product-Systems of Block-Triangular Form

In order to give some advice how to construct an appropriate r-transformation for a given system of pde we discuss the properties of block-triangular systems. This form turns out to be favourable for a product system.

Lemma 2.3.1. *Let R: $U_l \to F_l$ be a discrete operator with U_l and F_l from* (1.1.10). *Let R be of the form*

(2.3.1a) $$R = \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}$$

and assume with $\alpha, \beta > 0$, $\gamma \geqq 0$ in appropriate weak norms

(2.3.1b) $|R_{11}|_{-\alpha \leftarrow 0} \leqq C_1$,

(2.3.1c) $|R_{21}|_{-s \leftarrow t} \leqq C_2 h^{-\beta + t + s}$, *for* $t \in [0, \beta]$, $s \in [0, \beta - t]$,

and

(2.3.1d) $|R_{22}|_{0 \leftarrow 0} \leqq C_3 h^{-\gamma}$.

Let R_{11} satisfy the "discrete regularity"

(2.3.2) $$|R_{11}^{-1}|_{\alpha-\kappa\leftarrow-\kappa}\leqq C_4 \quad \textit{for } \kappa \textit{ from } (2.3.5\,\text{a}).$$

Further, split R into

(2.3.3 a) $$R=M-N, \qquad M \textit{ regular},$$

and

(2.4.3 b) $$S=M^{-1}N$$

with

(2.3.3 c) $$S=\begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22}\end{pmatrix}$$

and a similar blocking of M and N. Suppose these blocks satisfy

(2.3.4 a) $$|M_{11}^{-1}|_{0\leftarrow 0}\leqq C_5\, h^{\alpha},$$

(2.3.4 b) $$|M_{22}^{-1}|_{s\leftarrow s}\leqq C_6\, h^{\gamma}, \qquad s\in\{0,\lambda_0\}, \qquad \lambda_0 \textit{ from } (2.3.5\,\text{b})$$

and

(2.3.4 c) $$|M_{21}|_{0\leftarrow t}\leqq C_7\, h^{-\beta+t}, \qquad \textit{for } t\in[0,\beta].$$

Let the smoothing properties

(2.3.5 a) $$|R_{11}\,S_{11}^{\nu}|_{-\kappa\leftarrow 0}\leqq C_9\,\eta_{11}^{\vartheta}(\nu)\,h^{\kappa-\alpha}, \qquad \textit{for } 0\leqq\nu\leqq\nu_{11}'(h),$$
$$\textit{for some } \kappa\in[0,\alpha) \quad \textit{and} \quad \vartheta=1-\kappa/\alpha \quad \textit{and for } \kappa=0,$$

(2.3.5 b) $$|R_{22}\,S_{22}^{\nu}|_{0\leftarrow\lambda}\leqq C_{10}\,h^{\zeta}\eta_{22}^{\tau}(\nu), \qquad \textit{for } 0<\nu\leqq\nu_{22}'(h),$$
with suitable $\eta_{ii}(\nu)$ and $\nu_{ii}'(h)$ according to (1.1.12),
$$\lambda\in\{0,\lambda_0\}, \qquad \lambda_0:=\begin{cases}\max\{0,\alpha-\beta-\kappa\} & \textit{for } \gamma>0\\ 0 & \textit{for } \gamma=0\end{cases}, \qquad \zeta:=\lambda-\gamma,$$
$$\tau=\begin{matrix} 1-\dfrac{\lambda}{\gamma} & \textit{for } \gamma>0\\ 1 & \textit{for } \gamma=0\end{matrix}$$

be satisfied.

Let further $\eta_{ii}(\nu)$, $i=1,2$, from (2.3.5) *satisfy*

(2.3.6) $$\eta_{21}(\nu):=\eta_{11}^{\vartheta}(\nu) + \sum_{\mu=0}^{\nu-1}\eta_{22}^{\tau}(\mu)\,(\eta_{11}^{\vartheta}(\nu-\mu-1)+\eta_{22}^{1-\tau}(\mu)\,\eta_{11}^{\vartheta}(\nu-\mu-1))\to 0$$
$$\textit{for } \nu\to\infty, \vartheta, \tau \quad \textit{from } (2.3.5\,\text{b}).$$

Then there exists $\nu'_{21}(h)$ *according to* (1.1.12) *so that*

(2.3.7) $$|(RS^{\nu})_{21}|_{0\leftarrow 0} \leqq C_{11}\, h^{-\beta}\, \eta_{21}(\nu), \quad \text{for } 0<\nu\leqq \nu'_{21}(h),$$
for $0<h<\bar{h}$ *with* $\eta_{21}(\nu)$ *from* (2.3.6).

holds.

Proof. We have

(2.3.8) $$M^{-1} = \begin{pmatrix} M_{11}^{-1} & 0 \\ -(M_{22}^{-1}\, M_{21}\, M_{11}^{-1}) & M_{22}^{-1} \end{pmatrix}$$

and

(2.3.9a) $$S_{11} = M_{11}^{-1}\, N_{11},$$
(2.3.9b) $$S_{21} = M_{22}^{-1}\, N_{22},$$
(2.3.9c) $$S_{21} = M_{22}^{-1}\, N_{21} - M_{22}^{-1}\, M_{21}\, M_{11}^{-1}\, N_{11}.$$

Thus we obtain for S^{ν}, $\nu \in \mathbf{N}$,

(2.3.10a) $$S^{\nu} = \begin{pmatrix} S_{11}^{\nu} & 0 \\ S_{21}(\nu) & S_{22}^{\nu} \end{pmatrix},$$

with

(2.3.10b) $$S_{21}(\nu) = \sum_{\mu=0}^{\nu-1} S_{22}^{\mu}\, S_{21}\, S_{11}^{\nu-\mu-1}.$$

