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A (Θ)-stable approximations of abstract Cauchy problems

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Summary. We study the approximation of linear parabolic Cauchy problems by means of Galerkin methods in space and $A(\Theta)$-stable multistep schemes of arbitrary order in time. The error is evaluated in the norm of $L_t^2(H_x^1) \cap L_t^\infty(L_x^2)$.

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0. Introduction

The aim of this paper is to analyse the approximation of a linear parabolic Cauchy problem of the type:

$$
(0.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + Au = f & \text{in } \Omega \times]0, \infty[\\ u(x,0) = u_0(x) & \text{in } \Omega \\ u(x,t) = 0 & \text{in } \partial\Omega \times]0, \infty[\ , \end{cases}
$$

by using a Galerkin method in space and an $A(\Theta)$-stable linear multistep method of order $q \geqq 1$ in time. The use of a generic $A(\Theta)$-stable method (introduced by Widlund in [13]) allows us to discuss separately the space and the time discretization, and to overcome the second order Dahlquist barrier of the A-stable methods (see [5]).

We write (0.1) as an abstract Cauchy problem in an usual Hilbert triple $V \subset H \subset V^*$:

$$
(0.2) \qquad u(0) = u_0; \qquad u'(t) + A(t)\, u(t) = f(t), \quad \text{for } t > 0 \ ,
$$

and we study the error in the norm of $L^2(0, \infty; V) \cap L^\infty(0, \infty; H)$.

The time discretization by means of an implicit Euler scheme was studied in [12]. The error analysis in the case $u_0 = 0$ for Euler and Crank-Nicolson methods was carried out in [4], whose outline we follow. For a different approach see e.g. [3, 7].

We choose a Galerkin approximation family $\{V_h\}$ of V and a couple (ρ, σ) of polynomials which define the multistep method:

$$\rho(z) = \sum_{j=0}^{g} \alpha_j z^j, \qquad \sigma(z) = \sum_{j=0}^{g} \beta_j z^j \quad \in \mathbb{C}[z] .$$

For a discretization step $k > 0$ and a suitable choice of g initial values ${}^h u_0^k, \ldots, {}^h u_{g-1}^k$ in V_h, the fully discretized problem consists in the sequence of linear equations in the unknown ${}^h u_{n+g}^k \in V_h$:

$$\frac{1}{k} \sum_{j=0}^{g} \alpha_j ({}^h u_{n+j}^k, v) \;+\; \sum_{j=0}^{g} \beta_j a_{n+j}^k ({}^h u_{n+j}^k, v) \;=\; \sum_{j=0}^{g} \beta_j (f_{n+j}^k, v), \quad \forall v \in V_h, \ \forall n \geqq 0 ,$$

where $f_n^k = f(kn)$ and $a_n^k(u,v) = {}_{V^*}\langle A(kn)\, u, v\rangle_V$.

In particular we get the stabilty estimate:

$$k \sum_{n\in\mathbb{N}} \|{}^h u_n^k\|_V^2 + \sup_{n\in\mathbb{N}} \|{}^h u_n^k\|_H^2 \leqq C\left\{ k \sum_{n\in\mathbb{N}} \|f_n^k\|_{V^*}^2 + \sum_{j=0}^{g-1} \left(|{}^h u_j^k|_H^2 + k \|{}^h u_j^k\|_V^2 \right) \right\} .$$

If the multistep method is of order q and the data $\{f, u_0\}$ are sufficiently smooth and compatible, so that u belongs to $H^q(0,\infty; V) \cap H^{q+1}(0,\infty; V^*)$ and the initial values may be chosen opportunely, we have the error estimate:

$$\left\{ k \sum_{n\in\mathbb{N}} \|u(kn) - {}^h u_n^k\|_V^2 \right\}^{1/2} + \sup_{n\in\mathbb{N}} |u(kn) - {}^h u_n^k|_H \leqq C\{ e_h[u] + k^q \|u\|_{H^q(0,\infty;V)\cap H^{q+1}(0,\infty;V^*)} \} ,$$

where $e_h[u]$ is the best approximation error:

$$(0.3) \quad e_h[u] = \inf\{ \|u - {}^h v\|_{L^2(0,\infty;V)\cap L^\infty(0,\infty;H)};\ {}^h v \in L^2(0,\infty; V_h) \cap L^\infty(0,\infty; H) \} .$$

The paper can be outlined as follows: in Sect. 1 we make precise our hypotheses and state the theorems about stability and convergence in the "energy norm"; proofs are given in Sects. 2 and 3.

Error estimates in norms of type $L^2(0,\infty; V) \cap H^{1/2}(0,\infty; H)$ as showed in [11], are contained in a forthcoming paper.

1. The continuous problem and its discretization

Notations

Let:

$$V \overset{ds}{\hookrightarrow} H \equiv H^* \overset{ds}{\hookrightarrow} V^*$$

be a triple of separable Hilbert spaces, $\|\cdot\|$ the norm of V and $|\cdot|$ the norm of H, induced by the scalar products $((\cdot,\cdot))$ and $(\cdot,\cdot)$ respectively; we identify H and H^* and denote by $(\cdot,\cdot)$ again the antiduality between V^* and V. A density argument allows us to consider V^* as the completion of H with respect to the dual norm:

$$\|\cdot\|_* = \sup_{v\in V, \|v\|=1} (\cdot, v).$$

We shall also assume, without loss of generality, that $|v| \leqq \|v\|$, $\forall v\in V$.

Let $\mathscr{B}$ be a Banach space and let $n\in\mathbb{N}$. $H^n_+(\mathscr{B})$ and $W^{n,\infty}_+(\mathscr{B})$ are the usual Sobolev space of $\mathscr{B}$-valued distributions on the real half line $]0, +\infty[$.

We set also, for $n\in\mathbb{N}$:

$$H^{n+1}_+(V, V^*) = H^n_+(V) \cap H^{n+1}_+(V^*),$$

and we recall the continuous imbedding $H^{n+1}_+(V, V^*) \hookrightarrow W^{n,\infty}_+(H)$.

The continuous problem

Assume that we are given, for $t > 0$, a measurable family of linear continuous operators $A(t)$ from V to V^* and five constants $M, L, \alpha, \Theta, \delta > 0$, $\delta < \Theta \leqq \pi/2$, such that, for every $v\in V$, $t\in\mathbb{R}^+$:

(A1) $$\|A(t)\, v\|_* \leqq M\|v\|, \qquad \operatorname{Re}(A(t)\, v, v) \geqq \alpha\|v\|^2;$$

(A2) $$|\arg(A(t)v, v)| \leqq \Theta - \delta;$$

(A3) $$\sum_{j\in\mathbb{N}} \|A(t_{j+1}) - A(t_j)\|_{\mathscr{L}(V,V^*)} \leqq L, \qquad \forall t_0 < t_1 < \ldots < t_n < \ldots \in\mathbb{R}^+.$$

Remark 1.1. The values of Θ and δ influence the choice of the multistep method we consider; hypothesis (A1), which ensures the well-posedness of the successive Cauchy problem, implies that (A2) holds at least for $\Theta = \arccos(\alpha/M) + \delta$. (A3) is a supplementary hypothesis required by the stability of the discretizations; it simply means that A is of bounded variation.

