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# Improved Forms of the Alternating Direction Methods of Douglas, Peaceman, and Rachford for Solving Parabolic and Elliptic Equations

By

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## 1. Introduction

Consider the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{1}$$

for $u \equiv u(x, y, t)$ where $x, y$ are space co-ordinates, and $t$ is the time, subject to the boundary conditions $u(x, y, 0) = f(x, y)$ over the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and $u(x, y, t)$ given for all $t > 0$ where $x, y$ is a point on the boundary of the unit square.

The two implicit difference methods for solving (1) which have received most attention are the Peaceman Rachford (P.R.) method [3] and the Douglas Rachford (D.R.) method [1]. In order to describe these methods, a network of nodal points is illustrated in Fig. 1. For ease of reference the points are numbered and as seen from the diagram the mesh sizes are $\Delta x$, $\Delta y$, and $\Delta t$ in the $x$, $y$, and $t$ directions respectively.

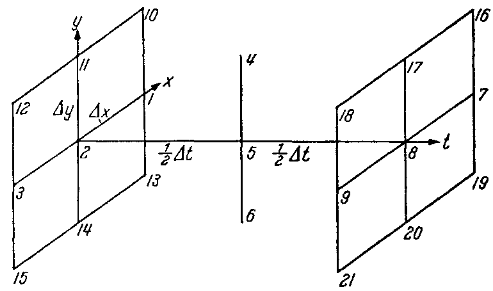

Fig. 1

The P.R. method is given by

$$\left(-\delta_y^2 + \frac{2}{r}\right) u^{(m+\frac{1}{2})} = \left(\delta_x^2 + \frac{2}{r}\right) u^{(m)},$$

$$\left(-\delta_x^2 + \frac{2}{r}\right) u^{(m+1)} = \left(\delta_y^2 + \frac{2}{r}\right) u^{(m+\frac{1}{2})},$$

where $\delta_x$, $\delta_y$ are the usual central difference operators in the $x$ and $y$ co-ordinates respectively, $r = \Delta t / h^2$ where $\Delta x = \Delta y = h$, and $u^{(m)}$, $u^{(m+\frac{1}{2})}$, $u^{(m+1)}$ are the values of $u$ at the nodes 2, 5, and 8 respectively. Elimination of $u^{(m+\frac{1}{2})}$ leads to

$$\left(-\delta_y^2 + \frac{2}{r}\right)\left(-\delta_x^2 + \frac{2}{r}\right) u^{(m+1)} = \left(\delta_y^2 + \frac{2}{r}\right)\left(\delta_x^2 + \frac{2}{r}\right) u^{(m)}, \tag{2}$$

where $\left(-\delta_y^2 + \frac{2}{r}\right)\left(-\delta_x^2 + \frac{2}{r}\right)$ and $\left(\delta_y^2 + \frac{2}{r}\right)\left(\delta_x^2 + \frac{2}{r}\right)$ are both nine-point operators. Similarly the D.R. method, which is given by

$$\left(-\delta_y^2+\frac{1}{r}\right)u^{*(m+1)}=\left(\delta_x^2+\frac{1}{r}\right)u^{(m)},$$

$$\left(-\delta_x^2+\frac{1}{r}\right)u^{(m+1)}=(-\delta_x^2)\,u^{(m)}+\frac{1}{r}u^{*(m+1)},$$

where $u^{*(m+1)}$ denotes an approximation to $u$ at the node 8, leads to

(3) $$\left(-\delta_y^2+\frac{1}{r}\right)\left(-\delta_x^2+\frac{1}{r}\right)u^{(m+1)}=\left(\delta_x^2\delta_y^2+\frac{1}{r^2}\right)u^{(m)}$$

after elimination of $u^{*(m+1)}$. The operators $\left(-\delta_y^2+\frac{1}{r}\right)\left(-\delta_x^2+\frac{1}{r}\right)$ and $\left(\delta_x^2\delta_y^2+\frac{1}{r^2}\right)$ are again nine-point operators.

