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On mixed finite element methods for linear elastodynamics ⋆

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Summary. We construct and analyze finite element methods for approximating the equations of linear elastodynamics, using mixed elements for the discretization of the spatial variables. We consider two different mixed formulations for the problem and analyze semidiscrete and up to fourth-order in time fully discrete approximations. L^2 optimal-order error estimates are proved for the approximations of displacement and stress.

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1 Introduction

The problem

The purpose of this paper is the construction and analysis of finite element methods for approximating the equations of linear elastodynamics using *mixed formulations* of the problem for the discretization of the spatial variables. For simplicity we shall consider the following initial-boundary value problem: Let Ω be a bounded domain $\mathbb{R}^N$ $(N=2, 3)$ with smooth boundary $\partial\Omega$ and let $0<T<\infty$. We seek a vector *displacement function* u: $\bar{\Omega}\times[0, T]\to\mathbb{R}^N$, satisfying – index notation and the summation convention will be employed throughout –

$$\begin{aligned} \ddot{u}_i - \partial_j(C_{ijkl}\,\partial_l u_k) &= f_i \quad \text{in } \bar{\Omega}\times[0, T], \\ u(x, t) &= 0 \quad \text{on } \partial\Omega\times[0, T], \\ u(x, 0) = u^0(x), \quad \dot{u}(x, 0) &= u^1(x) \quad \text{in } \bar{\Omega}, \end{aligned} \tag{1.1}$$

where the *elasticities* C_{ijkl}, $1\leqq i, j, k, l\leqq N$, the *body forces per unit volume* f_i, $1\leqq i\leqq N$, and the *initial displacement* and *velocity* u^0 and u^1 are given functions defined on $\bar{\Omega}$. (Here dots denote differentiation with respect to t and $\partial_j=\partial/\partial x_j$.) For a discussion of the physical background of the equations of elasticity (cf.

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⋆⋆ *Present address:* Department of Mathematics, University of Maryland, College Park, MD 20742, USA e.g. [16]). We shall consider two different approaches for the mixed finite element approximation of (1.1). First, in Sect. 2, we shall discretize (1.1) in space using its "displacement-stress" mixed formulation on appropriate finite element spaces and analyze the semidiscrete problem and fully discrete approximations thereof, based on up to fourth-order rational approximations to the cosine (cf. [6, 8]). Next, in Sect. 3, we shall consider the "velocity-stress" formulation of (1.1), proposed by Geveci [15], in the case of the scalar wave equation. We shall use mixed finite element discretizations for the space variables and base the time-stepping scheme on up to fourth-order accurate methods generated by rational approximations to the exponential (cf. [5, 7]).

Notation and preliminaries

Employing standard notation, for $s=1, 2, \ldots$ we shall use the symbol H^s to denote the Sobolev space $H^s(\Omega)^N$ or $H^s(\Omega)^{N\times N}$, as the case may be, with corresponding usual norm $\|\cdot\|_s$; we let H_0^s be the subspace of H^s with elements that vanish on $\partial\Omega$ in the sence of trace. Let $(\cdot,\cdot)$ be the inner product on $V=L^2(\Omega)^N$ or $L^2(\Omega)^{N\times N}$ with associated norm $\|\cdot\|$ and $H=H(\mathrm{div};\Omega) =\{\tau\in L^2(\Omega)^{N\times N};\ \mathrm{div}\,\tau\in V\}$, where

$$(\mathrm{div}\,\tau)_i=\partial_j\tau_{ij}, \qquad i=1, \ldots, N.$$

We let $R_N=\mathbb{R}^{N\times N}$ be the space of second-order real tensors ($N\times N$ matrices), R_s be the space of symmetric elements of R_N, and H_{sd} be the subspace of H with symmetric elements. Fouth-order tensors will usually be denoted by boldface capital letters.

We shall suppose that the elasticity tensor $\boldsymbol{C}=\boldsymbol{C}(x)$ corresponds to a *linear elastic material*, i.e. that its components C_{ijkl} – the elasticities – satisfy

$$C_{ijkl}=C_{jikl}=C_{klij}$$

in $\bar{\Omega}$. Hence, for antisymmetric second-order tensors ω, there holds $\boldsymbol{C}\omega=0$, and we have $\boldsymbol{C}[R_N]=\boldsymbol{C}[R_s]\subset R_s$. We shall also assume that the restriction of $\boldsymbol{C}$ to R_s is bounded and uniformly positive definite in $\bar{\Omega}$, i.e., that there exist positive constants μ_0 and μ_1 such that

$$\mu_0\,\tau_{ij}\tau_{ij}\leqq C_{ijkl}(x)\,\tau_{ij}\tau_{kl}\leqq\mu_1\,\tau_{ij}\tau_{ij}, \qquad \forall\tau\in R_s,\ x\in\bar{\Omega}.$$

