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Alternating Direction and Semi-Explicit Difference Methods for Parabolic Partial Differential Equations

By

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1. Introduction

In previous papers [*12*], [*13*], [*14*] the author developed a difference analogue of the energy method for determining the stability of difference approximations to partial differential equations with variable coefficients. The purpose of this paper is to apply this method to establish the unconditional stability of two types of difference approximations to parabolic differential equations, the (implicit) alternating direction methods of DOUGLAS, PEACEMAN, and RACHFORD [*3*], [*5*], [*15*], and a new semi-explicit method.

For the model problem, the first boundary value problem for the heat conduction equation in a rectangular domain, the unconditional stability of the alternating direction methods was proved in [*3*] and [*5*]. The proof consists in showing, with the aid of Fourier analysis, that the von Neumann stability condition [*4*], [*11*] is always satisfied. It can be shown [*1*], however, that this method of proof cannot be extended beyond the model problem.

With the aid of the energy method we prove that the results in [*3*] and [*5*] can be extended beyond the model problem. We first treat, as a typical case, the heat conduction equation in a cylindrical domain with an essentially arbitrary, bounded base. Then we indicate briefly the extension to parabolic equations with variable coefficients.

The second type of difference method we term the semi-explicit method, because it is an explicit method only for certain orderings of the net points. The idea for this difference method comes from the observation that there is a formal correspondence between parabolic difference equations and iterative methods for solving elliptic difference equations; the semi-explicit method corresponds to the well known method of successive displacements [*7*].

The only other known example of an unconditionally stable explicit difference method is due to DU FORT and FRANKEL [*6*]. Their method, however, requires two lines of initial data to start the solution, while the semi-explicit method is self-starting. On the other hand, both of these methods involve a similar local truncation error, and they must be subjected to a mild mesh ratio condition in order to be consistent [*11*] with the differential equation being approximated.

For other applications of the energy method to the stability problem for partial difference equations, see FRIEDRICHS [*8*], KREISS [*9*], LAX [*10*] and LEES [*12*], [*13*], [*14*].

In a paper to follow, the energy method will be applied to the problem of determining the rate of convergence of iterative methods for solving elliptic difference equations. It will include, in particular, a solution to this problem for the alternating direction methods relative to an arbitrary domain.

2. Notation and Definitions

In this section we describe certain preliminary concepts, necessary for our formulation of the difference schemes. We denote by Ω a bounded, open subset of E_N, with boundary $\dot{\Omega}$. The Euclidean length of a point $x=(x_1, x_2, \ldots, x_N)\in E_N$ will be denoted by $|x|$. Let $h=(h_1, h_2, \ldots, h_N)\in E_N$ have positive coordinates, and define G_h to be the set of all (net) points $(i_1 h_1, i_2 h_2, \ldots, i_N h_N)\in E_N$ the i_j being integers, positive, negative, or zero. Two points $x, y\in G_h$ are called neighbors if $|x-y|=h_i$, for some i, $i=1, 2, \ldots, N$. The points $x\in G_h\cap\Omega$ all of whose neighbors belong to $\bar{\Omega}$, the closure of Ω, we denote by Ω_h. The points $x\in G_h-\Omega_h$ with the property that at least one neighbor belongs to Ω_h we denote by $\dot{\Omega}_h$. Finally, we put $\bar{\Omega}_h=\Omega_h\cup\dot{\Omega}_h$.

If M is any subset of E_N, we define $\mathscr{C}(M)$ to be the collection of all real-valued functions defined on E_N whose support* is contained in M. Clearly, $\mathscr{C}(M)$ is a real linear space for the usual operations. In particular, $\mathscr{C}(\Omega_h)$ is a real finite-dimensional linear space, with dimension equal to the number of points in Ω_h.

Let $S=[0, \infty)$ and $S^0=S-\{0\}$. For each positive number k, the time step, we put

$$S_k=\{t\in S\,|\,t=m k,\ m=0, 1, \ldots,\}$$

and $S_k^0=S_k\cap S^0$.

We shall approximate the solutions of differential equations by functions $t\to u(t)$, defined on S_k, taking their values in $\mathscr{C}(\bar{\Omega}_h)$.

If $u\in\mathscr{C}(E_N)$ we define the linear translation operators $E^{\pm i}$ as follows:

$$E^{\pm i}[u](x)=u(x_1, \ldots, x_i\pm h_i, \ldots, x_N).$$

In terms of these we define the first order forward and backward difference operators:

$$\nabla_i u=h_i^{-1}[E^i u-u]$$

and

$$\bar{\nabla}_i u=h_i^{-1}[u-E^{-i}u].$$

Evidently, we have the relation

$$E^{-i}[\nabla_i u]=\bar{\nabla}_i u. \tag{2.1}$$

The first order centered difference operator $\hat{\nabla}_i$ is defined by

$$\hat{\nabla}_i u=\frac{1}{2}[\nabla_i+\bar{\nabla}_i]u=\frac{1}{2h_i}[E^i-E^{-i}]u. \tag{2.2}$$

Difference operators of higher order are defined in the obvious way, by repeated application of these formulas.

* The support of a function f is the closure of the set $\{x\,|\,f(x)\neq 0\}$.