By virtue of (2.3.10) we have

(2.3.11a) $$(RS^{\nu})_{ii} = R_{ii}\, S_{ii}^{\nu}, \quad \text{for } i \in \{1, 2\},$$
and
(2.3.11b) $$\begin{aligned}(RS^{\nu})_{21} &= R_{21}\, S_{11}^{\nu} + R_{22}\, S_{21}(\nu) \\ &= R_{21}\, S_{11}^{\nu} + R_{22} \sum_{\mu=0}^{\nu-1} S_{22}^{\mu}\, S_{21}\, S_{11}^{\nu-\mu-1}.\end{aligned}$$

With $\sigma = \alpha - \kappa$ the estimates

(2.3.12) $$\begin{aligned}|R_{21}\, S_{11}^{\nu}|_{0\leftarrow 0} &\leqq |R_{21}|_{0\leftarrow \sigma}\, |R_{11}^{-1}|_{\sigma\leftarrow -\kappa}\, |R_{11}\, S_{11}^{\nu}|_{-\kappa\leftarrow 0} \\ &\leqq C\, h^{-\beta}\, \eta_{11}^{\vartheta}(\nu)\end{aligned}$$

and

$$(2.3.13)\quad |R_{22}S_{21}(\nu)|_{0\leftarrow 0}=\left|R_{22}\sum_{\mu=0}^{\nu-1}S_{22}^{\mu}S_{21}S_{11}^{\nu-\mu-1}\right|_{0\leftarrow 0}$$

$$\leq\sum_{\mu=0}^{\nu-1}(|R_{22}S_{22}^{\mu}M_{22}^{-1}R_{21}S_{11}^{\nu-\mu-1}|_{0\leftarrow 0}$$

$$+|R_{22}S_{22}^{\mu}M_{22}^{-1}M_{21}M_{11}^{-1}R_{11}S_{11}^{\nu-\mu-1}|_{0\leftarrow 0})$$

are satisfied. The first term can be estimated as follows

$$|R_{22}S_{22}^{\mu}M_{22}^{-1}R_{21}S_{11}^{\nu-\mu-1}|_{0\leftarrow 0}$$
$$\leq|R_{22}S_{22}^{\mu}|_{0\leftarrow\lambda}|M_{22}^{-1}|_{\lambda\leftarrow\lambda}|R_{21}|_{\lambda\leftarrow\lambda+\beta}|R_{11}^{-1}|_{\lambda+\beta\leftarrow-\kappa}|R_{11}S_{11}^{\nu-\mu-1}|_{-\kappa\leftarrow 0}$$
$$\leq Ch^{-\beta}\eta_{22}^{\tau}(\mu)\eta_{11}^{\vartheta}(\nu-\mu-1).$$

Analogously we obtain for the second term in (2.3.13)

$$|R_{22}S_{22}^{\mu}M_{22}^{-1}M_{21}M_{11}^{-1}R_{11}S_{11}^{\nu-\mu-1}|_{0\leftarrow 0}\leq Ch^{-\beta}\eta_{22}(\mu)\eta_{11}(\nu-\mu-1),$$

and thus by virtue of (2.3.12)

$$|(RS^{\nu})_{21}|_{0\leftarrow 0}\leq Ch^{-\beta}\left(\eta_{11}^{\vartheta}(\nu)+\sum_{\mu=0}^{\nu-1}\eta_{22}^{\tau}(\mu)(\eta_{11}^{\vartheta}(\nu-\mu-1)\right.$$
$$\left.+\eta_{22}^{1-\tau}(\mu)\eta_{11}(\nu-\mu-1))\right)=Ch^{-\beta}\eta_{21}(\nu).$$

Usually, common smoothers satisfy the assumptions on M and N. Smoothing properties (2.3.5) and assumption (2.3.6) are discussed below.

Lemma 2.3.2. *Let a system of form* (2.3.1 a) *and a corresponding smoother satisfy* (2.3.1–4). *Let further smoothing properties* (2.3.5 a, b) *hold for the diagonal-blocks with*

$$(2.3.14)\qquad \eta_{ii}(\nu)\leq C\cdot\eta_0(\nu):=C\frac{\nu^{\nu}}{(\nu+1)^{\nu+1}},\quad \textit{for}\ \ \nu\geq\nu_0\in\mathbf{N}.$$

Then (2.3.7) *is valid with*

$$(2.3.15)\qquad \eta_{21}(\nu)\leq\begin{cases}C\nu^{q}, & \text{if } \vartheta,\quad \tau<1\\ C\nu^{q}\cdot\ln\nu, & \text{if } \vartheta\ \text{ or }\ \tau=1\end{cases}.$$
$$q=\max\{1-\vartheta-\tau,\ -\vartheta,\ -\tau\}$$

Proof. Because of (2.3.14) we have

$$(2.3.16)\qquad \eta_{21}(\nu)\leq C\sum_{\mu=0}^{\nu-1}\eta_0^{\tau}(\mu)(\eta_0^{\vartheta}(\nu-\mu-1)+\eta_0^{1-\tau}(\mu)\eta_0(\nu-\mu-1))$$
$$\leq C\sum_{\mu=0}^{\nu-1}\frac{1}{(\mu+1)^{\tau}}\left(\frac{1}{(\mu+1)^{1-\tau}(\nu-\mu)}+\frac{1}{(\nu-\mu)^{\vartheta}}\right).$$

By virtue of Euler's sum-formula (cf. [8]) we obtain

$$\text{(2.3.17)}\qquad \sum_{\mu=0}^{\nu-1} \frac{1}{(\mu+1)^{\tau}(\nu-\mu)^{\vartheta}} = \begin{cases} O(\nu^{q}), & \text{for } \vartheta, \tau < 1 \\ O(\nu^{q} \cdot \ln \nu), & \text{if } \vartheta = 1 \quad \text{or} \quad \tau = 1 \end{cases}. \qquad \text{qed}$$

Smoothing properties (2.3.5) in weak norms can be concluded using interpolation as given in the following lemma (see also [17] and [30]).

