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# Absolute and monotonic norms

By

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Studying the mechanism used to derive certain inclusion theorems [2] and exclusion theorems [1], one observes that the following property of certain bound norms is essential: the least upper bound of a diagonal matrix is the maximum of the moduli of the diagonal elements. In this paper, we characterize the class of these least upper bound norms and the class of vector norms to which they are subordinate, and show some of its properties. We complete some results given already in [1] and prepare the basis for the following paper [3]. Some of the concepts used in this paper have been considered in the general context of partially and lattice-ordered vector spaces (KANTOROVIC [9], FREUDENTHAL [7], BIRKHOFF [5]).

## 1.

Let $\operatorname{norm}(x)$ denote a norm[1] in an $n$-dimensional complex coordinate space. Let $|x|$ denote the vector the components of which are the moduli of the components of $x$. Inequalities between vectors are understood to hold componentwise. We then call $\operatorname{norm}(x)$

(1.1) monotonic[2]

if

$$|x| \leqq |y| \quad \text{implies} \quad \operatorname{norm}(x) \leqq \operatorname{norm}(y).$$

Further, a norm is called

(1.2) absolute

if it depends only on the moduli of its components, that is, if

$$\operatorname{norm}(x) = \operatorname{norm}(|x|) \quad \text{for all } x.$$

It is well known ([10], [8]) that every norm in the space of column vectors $x$ induces a dual norm

$$\operatorname{norm}^D(y^H) := \max_{x \neq 0} \frac{\operatorname{Re}(y^H x)}{\operatorname{norm}(x)} \tag{1.3}$$

in the dual space of row vectors $y^H$.

**Theorem 1.** *The dual of an absolute norm is again absolute.*

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[1] We require only weak homogenity, that is, $\operatorname{norm}(\alpha x) = \alpha \operatorname{norm}(x)$ for $\alpha \geqq 0$.

[2] FIEDLER and PTAK [6] use a slightly different notion of monotonicity. Compare also (3.1).

*Proof.* For every $y^H$ and $x$ there exists a vector $\bar{x}=\bar{x}(x,y)$ with $|\bar{x}|=|x|$ such that $y^H\bar{x}=|y|^H|x|$. We have

$$\operatorname{norm}^D(y^H)=\max_{x\neq 0}\frac{\operatorname{Re}(y^H x)}{\operatorname{norm}(x)}\geqq\max_{x\neq 0}\frac{\operatorname{Re}(y^H\bar{x})}{\operatorname{norm}(\bar{x})},$$

by restricting the set. Then $\operatorname{Re}(y^H\bar{x})$ may be replaced by $|y^H|\,|x|$, and $\operatorname{norm}(\bar{x})$ by $\operatorname{norm}(x)$, since this is an absolute norm. Thus

$$\operatorname{norm}^D(y^H)\geqq\max_{x\neq 0}\frac{|y^H|\,|x|}{\operatorname{norm}(x)}.$$

On the other hand,

$$\max_{x\neq 0}\frac{\operatorname{Re}(y^H x)}{\operatorname{norm}(x)}\leqq\max_{x\neq 0}\frac{|y^H|\,|x|}{\operatorname{norm}(x)},$$

and therefore,

$$\operatorname{norm}^D(y^H)=\max_{x\neq 0}\frac{|y^H|\,|x|}{\operatorname{norm}(x)}. \tag{1.4}$$

Thus, $\operatorname{norm}^D(y^H)$ depends only on $|y^H|$, q.e.d.

After this preparation, we prove

**Theorem 2.** *An absolute norm is monotonic and vice versa.*

*Proof.* Let $\operatorname{norm}(x)$ be absolute. Then its dual is absolute too (theorem 1), and formula (1.4) applied to $\operatorname{norm}^D(u^H)$ gives

$$\operatorname{norm}(x)=\max_{u\neq 0}\frac{|u^H|\,|x|}{\operatorname{norm}^D(u^H)}=\frac{|\bar{u}^H|\,|x|}{\operatorname{norm}^D(\bar{u}^H)} \tag{1.5}$$

for some maximizing $\bar{u}^H$. Now if $|x|\leqq|y|$, we have

$$\operatorname{norm}(x)\leqq\frac{|\bar{u}^H|\,|y|}{\operatorname{norm}^D(\bar{u}^H)}\leqq\max_{u\neq 0}\frac{|u^H|\,|y|}{\operatorname{norm}^D(u^H)}=\operatorname{norm}(y).$$