For a function $t \to u(t) \in \mathscr{C}(E_N)$, defined in S_k (resp. S_k^0), we use the notation

$$u_t(t) = k^{-1}[u(t+k) - u(t)]$$

and

$$u_{\bar{t}}(t) = k^{-1}[u(t) - u(t-k)].$$

3. The Linear Space $\mathscr{C}(\Omega_h)$

We provide $\mathscr{C}(\Omega_h)$ with an inner product defined as follows: if $u, v \in \mathscr{C}(\Omega_h)$,

$$(u, v) = h^N \sum_{x \in G_h} u(x)\, v(x),$$

where $h^N = h_1 h_2 \dots h_N$. Associated with this inner product is the norm $\|u\| = \sqrt{(u, u)}$.

The formula

$$\|u\|_1 = \Big(\sum_{i=1}^N \|\bar{\nabla}_i u\|^2\Big)^{\frac{1}{2}}$$

defines another norm for $\mathscr{C}(\Omega_h)$.

Since $\mathscr{C}(\Omega_h)$ is finite-dimensional, these norms are equivalent; that is, there exist two constants m and m_1 such that

$$m\|u\|^2 \leqq \|u\|_1^2 \leqq m_1 \|u\|^2,$$

for every $u \in \mathscr{C}(\Omega_h)$. Of course, m and m_1 depend on the dimension of $\mathscr{C}(\Omega_h)$ and therefore on the (net spacings) h_i.

We may assume that $\bar{\Omega}_h$ is contained in the rectangle

$$R: a_i \leqq x_i \leqq b_i,$$

where a_i and b_i are integral multiples of h_i.

Lemma 1. *If $u \in \mathscr{C}(\Omega_h)$ then*

$$m\|u\|^2 \leqq \|u\|_1^2,$$

where m is the minimal eigenvalue of the Laplace difference operator

$$\Delta_h(u) = \sum_{i=1}^N \bar{\nabla}_i \nabla_i u \tag{3.1}$$

relative to the net region Ω_h, and

$$m \geqq 4 \sum_{i=1}^N h_i^{-2} \sin^2\left[\frac{\pi h_i}{2(b_i - a_i)}\right].$$

Proof. The minimal eigenvalue m can be characterized (COURANT, FRIEDRICHS, and LEWY [2]) as follows:

$$m = \inf_{0 \neq u \in \mathscr{C}(\Omega_h)} \frac{\|u\|_1^2}{\|u\|^2},$$

and therefore

$$m \leqq \frac{\|u\|_1^2}{\|u\|^2},$$

for any $u \in \mathscr{C}(\Omega_h)$, not identically zero. This proves the first part of the lemma.

Now let $R_h = R^0 \cap G_h$, where R^0 is the interior of the rectangle R. Denoting by $m(R_h)$ the minimal eigenvalue of Δ_h relative to R_h, we have

$$\begin{aligned} m(R_h) &= \inf_{0 \neq u \in \mathscr{C}(R_h)} \frac{\|u\|_1^2}{\|u\|^2} \\ &\leqq \inf_{0 \neq u \in \mathscr{C}(\Omega_h)} \frac{\|u\|_1^2}{\|u\|^2} \\ &= m. \end{aligned}$$

But it is well known [7] that

$$m(R_h) = 4 \sum_{i=1}^{N} h_i^{-2} \sin^2 \left[\frac{\pi h_i}{2(b_i - a_i)}\right],$$

and this completes the proof of the lemma.

Lemma 2. *If* $u \in \mathscr{C}(\Omega_h)$ *then*

$$\|u\|_1^2 \leqq 4\left(\sum_{i=1}^{N} h_i^{-2}\right) \|u\|^2.$$

Proof. We have that

$$\begin{aligned} h_i^2 \|\bar{V}_i u\|^2 &= h^N \sum_{x \in G_h} [u(x) - E^{-i}[u](x)]^2 \\ &\leqq 2h^N \sum_{x \in G_h} [u^2(x) + E^{-i}[u^2](x)] \\ &= 4\|u\|^2. \end{aligned}$$

From this we obtain

$$\|u\|_1^2 = \sum_{i=1}^{N} \|\bar{V}_i u\|^2 \leqq 4\left(\sum_{i=1}^{N} h_i^{-2}\right) \|u\|_1^2,$$

which is what we set out to prove.

Remark. From lemma 1 we see that m is bounded away from zero, independently of h:

$$m \geqq 4 \sum_{i=1}^{N} (b_i - a_i)^{-2}.$$

On the other hand, m_1 is singular at $h=0$. This is to be expected since differentiation is an unbounded operator in L_2.

4. Difference Methods

In this section we describe the difference methods for the first boundary value problem for the parabolic equation

$$\frac{\partial U}{\partial t} = \sum_{i=1}^{N} \frac{\partial^2 U}{\partial x_i^2} \qquad (x \in \Omega,\ t > 0). \tag{4.1}$$

The function $t \to U(t) \in \mathscr{C}(\bar{\Omega})$, for $t \geqq 0$, is assumed to be sufficiently smooth and $U(t) \in C^P(\bar{\Omega})$ for a suitable integer P.