Lemma 2.3.3. *Let K_l be the discretization of a scalar differential operator of order $2m$ and let*

$$\text{(2.3.18)}\qquad |K_l S^{\nu}|_{0 \leftarrow 0} \leqq C h^{-2m} \eta(\nu)$$

with $\eta(\nu)$ according to (1.1.12) *hold in the spectral norm. Suppose further*

$$\text{(2.3.19)}\qquad |M|_{0 \leftarrow 0} \leqq C h^{-2m},$$

$$\text{(2.3.20)}\qquad |M^{-1}|_{0 \leftarrow 0} \leqq C h^{2m},$$

$$\text{(2.3.21)}\qquad |S^{\nu}|_{0 \leftarrow 0} \leqq C,$$

$$\text{(2.3.22)}\qquad |K_l|_{0 \leftarrow 2m} \leqq C,$$

and

$$\text{(2.3.22')}\qquad |K_l^T|_{0 \leftarrow 2m} \leqq C.$$

Then

$$\text{(2.3.23)}\qquad |K_l S^{\nu}|_{-\sigma \leftarrow \tau} \leqq C \eta(\nu)^{1-\frac{\sigma+\tau}{2m}} h^{\sigma+\tau-2m},$$

$$\textit{for } 0 \leqq \sigma, \tau \quad \textit{and } \sigma+\tau \leqq 2m.$$

Proof. The proof follows [17], crit. 4.3.*. Denote the spectral norm by $\|\,.\,\|$. We have

$$\begin{aligned} |K_l S^{\nu}|_{0 \leftarrow 2m} &\leqq \|K_l S^{\nu} K_l^{-1}\| \, |K_l|_{0 \leftarrow 2m} \leqq C \, \|(K_l S K_l^{-1})^{\nu}\| = C \, \|(N M^{-1})^{\nu}\| \\ &\leqq \operatorname{cond}_2(M) \cdot C \cdot \|S^{\nu}\| \leqq C. \end{aligned}$$

Analogously $|K_l S^{\nu}|_{-2m \leftarrow 0} \leqq |K_l|_{-2m \leftarrow 0} \|S^{\nu}\| \leqq C$ as $|K_l^T|_{0 \leftarrow 2m} = |K_l|_{-2m \leftarrow 0}$. Interpolation (see [20, 13, 30]) according to [13], lemma 6.2.6, yields for σ, $\tau \in [0, m)$

$$|K_l S^{\nu}|_{-\sigma \leftarrow \tau} \leqq |K_l S^{\nu}|_{0 \leftarrow 0}^{1-\frac{\sigma+\tau}{2m}} |K_l S^{\nu}|_{0 \leftarrow 2m}^{\frac{\sigma}{2m}} |K_l S^{\nu}|_{-2m \leftarrow 0}^{\frac{\tau}{2m}}$$

and thus the proposition. qed

Collecting the results of Lemmata 2.3.1, 2.3.2, and 2.3.3 we state

Theorem 2.3.4. *Consider a system of form* (2.3.1 a) *and a smoother satisfying* (2.3.1)–(2.3.4) *as well as the smoothing-property for the diagonal-blocks* (2.3.5 a, b) *with $\eta_{ii}(\nu) = O(1/\nu)$, $i = 1, 2$, and $\kappa = \lambda = 0$. Let futher* (2.3.19–22) *be satisfied for the* *diagonal blocks* $K_{l,ii}$, M_{ii}, *and* S_{ii}, $i=1,2$. *Then the smoothing-property holds for the complete system in the norms* (1.1.10a, b), *with*

(2.3.24) $$m_1=m_1'=\frac{\alpha}{2},\quad m_2=\beta-\frac{\alpha}{2},\quad m_2'=\gamma+\frac{\alpha}{2}-\beta,$$

provided $\gamma+\alpha\geqq 2\beta$, *as well as with*

(2.3.25) $$m_1=m_1'=\frac{\alpha}{2},\quad m_2'=\beta-\frac{\alpha}{2},\quad m_2=\gamma+\frac{\alpha}{2}-\beta,$$

and arbitrary α, β, *and* $\gamma\geqq 0$.

Proof. Follows immediately from Lemmata 2.3.1, 2.3.2 and 2.3.3. qed

Hence we recommend to construct the r-transformation so that the product-system is of block-triangular form up to terms of lower order and has reasonable diagonal blocks.

The above result was proven without assumptions on the order of the different blocks. If $\beta<\chi:=\min\{\alpha,\gamma\}$ we can consider R_{21} to be a term of lower order, however, consequently we have to require an uniform approximation-property of order χ. Thus in connection with r-transforming smoothing the above approach is better adapted to our problem as can be seen from theorem 3.1.3.

By virtue of theorem 2.3.4 we can easily give a convergence criterion for a two-grid method for a general block-triangular system. The corresponding approximation-property is discussed in the following proposition.