For every $f\in L^2_+(V^*)$, $u_0\in H$, we shall construct and study a family of approximations of the solution u of the abstract Cauchy problem:

(1.1) $$u(0) = u_0; \qquad u'(t) + A(t)\, u(t) = f(t), \quad \text{for } t > 0.$$

This function belongs to $H^1_+(V, V^*)$ and satisfies the "energy inequality" (see [2], for example):

(1.2) $$\|u\|_{L^2_+(V)\cap L^\infty_+(H)} \leqq C\{\|f\|_{L^2_+(V^*)} + |u_0|\}.$$

Moreover, when f belongs to $H^q_+(V^*)$, A belongs to $W^{q,\infty}_+(\mathscr{L}(V, V^*))$ and $\{f, A, u_0\}$ are related by suitable compatibility conditions, then u belongs to $H^{q+1}_+(V, V^*)$. These relations may be easily deduced by q-times differentiation of equation (1.1) and are expressed in terms of a vector $\boldsymbol{c}_q(f, u_0) = (c_0, \ldots, c_q)$ whose components are so defined:

(1.3) $$c_0 = u_0, \quad c_{m+1} = f^{(m)}(0) - \sum_{j=0}^{m} \binom{m}{j} A^{(j)}(0)\, c_{m-j}; \quad 0 \leqq m < q.$$

If we ask that $c_q \in V^q \times H$ we obtain:

$$\left\{\begin{array}{ll} u \in H^{q+1}_{+}(V, V^*), & u^{(j)}(0) = c_j(f, u_0), \quad 0 \leqq j \leqq q \\ \|u\|_{H^{q+1}_{+}(V,V^*)} \leqq C\{\|f\|_{H^q_+(V^*)} + \|c_q(f,u_0)\|_{V^q \times H}\} \end{array}\right. , \tag{1.4}$$

so that we may summarize our regularity hypotheses:

(A4) $\quad f \in H^q_+(V^*), \quad A \in W^{q,\infty}_+(\mathscr{L}(V, V^*)), \quad c_q(f, u_0) \in V^q \times H; \quad q \geqq 1 .$

The method

We discretize problem (1.1) by a g-step linear method. More precisely, we assign $2g + 2$ coefficients $\{\alpha_j, \beta_j\}_{j=0,\ldots,g}$ and we set, for every time step $k > 0$,

$$f^k_n = f(nk), \qquad A^k_n = A(kn); \quad n \in \mathbb{N}\,(^1) \tag{1.5}$$

Choosing g initial values $u^k_0, \ldots, u^k_{g-1} \in V$, we intend to construct an approximation u^k_n of the solution $u(nk)$ by the following algorithm:

$$\left\{\begin{array}{l} \forall n \geqq 0, \quad \text{find } u^k_{n+g} \in V \text{ such that:} \\ \dfrac{1}{k}\displaystyle\sum_{j=0}^{g} \alpha_j u^k_{n+j} + \sum_{j=0}^{g} \beta_j A^k_{n+j} u^k_{n+j} = \sum_{j=0}^{g} \beta_j f^k_{n+j} \end{array}\right. . \tag{1.6}$$

If $\mathrm{Re}[\alpha_g \bar{\beta}_g] > 0$ (^{2}), by (A1) and the Lax-Milgram lemma we can invert the operator:

$$\frac{1}{k}\alpha_g + \beta_g A^k_{n+g} , \tag{1.7}$$

for every $n \in \mathbb{N}$ and we can solve (1.6) with respect to u^k_{n+g}, once

$$u^k_n, \ldots, u^k_{n+g-1}, \qquad f^k_n, \ldots, f^k_{n+g}$$

are given. By induction we obtain existence and uniqueness for the sequence $\{u^k_n\}_{n \in \mathbb{N}}$

To solve (1.6) from the numerical point of view we introduce a Galerkin family $\{V_h\}$ of closed subspaces of V (^{3}), and consider the fully discretized problem:

$$\left\{\begin{array}{l} \text{Given } {}^h u^k_0, {}^h u^k_1, \ldots, {}^h u^k_{g-1} \in V_h, \text{ find } \{{}^h u^k_{n+g}\}_{n \in \mathbb{N}} \subset V_h \text{ such that:} \\ \left(\dfrac{1}{k}\displaystyle\sum_{j=0}^{g} \alpha_j {}^h u^k_{n+j} + \sum_{j=0}^{g} \beta_j A^k_{n+j} {}^h u^k_{n+j} - \sum_{j=0}^{g} \beta_j f^k_{n+j}, {}^h w\right) = 0 \quad \forall {}^h w \in V_h \end{array}\right. . \tag{1.8}$$

[1] By (A4) f and A are continuous, so this setting makes sense

[2] By (A2), $\alpha_g \bar{\beta}_g \neq 0$, $\arg[\alpha_g \bar{\beta}_g] \leqq \pi - \Theta$ would suffice. In fact these conditions are equivalent if the coefficients are real

[3] In practice, V_h are finite-dimensional

The stability and convergence properties of these methods (in the finite dimensional case) may be briefly expressed in terms of the two polynomials:

(1.9) $$\rho(z)=\sum_{j=0}^{g}\alpha_j z^j,\qquad \sigma(z)=\sum_{j=0}^{g}\beta_j z^j\quad \in\mathbb{C}[z];\quad |\alpha_g|^2+|\beta_g|^2>0\,,$$

which we may suppose prime. On (ρ,σ) we shall impose the following conditions (see for instance [10]) :

(P1) *strong $A(\Theta)$-stability*: for $|z|\geqq 1$ $\sigma(z)$ is different from 0 and the quotient $\rho(z)/\sigma(z)$ is contained in the closed sector:

(1.10) $$S_{\pi-\Theta}=\{\xi\in\mathbb{C}\colon |\arg\xi|\leqq\pi-\Theta\},\quad 0<\Theta\leqq\pi/2\,.$$

(P2) *order q*: when $z\to 0$ we have

(1.11) $$\rho(e^z)-z\sigma(e^z)=O(z^{q+1})$$

for an integer $q\geqq 1$; in particular this implies the consistency, i.e.:

(1.12) $$\rho(1)=0,\qquad \rho'(1)=\sigma(1)\neq 0$$

Remark 1.2. (P1) implies that $\alpha_g\bar{\beta}_g$ is different from 0 and is contained in $S_{\pi-\Theta}$; in other words, the method must be implicit ((ρ,σ) have degree g) and (1.7) can be inverted. Moreover, the possible unitary roots of ρ are simple.

Remark 1.3. When $\Theta=\pi/2$ we are dealing with an A-stable method, whose stability properties are well known (see [3, 5]). On the other hand, for these methods the "Dahlquist Barrier" forces $q\leqq 2$, so that the use of more general $A(\Theta)$-stable methods with $\Theta<\pi/2$ becomes necessary if we want to reach higher orders. We recall, for example, the Backward Differentiation Schemes of orders $\leqq 5$.

From now on we assume that (P1) and (P2) are satisfied for fixed Θ and q.

Stability estimates and approximation results

Theorem 1.4. *Let us assume that properties* (A1–3) and (P1) *hold; then the solution* ${}^h u_n^k$ *of* (1.8) *satisfies*:

(1.13) $$k\sum_{n\in\mathbb{N}}\|{}^h u_n^k\|^2+\sup_{n\in\mathbb{N}}|{}^h u_n^k|^2\leqq C\left\{k\sum_{n\in\mathbb{N}}\|f_n^k\|_*^2+\sum_{j=0}^{g-1}(k\|{}^h u_j^k\|^2+|{}^h u_j^k|^2)\right\},$$

where C depends only on the constants M, L, α, Θ, δ and on (ρ,σ) ([4]).

Remark 1.5. We have the estimate:

(1.14) $$k\sum_{n\in\mathbb{N}}\|f_n^k\|_*^2\leqq 2\|f\|^2_{H^1_+(V^*)}\,;$$

so, by (A4) the right hand member of (1.13) is finite.