A generalised formula which involves nine-point operators at each of two neighboring levels of $t$ is

(4) $$k[(u_8+aW+bX)+(cu_2+dY+eZ)]=0,$$

where

$$W=u_{16}+u_{18}+u_{19}+u_{21},$$

$$X=u_7+u_9+u_{17}+u_{20},$$

$$Y=u_{10}+u_{12}+u_{13}+u_{15},$$

$$Z=u_1+u_3+u_{11}+u_{14}$$

and $k, a, b, c, d,$ and $e$ are functions of $r$. The problem described by (1) and its associated boundary conditions can be solved by using (4) in an alternating direction form because of the commutative nature of the matrices involved (VARGA [5, p. 246, Ex. 6]). Formulae (2) and (3) can both be expressed in the form of (4). In fact the coefficients are

$$k=-\frac{(1+r)^2}{r},\quad a=\frac{r^2}{4(1+r)^2},\quad b=-\frac{r}{2(1+r)},$$

$$c=-\frac{(1-r)^2}{(1+r)^2},\quad d=-\frac{r^2}{4(1+r)^2},\quad e=-\frac{r(1-r)}{2(1+r)^2}$$

for the P.R. method and

$$k=-\frac{(1+2r)^2}{r},\quad a=\frac{r^2}{(1+2r)^2},\quad b=-\frac{r}{1+2r},$$

$$c=-\frac{1+4r^2}{(1+2r)^2},\quad d=-\frac{r^2}{(1+2r)^2},\quad e=\frac{2r^2}{(1+2r)^2},$$

for the D.R. method.

Also, by expanding $u_8$, $W$, $X$, $Y$, and $Z$ as Taylor series in terms of $u$ and its derivatives at the node 2, and replacing derivatives with respect to $t$ by using the relations

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\quad \frac{\partial^2 u}{\partial t^2}=\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4},$$

etc. from (1), it is easily shown that the principal parts of the truncation errors are $\frac{1}{12}h^4\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)$ for the P.R. method and $\frac{1}{2}\left(r+\frac{1}{6}\right)h^4\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)+r\,h^4\frac{\partial^4 u}{\partial x^2\partial y^2}$ for the D.R. method.

It is the purpose of the first section of this paper to examine formulae of type (4) and to determine values of $a$, $b$, $c$, $d$, and $e$ which eliminate the terms of order $h^4$ in formula (4) and further to rewrite the formula obtained as a pair of P.R. or D.R. type formulae.

## 2. Generalised P.R. and D.R. formulae

Since a formula of type (4) requires a solution of $N^2$ equations at each time step, where $Nh=1$, it is only of use as a means of solving (1) if it can be re-written as a pair of P.R. or D.R. type formulae i.e. a pair of formulae which utilises the same points as the P.R. or D.R. formulae.

Considering first the points used by the P.R. formulae, we can write (4) as

$$(5)\qquad \begin{aligned} \left(-\delta_y^2-\frac{1+2b}{b}\right)u^{(m+\frac{1}{2})} &= \frac{d(bc-e)}{be}\,k\left(\delta_x^2+\frac{e+2d}{d}\right)u^{(m)}\\ \left(-\delta_x^2-\frac{b+2a}{a}\right)u^{(m+1)} &= -\frac{be}{a(bc-e)k}\left(\delta_y^2+\frac{c+2e}{e}\right)u^{(m+\frac{1}{2})} \end{aligned}$$

where $b^2=a$, and $e^2=cd$. Next using the same points as the D.R. formulae, (4) can be written as

$$(6)\qquad \begin{aligned} \left(-\delta_x^2-\frac{1+2b}{b}\right)u^{*(m+1)} &= \frac{be-d}{b}\,k\left(\delta_x^2+\frac{(2d+e)-b(2e+c)}{d-eb}\right)u^{(m)}\\ \left(-\delta_x^2-\frac{b+2a}{a}\right)u^{(m+1)} &= \frac{d}{a}\left(\delta_x^2+\frac{e+2d}{d}\right)u^{(m)}-\frac{1}{ak}\,u^{*(m+1)} \end{aligned}$$

where $b^2=a$, and an asterisk again denotes an approximate value of $u$ at a node.

Formulae (5) and (6) are the generalised P.R. and D.R. type formulae respectively and formula (4) cannot in general be represented by (5) or (6). If however, $b^2=a$, (4) can be written as the D.R. type formulae (6), and if in addition $e^2=cd$, (4) can be written as the P.R. type formulae (5).

## 3. The Optimum Formula

We now expand $u_8$, $W$, $X$, $Y$, and $Z$ as Taylor series in terms of $u$ and its derivatives at the node 2. The derivatives with respect to $t$ are replaced by

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\qquad \frac{\partial^2 u}{\partial t^2}=\frac{\partial^4 u}{\partial x^4}+2\,\frac{\partial^4 u}{\partial x^2\,\partial y^2}+\frac{\partial^4 u}{\partial y^4}$$

etc. from (1).

