Numer. Math. 66, 123–137 (1993)

Numerische Mathematik

A nodal spline interpolant for the Gregory rule of even order

J.M. de Villiers
Department of Mathematics, University of Stellenbosch, Stellenbosch 7600, South Africa

Received July 28, 1992 / Revised version received February 1, 1993

Summary. The Gregory rule is a well-known example in numerical quadrature of a trapezoidal rule with endpoint corrections of a given order. In the literature, the methods of constructing the Gregory rule have, in contrast to Newton-Cotes quadrature, *not* been based on the integration of an interpolant. In this paper, after first characterizing an even-order Gregory interpolant by means of a generalized Lagrange interpolation operator, we proceed to explicitly construct such an interpolant by employing results from nodal spline interpolation, as established in recent work by the author and C.H. Rohwer. Nonoptimal order error estimates for the Gregory rule of even order are then easily obtained.

Mathematics Subject Classification (1991): 41A55, 41A15, 41A05, 65D32, 65D30, 65D07, 65D05

1. Introduction

Let n be a positive integer, and, for a given interval $[a, b]$ on the real line $\mathbb{R}$, consider the uniform partition $\{\xi_i;\ i = 0, 1, \ldots, n\}$ given by

$$(1.1)\qquad \xi_i = a + iH, \quad i = 0, 1, \ldots, n, \qquad H = \frac{b-a}{n}.$$

For a function $f \in C[a, b]$, and with the notation $I[f] := \int_a^b f(x)dx$ and $f_i := f(\xi_i)$, $i = 0, 1, \ldots, n$, we use the symbol $Q[f]$ to denote, for a given set $\{w_{n,i};\ i = 0, 1, \ldots, n\}$ of weights, a quadrature formula of the form

$$(1.2)\qquad Q[f] = \sum_{i=0}^{n} w_{n,i} f_i,$$

whereas the symbol $E[f] := I[f] - Q[f]$ will be used for the corresponding quadrature error. Both $Q[f]$ and $E[f]$ will, whenever necessary, be suitably indexed in order to distinguish between different quadrature formulas.

For a given nonnegative integer k, we write $\mathbb{P}^k$ for the set of polynomials of degree $\leqq k$. Also, we employ the usual notation $\|g\|_\infty := \max_{a \leqq x \leqq b}|g(x)|$ for the maximum norm of a function $g \in C[a, b]$.

In recent years, some attention in the literature (see, e.g., Förster (1987), Solak and Szydelko (1991)) has been devoted to the *Gregory rule*, which, for integers $k \geqq 0$ and $n \geqq k$, will be denoted here by $Q_{k,n}^{GR}[f]$, and was shown in Brass (1977, p. 210) to be expressible in the form

$$(1.3) \qquad Q_{k,n}^{GR}[f] = H\left\{ \sum_{i=0}^{n} f_i + \sum_{i=0}^{k} [(-1)^{i+1} \sum_{j=i}^{k} \binom{j}{i} L_{j+1}][f_i + f_{n-i}] \right\},$$

where the *Laplace coefficients* $\{L_1, L_2, L_3, \ldots\}$ can be computed from the recursion formula

$$(1.4) \qquad \sum_{\nu=1}^{\mu} \frac{L_\nu}{\mu - \nu + 1} = \frac{1}{\mu + 1}, \quad \mu = 1, 2, \ldots,$$

as given in Martensen (1973, p. 70). We call $Q_{k,n}^{GR}[f]$ the *Gregory rule of order* k, which is a *symmetric* quadrature rule in the sense that (1.3) can be written in the form (1.2), with the weights satisfying the symmetry condition

$$(1.5) \qquad w_{n,i} = w_{n,n-i}, \quad i = 0, 1, \ldots, n .$$

Note in particular from (1.3) and (1.4) that the case $k = 0$ yields the *trapezoidal rule* $Q_n^{TR}[f]$; we have

$$(1.6) \qquad Q_{0,n}^{GR}[f] = Q_n^{TR}[f] = H[\tfrac{1}{2} f_0 + f_1 + f_2 + \cdots + f_{n-1} + \tfrac{1}{2} f_n] .$$

It is clear from (1.3) and (1.6) that the Gregory rule can be interpreted as a *trapezoidal rule with endpoint corrections of order k.*

Whereas, for general nonnegative order k, the exactness condition $E_{k,n}^{GR}[p] = 0$, $p \in \mathbb{P}^k$, holds, an elementary result for symmetric quadrature rules, as proved in Brass (1977, Theorem 42), yields the improvement $E_{2m,n}^{GR}[p] = 0$, $p \in \mathbb{P}^{2m+1}$, $m = 1, 2, \ldots$, for the Gregory rule of *even* order $2m$, as is also implied by the optimal-order error estimate

$$(1.7) \qquad |E_{2m,n}^{GR}[f]| \leqq (b-a) L_{2m+2} H^{2m+2} \|f^{(2m+2)}\|_\infty, \quad f \in C^{2m+2}[a, b],$$
$$m = 0, 1, \ldots,$$

the proof of which is given in Martensen (1964, pp. 161–163).

Further notable features of the Gregory rule is that $Q_{k,n}^{GR}[f]$ is asymptotically optimal in a sense made precise in Babuska (1966) and Lötzbeyer (1972); also, in contrast to the composite Newton-Cotes and Romberg quadrature formulas, the integer n need *not* satisfy some given divisibility condition.

For a given quadrature formula $Q[f]$, the availability of an approximation f^* of f for which $Q[f] = I[f^*]$, often provides a useful aid in the study of the properties of $Q[f]$. For example, exploiting the resulting fact that $E[f] = I[f - f^*]$, the error analysis for $Q[f]$ can then be conducted by using possibly known expressions (or estimates) for the approximation error $f - f^*$. Whereas the Newton-Cotes (and the related Gauss) quadrature formulas are usually *defined* as the definite integral of the corresponding Lagrange polynomial interpolant, which then immediately yields such an approximation f^* of f, the construction in the literature of the Gregory rule has been based either, as in Ralston and Rabinowitz (1978, p. 140), on replacing the endpoint derivatives in the Euler-Maclaurin formula (see (2.2) below) by forward and backward differences up to a given order, or, as in Brass (1977, p. 210), by means of a certain polynomial identity. Neither of these two methods seems to suggest an obvious approximation f^* of f for which $Q_{k,n}^{GR}[f] = I[f^*]$.