As a consequence, if $\boldsymbol{A}=(a_{ijkl})$ is the inverse of $\boldsymbol{C}$ in R_s, it will satisfy, for some positive constants μ_0', μ_1', the inequality

$$\mu_0'\,\tau_{ij}\tau_{ij}\leqq a_{ijkl}(x)\,\tau_{ij}\tau_{kl}\leqq\mu_1'\,\tau_{ij}\tau_{ij}, \qquad \forall\tau\in R_s,\ x\in\bar{\Omega}.$$

Letting $\sigma=(\sigma_{ij})$ be the *stress tensor* defined by $\sigma_{ij}=C_{ijkl}\,\partial_l u_k$, we can write (1.1) as

$$\begin{aligned} \ddot{u}_i-\partial_j\sigma_{ij}&=f_i \quad \text{in } \bar{\Omega}\times[0,T], \\ \sigma_{ij}&=C_{ijkl}\,\partial_l u_k \quad \text{in } \bar{\Omega}\times[0,T], \\ u(x,t)&=0 \quad \text{on } \partial\Omega\times[0,T], \\ u(x,0)=u^0(x), &\quad \dot{u}(x,0)=u^1(x) \quad \text{in } \bar{\Omega}. \end{aligned} \tag{1.2}$$

It is well-known that the problem (1.1) – or (1.2) – has a unique solution (cf. e.g. [14]). Moreover, under standard smoothness and compatibility conditions on the data, one can prove e.g. that

$$u \in \bigcap_{k=0}^{m+1} C^{m+1-k}([0, T]; H^k \cap H_0^1),$$

where m is a nonnegative integer depending on the data. Thus, we shall assume in the sequel that the data of (1.1) are smooth and compatible enough to allow a unique, smooth enough for our purposes, classical solution u of (1.1) – or (1.2) – to exist.

Let now $w \in V$. The first equation of (1.2) gives

$$(\ddot{u}, w) - (\operatorname{div} \sigma, w) = (f, w).$$

If we denote by ε_{ij} the symmetric tensor $\varepsilon_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$, we have $\sigma_{ij} = C_{ijkl}\, \varepsilon_{ij}(u)$, and consequently $\varepsilon_{ij}(u) = a_{ijkl}\, \sigma_{kl}$. For $\chi \in H_{sd}$, by Green's formula and the symmetry of χ we can write the second equation of (1.2) as

$$a(\sigma, \chi) + (u, \operatorname{div} \chi) = 0, \qquad \chi \in H_{sd},$$

where

$$a(\sigma, \chi) := \int_\Omega a_{ijkl}\, \sigma_{kl}\, \chi_{ij} = (A\sigma, \chi).$$

Combining these relations we obtain the "displacement-stress" mixed formulation of the elastodynamics problem (1.1)–(1.2): Find (u, σ): $[0, T] \to V \times H_{sd}$ such that

$$(1.3) \qquad \begin{aligned} &(\ddot{u}, w) - (\operatorname{div} \sigma, w) = (f, w), \qquad \forall w \in V, \\ &a(\sigma, \chi) + (u, \operatorname{div} \chi) = 0, \qquad \forall \chi \in H_{sd}, \\ &u(x, 0) = u^0(x), \qquad \dot{u}(x, 0) = u^1(x) \quad \text{in } \bar{\Omega}. \end{aligned}$$

The finite element spaces

During the last decade many contributions have been made in the area of mixed finite element discretizations of the corresponding to (1.3) stationary problem, i.e. the equations of lineal elastostatistics (cf. e.g. [1–4, 17, 21–23]). A basic difficulty for the construction of effective mixed finite element spaces for this problem is the requirement of *symmetry* for the elements of H_{sd}. In [3] Arnold et al. have constructed a class of high order finite element spaces for $N = 2$. Letting V_h, H_h approximate V and H_{sd} respectively, they use for H_h *composite* elements to ensure that $H_h \subset H_{sd}$, and that V_h, H_h satisfy the so called *commutative diagram property* (cf. [3]) (see also [17, 21, 22]). An alternative approach has been taken in [2] and [23], where nonsymmetric approximations for the stress tensor are used and a weak symmetry condition is imposed using a Lagrange multiplier (cf. also [1]). Finally, in a recent paper, Arnold and Falk [4] change the mixed formulation introducing a new variable, the "pseudostress" ρ, which is not symmetric any more. If we know ρ then the stress tensor σ can be calculated directly without differentiation.