The expansions up to and including terms involving $h^6$ are

$$u_8=u+rA+\tfrac{1}{2}r^2B+r^2C+\tfrac{1}{6}r^3D+\tfrac{1}{2}r^3E,$$

$$W=4u+(4r+2)A+(2r^2+2r+\tfrac{1}{6})B+(4r^2+4r+1)C+ \\ +(\tfrac{2}{3}r^3+r^2+\tfrac{1}{6}r+\tfrac{1}{180})D+(2r^3+3r^2+\tfrac{7}{6}r+\tfrac{1}{12})E,$$

$$X=4u+(4r+1)A+(2r^2+r+\tfrac{1}{12})B+(4r^2+2r)C+ \\ +(\tfrac{2}{3}r^3+\tfrac{1}{2}r^2+\tfrac{1}{12}r+\tfrac{1}{360})D+(2r^3+\tfrac{3}{2}r^2+\tfrac{1}{12}r)E,$$

$$Y=4u+2A+\tfrac{1}{6}B+C+\tfrac{1}{180}D+\tfrac{1}{12}E,$$

$$Z=4u+A+\tfrac{1}{12}B+\tfrac{1}{360}D,$$

where

$$A = h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad B = h^4\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right), \quad C = h^4 \frac{\partial^4 u}{\partial x^2\, \partial y^2}.$$

$$D = h^6\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right), \quad \text{and} \quad E = h^6 \frac{\partial^4}{\partial x^2\, \partial y^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

If these expressions are substituted into formula (4) with $k=1$, values of $a, b, c, d$, and $e$ can be found which will eliminate $u, A, B, C$ and either $D$ or $E$. However, a formula of type (4) is only of use as a means of solving (1) if it can at least be written in the form (6). This will be possible only if $b^2 = a$. This condition on the coefficients can only be satisfied if values of $a, b, c, d$, and $e$ are obtained which eliminate $u, A, B$, and $C$ only and involve a parameter. This parameter is then chosen to satisfy $b^2 = a$. The coefficients which eliminate $u, A, B$, and $C$ are

$$a = \frac{3r+1+p}{4p}, \quad b = -\frac{3r+\frac{5}{2}+2p}{4p}, \quad c = -\frac{20r+4p}{4p},$$

$$d = -\frac{2r+1+p}{4p}, \quad e = \frac{7r+\frac{5}{2}+2p}{4p},$$

where $p$ is a parameter. If $b^2 = a$, then $p = -\frac{1}{6}(3r+\frac{5}{2})^2$, and so the coefficients become

$$a = \frac{(r-\frac{1}{6})^2}{4(r+\frac{5}{6})^2}, \quad b = -\frac{(r-\frac{1}{6})}{2(r+\frac{5}{6})}, \quad c = -\frac{(r-\frac{5}{6})^2}{(r+\frac{5}{6})^2},$$

$$d = -\frac{(r+\frac{1}{6})^2}{4(r+\frac{5}{6})^2}, \quad e = \frac{(r-\frac{5}{6})(r+\frac{1}{6})}{2(r+\frac{5}{6})^2}.$$

The formula of type (4) with $k=1$ and minimum truncation error which can be written in the form (6) is thus

$$u_8 + \frac{(r-\frac{1}{6})^2}{4(r+\frac{5}{6})^2} W - \frac{(r-\frac{1}{6})}{2(r+\frac{5}{6})} X - \frac{(r-\frac{5}{6})^2}{(r+\frac{5}{6})^2} u_2 - \frac{(r+\frac{1}{6})^2}{4(r+\frac{5}{6})^2} Y + \frac{(r-\frac{5}{6})(r+\frac{1}{6})}{2(r+\frac{5}{6})^2} Z = 0. \tag{7}$$

The principal part of the truncation error is in fact $-\frac{r(r^2-\frac{1}{20})}{12(r+\frac{5}{6})^2} h^6\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right)$, which is an order $h^2$ better than the original D.R. or P.R. formulae. It follows also that if $r$ is approximately $\frac{1}{2(5)^{\frac{1}{2}}}$, the truncation error may even be of order $h^8$. When $r = \frac{1}{6}$, (7) degenerates into the well known explicit formula

$$u_8 = \tfrac{4}{9} u_2 + \tfrac{1}{36} Y + \tfrac{1}{9} Z,$$

and so the method of alternating directions is inapplicable.