In this paper, after proving, in Sect. 2, a fundamental existence and uniqueness theorem for corrected trapezoidal rules, we proceed in Sect. 3 to characterize, in Theorem 3.1, a generalized Lagrange interpolation operator G: $C[a, b] \to C[a, b]$ which is assumed to have certain approximation properties, and which yields an interpolant for the Gregory rule of even order in the sense that, for given integers $m \geqq 1$ and $n \geqq 2m$, the desired relation $Q_{2m,n}^{GR}[f] = I[Gf]$ holds, with Gf interpolating f on the partition (1.1).

Next, in Sect. 4, we verify that the *nodal spline* interpolation operator which was developed in a sequence of papers by De Villiers and Rohwer (1987, 1991, 1992) and De Villiers (1992), and for which a fundamental existence and uniqueness theorem was proved by Dahmen, Goodman and Micchelli (1988, Theorem 4.2.1), actually satisfies, in the case of even-degree splines and uniformly spaced knots, the properties assumed in Theorem 3.1 of G, and thus yields an explicit even-order Gregory interpolant Gf. In similar work, Schoenberg and Sharma (1971), for a specific order, and Delvos (1986), for general order, constructed a spline interpolant for the Euler-Maclaurin quadrature formula.

Finally, in Sect. 5, we employ Jackson-type estimates for the nodal spline interpolation error, as established in De Villiers and Rohwer (1992), to derive nonoptimal order error estimates for the Gregory rule of even order $2m$ when applied to the integration of functions $f \in C^l[a, b]$, $l = 0, 1, \ldots, 2m + 1$. Analogous error estimates for Newton-Cotes and other quadrature formulas have been established by Stroud (1966; 1974, Sect. 3.12 and Appendix B).

Although the above-mentioned work by De Villiers and Rohwer on nodal spline interpolation was not restricted to even-degree splines, we observe that, in the case of splines of odd degree $\geqq 3$, the corresponding nodal splines have non-symmetric support intervals (see De Villiers and Rohwer (1987, equations (3.8) and (3.20)), so that Theorem 3.1 below is not applicable.

Also, it should perhaps be remarked that, since nodal spline interpolation was developed for *arbitrary* knot spacings, the resulting nodal spline quadrature formula could, in the case of even-degree splines, be regarded as a generalization (to partitions which are not necessarily uniform) of the even-order Gregory rule.

2. A fundamental existence and uniqueness theorem

As has already been pointed out in Sect. 1, the Gregory rule can, according to (1.3) and (1.6), be interpreted as a corrected trapezoidal rule, for which we now give the following precise definition.

Definition 2.1. For given integers $k \geqq 0$ and $n \geqq k$, a quadrature rule of the form (1.2) will be called a *trapezoidal rule with endpoint corrections of order* k, and denoted by $Q^{CT}_{k,n}[f]$, if and only if it has the form

$$(2.1)\qquad Q^{CT}_{k,n}[f] = Q^{TR}_n[f] + H\sum_{i=0}^{k} \alpha_{k,i}(f_i + f_{n-i}),$$

for a coefficient set $\{\alpha_{k,i};\ i = 0, 1, \ldots, k\}$ in which *each* $\alpha_{k,i}$ *is independent of* n.

We proceed to state and prove, for *even-order* k, a fundamental existence and uniqueness theorem for $Q^{CT}_{k,n}[f]$, according to which the coefficient set $\{\alpha_{k,i};\ i = 0, 1, \ldots, k\}$ in (2.1) is uniquely determined by the demand of exactness on the polynomial class $\mathbb{P}^k$. Although our result is immediately derivable from a more general theorem by Brass (1977, Theorem 116), we nevertheless give the complete proof there, since our specialization allows some significant simplifications.

We shall need the *Euler-Maclaurin formula*, which, according to Atkinson (1978, equation (5.93)), can be expressed in the form

$$(2.2)\qquad E^{TR}_n[f] = -\sum_{j=1}^{r} \frac{B_{2j}}{(2j)!}[f^{(2j-1)}(b) - f^{(2j-1)}(a)]H^{2j} - \frac{(b-a)B_{2r+2}}{(2r+2)!}H^{2r+2}f^{(2r+2)}(\xi)$$

for $f \in C^{2r+2}[a, b]$, $r = 0, 1, 2, \ldots,$ where ξ is a point in $[a, b]$, and with $\{B_{2j}, j = 1, 2, \ldots\}$ denoting the Bernoulli numbers of even order. In (2.2), as throughout the paper, we adopt the convention

$$\sum_{j=m}^{r} b_j = 0 \quad \text{if } r < m.$$

Theorem 2.1. *Let* m *and* n *be integers with* $m \geqq 0$ *and* $n \geqq 2m$. *Then there exists a unique trapezoidal rule with endpoint corrections of order* $2m$, $Q^{CT}_{2m,n}[f]$, *satisfying the exactness condition*

$$(2.3)\qquad E^{CT}_{2m,n}[p] = 0, \quad p \in \mathbb{P}^{2m}.$$

Proof. Let $p \in \mathbb{P}^{2m}$. Then, from (2.1) and (2.2),

$$(2.4)\qquad E^{CT}_{2m,n}[p] = -\sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}[p^{(2j-1)}(b) - p^{(2j-1)}(a)]H^{2j} - H\sum_{i=0}^{2m} \alpha_{2m,i}[p(\xi_i) + p(\xi_{n-i})].$$

Noting from (1.1) that $\xi_{n-i} = b - iH$, we next use Taylor expansions to obtain

$$(2.5)\qquad \sum_{i=0}^{2m} \alpha_{2m,i}[p(\xi_i) + p(\xi_{n-i})] = \sum_{j=0}^{2m} \frac{1}{j!}[p^{(j)}(a) + (-1)^j p^{(j)}(b)]\left[\sum_{i=0}^{2m} i^j \alpha_{2m,i}\right]H^j,$$

after having also changed the summation order. It is understood that $0^0 = 1$ in (2.5).