Proposition 2.3.5. *Let* R_l *be of form* (2.3.1a). *Denote* R_l^{-1} *by*

(2.3.26) $$R_l^{-1}=Z=\begin{pmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22}\end{pmatrix}$$

and let p and r have block-diagonal form. Let further

(2.3.27) $$0\leqq\beta\leqq\min\{\alpha,\gamma\}$$

and

(2.3.28) $$|r_1|_{0\leftarrow 0}<C,$$

(2.3.29a) $$|p_1|_{\beta\leftarrow\beta}<C,$$

(2.3.29b) $$|p_2|_{0\leftarrow 0}<C,$$

(2.3.30) $$|Z_{11}|_{t\leftarrow 0}<C,\quad \textit{for } t\in\{\alpha,\beta\},$$

(2.3.31) $$|Z_{22}|_{0\leftarrow -\gamma}<C,$$

(2.3.32) $$|R_{l,21}|_{s\leftarrow t}<C,\quad \textit{for } (s,t)\in\{(0,\beta),(-\gamma,0)\},\quad l\geqq 1,$$

(2.3.33) $$|R_{l-1,21}-r_2\,R_{l,21}\,p_1|_{-\gamma\leftarrow\alpha}<Ch^{\beta},\quad \textit{for } l\geqq 2,$$

(2.3.34a) $$|Z_{l,11}-p_1\,Z_{l-1,11}\,r_1|_{0\leftarrow 0}<Ch^{\alpha},\quad \textit{for } l\geqq 2,$$

(2.3.34b) $$|Z_{l,22}-p_2\,Z_{l-1,22}\,r_2|_{0\leftarrow 0}<Ch^{\gamma},\quad \textit{for } l\geqq 2.$$

Then the approximation-property for R_{21} holds:

(2.3.35) $$|Z_{l,21}-p_2 Z_{l-1,21} r_1|_{0\leftarrow 0}<C h^{\beta}, \quad \forall l \geqq 2.$$

Proof. The proof is similar to [13], Prop. 11.2.7. We have $Z_{11}=R_{l,11}^{-1}$, $Z_{21}=-R_{l,22}^{-1} R_{l,21} R_{l,11}^{-1}$, and $Z_{22}=R_{l,22}^{-1}$. Leaving out the subscript l and marking coarse-grid matrices by a prime, we obtain

$$\begin{aligned}|Z_{21}-p_2 Z'_{21} r_1|_{0\leftarrow 0} &= |p_2 R'^{-1}_{22} R'_{21} R'^{-1}_{11} r_1 - R_{22}^{-1} R_{21} R_{11}^{-1}|_{0\leftarrow 0}\\ &\leqq |p_2 R'^{-1}_{22}(R'_{21}-r_2 R_{21} p_1) R'^{-1}_{11} r_1|_{0\leftarrow 0}\\ &\quad + |(p_2 R'^{-1}_{22} r_2 - R_{22}^{-1}) R_{21} p_1 R'^{-1}_{11} r_1|_{0\leftarrow 0}\\ &\quad + |R_{22}^{-1} R_{21}(p_1 R'^{-1}_{11} r_1 - R_{11}^{-1})|_{0\leftarrow 0} \leqq C h^{\beta},\end{aligned}$$

as by virtue of assumptions (2.3.27–34)

$$\begin{aligned}&(-)|p_2 R'^{-1}_{22}(R'_{21}-r_2 R_{21} p_1) R'^{-1}_{11} r_1|_{0\leftarrow 0}\\ &\quad \leqq |p_2|_{0\leftarrow 0} |R'^{-1}_{22}|_{0\leftarrow -\gamma} |R'_{21}-r_2 R_{21} p_1|_{-\gamma\leftarrow\alpha} |R'^{-1}_{11}|_{\alpha\leftarrow 0} |r_1|_{0\leftarrow 0} \leqq C h^{\beta},\\ &(-)|(p_2 R'^{-1}_{22} r_2 - R_{22}^{-1}) R_{21} p_1 R'^{-1}_{11} r_1|_{0\leftarrow 0}\\ &\quad \leqq |p_2 R'^{-1}_{22} r_2 - R_{22}^{-1}|_{0\leftarrow 0} |R_{21}|_{0\leftarrow\beta} |p_1|_{\beta\leftarrow\beta} |R'^{-1}_{11}|_{\beta\leftarrow 0} |r_1|_{0\leftarrow 0} \leqq C h^{\gamma},\\ &(-)|R_{22}^{-1} R_{21}(p_1 R'^{-1}_{11} r_1 - R_{11}^{-1})|_{0\leftarrow 0}\\ &\quad \leqq |R_{22}^{-1}|_{0\leftarrow -\gamma} |R_{21}|_{-\gamma\leftarrow 0} |p_1 R'^{-1}_{11} r_1 - R_{11}^{-1}|_{0\leftarrow 0} \leqq C h^{\alpha}.\end{aligned}$$

qed

Now we are able to show two-grid convergence for a system of form (2.3.1).

Theorem 2.3.6. *Let the assumptions of Theorem* 2.3.4 *and Proposition* 2.3.5 *hold. Choosing m_i and m'_i from* (2.3.24), *the two-grid method for* (2.3.1 a) *with smoother S from* (2.3.3) *is convergent and we have in the norms from* (1.1.10 a, b)

(2.3.36 a) $$\| T_{2,l}(v,0)\|_{U_l\leftarrow U_l} \leqq C\eta(v), \quad \textit{for } 0<v<v'(h)$$

with

(2.3.36 b) $$\eta(v)=\begin{cases}\eta_{21}(v) \textit{ from } (2.3.6), & \textit{if } \alpha+\gamma=2\beta,\\ (1+\varepsilon)\max\{\eta_{11}(v),\eta_{22}(v)\}, & \textit{if } 2\beta<\alpha+\gamma\end{cases}$$

Proof. Follows immediately from Proposition 2.3.5 and Theorem 2.3.4. qed

3 Results for Stokes- and Navier-Stokes-Equations

3.1 The Linear Case

Consider the saddle-point problem (1.2.1). In [31] we construct r-transformations for (1.2.1) yielding two well-known iterations for (1.2.4), the SIMPLE methods (cf. [3, 21, 23, 24, 25]) and the DGS/TILU method (cf. [5, 30]). As r-transformation for (1.2.1) we take with the notations from (1.2).