Here we suppose that we have at our disposal a couple of finite element spaces V_h and H_h that satisfy the assumptions (*FE1*), (*FE2*) below. These are satisfied by the spaces of [3] and of [17, 21, 22]. Furthermore our error analysis can be applied, with proper modifications, if we choose for the space discretization the formulation of Arnold and Falk [4] (cf. below). It is also straightforward to see that all results and techniques of the present paper can be applied to mixed finite element discretizations of scalar wave equations.

In the sequel therefore, we shall deal with finite element spaces that satisfy the following properties. For $h>0$ (spatial discretization parameter) consider a family of couples V_h, H_h of finite dimensional subspaces of V and H_{sd}, respectively and suppose that:

(*FE1*) For $v\in H^1$, let $\psi=C\varepsilon(v)$. Assume that the pair of "elliptic projections" $P_1 v$, $\Pi_1\psi$, $(P_1 v, \Pi_1\psi)\in V_h\times H_h$, of v, ψ, exists uniquely a solution of the stationary problem in $V_h\times H_h$:

$$(\operatorname{div}\Pi_1\psi, w)=(\operatorname{div}\psi, w), \qquad \forall w\in V_h$$
$$a(\Pi_1\psi,\chi)+(P_1 v, \operatorname{div}\chi)=0, \qquad \forall\chi\in H_h.$$

(*FE2*) For r integer, $r\geqq 2$, the elliptic projections satisfy:

$$(\alpha)\quad \|P_1 v-v\|\leqq c h^s\|v\|_s, \qquad 1\leqq s\leqq r$$

and if $\psi=C\varepsilon(v)$

$$(\beta)\quad \|\Pi_1\psi-\psi\|\leqq c h^s\|\psi\|_s, \qquad 1\leqq x\leqq r.$$

Summary of results

In Sect. 2 we consider the semidiscrete problem resulting from (1.3) posed on $V_h\times H_h$ and prove that the error of the semidiscrete approximations (u_h, σ_h) is of optimal order in L^2, i.e. that

$$\|u-u_h\|+\|\sigma-\sigma_h\|\leqq c h^r,$$

provided $u_h(0)$ and $\dot u_h(0)$ are taken to be, respectively $P_1 u^0$ and $P_1 u^1$. Next we construct fully discrete schemes of second and fourth order of accuracy, that are based on rational approximations of the cosine (cf. [6, 8]). In each time step the calculation of the approximations $(U^n, \Sigma^n)\simeq(u(t_n), \sigma(t_n))$, $t_n=nk$, $n=0, 1, \ldots, J$ with $t_J=T$, requires the solution of a linear system with *positive definite* matrix. Moreover, we prove that the approximations satisfy

$$\max_{0\leqq n\leqq J}(\|u(t_n)-U^n\|+\|\sigma(t_n)-\Sigma^n\|)\leqq c(h^r+k^\nu), \qquad \nu=2,4,$$

provided the initial approximations are accurate enough. In [12] Cowsar et al. consider the analogous mixed formulation for the scalar wave equation. For the semidiscrete problem they prove an optimal-order L^2 error estimate, for the approximation of u, taking as $u_h(0)$ and $\dot u_h(0)$ the L^2-projections on (the analog of) V_h of their initial data. Moreover they show the stability of a conditionally stable two-step fully discrete scheme.

In Sect. 3 we consider the "velocity-stress" formulation of the problem (1.3): If we $v=\dot{u}$ then the pair (v,σ): $[0,T]\to V\times H_{sd}$ satisfies:

$$(1.4)\qquad \begin{aligned} &(\dot{v},w)-(\operatorname{div}\sigma,w)=(f,w), \quad \forall w\in V\\ &a(\dot{\sigma},\chi)+(v,\operatorname{div}\chi)=0, \quad \forall \chi\in H_{sd}.\\ &\sigma(x,0)=C\varepsilon(u^0(x)), \quad v(x,0)=u^1(x) \quad \text{in } \bar{\Omega}. \end{aligned}$$

This formulation has been proposed by Geveci [15] for the discretization of the scalar wave equation. In [15] Geveci analyzes the corresponding to this formulation semidiscrete problem for the wave equation and proves optimal convergence results for the error of the semidiscrete approximation. Here we first prove that the semidiscrete approximation of (1.4) in $V_h\times H_h$ satisfies an estimate of the form

$$\|v-v_h\|+\|\sigma-\sigma_h\|\leqq c\,h^r,$$

provided $\|v(0)-v_h(0)\|+\|\sigma(0)-\sigma_h(0)\|=O(h^r)$. Next we consider fully discrete schemes that are based on rational approximations to the exponential (cf. [5, 7]), of order $v=2$, 3 or 4. Let (V^n,Σ^n) be the approximation to $(v(t_n),\sigma(t_n))$ generated by these schemes. If in addition to (*FE1*), (*FE2*) we assume that

$$(FE3)\qquad \operatorname{div}H_h\subset V_h,$$

we can prove the following error estimate

$$\max_{0\leqq n\leqq J}(\|v(t_n)-V^n\|+\|\sigma(t_n)-\Sigma^n\|)\leqq C(h^r+k^v), \qquad v=2,3,4,$$

if accurate enough initial approximations are given. Note that (*FE3*) holds for the mixed finite element spaces that satisfy the commutative diagram property.