Next, we divide the sum over j in (2.5) into even and odd powers of H, and then insert the resulting expression into (2.4) to find

$$(2.6)\quad E^{CT}_{2m,n}[p] = \sum_{j=1}^{m} [p^{(2j-1)}(b) - p^{(2j-1)}(a)]\left[\sum_{i=0}^{2m} \frac{i^{2j-1}}{(2j-1)!}\alpha_{2m,i} - \frac{B_{2j}}{(2j)!}\right]H^{2j} - \sum_{j=0}^{m} [p^{(2j)}(a) + p^{(2j)}(b)]\left[\sum_{i=0}^{2m} \frac{i^{2j}}{(2j)!}\alpha_{2m,i}\right]H^{2j+1}.$$

Hence the exactness condition (2.3) is satisfied for a coefficient set $\{\alpha_{2m,i};\ i = 0, 1, \ldots, 2m\}$, with each $\alpha_{2m,i}$ independent of n, if and only if $\{\alpha_{2m,i};\ i = 0, 1, \ldots, 2m\}$ solves the $(2m+1)\times(2m+1)$ linear system

$$(2.7)\qquad \sum_{i=0}^{2m} i^j\alpha_{2m,i} = \begin{cases} \dfrac{B_{j+1}}{j+1}, & j = 1, 3, \ldots, 2m-1, \\ 0, & j = 0, 2, \ldots, 2m, \end{cases}$$

where the "only if" part can be shown by choosing $p(x) = (x-a)^{2m}$, for which, in (2.6), the factors $[p^{(2j-1)}(b) - p^{(2j-1)}(a)], j = 1, 2, \ldots, m$, $[p^{(2j)}(a) + p^{(2j)}(b)]$, $j = 0, 1, \ldots, m$, are nonzero and independent of n, and then recalling that (2.3) must hold for all values of $n(= (b-a)/H) \geqq 2m$.

The result of the theorem now follows by recognizing the coefficient matrix

$$(2.8)\qquad A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & 2m \\ 0 & 1 & 2^2 & \cdots & (2m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2^{2m} & \cdots & (2m)^{2m} \end{bmatrix}$$

corresponding to the linear system (2.7) as the transpose of a Vandermonde matrix, with $\det(A) = \prod_{r=0}^{2m} r! \neq 0$. □

Hence, recalling also (1.3), (1.6), Definition 2.1 and (1.7), we immediately deduce the following result.

Corollary 2.2. *The unique quadrature rule implied by Theorem 2.1 is precisely the Gregory rule of order 2m, $Q^{GR}_{2m,n}[f]$, as defined by setting $k = 2m$ in the representation* (1.3).

Setting $m = 1$ in Corollary 2.2 yields the *Lacroix rule* $Q^{LA}_n[f] := Q^{GR}_{2,n}[f]$, which, according to (1.3) and (1.4), is given by

$$(2.9)\quad Q^{LA}_n[f] = Q^{TR}_n[f] + H[-\tfrac{1}{8}(f_0 + f_n) + \tfrac{1}{6}(f_1 + f_{n-1}) - \tfrac{1}{24}(f_2 + f_{n-2})] \quad (n \geqq 2),$$

or, alternatively,

$$(2.10)\quad Q^{LA}_n[f] = H\left[\frac{3}{8}(f_0 + f_n) + \frac{7}{6}(f_1 + f_{n-1}) + \frac{23}{24}(f_2 + f_{n-2}) + \sum_{i=3}^{n-3} f_i\right] \quad (n \geqq 5).$$

The following relationship between the even-order Gregory and Newton-Cotes quadrature formulas should also be noted at this point. Suppose the integer $m \geqq 1$, and define the Newton-Cotes quadrature formula $Q_{2m}^{NC}[f] := I[p_{2m}]$, with $p_{2m} \in \mathbb{P}^{2m}$ denoting the (unique) Lagrange polynomial interpolant of f satisfying the interpolation conditions $p(\xi_i) = f_i$, $i = 0, 1, \ldots, 2m$. Since both $Q_{2m,2m}^{GR}[f]$ and $Q_{2m}^{NC}[f]$ have the form (1.2) with $n = 2m$, and are exact on $\mathbb{P}^{2m}$ (as well as on the larger class $\mathbb{P}^{2m+1}$), we can appeal to a standard uniqueness result in Davis and Rabinowitz (1984, p. 74) to deduce the relation

$$Q_{2m,2m}^{GR}[f] = Q_{2m}^{NC}[f], \quad m = 1, 2, \ldots, \tag{2.11}$$

which already provides a useful insight for the work of the next section.

3. Characterization of an even-order Gregory interpolant

In order to construct, for a given $f \in C[a, b]$, an approximation f^* of f for which $Q_{2m,n}^{GR}[f] = I[f^*]$, we first note from (2.11) that, in the case $n = 2m$, the required approximation f^* would have to be identical to the Lagrange polynomial interpolant p_{2m} which was used in the definition, as given in the argument leading to (2.11), of $Q_{2m}^{NC}[f]$. We shall now show how the uniqueness aspect of Theorem 2.1, together with Corollary 2.2, can be exploited to characterize a generalized Lagrange interpolation operator $G: C[a, b] \to C[a, b]$ from which the desired approximation f^* can be obtained by setting $f^* = Gf$.