Assume that $\operatorname{norm}(x)$ is monotonic. Put $y:=|x|$. Then $|x|=|y|$, which implies $\operatorname{norm}(x)\leqq\operatorname{norm}(y)=\operatorname{norm}(|x|)$, but also $\operatorname{norm}(y)=\operatorname{norm}(|x|)\leqq\operatorname{norm}(x)$. Therefore, $\operatorname{norm}(x)=\operatorname{norm}(|x|)$.

In particular, a monotonic norm is always strictly homogeneous[3].

## 2.

Let $\operatorname{norm}_{\mathrm{I}}(x)$ be a norm in the $n$-dimensional, and $\operatorname{norm}_{\mathrm{II}}(y)$ a norm in the $m$-dimensional complex coordinate space. Then

$$\operatorname{lub}_{\mathrm{II},\mathrm{I}}(A):=\max_{x\neq 0}\frac{\operatorname{norm}_{\mathrm{II}}(Ax)}{\operatorname{norm}_{\mathrm{I}}(x)} \tag{2.1}$$

defines a norm in the linear space of $m\times n$-matrices $A$, which is called the

(2.2) least upper bound norm

subordinate to $\operatorname{norm}_{\mathrm{I}}(x)$ and $\operatorname{norm}_{\mathrm{II}}(y)$. Both norms may be identical. The following theorem refers to this case.

**Theorem 3.** $\operatorname{norm}(x)$ *is absolute/monotonic if and only if for the subordinate bound norm,*

$$\operatorname{lub}(D)=\max_i|d_{ii}| \tag{2.3}$$

*holds for any diagonal matrix* $D=\operatorname{diag}(d_{ii})$.

[3] That is, $\operatorname{norm}(\alpha x)=|\alpha|\operatorname{norm}(x)$ is true for every $\alpha$.

Bound norms with property (2.3) have been called axis oriented in a previous paper [1].

*Proof.* $|Dx| \leqq (\max_i |d_{ii}|)|x|$ is true for any $x$. If $\operatorname{norm}(x)$ is monotonic, then $\operatorname{norm}(Dx) \leqq (\max_i |d_{ii}|) \operatorname{norm}(x)$ for every $x$, i.e.,

$$\operatorname{lub}(D) \leqq \max_i |d_{ii}|.$$

But, since[4] $\operatorname{lub}(D) \geqq |d_{ii}|$ for all $i$,

$$\operatorname{lub}(D) \geqq \max_i |d_{ii}|$$

and therefore (2.3) holds.

For each $x$ there exists a diagonal matrix $D_x$ such that

$$x = D_x|x| \quad \text{and} \quad |D_x| = I.$$

Now, if (2.3) holds for every diagonal matrix, we have

$$\operatorname{norm}(x) \leqq \operatorname{lub}(D_x) \operatorname{norm}(|x|) = \operatorname{norm}(|x|)$$
$$\operatorname{norm}(|x|) \leqq \operatorname{lub}(D_x^{-1}) \operatorname{norm}(x) = \operatorname{norm}(x),$$

and therefore, $\operatorname{norm}(x) = \operatorname{norm}(|x|)$.

## 3.

The bound norm $\operatorname{lub}_{\mathrm{II},\mathrm{I}}(A)$ of two monotonic norms is not necessarily a monotonic norm. Indeed, the euclidean norm is a monotonic norm, but the euclidean bound norm is not. However, monotonicity holds in a weaker sense.

We call $\operatorname{norm}(x)$

(3.1) monotonic in the positive orthant

if

$$0 \leqq x \leqq y \quad \text{implies } \operatorname{norm}(x) \leqq \operatorname{norm}(y).$$

**Theorem 4.** *The bound norm* $\operatorname{lub}_{\mathrm{II},\mathrm{I}}(A)$ *of two absolute norms is monotonic in the positive orthant.*