[4] From now on, we always denote with C such constants

We denote with H_h the closure of V_h in the H-norm and with V_h^* the antidual of V_h, so that V_h, H_h, V_h^* is a new Hilbert triple; P_h is the surjective "restriction" of V^* on V_h^*:

$$(1.15)\qquad {}_{V_h^*}\langle P_h v, {}^h w\rangle_{V_h} = (v, {}^h w), \quad \|P_h v\|_{V_h^*} \leqq \|v\|_*, \quad \forall v \in V^*, \quad \forall {}^h w \in V_h .$$

Moreover, we have the best approximation result:

$$\forall v \in H, \quad P_h v \in H_h, \quad |v - P_h v| = \min_{{}^h w \in H_h} |v - {}^h w| .$$

We assume that:

$$(\mathrm{G1})\qquad P_h(V) \subset V_h; \qquad \exists C > 0 : \|P_h v\| \leqq C\|v\|, \quad \forall v \in V$$

for a constant C independent of h. In particular, this implies that:

$$\forall {}^h w \in V_h, \quad \|v - P_h v\| \leqq \|v - {}^h w\| + \|{}^h w - P_h v\| = \|v - {}^h w\| + \|P_h({}^h w - v)\| \leqq (1 + C)\|v - {}^h w\| ,$$

so that P_h realizes:

$$(1.16)\qquad \|v - P_h v\| \leqq C' \min_{{}^h w \in V_h} \|v - {}^h w\| ,$$

and, for a function u in $L_+^2(V) \cap L_+^\infty(H)$:

$$(1.17)\qquad \|u - P_h u\|_{L_+^2(V) \cap L_+^\infty(H)} \leqq C e_h[u] ,$$

$e_h[u]$ given by (0.3). We denote the error on the initial values by:

$$(1.18)\qquad \varepsilon^2[u; {}^h u_0^k, \ldots, {}^h u_{g-1}^k] = \max_{0 \leqq j < g} |P_h u(kj) - {}^h u_j^k|^2 + k \sum_{j=0}^{g-1} \|P_h u(kj) - {}^h u_j^k\|^2$$

and we may suppose that the choice of the initial values satisfies the following requirement:

$$(\mathrm{I1})\qquad \varepsilon[u; {}^h u_0^k, \ldots, {}^h u_{g-1}^k] \leqq C k^q [\|f\|_{H^q(0,kg;V^*)} + \|c_q\|_{V^q \times H}] .$$

Remark 1.6. By (A4) we know from the equation the Taylor expansion of u around 0 up to the order q; so, a possible choice of the initial values is:

$$(1.19)\qquad u_j^k = \sum_{l=0}^{q-1} \frac{c_l}{l!}(jk)^l, \qquad {}^h u_j^k = P_h u_j^k; \quad 0 \leqq j < g .$$

We have:

Theorem 1.7. *Assume that* (A1–4), (P1–2), (G1) *and* (I1) *hold; then the solution* ${}^h u_n^k$ *of* (1.8) *satisfies*:

$$\left\{k \sum_{n \in \mathbb{N}} \|u(kn) - {}^h u_n^k\|^2\right\}^{1/2} + \sup_{n \in \mathbb{N}} |u(kn) - {}^h u_n^k| \leqq$$
$$C\{k^q \|u\|_{H_+^{q+1}(V,V^*)} + \|u - P_h u\|_{L_+^2(V) \cap L_+^\infty(H)} + \varepsilon[u; {}^h u_0^k, \ldots, {}^h u_{g-1}^k]\} \leqq$$
$$C\{k^q[\|f\|_{H_+^q(V^*)} + \|c_q(f, u_0)\|_{V^q \times H}] + e_h[u]\} ,$$

with C depending only on the various constants introduced but not on h, k.

2. Proof of the theorems: stability

Preliminary outline; sequences spaces

We try to find the estimates of the preceding theorems by rewriting equations (1.6) and (1.8) in a different form. Setting ${}^hA = P_hA$, equation (1.8) becomes formally equivalent to (1.6) in the new Hilbert triple V_h, H_h, V_h^*:

$$(2.1)\qquad \frac{1}{k}\sum_{j=0}^{g}\alpha_j\,{}^hu_{n+j}^k+\sum_{j=0}^{g}\beta_j\,{}^hA_{n+j}^k\,{}^hu_{n+j}^k=\sum_{j=0}^{g}\beta_jP_hf_{n+j}^k,\quad n\geqq 0\,;$$

moreover, the operator ${}^hA$ satisfies in this framework the same conditions (A1–3) and by (1.15) P_h is a contraction from V^* to V_h^*; so, concerning the study of stability, we may limit ourselves to consider equation (1.6), suppressing the index h.

We denote vector valued sequences with bold characters and suppress the index k too when this fact does not generate mistakes. If $\mathscr{H}$ is an Hilbert space, we introduce the operator E on $\mathscr{H}^{\mathbb{N}}$:

$$(2.2)\qquad (\mathrm{E}\boldsymbol{v})_n=v_{n+1}\,,$$

with its powers:

$$(2.3)\qquad (\mathrm{E}^j\boldsymbol{v})_n=v_{n+j},\qquad (\mathrm{E}^{-j}\boldsymbol{v})_n=\begin{cases}v_{n-j}, & \text{if } n\geqq j\\ 0, & \text{if } n<j\end{cases}\qquad \forall j\in\mathbb{N}\,.$$

E^{-j} is the right inverse of E^j: $\mathrm{E}^j\mathrm{E}^{-j}\boldsymbol{v}=\boldsymbol{v}$, for every sequence $\boldsymbol{v}$. For every polynomial $\tau(z)=\sum_{j=0}^{g}\gamma_jz^j$ we have consequently:

$$(2.4)\qquad (\tau(\mathrm{E})\boldsymbol{v})_n=\sum_{j=0}^{g}\gamma_jv_{n+j}\,.$$

Setting $(\boldsymbol{A}\boldsymbol{v})_n=A_nv_n$, for $\boldsymbol{v}\in V^{\mathbb{N}}$, we write:

$$\frac{1}{k}\sum_{j=0}^{g}\alpha_jv_{n+j}+\sum_{j=0}^{g}\beta_jA_{n+j}v_{n+j}=\left(\frac{\rho(\mathrm{E})}{k}\boldsymbol{v}+\sigma(\mathrm{E})\boldsymbol{A}\boldsymbol{v}\right)_n,\qquad \forall n\in\mathbb{N}\,.$$

We set also:

$$(2.5)\qquad \forall\,\boldsymbol{v}\in\mathscr{H}^{\mathbb{N}},\qquad \boldsymbol{v}|_j=\begin{cases}v_n, & \text{if } n\leqq j\\ 0, & \text{if } n>j\end{cases}$$

so that, if $\underline{\boldsymbol{u}}=(u_0,\dots,u_{g-1})\in V^g\subset V^{\mathbb{N}}$ is the vector of the initial values, (1.6) becomes:

$$(2.6)\qquad \begin{cases}\boldsymbol{u}|_{g-1}=\underline{\boldsymbol{u}},\\[2mm] \dfrac{\rho(\mathrm{E})}{k}\boldsymbol{u}+\sigma(\mathrm{E})\boldsymbol{A}\boldsymbol{u}=\sigma(\mathrm{E})\boldsymbol{f}\end{cases}$$

Finally, we call $T_k=k^{-1}\rho(\mathrm{E})+\sigma(\mathrm{E})\boldsymbol{A}$, and write (2.6) in the compact form:

$$(2.7)\qquad T_k\boldsymbol{u}=\sigma(\mathrm{E})\boldsymbol{f},\qquad \boldsymbol{u}|_{g-1}=\underline{\boldsymbol{u}}\,.$$

By linearity we may enclose the initial conditions in the equation and write it in terms of $\boldsymbol{u}^+ = \boldsymbol{u} - \underline{\boldsymbol{u}}$:

$$T_k \boldsymbol{u}^+ = \sigma(\mathrm{E})\boldsymbol{f} - T_k \underline{\boldsymbol{u}}, \quad \boldsymbol{u}^+|_{g-1} = 0 . \tag{2.8}$$

To complete our formulation, we specify the spaces where we set (2.8), taking into account the quantities arising in (1.13) which we shall deal with.