For given integers $m \geqq 1$, $n \geqq 2m$, we define the Lagrange interpolation operators $L: C[a, b] \to \mathbb{P}^{2m}$ and $\hat{L}: C[a, b] \to \mathbb{P}^{2m}$ by the conditions

$$\left.\begin{aligned} (Lf)(\xi_i) &= f_i, \\ (\hat{L}f)(\xi_{n-i}) &= f_{n-i}, \end{aligned}\right] \quad i = 0, 1, \ldots, 2m, \quad f \in C[a, b], \tag{3.1}$$

so that L and $\hat{L}$ can be represented by the formulas

$$\left.\begin{aligned} (Lf)(a + Ht) &= \sum_{i=0}^{2m} l_i(t) f_i, \\ (\hat{L}f)(a + Ht) &= \sum_{i=0}^{2m} l_i(n - t) f_{n-i}, \end{aligned}\right] \quad 0 \leqq t \leqq n, \; f \in C[a, b], \tag{3.2}$$

with

$$l_i(t) := \prod_{k=0, k \neq i}^{2m} \frac{t - k}{i - k}. \tag{3.3}$$

Also, we clearly have the polynomial reproduction properties

$$Lp = p, \quad p \in \mathbb{P}^{2m}, \qquad \hat{L}p = p, \quad p \in \mathbb{P}^{2m}. \tag{3.4}$$

We shall employ the usual symbol δ_{ij} for the Kronecker delta, whereas $\mathbb{Z}$ will denote the set of integers.

The following characterization theorem then holds.

Theorem 3.1. *Let the integer $m \geqq 1$, and suppose that there exists a function $\psi_m \in C(\mathbb{R})$ such that:*

(3.5) $$\psi_m(t) = 0, \quad t \notin [-m-1, m+1]\,;$$

(3.6) $$\psi_m(t) = \psi_m(-t), \quad t \in [-m-1, m+1]\,;$$

(3.7) $$\sum_{i \in \mathbb{Z}} \psi_m(t-i)p(i) = p(t), \quad t \in \mathbb{R},\, p \in \mathbb{P}^{2m}\,.$$

Then, if $n \geqq 2m$, the generalized Lagrange interpolation operator G defined for $f \in C[a, b]$ by

(3.8) $$(Gf)(a+Ht) = \begin{cases} (Lf)(a+Ht), & 0 \leqq t \leqq m\,, \\ \sum_{i=0}^{n} \psi_m(t-i)f_i, & m < t < n-m \; (n \geqq 2m+1)\,, \\ (\hat{L}f)(a+Ht), & n-m \leqq t \leqq n\,, \end{cases}$$

with L and $\hat{L}$ given as in (3.2), maps $C[a, b]$ into $C[a, b]$, and provides an interpolant Gf for the Gregory rule of even order $2m$ in the sense that

(3.9) $$(Gf)(\xi_i) = f(\xi_i), \quad i = 0, 1, \ldots, n,$$
(3.10) $$I[Gf] = Q^{GR}_{2m,n}[f], \qquad f \in C[a, b]\,.$$

Proof. (a) To prove (3.9), we fix $j \in \{-m, -m+1, \ldots, m\}$ and then choose, in the reproduction property (3.7),

$$p(t) = l_j(t), \quad t \in \mathbb{R}\,,$$

with $l_j \in \mathbb{P}^{2m}$ defined as in (3.3). Now set $t = m$ in the resulting equation to deduce

(3.11) $$\delta_{mj} = l_j(m) = \sum_{i=0}^{2m} l_j(i)\,\psi_m(m-i) = \psi_m(m-j)\,,$$

where the summation limits are obtained by virtue of the fact that $\psi_m(t) = 0$, $|t| \geqq m+1$, as is evident from (3.5) and the continuity condition $\psi_m \in C(\mathbb{R})$. It then follows from (3.11) that ψ_m satisfies the *nodal* property

(3.12) $$\psi_m(j) = \delta_{0j}, \quad j \in \mathbb{Z}\,,$$

from which, together with (3.8) and (3.1), the desired interpolation property (3.9) is easily shown.

(b) The property G: $C[a, b] \to C[a, b]$ can now be shown by exploiting the nodal property (3.12) of ψ_m, together with the fact that $\psi_m \in C(\mathbb{R})$, to deduce from (3.8) that

$$\lim_{x \to \xi_m^+} (Gf)(x) = \lim_{t \to m^+} (Gf)(a+Ht) = f_m = (Gf)(\xi_m)\,,$$

having noted also (3.1), and similarly at the point $x = \xi_{n-m}$.

(c) Next, setting $n = 2m$, we see from (3.1) and (3.8) that $G = L = \hat{L}$, and thus

$$I[Gf] = I[Lf] = Q^{NC}_{2m}[f] = Q^{GR}_{2m,2m}[f]\,,$$

by virtue also of (2.11), and therefore yielding (3.10) for $n = 2m$.

(d) For the rest of the proof, we assume that $n \geqq 2m+1$. Observing from (3.4), (3.5), (3.7) and (3.8) that the reproduction property

$$Gp = p, \quad p \in \mathbb{P}^{2m}, \tag{3.13}$$

holds, we shall, in paragraphs (e) and (f) below, show that $I[Gf]$ yields, in the sense of Definition 2.1, a trapezoidal rule with endpoint corrections of order $2m$,

$$I[Gf] = Q^{CT}_{2m,n}[f], \quad f \in C[a,b], \tag{3.14}$$

for which then, according to (3.13), $E^{CT}_{2m,n}[p] = I[p - Gp] = 0$, $p \in \mathbb{P}^{2m}$. The desired result (3.10) then follows by appealing to the uniqueness statement of Theorem 2.1, as well as Corollary 2.2.