Formula (7) can now be written as the D.R. type formulae

$$\begin{aligned} \left(-\delta_y^2 + \frac{2}{r-\frac{1}{6}}\right) u^{*(m+1)} &= -\frac{r(r+\frac{1}{6})}{(r+\frac{5}{6})^2 (r-\frac{1}{6})} \left(\delta_x^2 + \frac{2}{r+\frac{1}{6}}\right) u^{(m)}, \\ \left(-\delta_x^2 + \frac{2}{r-\frac{1}{6}}\right) u^{(m+1)} &= -\frac{(r+\frac{1}{6})^2}{(r-\frac{1}{6})^2} \left(\delta_x^2 + \frac{2}{r+\frac{1}{6}}\right) u^{(m)} - \frac{4(r+\frac{5}{6})^2}{(r-\frac{1}{6})^2} u^{*(m+1)} \end{aligned} \tag{8}$$

and since the optimum coefficients satisfy $e^2 = cd$ as well as $b^2 = a$, (7) can also be written as the P.R. formulae

$$\begin{aligned}\left(-\delta_y^2 + \frac{2}{r-\frac{1}{6}}\right) u^{(m+\frac{1}{2})} &= -\frac{r(r+\frac{1}{6})}{(r+\frac{5}{6})^2(r-\frac{1}{6})}\left(\delta_x^2 + \frac{2}{r+\frac{1}{6}}\right) u^{(m)}, \\ \left(-\delta_x^2 + \frac{2}{r-\frac{1}{6}}\right) u^{(m+1)} &= -\frac{(r+\frac{5}{6})^2(r+\frac{1}{6})}{r(r-\frac{1}{6})}\left(\delta_y^2 + \frac{2}{r+\frac{1}{6}}\right) u^{(m+\frac{1}{2})}.\end{aligned} \tag{9}$$

## 4. Stability

The stability of the scheme (4) is analyzed by the normal procedure of assuming that there exists an error $\varepsilon_{i,j,n}$ at each mesh point $i\Delta x$, $j\Delta y$, $n\Delta t$ ($i = 1, 2, \ldots, N-1$; $j = 1, 2, \ldots, N-1$; $n = 0, 1, 2, \ldots$) where $\Delta x = \Delta y = 1/N$. For a linear problem with constant coefficients, these errors grow according to equation (4) with $u$ replaced by $\varepsilon$. If the error is now expanded in the form

$$\varepsilon_{i,j,n} = \varrho_n \sin \pi p x_i \sin \pi q y_i \qquad (p, q = 1, 2, \ldots, N-1)$$

and substituted into equation (4) with $u$ replaced by $\varepsilon$, it follows that

$$\frac{\varrho_{n+1}}{\varrho_n} = -\frac{1}{d}\,\frac{cd + 2ed(\cos \pi p\,\Delta x + \cos \pi q\,\Delta y) + 4d^2 \cos \pi p\,\Delta x \cos \pi q\,\Delta y}{(1+2b\cos \pi p\,\Delta x)(1+2b\cos \pi q\,\Delta y)} \tag{10}$$

if $a = b^2$, the condition necessary for (4) to be written in D.R. form. If in addition, $e^2 = cd$, (4) can be written in P.R. form and the numerater of (10) factorizes to give

$$\frac{\varrho_{n+1}}{\varrho_n} = -\frac{1}{d}\,\frac{(e+2d\cos \pi p\,\Delta x)}{(1+2b\cos \pi p\,\Delta x)}\,\frac{(e+2d\cos \pi q\,\Delta y)}{(1+2b\cos \pi q\,\Delta y)}. \tag{11}$$

For the optimum formula (7), equation (11) becomes

$$\frac{\varrho_{n+1}}{\varrho_n} = \frac{2\left(r+\frac{1}{6}\right)\sin^2\frac{\pi p\,\Delta x}{2} - 1}{2\left(r-\frac{1}{6}\right)\sin^2\frac{\pi p\,\Delta x}{2} + 1}\;\frac{2\left(r+\frac{1}{6}\right)\sin^2\frac{\pi q\,\Delta y}{2} - 1}{2\left(r-\frac{1}{6}\right)\sin^2\frac{\pi q\,\Delta y}{2} + 1},$$

from which it follows that formula (7) is stable for all values of the mesh ratio $r$.