(3.1.1 a) $$\bar{K}_l = \begin{pmatrix} I_l & \bar{K}_{l,12} \\ 0 & \bar{K}_{l,22} \end{pmatrix}$$

with

(3.1.1 b) $$\bar{K}_{l,12} = A_l^{-1} C_l E_l^{-1} D_l,$$

(3.1.1 c) $$\bar{K}_{l,22} = -E_l^{-1} D_l,$$

where

(3.1.1 d) $$D_l\colon \bar{X}_{2,l}^{t+\gamma+(\alpha-2\beta)/2} \to X_{2,l}^{t+(\alpha-2\beta)/2}$$

has to be chosen properly, satisfying

(3.1.1 e) $$|D_l|_{0\leftarrow -\gamma} \leqq C.$$

The corresponding product-system reads

(3.1.2) $$K_l \bar{K}_l = \begin{pmatrix} A_l & 0 \\ B_l & D_l \end{pmatrix}.$$

Remark 3.1.1. $\bar{K}_l$ from (3.1.1 a) is regular with

(3.1.3 a) $$\bar{K}_l\colon X_1^{t+\frac{\alpha}{2}} \times \bar{X}_2^{t+\gamma-\beta+\frac{\alpha}{2}} \to X_1^{t+\frac{\alpha}{2}} \times X_2^{t+\beta-\frac{\alpha}{2}}$$

and

(3.1.3 b) $$\bar{K}_l^{-1} = \begin{pmatrix} I_l & A_l^{-1} C_l \\ 0 & -D_l^{-1} E_l \end{pmatrix}.$$

To apply Theorem 2.2.1 we have to ensure approximation properties (2.2.5) for problem (1.2.1), as given in the following lemma

Lemma 3.1.2. *Let a saddle-point problem with K_l from (1.2.1) be given. Let further the restrictions and prolongations p and r have block-diagonal form. Additionally, suppose a block-diagonal restriction r′ to exist, satisfying with $\sigma := 2\beta - \alpha$*

(3.1.4)
$$\begin{aligned}
&|I - p_1 r'_1|_{t\leftarrow s} \leqq C h^{s-t}, \quad \textit{for } (s,t) \in \{(\beta,0), (\beta,\beta-\gamma), (\alpha,0), (\alpha,\beta-\gamma)\},\\
&|I - p_2 r'_2|_{t\leftarrow s} \leqq C h^{s-t}, \quad \textit{for } (s,t) \in \{(\beta,\beta-\alpha), (\beta,\sigma-\gamma)\},\\
&|p_1|_{t\leftarrow t} \leqq C, \quad t \in \{0, \beta-\gamma, \alpha\},\\
&|p_2|_{t\leftarrow t} \leqq C, \quad t \in \{\sigma, \beta, \beta-\alpha, \sigma-\gamma\},\\
&|r_1|_{t\leftarrow t} \leqq C, \quad t \in \{0, -\alpha, \beta-\alpha, \beta-\alpha-\gamma\},\\
&|r_2|_{t\leftarrow t} \leqq C, \quad t \in \{0, -\beta, -\gamma\},\\
&|r'_1|_{t\leftarrow t} \leqq C, \quad t \in \{\beta, \alpha\},\\
&|r'_2|_{t\leftarrow t} \leqq C, \quad t \in \{\sigma, \beta\}.
\end{aligned}$$

Let the blocks of K_l satisfy

(3.1.5)
$$\begin{aligned}
|A_l|_{t-\alpha\leftarrow t} &\leqq C, & t&\in\{0,\beta-\gamma\},\\
|A_l^{-1}|_{t+\alpha\leftarrow t} &\leqq C, & t&\in\{0,-\alpha,\beta-\alpha-\gamma,\beta-\alpha\},\\
|B_l|_{t-\beta\leftarrow t} &\leqq C, & t&\in\{0,\alpha,\beta-\gamma\},\\
|C_l|_{t-\beta\leftarrow t} &\leqq C, & t&\in\{\sigma,\beta,-\beta,\sigma-\gamma,\beta-\alpha\},\\
|E_l^{-1}|_{t+\sigma\leftarrow t} &\leqq C, & t&\in\{0,-\beta,-\gamma,2\sigma-\gamma,\beta-\gamma,\alpha-\beta\}.
\end{aligned}$$

The necessary consistency assumptions are

(3.1.6)
$$\begin{aligned}
&|A_{l-1}-r_1 A_l p_1|_{t\leftarrow s}\leqq C h^{s-t-\alpha},\\
&\qquad \textit{for } (s,t)\in\{(\beta,-\alpha),(\beta,\beta-\alpha-\gamma),(\alpha,-\alpha),(\alpha,\beta-\gamma-\alpha)\},\\
&|B_{l-1}-r_2 B_l p_1|_{t\leftarrow s}\leqq C h^{s-t-\beta},\\
&\qquad \textit{for } (s,t)\in\{(\alpha,-\beta),(\alpha,-\gamma),(\beta,-\beta),(\beta,-\gamma)\},\\
&|C_{l-1}-r_1 C_l p_2|_{t\leftarrow s}\leqq C h^{s-t-\beta},\\
&\qquad \textit{for } (s,t)\in\{(\sigma,-\alpha),(\sigma,\beta-\gamma-\alpha),(\beta,-\alpha),(\beta,\beta-\alpha-\gamma)\}.
\end{aligned}$$