*Proof.* We have for $0 \leqq A$

$$\frac{\operatorname{norm}_{\mathrm{II}}(Ax)}{\operatorname{norm}_{\mathrm{I}}(x)} = \frac{\operatorname{norm}_{\mathrm{II}}(|Ax|)}{\operatorname{norm}_{\mathrm{I}}(|x|)} \leqq \frac{\operatorname{norm}_{\mathrm{II}}(A|x|)}{\operatorname{norm}_{\mathrm{I}}(|x|)}.$$

Therefore, an $x_0 \geqq 0$, $x_0 \neq 0$ exists such that

$$\operatorname{lub}_{\mathrm{II},\mathrm{I}}(A) = \max_{x \neq 0} \frac{\operatorname{norm}_{\mathrm{II}}(Ax)}{\operatorname{norm}_{\mathrm{I}}(x)} = \frac{\operatorname{norm}_{\mathrm{II}}(Ax_0)}{\operatorname{norm}_{\mathrm{I}}(x_0)}.$$

If now $0 \leqq A \leqq B$, then $|Ax_0| \leqq |Bx_0|$, and therefore, $\operatorname{norm}_{\mathrm{II}}(Ax_0) \leqq \operatorname{norm}_{\mathrm{II}}(Bx_0)$, or

$$\operatorname{lub}_{\mathrm{II},\mathrm{I}}(A) \leqq \frac{\operatorname{norm}_{\mathrm{II}}(Bx_0)}{\operatorname{norm}_{\mathrm{I}}(x_0)} \leqq \max_{x \neq 0} \frac{\operatorname{norm}_{\mathrm{II}}(Bx)}{\operatorname{norm}_{\mathrm{I}}(x)} = \operatorname{lub}_{\mathrm{II},\mathrm{I}}(B).$$

Any norm that is monotonic in the positive orthant coincides in that orthant with an absolute norm. Indeed, parallel to theorem 2, we have

[4] In general, let $Ax = \lambda x$, $x \neq 0$. Then $\operatorname{norm}(x) > 0$, and for a strictly homogeneous norm, $|\lambda| \operatorname{norm}(x) \leqq \operatorname{lub}(A) \operatorname{norm}(x)$, that is, $|\lambda| \leqq \operatorname{lub}(A)$.

**Theorem 5.** norm$(x)$ *is monotonic in the positive orthant if and only if*

$$f(x) := \operatorname{norm}(|x|)$$

*is a norm again.*

*Proof.* Obviously, $f(x)$ is definite and (strictly) homogeneous for any norm$(x)$. Moreover, if norm$(x)$ is monotonic in the positive orthant, $|x+y| \leqq |x|+|y|$ implies $f(x+y) = \operatorname{norm}(|x+y|) \leqq \operatorname{norm}(|x|+|y|)$, and from the convexity of norm$(x)$ it follows that

$$f(x+y) \leqq \operatorname{norm}(|x|+|y|) \leqq \operatorname{norm}(|x|) + \operatorname{norm}(|y|) = f(x) + f(y).$$

This proves[5] one half of the theorem. The other half is an immediate consequence of theorem 2 since $f(x)$, if it is a norm, is an absolute norm and coincides with norm$(x)$ in the positive orthant.

Since $\operatorname{lub}_{\mathrm{II},\mathrm{I}}(A)$, subordinate to two monotonic norms, is monotonic in the positive orthant, $\operatorname{lub}_{\mathrm{II},\mathrm{I}}(|A|)$ is a norm, too. Moreover

$$\operatorname{lub}_{\mathrm{II},\mathrm{I}}(A) \leqq \operatorname{lub}_{\mathrm{II},\mathrm{I}}(|A|) \tag{3.2}$$

holds, since $|A\,x| \leqq |A|\,|x|$ gives

$$\operatorname{norm}_{\mathrm{II}}(A\,x) \leqq \operatorname{norm}_{\mathrm{II}}(|A|\,|x|) \leqq \operatorname{lub}_{\mathrm{II},\mathrm{I}}(|A|)\operatorname{norm}_{\mathrm{I}}(x).$$

## 4.

Every norm satisfies together with its dual norm the Hölder inequality

$$\operatorname{norm}(x)\operatorname{norm}^D(y^H) \geqq \operatorname{Re}(y^H x). \tag{4.1}$$

We call $(x, y^H)$ a dual pair if equality holds in (4.1). For every vector $x$ there exists a dual vector $y^H$ such that $(x, y^H)$ are a dual pair.