## 5. Elliptic Equations

It is an old idea that provided $u(x, y, t) = u(x, y, 0)$ for all $t > 0$, where $x, y$ is a point on the boundary of the unit square, the steady state solution of equation (1) is the solution of the elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{12}$$

subject to the boundary condition of $u(x, y) = f(x, y)$ for $x, y$ a point on the boundary of the unit square.

There is no certainty, however, that the difference equation which gives the most accurate solution of (1) will also provide indirectly the best iteration technique for solving (12) using the time step as a parameter. In order to examine this point, we return to difference formulae of type (4). As shown previously, such formulae can be written in P.R. form if $b^2 = a$ and $e^2 = cd$.

In addition, these formulae represent (1) with a principal truncation error of order $h^4$ if the additional conditions

$$4a+4b+c+4d+4e=-1,$$
$$(4r+2)a+(4r+1)b+2d+e=-r$$

are satisfied. These four relations between the five coefficients enable us, after some simple manipulation, to obtain the values

$$a=b^2, \qquad b=b,$$
$$c=-[4(b+\tfrac{1}{2})r-1]^2, \qquad d=-[2(b+\tfrac{1}{2})r+b]^2,$$
$$e=[2(b+\tfrac{1}{2})r+b][4(b+\tfrac{1}{2})r-1].$$

If these values of the coefficients are substituted into (5), we obtain the P.R. type formulae

$$\text{(13)}\quad \begin{aligned} \left(-\delta_y^2-\frac{1+2b}{b}\right)u^{(m+\frac{1}{2})} &= \frac{(1+2b)^2[(1+2b)r+b]rk}{b}\left(\delta_x^2+\frac{1+2b}{(1+2b)r+b}\right)u^{(m)},\\ \left(-\delta_x^2-\frac{1+2b}{b}\right)u^{(m+1)} &= \frac{(1+2b)r+b}{b(1+2b)^2rk}\left(\delta_y^2+\frac{1+2b}{(1+2b)r+b}\right)u^{(m+\frac{1}{2})}. \end{aligned}$$

We now determine the value (or values) of the parameter $b$ which will make (13) the best iteration scheme for solving (12).

Following VARGA [5, pp. 214, 215], the spectral radius of the iteration matrix implicit in (13) is

$$\varrho(T_r)=\left\{\max_{1\le l\le N-1}\left|\frac{1-4\left(r+\frac{b}{2b+1}\right)\sin^2\frac{l\pi}{2N}}{1-4\frac{b}{2b+1}\sin^2\frac{l\pi}{2N}}\right|\right\}^2.$$

If we put

$$\text{(14)}\qquad b=-\frac{r+f}{2(r+f+1)},$$

where $f$ is a parameter, the spectral radius becomes

$$\varrho(T_r)=\left\{\max_{1\le l\le N-1}\left|\frac{r-\left(f+\frac{1}{2\sin^2\frac{l\pi}{2N}}\right)}{r+\left(f+\frac{1}{2\sin^2\frac{l\pi}{2N}}\right)}\right|\right\}^2.$$

This is now minimized as a function of the mesh ratio $r$ ($>0$) and the result

$$\text{(15)}\qquad \min_{r>0}\varrho(T_r)=\frac{1+f\sin^2\frac{\pi}{N}-\sin\frac{\pi}{N}\left(1+2f+f^2\sin^2\frac{\pi}{N}\right)^{\frac{1}{2}}}{1+f\sin^2\frac{\pi}{N}+\sin\frac{\pi}{N}\left(1+2f+f^2\sin^2\frac{\pi}{N}\right)^{\frac{1}{2}}}$$

is obtained. For each value of $f$, (15) gives the optimum convergence rate, and the value of $r$ necessary to give this optimum convergence is

$$\text{(16)}\qquad r^*=\frac{1}{\sin\frac{\pi}{N}}\left(1+2f+f^2\sin^2\frac{\pi}{N}\right)^{\frac{1}{2}}.$$

It should be emphasized that the above analysis depends on $r$ being kept constant during the iterations. When $r$ is allowed to vary, and take the value $r_i$ $(1 \leqq i \leqq m)$, for each of $m$ successive iterations, the situation is much more complicated, and references to methods for selecting the best acceleration parameters $r_i$ can be found in VARGA [5], TODD [4, Ch. 1] and WACHSPRESS [6].