Then

(3.1.7)
$$\begin{aligned}
&|(K_l^{-1})_{11}-p_1(K_{l-1}^{-1})_{11} r_1|_{t\leftarrow 0}\leqq C h^{\alpha-t}, && \textit{for } t\in\{0,\beta-\gamma\},\\
&|(K_l^{-1})_{12}-p_1(K_{l-1}^{-1})_{12} r_2|_{t\leftarrow 0}\leqq C h^{\beta-t}, && \textit{for } t\in\{0,\beta-\gamma\},\\
&|(K_l^{-1})_{21}-p_2(K_{l-1}^{-1})_{21} r_1|_{t\leftarrow 0}\leqq C h^{\beta-t}, && \textit{for } t\in\{\beta-\alpha,\sigma-\gamma),\\
&|(K_l^{-1})_{22}-p_2(K_{l-1}^{-1})_{22} r_2|_{t\leftarrow 0}\leqq C h^{\sigma-t}, && \textit{for } t\in\{\beta-\alpha,\sigma-\gamma).
\end{aligned}$$

Proof. The proof uses the same splittings as Hackbusch's proof of the approximation property for the Stokes-problem, prop. 11.2.7 in [13], and is carried out entirely in parallel. qed

Now we are ready to prove two-grid convergence for (1.2.1).

Theorem 3.1.3. *Let A_l from* (1.2.2) *and D_l from* (3.1.1d, e) *be symmetric and positive definite, let* (2.3.1) *and* (2.3.2) *be satisfied for* (3.1.2). *Let further* (3.1.2) *be decomposed according to* (2.3.3), *and suppose* (2.3.4) *and* (2.3.5a, b) *with $\eta_{ii}(v) = O(1/v)$ to be satisfied. In addition, let* (3.1.4)–(3.1.6) *with $\beta>0$ hold for the original system* (1.2.1). *Then with*

(3.1.8)
$$\bar{m}_1=m_1=m_1'=\frac{\alpha}{2},$$

$$m_2=m_2'=\beta-\frac{\alpha}{2},$$

$$\bar{m}_2=\gamma+\frac{\alpha}{2}-\beta,$$

the two-grid method with r-transforming smoother S_l corresponding to S via $\bar{K}_l$ from (3.1.1) *converges independently of h. More precisely, we have in the norms from* (2.2.3)

$$\text{(3.1.9)} \qquad |||T_{2,l}(v,0)|||_{U_l\leftarrow U_l} \leqq C\eta(v), \quad \textit{with } \eta(v)=\max\{\eta_{11}(v),\eta_{21}(v),\eta_{22}(v)\}, \quad \textit{and } \eta_{21}(v) \textit{ from } (2.3.6).$$

Proof. Because of $\beta>0$ and assumptions (2.3.1)–(2.3.5) Theorem 2.3.4 yields smoothing property (2.2.6) in the norms from (2.2.3) based on (3.1.8). Lemma 3.1.2 yields (3.1.7) corresponding to (2.2.5). It remains to prove (2.2.4). We have

$$|(\bar{K}_l^{-1})_{12}|_{0\leftarrow\beta-\alpha} = |A_l^{-1}C_l|_{0\leftarrow\beta-\alpha} \leqq |A_l^{-1}|_{0\leftarrow-\alpha}|C_l|_{-\alpha\leftarrow\beta-\alpha} \leqq C$$

and by virtue of (3.1.1e) and (3.1.5)

$$\begin{aligned}|(\bar{K}_l^{-1})_{22}|_{0\leftarrow2\beta-\alpha-\gamma} &= |D_l^{-1}E_l|_{0\leftarrow2\beta-\alpha-\gamma} \leqq |D_l^{-1}|_{0\leftarrow-\gamma}|E_l|_{-\gamma\leftarrow2\beta-\alpha-\gamma}\\ &\leqq |D_l^{-1}|_{0\leftarrow-\gamma}|B_l|_{-\gamma\leftarrow\beta-\gamma}|A_l^{-1}|_{\beta-\gamma\leftarrow\beta-\gamma-\alpha}|D_l|_{\beta-\gamma-\alpha\leftarrow2\beta-\gamma-\alpha} \leqq C.\end{aligned}$$

The proposition then follows by Theorem 2.2.1. qed

Now consider Stokes-equations (1.2.5). There we have

$$\text{(3.1.10)} \qquad A_l = -\Delta_{h_l}, \qquad B_l = -\mathrm{div}_{h_l}, \qquad C_l = B_l^T = \mathrm{grad}_{h_l},$$

and $\alpha=2\beta=2$. Suppose the discretization satisfies Brezzi's condition (cf. [6, 12]), so that K_l is regular according to Sect. 1.2. Then $\bar{K}_l$ from (3.1.1a) has the form

$$\text{(3.1.11)} \qquad \bar{K}_l = \begin{pmatrix} I & \Delta_{h_l}^{-1}\nabla_{h_l}E_l^{-1}D_l \\ 0 & -E_l^{-1}D_l \end{pmatrix}$$

where $E_l=\mathrm{div}_{h_l}\Delta_{h_l}^{-1}\mathrm{grad}_{h_l}$ and D_l, regular, has to be chosen properly. Setting

$$\text{(3.1.12)} \qquad D_l = E_l,$$

we obtain

$$\text{(3.1.13)} \qquad \bar{K}_l = \begin{pmatrix} I & \Delta_{h_l}^{-1}\nabla_{h_l} \\ 0 & -I \end{pmatrix}.$$

This r-transformation yields the SIMPLE-method (cf. [30, 31]). Using theorem 3.1.3 we prove

Theorem 3.1.4. *Let $\Omega\subset\mathbf{R}^2$ be bounded and convex. Let the Stokes-equations* (1.2.5) *be discretized on Ω using quadratic finite-elements for the velocities and linear ones for the pressure or linear elements for the velocities and linear ones on triangles with twice the size as described by Taylor and Hood* (cf. [27]). *Suppose* *some symmetric incomplete factorization of* A_l *and* E_l *exists with prescribed sparsity pattern such that* $M = LL^T$ *with*