If norm$(x)$ is strictly homogeneous we have

$$\operatorname{norm}(x)\operatorname{norm}^D(y^H) \geqq |y^H x|. \tag{4.2}$$

But, the following even stronger version of the Hölder inequality is valid for absolute norms,

$$\operatorname{norm}(x)\operatorname{norm}^D(y^H) \geqq |y^H|\,|x|. \tag{4.3}$$

**Theorem 6.** norm$(x)$ *is absolute/monotonic if and only if the strong version* (4.3) *of the Hölder inequality holds for every pair of vectors* $x, y^H$.

*Proof.* Obviously,

$$\operatorname{norm}(x) = \max_{y \neq 0} \frac{\operatorname{Re}(y^H x)}{\operatorname{norm}^D(y^H)} \leqq \max_{y \neq 0} \frac{|y^H|\,|x|}{\operatorname{norm}^D(y^H)}.$$

Now the strong Hölder inequality (4.3) implies

$$\operatorname{norm}(x) \geqq \max_{y \neq 0} \frac{|y^H|\,|x|}{\operatorname{norm}^D(y^H)}$$

and therefore,

$$\operatorname{norm}(x) = \max_{y \neq 0} \frac{|y^H|\,|x|}{\operatorname{norm}^D(y^H)}.$$

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[5] This proof is essentially the same as given by OSTROWSKI [*11*] in the more general case of compound norms.

Hence, norm$(x)$ depends only on $|x|$. On the other hand, if norm$(x)$ is absolute then by theorem 1 its dual is absolute, too. Therefore

$$\operatorname{norm}(x)\operatorname{norm}^D(y^H)=\operatorname{norm}(|x|)\operatorname{norm}^D(|y^H|)\geqq|y^H|\,|x|$$

holds for arbitrary vectors $x$, $y^H$, q.e.d.

It is plain that versions (4.2) and (4.3) of the Hölder inequality are both-sided sharp. Indeed, for dual pairs $(x, y^H)$ we have in case of strictly homogeneous norms

$$\operatorname{norm}(x)\operatorname{norm}^D(y^H)=y^H x=|y^H x|, \tag{4.4}$$

and in case of absolute norms

$$\operatorname{norm}(x)\operatorname{norm}^D(y^H)=y^H x=|y^H|\,|x|. \tag{4.5}$$

Note, that equality in (4.3) or (4.2) does not imply equality in (4.1). Thus it must be remembered that we define duality of vectors with respect to (4.1), i.e., a dual pair of vectors is required to realize equality in (4.1).

## 5.

In this part we shall study properties of vectors which are a dual pair with respect to an absolute norm. Then the strong Hölder inequality (4.3) holds and implies (4.5). This leads at once to

**Theorem 7.** *If* norm$(x)$ *is absolute/monotonic, then for every dual pair* $(x, y^H)$, $x>0$ *implies* $y^H\geqq 0$[6].

If some components of a nonnegative vector $x$ vanish, then the corresponding components of some dual $y^H$ may be negative or not even real. But if so, then $|y^H|$, too, is dual to $x$ since

$$\operatorname{norm}(x)\operatorname{norm}^D(|y^H|)=\operatorname{norm}(x)\operatorname{norm}^D(y^H)=|y^H|\,|x|=\operatorname{Re}(|y^H|x).$$

We may sharpen this result.

**Theorem 8.** *If* norm$(x)$ *is absolute/monotonic, then for each nonvanishing* $x\geqq 0$ *exists a dual* $\bar{y}^H\geqq 0$ *such that* $x_i=0$ *implies* $\bar{y}_i=0$. *(*$x_i$ *and* $\bar{y}_i$ *denote components of* $x$ *and* $\bar{y}^H$.*)*

*Proof.* Choose any dual $y^H$. Consider the vector $\bar{y}^H$ the components of which vanish if the corresponding components of $x$ vanish, and which equals $|y^H|$ otherwise. We shall show that $\bar{y}^H$ is dual to $x$. This will complete the proof.

We note

$$\operatorname{Re}(\bar{y}^H x)=\operatorname{Re}(y^H x)=\operatorname{norm}(x)\operatorname{norm}^D(y^H). \tag{5.1}$$

The last product of norms does not vanish, since $x\neq 0$ by hypothesis, and $y^H\neq 0$ by definition of a dual vector. Hence

$$\bar{y}^H\neq 0.$$

It remains to verify that $\bar{y}^H$ and $x$ satisfy the equality sign in the Hölder inequality.