A function $t \to u(t) \in \mathscr{C}(\bar{\Omega}_h)$, defined for $t \in S_k$ will be called admissible if $u(0) = U(0)$, and $u = U$ on $\dot{\Omega}_h \times S_k^0$. In general, U is not known in $\dot{\Omega}_h \times S_k^0$, so that we make the assumption: there exists a null sequence $\{h^\beta\}$ of net spacings such that $\dot{\Omega}_{h^\beta} \subset \dot{\Omega}$. We always assume that h belongs to the sequence $\{h^\beta\}$.

With this assumption, an admissible function is uniquely specified in the complement of $\Omega_h \times S_k^0$ (relative to G_h), in terms of the initial and boundary values of U.

In the semi-explicit difference method for (4.1), one determines an admissible function $t \to u(t)$ such that, in $\Omega_h \times S_k$, the following difference equation is satisfied:

$$u_t = \Delta_h(u) - k \sum_{i=1}^{N} h_i^{-1} \overline{V}_i u_t, \tag{4.2}$$

where Δ_h is the Laplace difference operator defined in (3.1).

The classical explicit difference equation

$$u_t = \Delta_h(u) \tag{4.3}$$

is known to be conditionally stable [7], the condition being that the mesh ratio

$$\lambda = k \sum_{i=1}^{N} h_i^{-2}$$

satisfy $2\lambda \leqq 1$. We shall prove in section 5 that, by adding the "stabilizing" term

$$-k \sum_{i=1}^{N} h_i^{-1} \overline{V}_i u_t$$

to (4.1), we obtain an unconditionally stable difference equation.

It is not difficult to verify that (4.2) is an explicit difference equation when the net points in Ω_h are suitably ordered, for example, lexicographically.

A straightforward application of Taylor's theorem shows that U satisfies the difference equation (4.2) to within a term of order $k + |h|^2 + k \sum_{i=1}^{N} h_i^{-1}$. Therefore, [11] (4.2) is consistent with (4.1) if

$$k = o(1) \left(\sum_{i=1}^{N} h_i^{-1} \right)^{-1},$$

as $h \to 0$.

We now turn to the alternating direction methods. In the first alternating direction method for (4.1), one determines an admissible function $t \to u(t)$ such that

$$k^{-1}[u^{(1)}(t) - u(t-k)] = \overline{V}_1 V_1 u^{(1)}(t) + \sum_{i=2}^{N} \overline{V}_i V_i u(t-k),$$

$$k^{-1}[u^{(i+1)}(t) - u^{(i)}(t)] = \overline{V}_{i+1} V_{i+1} u^{(i+1)}(t) - \overline{V}_{i+1} V_{i+1} u(t-k),$$

for $i = 1, 2, \ldots, N-2$, and

$$k^{-1}[u(t) - u^{(N-1)}(t)] = \overline{V}_N V_N u(t) - \overline{V}_N V_N u(t-k),$$

where the auxiliary functions $t \to u^{(i)}(t) \in \mathscr{C}(\overline{\Omega}_h)$, defined for $t \in S_k^0$, are such that $u^{(i)} = u$ on $\dot{\Omega}_h \times S_K^0$. We assume, of course, that $N \geqq 2$, otherwise the method is undefined.

This difference scheme, for $N \leqq 3$, was investigated in [5], where it was shown that the operation taking $u(t-K)$ into $u(t)$ involves the inversion of $N\sqrt{p}$ tridiagonal matrices, where p is the dimension of $\mathscr{C}(\Omega_h)$.

Following DOUGLAS and RACHFORD, we eliminate from these equations the auxiliary functions $u^{(i)}$. Adding the equations, we obtain

$$u_{\bar t} = \bar V_N V_N u + \sum_{i=1}^{N} \bar V_i V_i u^{(i)}. \tag{4.4}$$

From the last equation

$$u^{(N-1)} = u - k \bar V_N V_N u_{\bar t}. \tag{4.5}$$

Similarly, solving the next to last equation for $u^{(N-2)}$, we find that

$$u^{(N-2)}(t) = u(t) - k \bar V_{N-1} V_{N-1} u^{(N-1)}(t) + k \bar V_{N-1} V_{N-1} u(t-k),$$

which, in view of (4.5), becomes

$$u^{(N-2)} = u - K^2 \bar V_N V_N u_{\bar t} + K^3 \bar V_N V_N \bar V_{N-1} V_{N-1} u_{\bar t}.$$

It is clear that, by continuing this process of elimination, we can determine the functions $u^{(i)}$ as a linear combination of u and certain of its difference quotients. When the resulting expressions for the $u^{(i)}$ are inserted into (4.4), we find that u is a solution of a single difference equation of order $2N+1$:

$$u_{\bar t} = \Delta_h(u) + \sum_{j=2}^{N} (-1)^{j+1} k^j D^j(u_{\bar t}), \tag{4.6}$$

where the difference operators D^j are defined by

$$D^j(u) = \sum{}^* \bar V_{i_1} V_{i_1} \bar V_{i_2} V_{i_2} \dots \bar V_{i_j} V_{i_j} u,$$

and the sum $\sum^*$ is extended over the $\binom{N}{j}$ different j-tuples $(i_1, i_2, \dots, i_j)$ formed from the first N positive integers.