(e) Hence, we proceed to verify the result (3.14). Using (3.8) and (3.2), we derive the expression

$$I[Gf] = H\int_0^n (Gf)(a+Ht)\,dt = H\left[\sum_{i=0}^{n} \beta_i f_i + \sum_{i=0}^{2m} \gamma_i (f_i + f_{n-i})\right], \tag{3.15}$$

with

$$\beta_i := \int_{m-i}^{n-m-i} \psi_m(t)dt, \quad i = 0, 1, \ldots, n, \tag{3.16}$$

$$\gamma_i := \int_0^m l_i(t)dt, \quad i = 0, 1, \ldots, 2m, \tag{3.17}$$

and where, in (3.16), the symmetry property

$$\beta_{n-i} = \beta_i, \quad i = 0, 1, \ldots, n, \tag{3.18}$$

can easily be verified by means of (3.6).

Now restrict the integer n further by the inequality $n \geqq 4m+2$. Then (3.15) can be rewritten, by virtue of (3.18), in the form

$$I[Gf] = H\left[\sum_{i=2m+1}^{n-(2m+1)} \beta_i f_i + \sum_{i=0}^{2m} (\beta_i + \gamma_i)(f_i + f_{n-i})\right]. \tag{3.19}$$

Since the integration limits in (3.16) satisfy

$$\left.\begin{array}{l} m - i \leqq -m-1, \\ n - m - i \geqq m+1, \end{array}\right] \quad 2m+1 \leqq i \leqq n-(2m+1),$$

it follows from (3.5) that

$$\beta_i = \int_{-m-1}^{m+1} \psi_m(t)dt, \quad i = 2m+1, \ldots, n-(2m+1). \tag{3.20}$$

But

$$\int_{-m-1}^{m+1} \psi_m(t)dt = \sum_{j=-m-1}^{m} \int_0^1 \psi_m(t+j)dt = \int_0^1 \sum_{j=-m}^{m+1} \psi_m(t-j)\,dt, \tag{3.21}$$

and since the choice $p(t) = 1, t \in \mathbb{R}$, in the reproduction property (3.7) gives, together with (3.5), the value

$$\sum_{j=-m}^{m+1} \psi_m(t-j) = 1, \quad 0 \leqq t \leqq 1,$$

it follows from (3.21) that

$$\int_{-m-1}^{m+1} \psi_m(t)dt = 1 . \tag{3.22}$$

Comparing Definition 2.1 with (3.19), (3.20) and (3.22), we see that, to prove the relation (3.14) for $n \geqq 4m+2$, it remains to show that, for these values of n, the coefficients β_i and γ_i are independent of n for $i = 0, 1, \ldots, 2m$. In the case of β_i, we note in (3.16) that

$$n - m - i \geqq (4m+2) - m - 2m = m + 2, \quad i = 0, 1, \ldots, 2m ,$$

and thus, from (3.5),

$$\beta_i = \int_{m-i}^{m+1} \psi_m(t)dt, \quad i = 0, 1, \ldots, 2m ,$$

which is clearly independent of n, whereas, for γ_i, the desired independence of n is an immediate consequence of (3.17) and (3.3).

(f) Finally, we show that (3.14) also holds for $2m+1 \leqq n \leqq 4m+1$, by proving that, in (3.15),

$$\sum_{i=0}^{n} \beta_i f_i = \sum_{i=0}^{n} f_i + \sum_{i=0}^{2m} \delta_i (f_i + f_{n-i}), \quad 2m+1 \leqq n \leqq 4m+1 , \tag{3.23}$$

with

$$\delta_i := -\int_{-m-1}^{m-i} \psi_m(t)\,dt, \quad i = 0, 1, \ldots, 2m . \tag{3.24}$$

A careful inspection of (3.23) reveals that it will suffice to verify, for a fixed integer $k \in \{1, 2, \ldots, 2m+1\}$, and for $n = 2m + k$, the relations

$$\beta_i - 1 = \begin{cases} \delta_i, & i = 0, 1, \ldots, k-1 , \\ \delta_i + \delta_{2m+k-i}, & i = k, k+1, \ldots, 2m \ (k \leqq 2m) , \\ \delta_{2m+k-i}, & i = 2m+1, 2m+2, \ldots, 2m+k . \end{cases} \tag{3.25}$$

But, from (3.16) and (3.24),

$$\beta_i - \delta_i = \int_{-m-1}^{m+k-i} \psi_m(t)dt = \int_{-m-1}^{m+1} \psi_m(t)dt, \quad i = 0, 1, \ldots, k-1 ,$$

having used also (3.5), and the top line of (3.25) follows from (3.22). Similarly, for $k \leqq 2m$, we get

$$\beta_i - \delta_i - \delta_{2m+k-i} = \int_{-m-1}^{m+k-i} \psi_m(t)dt - \delta_{2m+k-i}, \quad i = k, \ldots, 2m , \tag{3.26}$$

where, from (3.24),

$$-\delta_{2m+k-i} = \int_{-m-1}^{-m-k+i} \psi_m(t)dt = \int_{m+k-i}^{m+1} \psi_m(t)dt, \quad i = k, \ldots, 2m+k , \tag{3.27}$$

by virtue of (3.6), and the middle line of (3.25) follows by combining (3.26), (3.27) and (3.22). For the bottom line of (3.25), we note from (3.16) and (3.27) that

$$\beta_i - \delta_{2m+k-i} = \int_{m-i}^{m+1} \psi_m(t)dt = \int_{-m-1}^{m+1} \psi_m(t)dt, \quad i = 2m+1, \ldots, 2m+k,$$

from (3.5), and then employ (3.22) once again. □

Remarks. (a) In the case $n = 2m$ we have, as noted in paragraph (c) of the proof above, the equalities $G = L = \hat{L}$, whereas if $n \geqq 2m+1$, the approximation operator G is, in the sense of (3.8), a continuous extension of the Lagrange interpolation operator, where, according to (3.9) and (3.13), the Lagrangian properties of interpolation and optimal order polynomial reproduction (cf. (3.1) and (3.4)) have been preserved by G.

(b) The function ψ_m can be associated with the concept of a *superfunction*, in accordance with the usage by Strang and Fix (1973, pp. 141–143).