We call $l_k^p(\mathscr{H})$ the Banach space of the $\mathscr{H}$-valued sequences $\boldsymbol{v}$ such that:

$$\|\boldsymbol{v}\|^p_{l_k^p(\mathscr{H})} = k \sum_{n\in\mathbb{N}} \|v_n\|^p_{\mathscr{H}} < \infty, \quad 1 \leqq p < \infty , \tag{2.9}$$

and $l_k^\infty(\mathscr{H}) = l^\infty(\mathscr{H})$ the Banach space of the bounded sequences with the sup-norm; we observe that there is a natural antiduality between $l_k^p(\mathscr{H})$ and $l_k^{p'}(\mathscr{H}^*)$:

$$_{l_k^p(\mathscr{H})}\langle \boldsymbol{v}, \boldsymbol{w}\rangle_{l_k^{p'}(\mathscr{H}^*)} = k \sum_{n\in\mathbb{N}} \hat{\mathscr{H}}(v_n, w_n)_{\mathscr{H}^*}; \quad \frac{1}{p} + \frac{1}{p'} = 1 ; \tag{2.10}$$

finally, we indicate with $\dot{l}_k^p(\mathscr{H})$ the closed subspace of $l_k^p(\mathscr{H})$ given by the sequences $\boldsymbol{v}$ with $\boldsymbol{v}|_{g-1} = 0$. The operator E is well defined on these spaces and its norm is 1.

Theorem 1.4 may be so restated:

Theorem 2.1. *Assume that $\boldsymbol{u}^+$ is a solution of (2.8) with $\boldsymbol{f}\in l_k^2(V^*)$. Then $\boldsymbol{u}^+$ satisfies the stability estimate*:

$$\|\boldsymbol{u}^+\|_{l_k^2(V)\cap l^\infty(H)} \leqq C\{\|\boldsymbol{f}\|_{l_k^2(V^*)} + \|\underline{\boldsymbol{u}}\|_{l^\infty(H)\cap l_k^2(V)}\} . \tag{2.11}$$

Remark 2.2. As we have already noticed, this result gives an analogous bound for the solution of (2.1): we call ${}^hT_k$ the operator $P_h T_k$ and consider ${}^h\boldsymbol{u}^+$, solution of:

$${}^hT_k\, {}^h\boldsymbol{u}^+ = \sigma(\mathrm{E})P_h \boldsymbol{f} - {}^hT_k\, {}^h\underline{\boldsymbol{u}} ,$$

we have:

$$\|{}^h\boldsymbol{u}^+\|_{l_k^2(V)\cap l^\infty(H)} \leqq C\{\|\boldsymbol{f}\|_{l_k^2(V^*)} + \|{}^h\underline{\boldsymbol{u}}\|_{l^\infty(H)\cap l_k^2(V)}\} . \tag{2.12}$$

Up to now we have only changed our notations; we shall show how these are really more convenient. The basic tool of our proof is explained in the following section; we state first a lemma on inversion of operators like (2.4):

Lemma 2.3. *Assume that the roots of the polynomial $\tau(z) = \sum_{j=0}^g \gamma_j z^j$ have modulus < 1; then there exists a sequence of complex numbers $\{\gamma'_j\}_{j\in\mathbb{N}+g}$ such that*:

$$\sum_{j\geqq g} |\gamma'_j| = |\tau^{-1}| < \infty ,$$

and $\forall\, \boldsymbol{w}\in\mathscr{H}^{\mathbb{N}}$:

$$\boldsymbol{v}|_{g-1} = 0, \quad \tau(\mathrm{E})\boldsymbol{v} = \boldsymbol{w} \Leftrightarrow v_n = \sum_{j=g}^{n} \gamma'_j w_{n-j}, \quad \forall\, n \geqq g . \tag{2.13}$$

Moreover:

$$\boldsymbol{w}\in l_k^p(\mathscr{H}) \Rightarrow \boldsymbol{v}\in l_k^p(\mathscr{H}), \qquad \|\boldsymbol{v}\|_{l_k^p(\mathscr{H})} \leqq |\tau^{-1}|\, \|\boldsymbol{w}\|_{l_k^p(\mathscr{H})} .$$

Proof. Thanks to the hypothesis on τ, $\tau(z)^{-1}$ is a holomorphic function in $|z| > 1 - \varepsilon$ for an $\varepsilon > 0$ and we can write its power series development around ∞:

$$\tau(z)^{-1} = \sum_{j \geqq g} \gamma'_j z^{-j}, \qquad \sum_{j \geqq g} |\gamma'_j| = |\tau^{-1}| < \infty . \tag{2.15}$$

We denote with $\tau^{-1}(\mathrm{E})$ the linear operator:

$$\boldsymbol{w} \to \tau^{-1}(\mathrm{E})\boldsymbol{w} = \boldsymbol{v}, \qquad v_n = \sum_{j=g}^{n} \gamma'_j w_{n-j}$$

which is uniformly bounded in every $l^p_k(\mathscr{H})$ by $|\tau^{-1}|$.

It remains to prove (2.13); by definition, the coefficients γ_j' satisfy the algebraic relations:

$$\sum_{j=0}^{g} \gamma_j \gamma'_{n+j} = \delta_{0,n} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0 \end{cases} \qquad \forall n \in \mathbb{N} ,$$

which imply that:

$$(\tau(\mathrm{E})\tau^{-1}(\mathrm{E})\boldsymbol{w})_n = \sum_{j=0}^{g} \gamma_j (\tau(\mathrm{E})^{-1}\boldsymbol{w})_{n+j} = \sum_{j=0}^{g} \gamma_j \sum_{i=j}^{n+j} \gamma'_i w_{n+j-i} =$$

$$(i = j + l) \quad = \sum_{l=0}^{n} \left(\sum_{j=0}^{g} \gamma_j \gamma'_{j+l} \right) w_{n-l} = w_n$$

Remark 2.4. It is obvious that $\tau(\mathrm{E})$ is bounded on every $l^p_k(\mathscr{H})$, with norm $\leqq |\tau| = \sum_{j=0}^{g} |\gamma_j|$.