(3.1.14)
$$\|M_{ii}\| \leqq C \cdot h^{-s(i)}$$
$$\text{with } s(i) := \begin{cases} 2 & \text{for } i=1 \\ 0 & \text{for } i=2 \end{cases}$$

and for smoothing use a modified ILU-scheme as introduced in [33] *with modification parameter* β' *in r-transforming manner with transformation* $\bar{K}_l$ *from* (3.1.13). *Then there exists* $\beta_\vartheta > 0$ *such that for* $\beta' > \beta_\vartheta$ *the corresponding two-grid method converges and with*

(3.1.15)
$$\bar{m}_1 = m_1 = m'_1 = 1,$$
$$\bar{m}_2 = m_2 = m'_2 = 0,$$

we have

(3.1.16)
$$|||T_{2,l}(\nu, 0)|||_{U_l \leftarrow U_l} \leqq C\eta(\nu)$$
$$\text{with } \eta(\nu) = O\left(\frac{\ln \nu}{\sqrt{\nu}}\right)$$

in the norms from (2.2.3).

Proof. A_l, D_l and E_l are symmetric and positive definite. (2.3.1) and (2.3.2) are valid with $\gamma = 0$, $\alpha = 2$, and $\beta = 1$. Since $M_{\beta'} = LL^T + \beta' \|N\| I$, $M_{ii,\beta'} \geqq c > 0$ holds independently of h, yielding (2.3.4a, b). (2.3.4c) is satisfied as $M_{21} = (K_l \bar{K}_l)_{21} (I + D_{11}^{-1} U_{11})$ where D_{11} and U_{11} are blocks of the diagonal-matrix D and of the strictly upper triangular matrix U from the incomplete decomposition of $K_l \bar{K}_l = (L + D) D^{-1} (U + D) - N$. According to Theorem 3.1.5 from [33] smoothing properties (2.3.5a, b) hold with $\kappa = \lambda = 0$. Interpolation using Lemma 2.3.3 yields (2.3.5a, b) with $\kappa = 1$ and $\lambda = 0$. As scaling (3.1.15) is the same as the one used for Stokes-equations in [13], Lemma 3.1.2 is equivalent to Proposition 11.2.7 from [13]. (3.1.4) and (3.1.5) are immediate, (3.1.6) is satisfied according to the estimates given in [10] (cf. ex. 11.2.8 in [13]). Because of $\beta = 1 > 0$ we can apply Theorem 3.1.3 yielding the proposition. qed

Remark 3.1.5. Theorem 3.1.4 remains valid if the Taylor-Hood element is replaced by the mini-element (see [2, 16]).

Proof. By the same arguments as in the proof of Theorem 3.1.4 (2.3.1)–(2.3.5), (3.1.4) and (3.1.5) are still valid. According to [10] and [15] (3.1.6) is fulfilled too, yielding the proposition. qed

Remark 3.1.6. The above results remain valid, if a r-transforming damped Jacobi-smoother with suitable damping parameter (see [13]) or a modified symmetric Gauß-Seidel scheme with positive modification parameter is employed.

Proof. For the damped Jacobi the assertion follows immediately by [13], Prop. 6.2.14 and the above considerations. For the modified symmetric Gauß-Seidel it follows from [32], Thm. 2.2.4. qed

$\bar{K}_l$ from (3.1.13) requires Δ_h^{-1}, which is not explicitly available. Hence for practical purposes Δ_h^{-1} is replaced by an "approximate inverse" G_l introducing perturbations into the product-system.

The second choice of D_l from (3.1.1e) is $D_l = \Delta_{h_l}^{(p)}$, (p) denoting that the operator has to be taken w.r.t. the pressure. The resulting transformation yields the DGS/TILU methods as described in [5, 30, 31]. $\bar{K}_l$ then reads

$$\bar{K}_l = \begin{pmatrix} I & \Delta_{h_l}^{-1} \nabla_{h_l} E_l^{-1} \Delta_{h_l}^{(p)} \\ 0 & -E_l^{-1} \Delta_{h_l}^{(p)} \end{pmatrix}. \tag{3.1.17}$$

Now we have $\gamma = 2$, implying that for Lemma 3.1.2 consistency and interpolation of third order are required. Using a Galerkin-ansatz, however, we obtain.

Theorem 3.1.7. *Let $\Omega \subset \mathbf{R}^2$ be a convex and bounded domain. Let the Stokes-equations* (1.2.5) *be discretized on staggered grids with equidistant mesh-size as described by Harlow and Welch (see* [18]*) with lexicographical ordering of the grid-points and linear interpolation at the boundary* (cf. [16]). *Further let the prolongation p be a biquadratic interplation, the restriction $r = p^*$ and let the coarse-grid matrices be computed by Galerkin-ansatz* (2.2.8).

Then the two-grid method with r-transforming damped Jacobi with suitable damping factor, symmetric Gauß-Seidel or 5-point $ILU_{\beta'}$-smoothing, for $\beta' \geqq 0$ (cf. [32]), *resp. based on $\bar{K}_l$ from* (3.1.17) *converges. With*

$$\bar{m}_1 = m_1 = m'_1 = 1, \quad m_2 = m'_2 = 0, \quad \bar{m}_2 = 2, \tag{3.1.18}$$

$$|||T_{2,l}(\nu, 0)|||_{U_l \leftarrow U_l} \leqq C\eta(\nu) \tag{3.1.19}$$

$$\textit{with } \eta(\nu) = \begin{cases} O\left(\frac{1}{\sqrt{\nu}}\right), & \textit{for damped Jacobi,} \\ O\left(\frac{\ln \nu}{\sqrt{\nu}}\right), & \textit{else,} \end{cases}$$

holds.