The usual argument involving Taylor's theorem shows that U satisfies (4.6) to within a term of order $k+|h|^2$. Hence (4.6) is consistent with (4.1).

For the second alternating direction method we restrict ourselves to the case $N=2$. The method consists in determining an admissible function $t \to u(t)$ such that, in $\Omega_h \times S_K^0$,

$$2k^{-1}[v(t) - u(t-k)] = \bar V_1 V_1 v(t) + \bar V_2 V_2 u(t-k)$$

and

$$2k^{-1}[u(t) - v(t)] = \bar V_1 V_1 v(t) + \bar V_2 V_2 u(t),$$

the auxiliary function $t \to v(t) \in \mathscr{C}(\bar\Omega_h)$, defined for $t \in S_k^0$, being such that $u=v$ on $\dot\Omega_h \times S_k^0$.

Following PEACEMAN and RACHFORD, we eliminate the function v between these equations and find that u satisfies the fourth order difference equation

$$u_{\bar t} = \Delta_h(u) - \frac{k}{2} \Delta_h(u_{\bar t}) - \frac{k^2}{4} \bar V_1 V_1 \bar V_2 V_2 u_{\bar t}. \tag{4.7}$$

As before, we see that U satisfies (4.7) to within a term of order $k^2+|h|^2$.

In the following sections we shall prove that these difference equations are unconditionally stable in the following sense: if $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_k$, is a solution of (4.2), (4.6), or (4.7), then

$$\|u(t)\| \leq C \|u(0)\|,$$

where C depends only on N and the mesh ratio $\lambda = k\left(\sum_{i=1}^{N} h_i^{-2}\right)$.

5. Stability of the Semi-Explicit Method

Before stating the main theorem of this section, we prove several lemmas. These will be seen to be difference analogous of the usual quadratic differential identities and inequalities which are basic to the energy method.

Lemma 3. *The operators ∇_i and $-\overline{\nabla}_i$ are adjoints for functions in $\mathscr{C}(\Omega_h)$; that is, for any v and w in $\mathscr{C}(\Omega_h)$*

$$(v, \nabla_i w) = -(\overline{\nabla}_i v, w).$$

Proof. We have the identity

$$v \nabla_i w = \nabla_i [E^{-i}(v) w] - E^{-i}(\nabla_i v) w. \tag{5.1}$$

By (2.1), $E^{-i}(\nabla_i v) = \overline{\nabla}_i v$. Since $v, w \in \mathscr{C}(\Omega_h)$,

$$\sum_{x \in G_h} \nabla_i [E^{-i}(v) w] = 0,$$

and we conclude from (5.1) that

$$h^N \sum_{x \in G_h} v \nabla_i w = -h^N \sum_{x \in G_h} (\overline{\nabla}_i v) w,$$

which is equivalent to the conclusion of the lemma.

Lemma 4. *If $u \in \mathscr{C}(\Omega_h)$ then*

$$(u, \Delta_h(u)) = -\|u\|_1^2.$$

Proof. Taking $v = u$ and $w = \overline{\nabla}_i u$ in lemma 3, we find that

$$(u, \nabla_i \overline{\nabla}_i u) = -\|\overline{\nabla}_i u\|^2,$$

from which the desired result follows, by summation with respect to i.

Lemma 5. *For any function $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_K$, we have*

$$2(u, u_t) = (\|u\|^2)_t - k\|u_t\|^2$$

and

$$2(u, u_{\bar{t}}) = (\|u\|^2)_{\bar{t}} + k\|u_{\bar{t}}\|^2.$$

Proof. These identities are immediate consequences of

$$2u\, u_t = (u^2)_t - k(u_t)^2$$

and

$$2u\, u_{\bar{t}} = (u^2)_{\bar{t}} + k(u_{\bar{t}}^2).$$

(See [*14*].)

Lemma 6. *If* $u \in \mathscr{C}(\Omega_h)$ *then*

$$2\left(u, \sum_{i=1}^{N} h_i^{-1} \bar{\nabla}_i u\right) = \|u\|_1^2.$$

Proof. From lemma 3, with $v = w = u$, we obtain the relation

$$(u, \nabla_i u) = -(\bar{\nabla}_i u, u).$$

Since $\nabla_i - \bar{\nabla}_i = h_i \nabla_i \bar{\nabla}_i$, we conclude from this that

$$2(u, \bar{\nabla}_i u) = -h_i (u, \nabla_i \bar{\nabla}_i u).$$

Consequently,

$$\begin{aligned} 2\left(u, \sum_{i=1}^{N} h_i^{-1} \nabla_i u\right) &= -\left(u, \Delta_h(u)\right) \\ &= \|u\|_1^2, \end{aligned}$$

by lemma 4.

Lemma 7. *For any function* $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_k$, *we have*

$$2\left(u_t, \Delta_h(u)\right) = -\left(\|u\|_1^2\right)_t + k\|u_t\|_1^2.$$

Proof. From lemma 3

$$\begin{aligned} 2(u_t, \nabla_i \bar{\nabla}_i u) &= -2(\bar{\nabla}_i u_t, \bar{\nabla}_i u) \\ &= -\left(\|\bar{\nabla}_i u\|^2\right)_t + k\|\bar{\nabla}_i u_t\|^2, \end{aligned}$$

by lemma 5. The result follows by summation over i.