Corollary 2.5. *Suppose that* $\boldsymbol{v}$ *satisfies*:

$$\boldsymbol{v}|_{g-1} = \underline{\boldsymbol{v}}, \qquad \tau(\mathrm{E})\boldsymbol{v} = \boldsymbol{w} \in l^p_k(\mathscr{H}) .$$

Then we have:

$$\|\boldsymbol{v}\|_{l^p_k(\mathscr{H})} \leqq |\tau^{-1}| \, \|\boldsymbol{w}\|_{l^p_k(\mathscr{H})} + |\tau^{-1}| \, |\tau| \, \|\underline{\boldsymbol{v}}\|_{l^p_k(\mathscr{H})} \tag{2.16}$$

Proof. Writing $\boldsymbol{v}^+ = \boldsymbol{v} - \underline{\boldsymbol{v}}$ we observe that $\boldsymbol{v}^+$ satisfies:

$$(\boldsymbol{v}^+)|_{g-1} = 0; \qquad \tau(\mathrm{E})\boldsymbol{v}^+ = \boldsymbol{w} - \tau(\mathrm{E})\underline{\boldsymbol{v}} ,$$

and conclude by the previous lemma. □

A basic isomorphism

Let $\boldsymbol{U}$ be the subset of the extended complex plane: $\{z \in \mathbb{C}: |z| > 1\} \cup \{\infty\}$ and consider the Hardy space $H^2(\boldsymbol{U}; \mathscr{H})$ of the $\mathscr{H}$-valued holomorphic functions g on $\boldsymbol{U}$ such that:

$$\exists \lim_{r \to 1^+} \frac{1}{2\pi} \int_{-\pi}^{\pi} \|g(r\mathrm{e}^{\mathrm{i}\theta})\|^2_{\mathscr{H}} \, \mathrm{d}\theta = \|g\|^2_{H^2(U;\mathscr{H})} . \tag{2.17}$$

Every g in $H^2(U;\mathscr{H})$ admits a trace (still denoted with g) on $\partial U = \{z \in \mathbb{C}: |z| = 1\}$ which belongs to $L^2(\partial U; \mathscr{H})$. The coefficients of the Laurent expansion around ∞ are given by the Fourier coefficients of g in $L^2(\partial U; \mathscr{H})$:

$$(2.18) \qquad g_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}n\theta}\, d\theta, \qquad g(z) = \sum_{n \in \mathbb{N}} g_n z^{-n}\,.$$

We have the fundamental relation:

$$(2.19) \qquad \|g\|^2_{H^2(U;\mathscr{H})} = \|g\|^2_{L^2(\partial U;\mathscr{H})} = \sum_{n \in \mathbb{N}} \|g_n\|^2_{\mathscr{H}}\,.$$

So, $H^2(U;\mathscr{H})$ is a Hilbert space isomorphic to $l^2_k(\mathscr{H})$ by the transformation:

$$(2.20) \qquad \mathbf{g} \in l^2_k(\mathscr{H}) \to \hat{g}(z) = \sum_{n \in \mathbb{N}} g_n z^{-n}; \qquad \|\mathbf{g}\|^2_{l^2_k(\mathscr{H})} = k\|\hat{g}\|^2_{H^2(U;\mathscr{H})}\,.$$

The most interesting fact for us is given by the following rules:

$$(2.21) \qquad \text{if } g_0 = 0 \quad \text{then } \widehat{\mathrm{E}\mathbf{g}}(z) = z\hat{g}(z)\,;$$

$$(2.22) \qquad \mathbf{A} \equiv A \text{ constant} \Rightarrow \widehat{\mathbf{A}\mathbf{g}}(z) = A\hat{g}(z)\,.$$

For a sequence $\mathbf{v} \in \dot{l}^2_k(V)$ we have:

$$\widehat{\rho(\mathrm{E})\mathbf{v}}(z) = \rho(z)\,\hat{v}(z), \qquad \widehat{\sigma(\mathrm{E})\mathbf{v}}(z) = \sigma(z)\,\hat{v}(z)\,,$$

and:

$$(2.23) \qquad \widehat{T_k\mathbf{v}}(z) = \frac{\rho(z)}{k}\,\hat{v}(z) + A\sigma(z)\hat{v}(z) = \hat{T}_k\hat{v}(z)$$

when $\mathbf{A}$ is constant.

Proof of Theorem 2.1. The case $\mathbf{A} \equiv A$ constant.

We call $\mathbf{g}_1 = \sigma(\mathrm{E})[\mathbf{f} - \mathbf{A}\underline{\mathbf{u}}]$, $\mathbf{g}_2 = \|k^{-1}\rho(\mathrm{E})\underline{\mathbf{u}}$ with the obvious bounds:

$$\|\mathbf{g}_1\|_{l^2_k(V^*)} \leqq |\sigma|\{\|\mathbf{f}\|_{l^2_k(V^*)} + M\|\underline{\mathbf{u}}\|_{l^2_k(V)}\}, \qquad \|\mathbf{g}_2\|_{l^1_k(H)} \leqq g|\rho|\,\|\underline{\mathbf{u}}\|_{l^\infty(H)}$$

We split correspondingly $\mathbf{u}^+$ into the sum $\mathbf{u}_1 + \mathbf{u}_2$, with:

$$(\mathbf{u}_j)|_{g-1} = 0, \qquad T_k\mathbf{u}_j = \mathbf{g}_j, \quad j = 1, 2$$

and study separately these sequences.

Claim 2.6.

$$(2.24) \qquad \|\mathbf{u}_1\|_{l^2_k(V)} \leqq C\|\mathbf{g}_1\|_{l^2_k(V^*)}\,.$$

By (2.23), $\mathbf{u}_1$ belongs to $l^2_k(V)$ if and only if there exists a solution $\hat{u}_1$ in $H^2(U; V)$ of the equation:

$$(2.25) \qquad \hat{T}_k\hat{u}_1(z) = \frac{\rho(z)}{k}\,\hat{u}(z) + A\sigma(z)\,\hat{u}(z) = \hat{g}_1(z)\,.$$

We know that, for $|z| \geqq 1$, is $\sigma(z) \neq 0$; denoting by $\gamma(z)$ the rational function $\rho(z)/\sigma(z)$, $\gamma(z)$ is holomorphic in U and continuous on ∂U. We may rewrite as follows:

$$\frac{\gamma(z)}{k}\hat{u}_1(z) + A\hat{u}(z) = \sigma(z)^{-1}\hat{g}_1(z). \tag{2.26}$$

If $\hat{g}_1(z)$ is in $H^2(U; V^*)$ also $\hat{g}_1(z)/\sigma(z)$ belongs to $H^2(U; V^*)$ and its norm is bounded by $C_\sigma \|g_1\|_{H^2(U;V^*)}$, with: $C_\sigma = \max_{|z|=1} |\sigma(z)^{-1}|$.

It remains to study the invertibility of $\gamma(z) + A$. But (A1–2) imply that the operator $\zeta + A$ is invertible from V^* to V if $\zeta \in S_{\pi-\Theta}$ with the bound:

$$\zeta v + Av = f \Rightarrow \|v\| \leqq \frac{1}{\alpha \sin\delta} \|f\|_* \tag{2.27}$$

By (P1) $\gamma(z)$ belongs to $S_{\pi-\Theta}$ when $|z| \geqq 1$, so the mapping:

$$z \to \left[\frac{\gamma(z)}{k} + A\right]^{-1}$$

is well defined, bounded and continuous from $\bar{U}$ to $\mathscr{L}(V^*, V)$ and holomorphic in U. It follows that $[k^{-1}\gamma(z) + A]^{-1}\sigma(z)^{-1}\hat{g}_1(z)$ is holomorphic in U, has a 0 of order g in ∞ and satisfies the estimate:

$$\|\hat{u}(z)\| \leqq \frac{1}{\alpha \sin\delta |\sigma(z)|} \|\hat{g}_1(z)\|_*. \tag{2.28}$$