(c) The bounded support property (3.5) of ψ_m ensures that the approximation operator G is *local* in the sense that, for fixed $j \in \{0, 1, \ldots, n-1\}$, the value of Gf at a point $x \in [\xi_j, \xi_{j+1}]$ depends on the values of at most $2m$ neigbouring values of f on the point set $\{\xi_i, i = 0, 1, \ldots, n\}$.

(d) It can easily be verified that the interpolation property (3.9) of G holds *if and only if* the function ψ_m satisfies the nodal property (3.12).

4. A nodal spline interpolant

In this section we proceed to demonstrate constructively the existence of a polynomial spline satisfying the properties assumed of ψ_m in Theorem 3.1, and thus generating a generalized Lagrange interpolant Gf of the type (3.8) for the Gregory rule of even order.

First, to introduce some notation for polynomial splines, we consider, for positive integers k and n, the uniform partition $\Delta_{k,n} := \{x_i;\ i = 0, 1, \ldots, kn\}$ of $[a, b]$, as given by

(4.1) $$x_i = a + ih, \quad i = 0, 1, \ldots, kn, \qquad h = \frac{b-a}{kn}.$$

Hence, comparing (4.1) and (1.1), we see that

(4.2) $$\xi_i = x_{ki}, \quad i = 0, 1, \ldots, n, \qquad H = kh.$$

In the context of the work on nodal spline interpolation by De Villiers and Rohwer (1987, 1991, 1992) and De Villiers (1992), the points of $\{\xi_i, i = 0, 1, \ldots, n\}$ were called *primary knots*, whereas the points of the set $\Delta_{k,n} \backslash \{\xi_i;\ i = 0, 1, \ldots, n\}$ were referred to as *secondary* (or *additional*) *knots*.

For a given set Δ of distinct points in $\mathbb{R}$, and for k a positive integer, we denote by $S_k(\Delta)$ the set of polynomial splines of order $k+1$ (degree $\leq k$) and with simple knots, so that $S_k(\Delta) \subset C^{k-1}$. In particular, we define $S_{k,n} := S_k(\Delta_{k,n})$ and $S_k := S_k(\mathbb{Z})$. We employ the symbol N_k for the normalized B-spline in S_k, which, according to

Schumaker (1981, p. 135), has support interval [0, $k+1$], and is given by the formula

(4.3) $$N_k(\tau) = \frac{1}{k!}\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}(\tau - i)_+^k, \quad \tau\in\mathbb{R},$$

in terms of the truncated power function defined by

$$(\tau - y)_+^k := \left[\begin{array}{ll} 0, & \tau < y, \\ (\tau - y)^k, & \tau \geqq y. \end{array}\right.$$

The following connection between nodal spline interpolation and the Gregory rule can now be proved.

Theorem 4.1. *For a given positive integer m, let the nodal spline* $s_{2m}\in S_{2m}$ *be defined by the B-spline series*

(4.4) $$s_{2m}(t) = \sum_{j=-m-1}^{m-1}\sum_{r=0}^{2m-1} a_{j,r}N_{2m}(2m(t-j)-r), \quad t\in\mathbb{R},$$

where

(4.5) $$a_{j,r} = \frac{1}{(2m)!}\sum_{1\leqq v_l\leqq 2m;\ l\in M_j;\ v_l \text{ distinct}}\ \prod_{l=-m-1,\ l\neq j}^{m-1}\frac{l+\dfrac{v_l+r}{2m}}{l-j},$$

with

(4.6) $$M_j := \{-m-1, -m, \ldots, m-1\}\backslash\{j\}.$$

Then, in Theorem 3.1, *the choice* $\psi_m = s_{2m}$ *is an admissible one, such that, in particular,* $G\colon C[a,b]\to S_{2m,n}\subset C^{2m-1}[a,b]$, *and which yields a nodal spline interpolant Gf, in the sense of* (3.9) *and* (3.10), *for the Gregory rule of even order 2m.*

Proof. Appealing to the results of De Villiers and Rohwer (1987, p. 110; 1991, Theorem 2.1 and Corollary 2.2) and De Villiers (1992, Theorem 3.1), we conclude that the *nodal spline* s_{2m}, which belongs to $S_{2m}\subset C^{2m-1}(\mathbb{R})$, satisfies the properties (3.5) and (3.7), with ψ_m replaced (in both cases) by s_{2m}; also, with the choice $\psi_m = s_{2m}$ in (3.8), the approximation operator G maps $C[a,b]$ into $S_{2m,n}\subset C^{2m-1}[a,b]$. Hence, according to Theorem 3.1, it remains to verify the even function property

(4.7) $$s_{2m}(t) = s_{2m}(-t), \quad t\in\mathbb{R}.$$

First, we note that the B-spline coefficients in (4.4) satisfy the symmetry relation

(4.8) $$a_{j,r} = a_{-j-2,2m-1-r}, \qquad r = 0, 1, \ldots, 2m-1,$$
$$j = -m-1, -m, \ldots, m-1,$$

which is proved by using (4.5) to obtain

(4.9) $$a_{-j-2,2m-1-r} = \frac{1}{(2m)!}\sum_{1\leqq v_{-l-2}\leqq 2m,\ -l-2\in M_{-j-2},\ v_{-l-2} \text{ distinct}}\ \prod_{l=-m-1,\ l\neq j}^{m-1}\frac{l+\dfrac{[(2m+1-v_{-l-2}]+r}{2m}}{l-j},$$

yielding the desired relation (4.8), where, comparing (4.9) and (4.5), we have observed that the properties $-l-2\in M_{-j-2}$ and $l\in M_j$ are equivalent by virtue of the definition (4.6), and also that the symbol ν_{-l-2} in (4.9) can clearly be replaced, throughout that equation, by ν_l to argue the equivalence of $1 \leqq (2m+1) - \nu_{-l-2} \leqq 2m$ and $1 \leqq \nu_l \leqq 2m$.