[6] "$>$" between vectors is understood to hold componentwise.

(5.1), the monotonicity of $\text{norm}^D$, and finally the Hölder inequality (4.1) imply

$$\operatorname{Re}(\bar{y}^H x) = \text{norm}(x)\,\text{norm}^D(y^H) \geqq \text{norm}(x)\,\text{norm}^D(\bar{y}^H) \geqq \operatorname{Re}(\bar{y}^H x).$$

Therefore,

$$\text{norm}(x)\,\text{norm}^D(\bar{y}^H) = \operatorname{Re}(\bar{y}^H x), \quad \text{q.e.d.}$$

The geometric significance of theorem 8 can be seen more clearly in the case of a real coordinate space. Consider a dual pair $(x, y^H)$ which is scaled such that

$$\text{norm}(x) = \text{norm}^D(y^H) = 1.$$

Then $y^H$ characterizes a hyperplane supporting the norm-convex

$$K := \{u : \text{norm}(u) \leqq 1\}$$

at the boundary point $x$ of $K$. Now, if an $x$ with $\text{norm}(x) = 1$ belongs to a proper coordinate subspace $S$, then theorem 8 shows the existence of a supporting hyperplane $H$ through $x$ such that the angle $\sphericalangle(S, H)$ is a right one.

Moreover, from theorem 8 it follows immediately

**Theorem 9.** *For absolute/monotonic norms, axis vectors $e^i$ are self-dual, i.e.,*

$$\text{norm}(e^i)\,\text{norm}^D\left((e^i)^H\right) = (e^i)^H e^i = 1.$$

A consequence of theorem 9 has been used in [1] to derive lower bounds for the condition of a matrix of eigenvectors. Another consequence is the inequality

$$\max_i \left(|x_i|\,\text{norm}(e^i)\right) \leqq \text{norm}(x) \tag{5.2}$$

for absolute norms. Indeed, using theorem 6 we get

$$\text{norm}(x) \geqq \max_i \frac{|e^i|^H |x|}{\text{norm}^D((e^i)^H)} = \max_i \left(|x_i|\,\text{norm}(e^i)\right).$$

## 6.

An elegant direct proof of theorem 9 may be based on theorem 3 and the useful

**Lemma I.** *If* $\text{norm}(x)$ *is strictly homogeneous, then*

$$\text{lub}(x y^H) = \text{norm}(x)\,\text{norm}^D(y^H).$$

We then have

$$\text{norm}(e^i)\,\text{norm}^D\left((e^i)^H\right) = \text{lub}\left(e^i (e^i)^H\right) = (e^i)^H e^i = 1$$

since $e^i (e^i)^H$ is a diagonal matrix with the only nonvanishing element $d_{ii} = (e^i)^H e^i = 1$. This argument justifies the denotation "axis oriented" [1] for the bound norm in theorem 3.

*Proof of lemma I.* By (4.4) we have

$$\text{lub}(x y^H) = \max_{u \neq 0} \frac{\text{norm}(x y^H u)}{\text{norm}(u)} = \text{norm}(x) \cdot \max_{u \neq 0} \frac{|y^H u|}{\text{norm}(u)}$$
$$= \text{norm}(x)\,\text{norm}^D(y^H).$$

A lemma similar to lemma I holds for the duals of strictly homogeneous bound norms.

**Lemma II.** *If* norm$(x)$ *is strictly homogeneous then*

$$\operatorname{norm}(x)\operatorname{norm}^D(y^H) \geqq \operatorname{lub}^D(x\,y^H) \geqq \frac{|y^H x|^2}{\operatorname{norm}(x)\operatorname{norm}^D(y^H)}$$

*holds for* $x \neq 0$ *and* $y^H \neq 0$.