Lemma 8. *Let* $\mu = \left(\sum_{i=1}^{N} h_i^{-2}\right)^{\frac{1}{2}}$. *Then for any function* $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_k$, *we have*

$$2\left(u, \sum_{i=1}^{N} h_i^{-1} \bar{\nabla}_i u_t\right) \leqq \varepsilon \mu \|u_t\|^2 + \frac{1}{\varepsilon} \mu \|u\|_1^2,$$

where $\varepsilon > 0$ *is arbitrary.*

Proof. From lemma 3

$$\left(u, \sum_{i=1}^{N} h_i^{-1} \bar{\nabla}_i u_t\right) = \left(u_t, \sum_{i=1}^{N} h_i^{-1} \nabla_i u\right).$$

Applying Schwarz' inequality to the right side of this, we obtain

$$\begin{aligned} \left(u, \sum_{i=1}^{N} h_i^{-1} \bar{\nabla}_i u_t\right) &\leqq \|u_t\| \left\| \sum_{i=1}^{N} h_i^{-1} \nabla_i u \right\| \\ &\leqq \sum_{i=1}^{N} \|u_t\| h_i^{-1} \|\bar{\nabla}_i u\|, \end{aligned}$$

by the triangle inequality and the fact that $\|\nabla_i u\| = \|\bar{\nabla}_i u\|$.

Using Schwarz' inequality again, we find that

$$\left(u, \sum_{i=1}^{N} h_i^{-1} \bar{\nabla}_i u_t\right) \leqq \|u_t\| \mu \|u\|_1.$$

To the right side of this inequality we apply the generalized arithmetic-geometric mean inequality:

$$2\,a\,b\,\varepsilon \leqq a^2 + \frac{1}{\varepsilon}\,b^2 \qquad (\varepsilon > 0), \tag{5.2}$$

to obtain the desired inequality.

Theorem 1. *Let the function* $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_k$, *be a solution of the semi-explicit difference equation* (4.2). *Then*

$$\|u(t)\|^2 + k \sum_{\eta=0}^{t-k} \|u(\eta)\|_1^2 \leqq \big(1 + 2\lambda(1+\lambda)\big)\,\|u(0)\|^2,$$

which implies that (4.2) *is unconditionally stable.*

Proof. The difference equation (4.2) is satisfied only in $\Omega_h \times S_k$. But since $u(t) \in \mathscr{C}(\Omega_h)$, u vanishes on $\dot{\Omega}_h \times S_k$, and hence we may form the inner product of (4.2) with u to get

$$(u, u_t) = \big(u, \Delta_h(u)\big) - k\Big(u, \sum_{i=1}^{N} h_i^{-1}\,\overline{V}_i\,u_t\Big),$$

valid for $t \in S_k$. From this and lemmas 4, 5 and 8, we obtain, after multiplication by 2, the inequality

$$\begin{aligned}(\|u\|^2)_t - k\,\|u_t\|^2 + 2\,\|u\|_1^2 \leqq \varepsilon\,k\,\mu\,\|u_t\|^2 + \\ + \frac{1}{\varepsilon}\,k\,\mu\,\|u\|_1^2.\end{aligned} \tag{5.3}$$

If we choose ε so that $k\,\mu = \varepsilon$, then (5.3) becomes

$$(\|u\|^2)_t - k(1 + K\mu^2)\,\|u_t\|^2 + \|u\|_1^2 \leqq 0. \tag{5.4}$$

Similarly, we may form the inner product of (4.2) with u_t to obtain

$$\|u_t\|^2 = \big(u_t, \Delta_h(u)\big) - k\Big(u_t, \sum_{i=1}^{N} h_i^{-1}\,\overline{V}_i\,u_t\Big).$$

In view of lemmas 6 and 7, this becomes

$$2\,\|u_t\|^2 = -\,(\|u\|_1^2)_t.$$

When this result is combined with (5.4), we obtain the inequality

$$(\|u\|^2)_t + \frac{k}{2}\,(1 + k\,\mu^2)\,(\|u\|_1^2)_t + \|u\|_1^2 \leqq 0.$$

Since

$$k \sum_{\eta=0}^{t-k} (\|u\|^2)_t = \|u(t)\|^2 - \|u(0)\|^2$$

we have from this that

$$\|u(t)\|^2 + k \sum_{\eta=0}^{t-k} \|u(\eta)\|_1^2 \leqq \|u(0)\|^2 + \frac{k}{2}\,(1+\lambda)\,\|u(0)\|_1^2,$$

since $k\,\mu^2 = \lambda$. The desired result follows by applying lemma 2 to the last term.