Now substitute (4.8) into (4.4) to find that, for $t\in\mathbb{R}$,

$$s_{2m}(t) = \sum_{j=-m-1}^{m-1} \sum_{r=0}^{2m-1} a_{j,r} N_{2m}(2m+1+2m(t+j)+r),$$

so that the desired relation (4.7) follows by applying the symmetry property $N_{2m}(2m+1-\tau) = N_{2m}(\tau), \tau\in\mathbb{R}$, of the B-spline N_{2m}, as can easily be established by means of the standard recursion formula

$$N_{k+1}(\tau) = \frac{\tau N_k(\tau) + (k+1-\tau)N_k(\tau-1)}{k}, \quad \tau\in\mathbb{R},\ k = 1, 2, \ldots,$$

obtained from Schumaker (1981, p. 136). □

Setting $m = 1$ in the formulas (4.4), (4.5), (4.6) and (4.3) above, we calculate (cf. De Villiers and Rohwer (1991, p. 208)) that, on its support interval $[-2, 2]$, the *quadratic nodal spline* $s_2 \in S_2$ is given by the formulas

$$\text{(4.10)} \qquad \begin{cases} s_2(t) = \dfrac{1}{4}\begin{bmatrix} 4-7t^2, & 0 \leqq t \leqq \frac{1}{2}, \\ (t-1)(5t-7), & \frac{1}{2} \leqq t \leqq 1, \\ (t-1)(3t-5), & 1 \leqq t \leqq \frac{3}{2}, \\ -(t-2)^2, & \frac{3}{2} \leqq t \leqq 2, \end{bmatrix} \\ s_2(t) = s_2(-t), \quad -2 \leqq t \leqq 0, \end{cases}$$

having used also (4.7). Note that, in accordance with (3.12), the nodal property $s_2(i) = \delta_{0i}$, $i\in\mathbb{Z}$, is clearly illustrated by the representation (4.10). An illustrative graph of s_2 is drawn in Fig. 1.

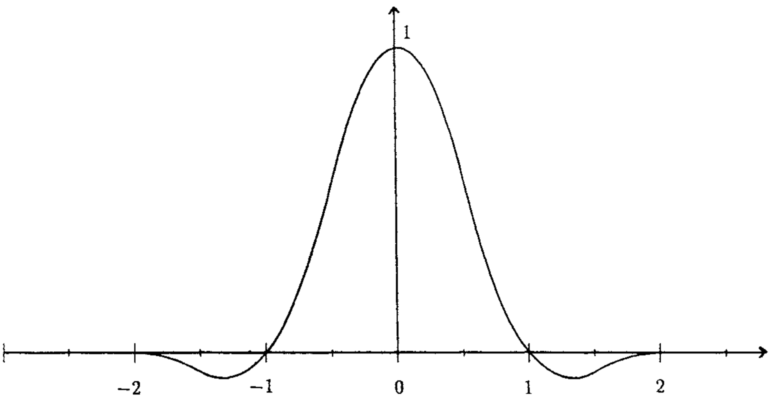

Fig. 1. The quadratic nodal spline s_2

If we now choose, for the case $m = 1$ in the definition (3.8) of the operator G, the function $\psi_1 = s_2$, with s_2 given by (4.10), we have obtained, according to Theorem 4.1, an interpolant Gf for the Lacroix rule $Q_n^{\mathrm{LA}}[f]$, as given by the formulas (2.9), (2.10).

5. Nonoptimal order error estimates

Whereas the optimal-order error estimate (1.7) for the even-order Gregory rule $Q_{2m,n}^{\mathrm{GR}}[f]$ holds for $f \in C^{2m+2}[a,b]$, we next exploit the availability of the Gregory interpolant Gf of Theorem 4.1 to establish *nonoptimal order error estimates* of the form

$$(5.1)\quad |E_{2m,n}^{\mathrm{GR}}[f]| \leqq (b-a)\begin{bmatrix} c_m\omega(f;\, k_m H), & f\in C[a,b] \;, \\ c_{m,l}H^l \|f^{(l)}\|_\infty, & f\in C^l[a,b],\, l = 1,2,\ldots,2m+1 \;, \end{bmatrix}$$

with c_m, k_m and $c_{m,l}$ denoting positive numbers which are independent of n, and where the *modulus of continuity* ω is defined as usual by

$$\omega(g;\delta) := \sup\{\,|g(x) - g(y)|;\; x, y\in[a,b], |x-y| \leqq \delta\} \,.$$

Since clearly

$$(5.2)\qquad |E_{2m,n}^{\mathrm{GR}}[f]| = |I[f - Gf]| \leqq (b-a)\|f - Gf\|_\infty \quad f\in C[a,b] \,,$$

we now recall from De Villiers and Rohwer (1992, Theorem 4.3) the Jackson-type estimates

$$(5.3)\qquad \|f - Gf\|_\infty \leqq \begin{bmatrix} \|G\|_\infty \omega(f;\,(2m+1)H), & f\in C[a,b] \;, \\ (1 + \|G\|_\infty)c_{m,l}H^l\|f^{(l)}\|_\infty, & f\in C^l[a,b] \;, \\ & l = 1,2,\ldots,2m+1 \;, \end{bmatrix}$$

where the positive numbers $c_{m,l}$ are bounded by

$$(5.4)\qquad c_{m,l} \leqq \begin{bmatrix} \dfrac{[\pi(2m+1)]^l(2m+1-l)!}{2^{2l}(2m+1)!}, & l = 1,2,\ldots,2m \;, \\[2ex] \dfrac{(2m+1)^{2m+1}}{2^{4m+1}(2m+1)!}, & l = 2m+1 \;. \end{bmatrix}$$

In (5.3), $\|G\|_\infty$ denotes the *Lebesgue constant* of the nodal spline operator G, as defined by $\|G\|_\infty := \sup\{\|Gf\|_\infty; f\in C[a,b], \|f\|_\infty \leqq 1\}$, and which, according to De Villiers and Rohwer (1992, Theorem 4.1), is bounded above by

$$(5.5)\qquad \|G\|_\infty \leqq (2m+2)(2m)^{2m} \,.$$

Combining (5.2), . . . , (5.5) then yields the desired estimates (5.1).