Because of (2.19) we get:

$$\|\boldsymbol{u}\|_{l_k^2(V)} \leqq \frac{C_\sigma}{\alpha \sin\delta} \|\boldsymbol{g}_1\|_{l_k^2(V^*)},$$

that is (2.24). □

Claim 2.7. *There exists a polynomial $\lambda(z)$ of degree g such that:*

$$\sup_{n\in\mathbb{N}} \left\{ \operatorname{Re}\widehat{_{l_k^2(H)}}\left\langle \frac{\rho(\mathrm{E})\boldsymbol{v}}{k}, [\lambda(\mathrm{E})\boldsymbol{v}]|_n \right\rangle_{l_k^2(H)} \right\} \geqq \|\boldsymbol{v}\|^2_{l^\infty(H)}, \quad \forall\, \boldsymbol{v} \in \dot{l}_k^2(H); \tag{2.29}$$

in particular, this implies:

$$\|\boldsymbol{u}_1\|_{l^\infty(H)} \leqq C \|\boldsymbol{g}_1\|_{l_k^2(V^*)}. \tag{2.30}$$

We denote with Z_ρ the set of the unitary roots of ρ, and set:

$$\rho_u(z) = \prod_{\xi \in Z_\rho} (z - \xi), \qquad \rho_0 = \rho/\rho_u, \qquad \rho_\xi(z) = \frac{\rho_u(z)}{z - \xi}.$$

We call $\boldsymbol{w} = \rho_0(\mathrm{E})\boldsymbol{v}$; by Lemma 2.3 there exists a constant $\beta = |\rho_0^{-1}| > 0$ only depending on ρ_0 such that:

$$\|\boldsymbol{v}\|_{l^\infty(H)} \leqq \beta \|\boldsymbol{w}\|_{l^\infty(H)}. \tag{2.31}$$

We note that, by Remark 1.2, there exist constants $\{c_\xi\}_{\xi \in Z_\rho}$ such that:

$$1 = \sum_{\xi \in Z_\rho} c_\xi \rho_\xi(z) \Rightarrow \boldsymbol{w} = \sum_{\xi \in Z_\rho} c_\xi \rho_\xi(\mathrm{E})\boldsymbol{w};$$

setting $c = \sum_{\xi\in Z_p}|c_\xi|^2$ and $\boldsymbol{v}^\xi = \rho_\xi(\mathrm{E})\boldsymbol{w} = [\rho_\xi\rho_0](\mathrm{E})\boldsymbol{v}$, we have:

$$(2.32)\qquad |w_n|^2 \leqq c \sum_{\xi\in Z_p} |v_n^\xi|^2; \qquad \|\boldsymbol{w}\|^2_{l^\infty(H)} \leqq c \sup_{n\in\mathbb{N}} \sum_{\xi\in Z_p} |v_n^\xi|^2 .$$

We say that:

$$(2.23)\quad \lambda(z) = 2\beta c z\rho_0(z) \sum_{\xi\in Z_p} \rho_\xi(z), \qquad \lambda(\mathrm{E})\boldsymbol{v} = 2\beta c \sum_{\xi\in Z_p} \rho_\xi(\mathrm{E})\mathrm{E}\boldsymbol{w} = 2\beta c \sum_{\xi\in Z_p} \mathrm{E}\boldsymbol{v}^\xi$$

is a good choice for (2.29). Recalling that $\rho(\mathrm{E})\boldsymbol{v} = \mathrm{E}\boldsymbol{v}^\xi - \xi\boldsymbol{v}^\xi$ and observing that $\rho_\xi\rho_0$ has degree $g-1$ and consequently $v_0^\xi = 0$, we have:

$$\mathrm{Re}\ {}_{l^2_k(H)}\left\langle \frac{\rho(\mathrm{E})\boldsymbol{v}}{k}, [\lambda(\mathrm{E})\boldsymbol{v}]|_n \right\rangle_{l^2_k(H)}$$

$$= \frac{2\beta c}{k}\mathrm{Re}\sum_{\xi\in Z_p} {}_{l^2_k(H)}\langle \mathrm{E}\boldsymbol{v}^\xi - \xi\boldsymbol{v}^\xi, (\mathrm{E}\boldsymbol{v}^\xi)|_n\rangle_{l^2_k(H)}$$

$$= 2\beta c\,\mathrm{Re}\sum_{\xi\in Z_p}\sum_{j=0}^{n} (v_{j+1}^\xi - \xi v_j^\xi, v_{j+1}^\xi)$$

$$\geqq \beta c \sum_{\xi\in Z_p}\sum_{j=0}^{n} |v_{j+1}^\xi|^2 - |v_j^\xi|^2 = \beta c \sum_{\xi\in Z_p} |v_{n+1}^\xi|^2.$$

By (2.32) and (2.31) we get (2.29); (2.30) follows by taking the duality of equation $T_k\boldsymbol{u}_1 = \boldsymbol{g}_1$ with $\lambda(\mathrm{E})\boldsymbol{u}_1|_n$ and recalling (2.24). □

Claim 2.8.

$$(2.34)\qquad \|\boldsymbol{u}_2\|_{l^2_k(V)} \leqq C\|\boldsymbol{g}_2\|_{l^1_k(H)} .$$

We use a duality argument; first we establish a transposition formula. Suppose that $\boldsymbol{u}|_{g-1} = \boldsymbol{v}|_{g-1} = 0$ and consider the symmetry:

$$\mathrm{S}_N\colon \boldsymbol{w} \to \mathrm{S}_N\boldsymbol{w} = \boldsymbol{w}', \qquad (\boldsymbol{w}')_n = \begin{cases} w_{N-n} & \text{if } 0 \leqq n \leqq N \\ 0 & \text{if } n > N \end{cases} .$$

For a polynomial $\tau(z) = \sum_{j=0}^{g}\gamma_j z^j$ we have:

$$(2.35)\qquad {}_{l^2_k(\mathscr{H})}\langle \tau(\mathrm{E})\boldsymbol{u}, \boldsymbol{v}'\rangle_{l^2_k(\mathscr{H})} = {}_{l^2_k(\mathscr{H})}\langle \boldsymbol{u}', \bar\tau(\mathrm{E})\boldsymbol{v}\rangle_{l^2_k(\mathscr{H})} ,$$

where we called $\bar\tau(z) = \overline{\tau(\bar z)} = \sum_{j=0}^{g}\bar\gamma_j z^j$. In fact we have:

$${}_{l^2_k(\mathscr{H})}\langle \tau(\mathrm{E})\boldsymbol{u}, \boldsymbol{w}\rangle_{l^2_k(\mathscr{H})} = k\sum_{n=0}^{N}\left(\sum_{j=0}^{g}\gamma_j u_{n+j}, w_{N-n}\right) = k\sum_{n=0}^{N}\sum_{j=0}^{g}(u_{n+j}, \bar\gamma_j w_{N-n})$$

$$(n = N-m-j)\quad = k\sum_{j=0}^{g}\sum_{n=0}^{N-g}(u_{n+j}, \bar\gamma_j w_{N-n}) = k\sum_{j=0}^{g}\sum_{m=0}^{N-g}(u_{N-m}, \bar\gamma_j w_{m+j})$$

$$= k\sum_{j=0}^{g}\sum_{m=0}^{N}(u'_m, \bar\gamma_j w_{m+j}) = {}_{l^2_k(H)}\langle \boldsymbol{u}', \bar\tau(\mathrm{E})\boldsymbol{w}\rangle_{l^2_k(H)} .$$