Proof. A_l and D_l are symmetric and positive definite. (2.3.1)–(2.3.3) hold with $\alpha = \gamma = 2$, $\beta = 1$. A_l as well as D_l satisfy (3.1.16), implying (2.3.4a, b) with $\lambda = 0$ for the symmetric Gauß-Seidel as well as the damped Jacobi smoother with suitable damping factor. For ILU_0 (2.3.4a, b) are readily veryfied for $\lambda = 0$ using Lemma 2.1.4 from [32]. (2.3.4c) follows by the same argument as in the proof of Theorem 3.1.4. According to [13], Prop. 6.2.27 and [32], Prop. 2.1.6 (see also [30], Prop. 5.3.1), smoothing-properties (2.3.5a, b) hold for $\kappa = \lambda = 0$. By virtue of Lemma 2.3.3 they are easily established for $\lambda = \kappa = 1/2$ and $\kappa = 1$. For Jacobi-type smoothers (2.3.4a, b) is evident for $\lambda = 1/2$. By virtue of the Galerkin-ansatz (2.2.8) consistency assumptions (3.1.6) are satisfied automatically (see [13]). (3.1.4) and (3.1.5) hold too. Thus the proposition follows from Theorem 3.1.3. qed

$\bar{K}_l$ from (3.1.17) needs $\Delta_{h_l}^{-1}$ and E_l^{-1} which are both not available in practice. For practical purposes $\bar{K}_l$ is replaced by

$$\tilde{K}_l = \begin{pmatrix} I & \nabla_{h_l} \\ 0 & -\Delta_{h_l} \end{pmatrix}, \tag{3.1.20}$$

introducing perturbations into the product system. Handling these perturbations will be the subject of a forthcoming paper.

The asymptotic behaviour of $\eta(v)$ from (3.1.19) for damped Jacobi is better than the one of SGS and $ILU_{\beta'}$. However, this is only a matter of proof. To obtain $O(1/v^{1/2})$ instead of $O(\ln v/v^{1/2})$, we need

$$|M_{22}^{-1}|_{\frac{1}{2}\leftarrow\frac{1}{2}}\leqq C. \tag{3.1.21}$$

This is straightforward for Jacobi, but not at all for the other smoothers.

The asymptotic behaviour of $\eta(v)$ from (3.1.16) and (3.1.19) is worse than the usual $O(1/v)$. Now, practical tests for TILU with both transformations show a behaviour of $\eta(v)\approx O(1/v)$ (see [31]). Thus (3.1.19) is an overestimate caused by abandoning symmetry in our product-system.

3.2 The Nonlinear Case

Now consider the Navier-Stokes equations (1.2.4). We linearize it by

$$Q_l(u_l^0)\,u_l=-\Delta_{h_l}u_l+\mathrm{Re}\cdot N_l(u_l^0)\,u_l \tag{3.2.1}$$

with

$$N_l(u_l^0)=\begin{cases}u_l^0\,\nabla_{h_l}u_l+u_l\,\nabla_{h_l}u_l^0, & \text{for Newton's method}\\ u_l^0\,\nabla_{h_l}u_l, & \text{for a simplified Newton-method}\end{cases}$$

and use one of these linerarizations within an outer Newton-like algorithm. The linear systems within each step can easily be solved by the linear multi-grid method used for (1.2.5). For details see [31].

Theorem 3.2.1. *Let $\Omega\subset\mathbf{R}^2$ be a convex and bounded domain. Let the linearized Navier-Stokes equations with $A_l=Q_l$ from* (3.2.1) *be discretized by the Taylor-Hood element or the mini-element as mentioned above and assume the discretization to be stable* w.r.t. *the convective terms. Then there exists $h_0=h_0(\mathrm{Re})$, such that the two-grid method with r-transforming $ILU_{\beta'}$-smoothing, $\beta'>1$, using $\bar{K}_l$ from* (3.1.13) *satisfies smoothing-property* (2.2.6) *in the norms from Theorem* 3.1.4, *provided $h_l<h_0$, $l\geqq 2$.*

Proof. Split $K_l=K_l'+K_l''$, where K_l'' contains the linearized convective terms while K_l' contains the Stokes-operator. Correspondingly S_l can be split into $S_l=S_l'-S_l''$ with $S_l'=I-M_l'^{-1}K_l'$ and

$$S_l''=\sum_{i=1}^{\infty}(-M_l'^{-1}M_l'')^i\,M_l'^{-1}(K_l'+K_l'')+M_l'^{-1}K_l''.$$

The sum converges for h sufficiently small, as M_l'' is of lower order. Further, we have $\|S_l'^{\nu}\|\leqq C\cdot\nu$ as $S_{l,ii}'$ is convergent. Further $\lim_{l\to\infty}h_l^2\|K_l''\|=0$, as K_l'' is of first order, and

$$\|(S_l'')^{\nu}\|\leqq c_1(\nu)\cdot h_l^{\nu}\left(\sum_{i=0}^{\infty}\left\|M_l'^{-1}M_l''\right\|^{\nu}c_2+c_3\right)\to 0,$$

for $h_l \to 0$ and fixed ν.

Based on the results from Sec. 3.1 we can apply criterion 6.2.7 from [13], completing the proof. qed

Remark 3.2.2. The same result holds for the staggered-grid discretization, r-transforming ILU_β and symmetric Gauß-Seidel smoothing, based on $\bar{K}_l$ from (3.1.17).

Proof. Analogously to the proof of Theorem 3.2.1. qed

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