Returning to formula (16) it can be seen that $r^*$ is real if

$$-\left(2\cos^2\frac{\pi}{2N}\right)^{-1} \leqq f < +\infty. \tag{17}$$

This is also the condition for (13), with $b$ replaced by $f$ using (14), to be stable for all $r$. The convergence rate given by (15) takes the value unity when $f = -\left(2\cos^2\frac{\pi}{2N}\right)^{-1}$, and tends to zero, as $f$ tends to infinity. For $N=30$, 100, and 1000, the convergence rates for various values of $f$ within the permissible range given by (17), are shown in the Table. It appears that the best convergence rate for a given value of $N$ is obtained when $f$ is positive and as large as possible. It can also be seen from the Table, that the original P.R. formula ($f=0$) has a better convergence rate than the optimum P.R. type formula ($f=-\frac{1}{6}$) derived previously to solve (1).

Table

| $N$ | $f$ | | | | |
|---|---|---|---|---|---|
| | $-\frac{1}{6}$ | 0 | 1 | 10 | 100 |
| 30 | 0.842 | 0.811 | 0.696 | 0.386 | 0.0639 |
| 100 | 0.950 | 0.939 | 0.897 | 0.750 | 0.413 |
| 1000 | 0.995 | 0.994 | 0.990 | 0.970 | 0.915 |

However the substational improvement in convergence implicit in (15) which arises from the choice of large positive values of the parameter $f$ may be accompanied by a certain loss of accuracy. In terms of $f$, using (14), equations (13) become

$$\begin{aligned}\left(-\delta_y^2+\frac{2}{r^*+f}\right)u^{(m+\frac{1}{2})} &= -\frac{(r^*-f)\,r^*k}{(r^*+f+1)^2(r^*+f)}\left(\delta_x^2+\frac{2}{r^*-f}\right)u^{(m)}\\ \left(-\delta_x^2+\frac{2}{r^*+f}\right)u^{(m+1)} &= -\frac{(r^*-f)\,(r^*+f+1)^2}{(r^*+f)\,r^*k}\left(\delta_y^2+\frac{2}{r^*-f}\right)u^{(m+\frac{1}{2})},\end{aligned} \tag{18}$$

where $r^*$ is given by (16) and $k$ is arbitrary. Formulae (18) of course reduce to the original P.R. formulae (2) if $f=0$ and $k=-\frac{(1+r)^2}{r}$ and to the optimum P.R. type formulae (9) if $f=-\frac{1}{6}$ and $k=1$. The principal part of the truncation error for (18) is $-\frac{(f+\frac{1}{6})\,r^*k}{2(r^*+f+1)^2}h^4\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)$. If, $u^{(m+\frac{1}{2})}$ is now eliminated from (18), the formula

$$\begin{aligned}&\left(-\delta_y^2+\frac{2}{r^*+f}\right)\left(-\delta_x^2+\frac{2}{r^*+f}\right)u^{(m+1)}\\ &\qquad=\frac{(r^*-f)^2}{(r^*+f)^2}\left(\delta_y^2+\frac{2}{r^*-f}\right)\left(\delta_x^2+\frac{2}{r^*-f}\right)u^{(m)}\end{aligned} \tag{19}$$

is obtained. The iteration procedure described by (19) converges if $u^{(m+1)} = u^{(m)} = u$, for $m$ sufficiently large, and so (19) reduces to

$$[(\delta_x^2+\delta_y^2) - f(\delta_x^2\,\delta_y^2)]\,u = 0. \tag{20}$$

However, equation (12) can be replaced by

$$[(\delta_x^2 + \delta_y^2) + \tfrac{1}{6}(\delta_x^2\,\delta_y^2)]\,u = 0$$

correct to fourth differences, and so it follows from (20) that equations (18) are most accurate when $f = -\frac{1}{6}$, a result obtained previously, and that there is a loss in accuracy when $f$ is large. In fact as $f$ tends to infinity, equation (20) degenerates to

$$\delta_x^2\,\delta_y^2 = 0,$$

which is no longer a difference approximation to Laplace's equation. Accordingly a balance is required between the rate of convergence (an optimum when $f$ is infinite) and the accuracy of the process (an optimum when $f = -\frac{1}{6}$).

Finally, the convergence rates for various values of $N$ and $f$ given by (15) will almost certainly be reduced when the region under consideration departs from the rectangular (cf. Douglas and Pearcy [2]).

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