Consider now A^*, the adjoint of A, and set:

$$\bar{T}_k = \frac{\bar{\rho}(\mathrm{E})}{k} + \bar{\sigma}(\mathrm{E})A^* ;$$

$\bar{T}_k$ has the same property of T_k, since $(\bar{\rho}, \bar{\sigma})$ satisfies (P1) and A^* satisfies (A1–2). In particular:

$$\|w\|_{l^\infty(H)} \leqq C \|\bar{T}_k w\|_{l_k^2(V^*)}, \quad \forall w \in \mathring{l}_k^2(V) .$$

and, by (2.35):

$$(2.36) \qquad {}_{l_k^2(V^*)}\langle T_k \boldsymbol{u}, \mathrm{S}_N \boldsymbol{v}\rangle_{l_k^2(V)} = {}_{l_k^2(V)}\langle \mathrm{S}_N \boldsymbol{u}, \bar{T}_k \boldsymbol{v}\rangle_{l_k^2(V^*)}$$

On the other hand we have:

$$\|\boldsymbol{u}_2\|_{l_k^2(V)} = \sup_{N\in\mathbb{N}} \|\mathrm{S}_N \boldsymbol{u}_2\|_{l_k^2(V)} ,$$

and:

$$\begin{aligned}
\|\mathrm{S}_N \boldsymbol{u}_2\|_{l_k^2(V)} &= \sup_{w\in \mathring{l}_k^2(V)\setminus\{0\}} \frac{{}_{l_k^2(V)}\langle \mathrm{S}_N \boldsymbol{u}_2, \bar{T}_k w\rangle_{l_k^2(V^*)}}{\|\bar{T}_k w\|_{l_k^2(V^*)}} \\
&= \sup_{w\in \mathring{l}_k^2(V)\setminus\{0\}} \frac{{}_{l_k^2(V^*)}\langle T_k \boldsymbol{u}_2, \mathrm{S}_N w\rangle_{l_k^2(V)}}{\|\bar{T}_k w\|_{l_k^2(V^*)}} \\
&= \sup_{w\in \mathring{l}_k^2(V)\setminus\{0\}} \frac{{}_{l_k^1(H)}\langle \boldsymbol{g}_2, \mathrm{S}_N w\rangle_{l^\infty(H)}}{\|\bar{T}_k w\|_{l_k^2(V^*)}} \\
&\leqq \|\boldsymbol{g}_2\|_{l_k^1(H)} \sup_{w\in \mathring{l}_k^2(V)\setminus\{0\}} \frac{\|w\|_{l^\infty(H)}}{\|\bar{T}_k w\|_{l_k^2(V^*)}} \leqq C\|\boldsymbol{g}_2\|_{l_k^1(H)} \qquad \square
\end{aligned}$$

Claim 2.9.

$$\|\boldsymbol{u}_2\|_{l^\infty(H)} \leqq \|\boldsymbol{g}_2\|_{l_k^1(H)} .$$

We repeat the same technique of 2.7. □

Remark 2.10. It may seem that notations like $\|\cdot\|_{l_k^2(V)\cap l^\infty(H)}$ are superfluous, being $l_k^2(V) \subsetneq l^\infty(H)$; actually the norm of this immersion tends to ∞ when k goes to 0, whereas our constants C are independent of k.

Proof of Theorem 2.1. A depending on time.

The discussion of this more general case is based on the simple remark that the values of the truncated sequence $\boldsymbol{u}|_N$ of (2.7) depend only on $\underline{\boldsymbol{u}}$ and $\boldsymbol{f}|_N$. Observing that $\boldsymbol{u}$ satisfies:

$$(2.37) \qquad \frac{\rho(\mathrm{E})}{k}\boldsymbol{u} + \sigma(\mathrm{E})A_N\boldsymbol{u} = \sigma(\mathrm{E})\boldsymbol{f} - T_k\underline{\boldsymbol{u}} + \sigma(\mathrm{E})[(A_N - A)\boldsymbol{u}], \quad \forall N\in\mathbb{N} ,$$

we get consequently the estimate:

$$\begin{aligned}(2.38) \qquad \|\boldsymbol{u}|_N\|^2_{l_k^2(V)\cap l^\infty(H)} \leqq C[\|\boldsymbol{f}|_N\|^2_{l_k^2(V^*)} &+ \|(A_N - A)\boldsymbol{u}|_N\|^2_{l_k^2(V^*)} \\ &+ \|\underline{\boldsymbol{u}}\|^2_{l_k^2(V)\cap l^\infty(H)}] ;\end{aligned}$$

the last term may be controlled in the following way (we set $u|_{-1} = 0$):

$$\|(A_N - A)u|_N\|^2_{l^2_k(V^*)} \leqq k \sum_{j=0}^{N} \|A_N - A_j\|^2 \cdot \|u_j\|^2$$

$$\leqq \sum_{j=0}^{N} \|A_N - A_j\|^2 \cdot (\|u|_j\|^2_{l^2_k(V)} - \|u|_{j-1}\|^2_{l^2_k(V)})$$

$$\leqq \sum_{j=0}^{N-1} \|A_N - A_j\|^2 \cdot \|u|_j\|^2_{l^2_k V}$$

$$- \sum_{j=0}^{N-1} \|A_N - A_{j+1}\|^2 \cdot \|u|_j\|^2_{l^2_k(V)}$$

$$\leqq 4M \sum_{j=0}^{N-1} |\|A_N - A_j\| - \|A_N - A_{j+1}\|| \cdot \|u|_j\|^2_{l^2_k(V)}$$

$$\leqq 4M \sum_{j=0}^{N-1} \|A_j - A_{j+1}\| \cdot \|u|_j\|^2_{l^2_k(V)}$$

From (2.38), denoting with X_N the square of the norm of $u|_N$ in $l^2_k(V) \cap l^\infty(H)$, we get the recurrent relation:

(2.39) $$X_N \leqq C\{\|f\|^2_{l^2_k(V^*)} + \|\underline{u}\|^2_{l^2_k(V) \cap l^\infty(H)}\} + \sum_{j=0}^{N-1} a_j X_j,$$

$$a_j = 4M \|A_{j+1} - A_j\|_{\mathscr{L}(V,V^*)} .$$

Since $\sum_{j \in \mathbb{N}} a_j \leqq 4ML < \infty$, by an easy application of a Gronwall-like lemma, we have:

$$\|u\|_{l^2_k(V) \cap l^\infty(H)} \leqq C\{\|f\|_{l^2_k(V^*)} + \|\underline{u}\|_{l^2_k(V) \cap l^\infty(H)}\} \quad \square$$

3. Proof of the theorems: convergence

Approximation lemmata

We shall compare the approximate solution ${}^h u$ of (1.8) with the discretized continuous solution u; we set:

(3.1) $$(\Pi u)_n = u(kn), \qquad ({}^h\Pi u)_n = P_h u(kn) = (\Pi P_h u)_n .$$

On Π we have the following results (see [1, 9]):

Lemma 3.1. *There exists a constant $C > 0$ such that:*

(3.2) $$\forall v \in H^q_+(\mathscr{H}), \quad \|\Pi v\|_{l^2_k(\mathscr{H})} \leqq C\{\|v\|_{L^2_+(\mathscr{H})} + k^q \|D^q v\|_{L^2_+(\mathscr{H})}\}$$

Corollary 3.2. *If v belongs to $H^q_+(V)$ and* (G1) *holds true, we have:*

(3.3) $$\|\Pi v - {}^h\Pi v\|_{l^2_k(V) \cap l^\infty(H)} \leqq C\{k^q \|v\|_{H^q_+(V)} + \|v - P_h v\|_{L^2_+(V) \cap L^\infty(H)}\}$$