If we denote by e the error $U-u$ for the semi-explicit method, then $e(0)=0$, and $e(t)\in\mathscr{C}(\Omega_h)$ for $t\in S_k$. Also, e satisfies, in $\Omega_h\times S_k$, the difference equation

$$e_t=\Delta_h(e)-k\sum_{i=1}^{N}h_i^{-1}\overline{V}_i e_t+T,$$

where the function $t\to T(t)\in\mathscr{C}(\Omega_h)$ is the local truncation error. It satisfies

$$\Big(k\sum_{\eta=0}^{t-K}\|T(\eta)\|^2\Big)^{\frac{1}{2}}=O\Big(k+|h|^2+k\sum_{i=1}^{N}h_i^{-1}\Big),$$

where the constant implied in this relation is independent of k, h and λ.

The following result can be proved along the lines of theorem 1.

Theorem 2. *The error $e=U-u$ for the semi-explicit method for* (4.1) *satisfies the inequality*

$$\|e(t)\|^2+k\sum_{\eta=0}^{t-k}\|e(\eta)\|_1^2\leq Mk\sum_{\eta=0}^{t-k}\|T(\eta)\|^2,$$

where

$$M=\frac{2}{m}+k+2k^2\sum_{i=1}^{N}h_i^{-2}.$$

Proof. As in the proof of theorem 1, we obtain

$$(\|e\|^2)_t-k\|e_t\|^2+2\|e\|_1^2\leq\varepsilon k\mu\|e_t\|^2+\frac{1}{\varepsilon}k\mu\|e\|_1^2+2(e,T). \tag{5.5}$$

In view of Schwarz' inequality and (5.2), with ε replaced by ε', we have

$$2(e,T)\leq\varepsilon'\|e\|^2+\frac{1}{\varepsilon'}\|T\|^2.$$

According to lemma 1, $m\|e\|^2\leq\|e\|_1^2$, so that

$$2(e,T)\geq\frac{\varepsilon'}{m}\|e\|_1^2+\frac{1}{\varepsilon'}\|T\|^2,$$

and when this is combined with (5.5), we obtain

$$(\|e\|)_t^2-k(1+\varepsilon k\mu)\|e_t\|^2+\Big(2-\frac{k}{\varepsilon}\mu-\frac{\varepsilon'}{m}\Big)\|e\|_1^2\leq\frac{1}{\varepsilon'}\|T\|^2.$$

We now choose ε and ε' so that $2k\mu=\varepsilon$ and $2\varepsilon'=m$. Then

$$(\|e\|^2)_t-k(1+2\lambda)\|e_t\|^2+\|e\|_1^2\leq\frac{2}{m}\|T\|^2. \tag{5.6}$$

As before, we have

$$\|e_t\|^2=-\tfrac{1}{2}(\|e\|_1^2)_t+(e_t,T)$$

which, after invoking Schwarz' inequality, becomes

$$\|e_t\|^2\leq-(\|e\|^2)_t+\|T\|^2.$$

This combined with (5.6) gives

$$\left(\|e\|^2\right)_t + k(1+2\lambda)\left(\|e\|_1^2\right)_t + \|e\|_1^2 \leq \left[\frac{2}{m} + k(1+2\lambda)\right] \|T\|_1^2,$$

from which the desired result follows, since $\|e(0)\| = \|e(0)\|_1 = 0$.

6. Generalizations

We now indicate briefly how the results of the preceding section can be extended to parabolic equations of the form

$$\frac{\partial U}{\partial t} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a^{ij}(x,t)\frac{\partial U}{\partial x_j}\right), \tag{6.1}$$

for $x \in \Omega$ and $0 < t \leqq t_0$. We assume that $a_{ij} = a_{ji}$ and that, for $x \in \bar{\Omega}$, $0 \leqq t \leqq t_0$,

$$\varrho_1 |\xi|^2 \geqq \sum_{i,j=1}^{N} a^{ij}(x,t)\,\xi_i \xi_j \geqq \varrho |\xi|^2,$$

where ξ is any real N-vector and $\varrho, \varrho_1 > 0$.

We approximate (6.1) by the semi-explicit difference equation (see (2.2))

$$\begin{aligned} u_t = \sum_{i=1}^{N} \bar{\nabla}_i (\tilde{a}^{ij} \nabla_i u) + \sum_{j \neq i}^{N} \hat{\nabla}_t (a^{ij} \hat{\nabla}_j u) - \\ - k \sum_{i=1}^{N} (\varrho_2 + \varrho_1 h_i^{-1}) \bar{\nabla}_i u_t, \end{aligned} \tag{6.2}$$

where

$$\tilde{a}^{ii}(x,t) = a^{ii}\left(x_1, \ldots, x_i + \frac{h_i}{2}, \ldots, x_N, t\right)$$

and

$$\varrho_2 = \sup_{\substack{x \in \bar{\Omega} \\ 0 \leqq t \leqq t_0}} \left|\frac{\partial a^{ii}}{\partial x_i}\right|.$$

Denote by $L_h(u)$ the sum of the first two terms on the right side of (6.2). It can be shown that

$$\left(u, L_h(u)\right) \geqq -\left[\varrho + O(|h|)\right] \|u\|_1^2, \tag{6.3}$$

for all $u \in \mathscr{C}(\Omega_h)$.