Moreover, it was shown by De Villiers and Rohwer (1992, Corollary 5.2) that, in the *quadratic* case $m = 1$, the rather crude (general-order) estimate (5.5) can be sharpened considerably by means of explicit calculations to yield the quadratic

Lebesgue constant $\|G\|_\infty = 1.25$. Hence, from (5.2), and setting also $m = 1$ in the bounds (5.3) and (5.4), we find, for the Lacroix rule $Q_n^{\mathrm{LA}}[f]$, the error estimates

$$|E_n^{\mathrm{LA}}[f]| \leqq (b-a) \begin{cases} \frac{4}{5}\omega(f;\ 3H), & f \in C[a,b]\ , \\ \frac{9\pi}{16} H \|f'\|_\infty, & f \in C^1[a,b]\ , \\ \frac{27\pi^2}{128} H^2 \|f''\|_\infty, & f \in C^2[a,b]\ , \\ \frac{81}{256} H^3 \|f'''\|_\infty, & f \in C^3[a,b]\ , \end{cases}$$

whereas (1.7) and (1.4) gives

$$|E_n^{\mathrm{LA}}[f]| \leqq (b-a)\frac{19}{720} H^4 \|f^{(iv)}\|_\infty, \quad f \in C^4[a,b]\ .$$

The nonoptimal order error estimates for $E_{2m,n}^{\mathrm{GR}}[f]$, as given above, can be sharpened by employing a method which, instead of using (5.2), is based on the Peano kernel theorem, as proved in Davis and Rabinowitz (1984, pp. 286–287), in conjunction with the explicitly constructed Gregory interpolant Gf of Theorem 4.1. This approach will be pursued in subsequent work.

Finally, it is interesting to note that, for fixed m, the convergence result

$$E_{2m,n}^{\mathrm{GR}}[f] \to 0, n \to \infty, \quad f \in C[a,b]\ ,$$

as implied by an application of the Pólya theorem (see Davis and Rabinowitz (1984, p. 130)), can alternatively be deduced directly from the estimate

$$|E_{2m,n}^{\mathrm{GR}}[f]| \leqq (b-a)(2m+2)(2m)^{2m}\omega(f;\ (2m+1)H), \quad f \in C[a,b]\ ,$$

which is obtained by combining (5.2), the top line of (5.3) and (5.5).

Acknowledgements. The author wishes to express his most sincere gratitude to his friend and research colleague, Dr. Carl Rohwer, for many stimulating discussions during the preparation of this paper.

References

Atkinson, K.E. (1978): An Introduction to Numerical Analysis. Wiley, New York

Babuška, I. (1966): Über die optimale Berechnung der Fourierschen Koeffizienten, Apl. Mat. **11**, 113–121

Brass, H. (1977): Quadraturverfahren. Vandenhoeck & Ruprecht, Göttingen

Dahmen, W., Goodman, T.N.T., Micchelli, C.A. (1988): Compactly supported fundamental functions for spline interpolation, Numer. Math. **52**, 639–664

Davis, P.J., Rabinowitz, P. (1984): Methods of Numerical Integration, Academic Press, Orlando

Delvos, F.-J. (1986): Interpolation of even periodic functions. In: C.K. Chui, L.L. Schumaker, J.D. Ward, eds., Approximation Theory V, pp. 315–318. Academic Press, Orlando

De Villiers, J.M. (1992): A convergence result in nodal spline interpolation, J. Approximation Theory (to appear)

De Villiers, J.M., Rohwer, C.H. (1987): Optimal local spline interpolants, J. Comput. Appl. Math. **18**, 107–119

De Villiers, J.M., Rohwer, C.H. (1991): A nodal spline generalization of the Lagrange interpolant. In: P. Nevai, A. Pinkus, eds., Progress in approximation theory, pp. 201–211. Academic Press, San Diego

De Villiers, J.M., Rohwer, C.H. (1992): Approximation properties of a nodal spline space (submitted for publication)

Förster, K.-J. (1987): Über Monotonie und Fehlerkontrolle bei den Gregoryschen Quadraturverfahren, Z. Angew. Math. Mech. **67**, 257–266

Lötzbeyer, W. (1972): Asymptotische Eigenschaften linearer und nichtlinearer Quadraturformeln, Z. Angew. Math. Mech. **52**, T211–T214

Martensen, E. (1964): Optimale Fehlerschranken für die Quadraturformel von Gregory. Z. Angew. Math. Mech. **44**, 159–168

Martensen, E. (1973): Darstellung und Entwicklung des Restgliedes der Gregoryschen Quadraturformel mit Hilfe von Spline-Funktionen, Numer. Math. **21**, 70–80

Ralston, A., Rabinowitz, P. (1978): A First Course in Numerical Analysis. McGraw-Hill, New York

Schoenberg, I.J., Sharma, A. (1971): The interpolatory background of the Euler-Maclaurin quadrature formula, Bull. Amer. Math. Soc. **77**, 1034–1038

Schumaker, L.L. (1981): Spline Functions: Basic Theory. Wiley, New York

Solak, W., Szydelko, Z. (1991): Quadrature rules with Gregory-Laplace end corrections. J. Comput. Appl. Math. **36**, 251–253

Strang, G., Fix, G.J. (1973): An Analysis of the Finite Element Method. Prentice Hall, Englewood Cliffs, New Jersey

Stroud, A.H. (1966): Estimating quadrature errors for functions with low continuity. SIAM J. Numer. Anal. **3**, 420–424

Stroud, A.H. (1974): Numerical Quadrature and Solution of Ordinary Differential Equations. Appl. Math. Sci. **10**, Springer, Berlin Heidelberg New York