*Proof.* The bound norm satisfies version (4.2) of the Hölder inequality

$$\operatorname{lub}(A)\operatorname{lub}^D(B^H) \geqq |\operatorname{trace}(B^H A)|.$$

This yields

$$\operatorname{lub}(x\,y^H)\operatorname{lub}^D(x\,y^H) \geqq |\operatorname{trace}(x\,y^H x\,y^H)| = |y^H x|\,|\operatorname{trace}(x\,y^H)| = |y^H x|^2,$$

and by lemma I

$$\operatorname{norm}(x)\operatorname{norm}^D(y)\operatorname{lub}^D(x\,y^H) \geqq |y^H x|^2.$$

This yields a lower bound of $\operatorname{lub}^D(x\,y^H)$.

On the other hand, choose a matrix $A \neq 0$ which is lub-dual to $x\,y^H$. We then have

$$\operatorname{lub}(A)\operatorname{lub}^D(x\,y^H) = \operatorname{trace}(x\,y^H A) = y^H A\,x \leqq \operatorname{norm}^D(y^H)\operatorname{lub}(A)\operatorname{norm}(x).$$

This establishes an upper bound of $\operatorname{lub}^D(x\,y^H)$.

As an immediate consequence of lemma II we have

**Lemma III.** *If* norm$(x)$ *is strictly homogeneous then for every dual pair* $(x, y^H)$

$$\operatorname{lub}^D(x\,y^H) = \operatorname{norm}(x)\operatorname{norm}^D(y^H) = y^H x$$

*holds, and* $(x\,y^H, y\,x^H)$ *are a dual pair with respect to the bound norm, i.e.*

$$\operatorname{lub}(x\,y^H)\operatorname{lub}^D(x\,y^H) = \operatorname{trace}(x\,y^H x\,y^H) = (y^H x)^2.$$

## 7.

The following theorem may be regarded as the dual of theorem 3.

**Theorem 10.** *If* norm$(x)$ *is absolute/monotonic then*

$$\operatorname{lub}^D(D^H) = \sum_i |d_{ii}|$$

*holds for any diagonal matrix* $D = \operatorname{diag}(d_{ii})$.

*Proof.* For each diagonal matrix $D = \operatorname{diag}(d_{ii})$ there exists a diagonal matrix $T = \operatorname{diag}(t_{ii})$ such that

$$D^H T = |D| \quad \text{and} \quad |T| = I. \tag{7.1}$$

Theorem 3 implies $\operatorname{lub}(T) = 1$. Therefore,

$$\operatorname{lub}^D(D^H) = \operatorname{lub}(T)\operatorname{lub}^D(D^H) \geqq \operatorname{trace}(D^H T) = \sum_i |d_{ii}|.$$

Denoting axis vectors by $e^i$ we have $D = \sum_i d_{ii} e^i (e^i)^H$, and therefore

$$\operatorname{lub}^D(D^H) \leqq \sum_i |d_{ii}|\operatorname{lub}^D\left(e^i (e^i)^H\right).$$

Now if norm$(x)$ is absolute then theorem 9 shows that axis-vectors $e^i$ are self-dual. Hence by lemma III

$$\mathrm{lub}^D\left(e^i (e^i)^H\right) = (e^i)^H e^i = 1$$

holds for every $i$. This completes the proof.

For the euclidean bound norm

$$\mathrm{lub}(A) = \max_i \omega_i(A) \quad \text{and} \quad \mathrm{lub}^D(A^H) = \sum \omega_i(A),$$

where $\omega_i(A)$ are the singular values of $A$. This is a special case of a more general result of VON NEUMANN [*10*]. It follows from theorem 3 and theorem 10 by virtue of the invariance property of the euclidean bound norm,

$$\mathrm{lub}(U A V) = \mathrm{lub}(A), \quad U, V \text{ unitary},$$

and the immediate consequence of this

$$\mathrm{lub}^D(U A V) = \mathrm{lub}^D(A^H),$$

since $UAV = \mathrm{diag}(\omega_i)$ for suitable $U$, $V$.

## 8.

From theorem 10 we may conclude that every nonnegative real diagonal matrix $D \neq 0$ is lub-dual to the identity matrix $I$. Hence, for absolute norms, the surface of the lub-convex $\{A\colon \mathrm{lub}(A) \leqq 1\}$ is not smooth at the point $I$. Actually, this may be shown for arbitrary bound norms [*4*].

The set of all matrices being dual to the identity matrix $I$ with respect to the euclidean bound norm is the convex cone of all positive semidefinite matrices. This follows from VON NEUMANN'S results [*10*].

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