Similarly, for any function $t \to u(t) \in \mathscr{C}(\Omega_h)$,

$$\begin{aligned} \left(u_t, L_h(u)\right) \leqq (\Lambda u) + \varrho_3 \|u\|_1^2 + \\ + \frac{k}{2}\left(\varrho_2 + |h|\,\varrho_1\right) \|u_t\|^2, \end{aligned} \tag{6.4}$$

where

$$\varrho_3 = \sup_{\substack{x \in \bar{\Omega} \\ 0 \leqq t \leqq t_0}} \left|\frac{\partial a^{ij}}{\partial t}\right|,$$

and Λu is bounded, both above and below, by a constant multiple of

$$\left[1 + O\left(k + |h|\right)\right] \|u\|_1^2.$$

With the aid of the inequalities (6.3) and (6.4) the arguments employed in theorems 1 and 2 can be carried over to prove that the semi-explicit difference equation (6.2) is unconditionally stable, provided k and $|h|$ are sufficiently small.

7. Stability of the First Alternating Direction Method

For $j>1$ put

$$\|u\|_j = \left(\sum{}^* \|\bar{\nabla}_{i_1} \bar{\nabla}_{i_2} \dots \bar{\nabla}_{i_j} u\|^2\right)^{\frac{1}{2}},$$

for $u \in \mathscr{C}(\Omega_h)$. Then we have

Lemma 9. *If* $u \in \mathscr{C}(\Omega_h)$ *then*

$$\left(u, \sum_{j=2}^{N} (-1)^{j+1} k^j D^j(u)\right) = -\sum_{j=2}^{N} k^j \|u\|_j^2.$$

Proof. From the adjointness of the operators ∇_i and $-\bar{\nabla}_i$ we have, for $1 \leqq r \leqq j$,

$$(u, \nabla_{i_1} \bar{\nabla}_{i_1} \dots \nabla_{i_j} \bar{\nabla}_{i_j} u) = -(\bar{\nabla}_{i_r} u, \nabla_{i_1} \bar{\nabla}_{i_1} \dots \nabla_{i_r} \dots \nabla_{i_j} \bar{\nabla}_{i_j} u).$$

From this we obtain the relation

$$(u, \nabla_{i_1} \bar{\nabla}_{i_1} \dots \nabla_{i_j} \bar{\nabla}_{i_j} u) = (-1)^j \|\bar{\nabla}_{i_1} \dots \bar{\nabla}_{i_j} u\|^2.$$

Summing both sides of this relation over the $\binom{N}{j}$ j-tuples, we find that

$$\left(u, D^j(u)\right) = (-1)^j \|u\|_j^2.$$

The desired result follows from this by multiplication by $(-1)^{j+1} k^j$ and summation over j.

The arguments of lemmas 7 and 10 can be employed to prove the

Lemma 10. *For any function* $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_k$, *we have*

$$2\left(u, \sum_{j=2}^{N} (-1)^{j+1} k^j D^j u_{\bar{t}}\right) = -\sum_{j=2}^{N} k^j (\|u\|_j^2)_{\bar{t}} - \sum_{j=2}^{N} k^{j+1} \|u_{\bar{t}}\|_j^2.$$

Theorem 3. *If the function* $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_k$, *is a solution of the difference equation* (4.6), *then*

$$\|u(t)\|^2 + 2k \sum_{\eta=k}^{t} \|u(\eta)\|_1^2 \leqq c \|u(0)\|^2,$$

where

$$c = (1 + 4\lambda)^N - 4\lambda.$$

Hence, the first alternating direction method is unconditionally stable.

Proof. Since $u(t) \in \mathscr{C}(\Omega_h)$ for each $t \in S_k^0$, we have

$$(u, u_{\bar{t}}) = (u, \Delta_h u) + \left(u, \sum_{j=2}^{N} (-1)^{j+1} k^j D^j u_{\bar{t}}\right),$$

valid for $t \in S_K^0$. Using lemmas 4, 5, and 10, this becomes

$$(\|u\|^2)_{\bar{t}} + k\|u_{\bar{t}}\|^2 + 2\|u\|_1^2 + \sum_{j=2}^{N} k^j (\|u\|_j^2)_{\bar{t}} = -\sum_{j=2}^{N} k^{j+1} \|u_{\bar{t}}\|_j^2.$$

Dropping the second term on the left and the negative term on the right, we find that

$$\|u(t)\|^2 + 2k\sum_{\eta=k}^{t}\|u(\eta)\|_1^2 \le \|u(0)\|^2 + \sum_{j=2}^{N} k^j \|u(0)\|_j^2. \tag{7.2}$$

But, from lemma 2, we find that

$$\begin{aligned} k^j\|u(0)\|_j^2 &= k^j \sum{}^* \|\bar{V}_{i_1}\ldots\bar{V}_{i_j} u(0)\|^2 \\ &\le 4^j k^j \|u(0)\|^2 \sum{}^* h_{i_1}^{-2}\ldots h_{i_j}^{-2} \\ &\le 4^j \|u(0)\|^2 \Delta^j \binom{N}{j}, \end{aligned}$$

where

$$\Delta = \max_i k h_i^{-2}.$$

Therefore,

$$\begin{aligned} \sum_{j=2}^{N} k^j \|u(0)\|_j^2 &\le [(1+4\Delta)^N - 4N\Delta - 1]\,\|u(0)\|^2 \\ &\le [(1+4\lambda)^N - 1 - 4\lambda]\,\|u(0)\|^2, \end{aligned}$$

and this, together with (7.2), completes the proof of the theorem.