Lemma 3.3. *Assume that* $v \in H_+^{q+1}(\mathscr{H})$ *and consider the local truncation error:*

$$G_k[v](t) = \frac{1}{k} \sum_{j=0}^{g} \alpha_j v(t+jk) - \sum_{j=0}^{g} \beta_j v'(t+jk), \quad t \geqq 0 . \tag{3.4}$$

There exists a constant $C > 0$ *such that:*

$$\|G_k[v]\|_{L_+^2(\mathscr{H})} + k^q \|D^q G_k[v]\|_{L_+^2(\mathscr{H})} \leqq C k^q \|v\|_{H_+^{q+1}(\mathscr{H})} , \tag{3.5}$$

and:

$$\|\Pi G_k[v]\|_{l_k^2(\mathscr{H})} \leqq C k^q \|u\|_{H_+^{q+1}(\mathscr{H})} . \tag{3.6}$$

Proof. (3.6) is an immediate consequence of (3.5) and (3.2); so, we may limit ourselves to prove (3.5), or equivalently:

$$\|D^j G_k[v]\|_{L_+^2(\mathscr{H})} \leqq C k^{q-j} \|v\|_{H_+^{q+1}(\mathscr{H})}, \quad 0 \leqq j \leqq q . \tag{3.7}$$

Let $r_{[0,\infty[}$ be the restriction operator from $L^2(\mathscr{H})$ to $L_+^2(\mathscr{H})$ and let p be a linear extension operator with the properties:

$$p \in \mathscr{L}(L_+^2(\mathscr{H}), L^2(\mathscr{H})) \cap \mathscr{L}(H_+^{q+1}(\mathscr{H}), H^{q+1}(\mathscr{H})); \quad \forall f \in L_+^2(H), \tag{3.8}$$
$$r_{[0,\infty[}(pf) = f .$$

Still denoting by G_k the operator (3.4) on the whole real line, we have:

$$r_{[0,\infty[} G_k[p(v)] = G_k[v] ,$$

so that:

$$\|G_k[v]\|_{L_+^2(\mathscr{H})} = \|r_{[0,\infty[}[G_k[p(v)]]\|_{L_+^2(\mathscr{H})} \leqq \|G_k[p(v)]\|_{L^2(\mathscr{H})} ;$$

therefore we have only to prove (3.7) for $\mathbb{R}$-defined functions.

By applying the Fourier transform ([5]) to $G_k[v]$ we obtain:

$$\mathscr{F}[G_k[v]](\xi) = k^{-1}\{\rho(e^{2\pi i k\xi}) - 2\pi i k\xi\sigma(e^{2\pi i k\xi})\}\mathscr{F}[v](\xi) .$$

By (P2) we get:

$$|\rho(e^{ix}) - ix\sigma(e^{ix})| \leqq C|x|^{q+1}, \quad x \in \mathbb{R} ,$$

so that:

$$\|\mathscr{F}[G_k[v]](\xi)\|_{L^2(\mathscr{H})} \leqq C k^q \| |\xi|^{q+1} \mathscr{F}[v](\xi)\|_{L^2(\mathscr{H})} \leqq C k^q \|v\|_{H^{q+1}(\mathscr{H})} .$$

(3.7) for $j > 0$ follows immediately by the identity $D^j G_k[v] = G_k[D^j v]$. □

Remark 3.4. We observe that:

$$\Pi G_k[v] = \frac{\rho(\mathrm{E})}{k} \Pi v - \sigma(\mathrm{E}) \Pi v' .$$

Convergence theorem

With new notations, Theorem 1.7 becomes:

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[5] We denote with $\mathscr{F}$ the Fourier transform in $L^2(\mathscr{H})$:

$$\mathscr{F}[v](\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi t} v(t)\, dt; \qquad \|\mathscr{F}[v]\|_{L^2(\mathscr{H})} = \|v\|_{L^2(\mathscr{H})} .$$

Theorem 3.5. *Assume that* (A1–4), (P1–2), (G1) *and* (I1) *hold true*; *the solution* ${}^h\boldsymbol{u}$ *of*:

(3.9) $$ {}^hT_k{}^h\boldsymbol{u} = \sigma(\mathrm{E})^h\Pi f, \quad {}^h\boldsymbol{u}|_{g-1} = {}^h\underline{\boldsymbol{u}} $$

satisfies:

$$ \|{}^h\mathbf{u} - \Pi u\|_{l^2_k(V)\cap l^\infty(H)} \leqq C\{k^q\|u\|_{H^{q+1}_+(V,V^*)} + \|u - P_h u\|_{L^2_+(V)\cap L^\infty_+(H)} + \varepsilon[u;{}^h\underline{\boldsymbol{u}}]\} $$

(3.10) $$ \leqq C\{k^q[\|f\|_{H^q_+(V^*)} + \|c_q(f,u_0)\|_{V^q\times H}] + e_h[u]\} $$

Proof. We have the following decomposition:

$$ \Pi u - {}^h\boldsymbol{u} = (\Pi u - {}^h\Pi u) + ({}^h\Pi u - {}^h\boldsymbol{u}) $$

so that, by applying Corollary 3.2, it remains to study the difference ${}^h\boldsymbol{d} = {}^h\Pi u - {}^h\boldsymbol{u}$ which is contained in $l^2_k(V_h) \cap l^\infty(H_h)$.

Our purpose is to write a difference equation satisfied by ${}^h\boldsymbol{d}$ and to apply the preceding stability estimates. We observe that:

$$ \|{}^h\boldsymbol{d}|_{g-1}\|_{l^2_k(V)\cap l^\infty(H)} = \|({}^h\Pi u)|_{g-1} - {}^h\underline{\boldsymbol{u}}\|_{l^2_k(V)\cap l^\infty(H)} = \varepsilon[u;{}^h\underline{\boldsymbol{u}}] $$

so that, by (I1) :

(3.11) $$ \|{}^h\boldsymbol{d}|_{g-1}\|_{l^2_k(V)\cap l^\infty(H)} \leqq k^q[\|f\|_{H^q(0,kg;V^*)} + \|\boldsymbol{c}_q\|_{V^q\times H}] . $$

If we apply operator ${}^h\Pi$ to (1.1), we obtain ${}^h\Pi u' + {}^hA\Pi u = {}^h\boldsymbol{f}$, with ${}^hA = P_hA$, ${}^hf = P_hf$, and:

$$ {}^hT_k[{}^h\Pi u] = \sigma(\mathrm{E})\,{}^h\Pi f + P_h\left\{\frac{\rho(\mathrm{E})}{k}\Pi u - \sigma(\mathrm{E})\Pi u'\right\} + \sigma(\mathrm{E})^hA\Pi(P_hu - u) . $$

Taking the difference with (3.9), we get:

$$ {}^hT_k{}^h\boldsymbol{d} = {}^h\Pi G_k[u] + \sigma(\mathrm{E})^hA\,\Pi(P_hu - u) . $$

By Lemma 3.3

$$ \|{}^h\Pi G_k[u]\|_{l^2_k(V_h^*)} \leqq C k^q\|u\|_{H^{q+1}_+(V^*)} , $$

and by Corollary 3.2 we have:

$$ \|{}^hA\Pi(P_hu - u)\|_{l^2_k(V_h^*)} \leqq M\|{}^h\Pi u - \Pi u\|_{l^2_k(V)} $$

$$ \leqq C\{\|P_hu - u\|_{L^2_+(V)} + k^q\|u\|_{H^q_+(V)}\} ; $$

taking into account (3.11) and applying Theorem 1.4, we conclude our proof. □

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