As before, we can prove the

Theorem 4. *If $e = U - u$ is the error for the first alternating direction method, then*

$$\|e(t)\|^2 + k\sum_{\eta=k}^{t}\|e(\eta)\|_1^2 \le \frac{1}{m}\,k\sum_{\eta=k}^{t}\|T(\eta)\|^2,$$

where $t \to T(t) \in \mathscr{C}(\Omega_h)$ is the local truncation error, and $T = O(k^2 + |h|^2)$.

Similar results can be proved for parabolic equations of the form

$$\frac{\partial U}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a^{ii}(x,t)\frac{\partial U}{\partial x_i}\right). \tag{7.3}$$

8. Stability of the Second Alternating Direction Method

Lemma 11. *For any function $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_k$, we have*

$$2(u_{\bar{t}}, \Delta_h u) = -\left(\|u\|_1^2\right)_{\bar{t}} - k\|u_{\bar{t}}\|_1^2.$$

The proof is similar to that of lemma 7 and therefore will be omitted.

Theorem 5. *If the function $t \to u(t) \in \mathscr{C}(\Omega_h)$, $t \in S_k$, is a solution of the difference equation* (4.7), *then*

$$\|u(t)\|^2 + 2k\sum_{\eta=k}^{t}\|u(\eta)\|_1^2 \le [4\zeta - 8\lambda + (1+4\lambda)^2]\,\|u(0)\|^2,$$

where $\zeta = \frac{1}{2}$ if $4\lambda \le 1$ and $\zeta = 4\lambda - \frac{1}{2}$ if $4\lambda \ge 1$. Hence, the second alternating direction method is unconditionally stable.

Proof. From (4.7) we have, as before, that

$$\begin{aligned} (u, u_{\bar{t}}) &= (u, \Delta_h u) - \frac{k}{2}(u, \Delta_h u_{\bar{t}}) \\ &\quad - \frac{k^2}{4}(u, \bar{V}_1 V_1 \bar{V}_2 V_2 u_{\bar{t}}). \end{aligned}$$

In view of lemmas 4, 5, and 10, this becomes

$$\left(\|u\|^2\right)_{\bar t} + k\|u_{\bar t}\|^2 + 2\|u\|_1^2 = \frac{k}{2}\left(\|u\|_1^2\right)_{\bar t} + \frac{k^2}{2}\|u_{\bar t}\|_1^2 - \frac{k^2}{4}\left(\|u\|_2^2\right)_{\bar t} - \frac{k^3}{4}\|u_{\bar t}\|_2^2.$$

Taking the inner product of (4.7) with $u_{\bar t}$, we find that

$$\|u_{\bar t}\|^2 = -\frac{1}{2}\left(\|u\|_1^2\right)_{\bar t} - \frac{k^2}{4}\|u_{\bar t}\|_2^2. \tag{8.2}$$

We now multiply (8.2) through by $2\zeta k$ and add the result to (8.1) to obtain

$$\begin{aligned}\left(\|u\|^2\right)_{\bar t} + k(2\zeta+1)\|u_{\bar t}\|u^2 + 2\|u\|_1^2 + (\zeta - \tfrac{1}{2})k\left(\|u\|_1^2\right)_{\bar t} \\ \leqq \frac{k^2}{2}\|u_{\bar t}\|_1^2 - \frac{k^2}{4}\left(\|u\|_2^2\right)_{\bar t}.\end{aligned} \tag{8.3}$$

According to lemma 2,

$$k^2\|u_{\bar t}\|_1^2 \leqq 4\lambda\|u_{\bar t}\|^2,$$

so that (8.3) implies that

$$\left(\|u\|^2\right)_{\bar t} + (2\zeta + 1 - 4\lambda)\,k\|u_{\bar t}\|^2 + 2\|u\|_1^2 + \left(\zeta - \frac{1}{2}\right)k\left(\|u\|_1^2\right)_{\bar t} + \frac{k^2}{4}\left(\|u\|_2^2\right)_{\bar t} \leqq 0.$$

In view of the definition of ζ, we have from this that

$$\|u(t)\|^2 + 2k\sum_{\eta=k}^{t}\|u(\eta)\|_1^2 \leqq \|u(0)\|^2 + \left(\zeta - \frac{1}{2}\right)k\|u(0)\|_1^2 + \frac{k^2}{4}\|u(0)\|_2^2,$$

and this, with the aid of lemma 2, gives the desired result.

Similarly, we have the

Theorem 6. *If $e = U - u$ is the error for the second alternating direction method, then*

$$\|e(t)\|^2 + k\sum_{\eta=k}^{t}\|e(\eta)\|_1^2 \leqq \left(\frac{1}{m} + \zeta\right)k\sum_{\eta=k}^{t}\|T(\eta)\|^2,$$

where ζ is defined in theorem 5, and $t \to T(t) \in \mathscr{C}(\Omega_h)$ is the local truncation error: $T = O(k^2 + |h|^2)$.

Similar results can be proved for the parabolic equation (7.3).

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