In [13] Douglas and Gupta analyze the superconvergence of semidiscrete mixed finite elements for the equations of elastodynamics. Finally for work on semidiscrete mixed finite element approximations to parabolic problems cf. Johnson and Thomée [18].

Arnold-Falk discretization

In [4] Arnold and Falk approximate instead of σ the (nonsymmetric) pseudo-stress tensor ρ defined by

$$\rho=(C+\beta D)\operatorname{grad}u,$$

(recall that $\sigma=C\operatorname{grad}u$), where $D\tau=\operatorname{tr}(\tau)I-\tau^T$, $\operatorname{tr}(\tau)=\sum_{i=1}^{N}\tau_{ii}$, and β is a positive constant that has been chosen so that $C+\beta D$ is invertible and so that the inverse B satisfies

$$\mu_2\,\tau_{ij}\tau_{ij}\leqq B_{ijkl}(x)\,\tau_{ij}\tau_{kl}\leqq\mu_3\,\tau_{ij}\tau_{ij},\ \forall\tau\in R_N, \quad x\in\bar{\Omega},$$

where μ_2, μ_3 are positive constants. Since now div $\boldsymbol{D}\tau=0$, $\tau\in R_N$, the first equation of (1.2) yields

$$\ddot{u}_i-\partial_j\rho_{ij}=f_i,$$

so that $(u,\rho)\in V\times H$ $(H=H(\mathrm{div};\Omega))$ satisfies

$$(1.5)\qquad \begin{aligned}&(\ddot{u},w)-(\mathrm{div}\,\rho,w)=(f,w), && \forall w\in V,\\ &(\boldsymbol{B}\rho,\chi)+(u,\mathrm{div}\,\chi)=0, && \forall \chi\in H.\end{aligned}$$

For the space discretization of these equations one may use tensor products of the spaces used in the mixed finite element discretization of scalar elliptic equations (cf. e.g. [10, 11, 20] and the references in [9]). There exist such spaces satisfying the assumptions *(FE1)–(FE3)*. Consequently our analysis of Sects. 2 and 3 can be applied – with the obvious modifications – to analyze semidiscrete and fully discrete temporal approximations of (1.5).

2 Displacement-stress discretization

We first consider the continuous in time finite element approximation of the solution of the problem (1.3). Let (u_h,σ_h): $[0,T]\to V_h\times H_h$ be such that

$$(2.1)\qquad \begin{aligned}&(\ddot{u}_h,w)-(\mathrm{div}\,\sigma_h,w)=(f,w), && \forall w\in V_h,\\ &a(\sigma_h,\chi)+(u_h,\mathrm{div}\,\chi)=0, && \forall \chi\in H_h,\\ &u_h(0)=u_h^0,\quad \dot{u}_h(0)=u_h^1, &&\end{aligned}$$

where u_h^0 and u_h^1 are approximations of u^0 and u^1, respectively, in V_h.

Discretization operators

For the purposes of the error analysis we introduce two discretization operators D and $\tilde{D}$, as follows: Let $\tilde{D}$: $V\to H_h$ be defined for $v\in V$ by

$$(2.2)\qquad (\tilde{D}v,\chi)=(v,\mathrm{div}\,\chi)\quad \forall\chi\in H_h.$$

It is straightforward to see that the above relation defines $\tilde{D}$ uniquely. In fact, the element $\tilde{D}v\in H_h$ is the solution of a linear system with a $\dim H_h\times\dim H_h$ matrix with elements (ψ_i,ψ_j), where $\{\psi_i\}$ is a basis of H_h. We also define the operator D: $H_{\mathrm{sd}}\to V_h$, given for $\tau\in H_{\mathrm{sd}}$ by

$$(2.3)\qquad (D\tau,w)=(\mathrm{div}\,\tau,w),\quad \forall w\in V_h.$$

We consider now the operators

$$D\tilde{D}\colon V\to V_h\quad\text{and}\quad \tilde{D}D\colon H_{\mathrm{sd}}\to H_h.$$

Note that $D\tilde{D}$ and $\tilde{D}D$ are symmetric and positive semidefinite with respect to the L^2-inner product in V_h and H_h, respectively. Indeed, (2.2) and (2.3) give for $v, w \in V_h$

$$(D\tilde{D}v, w) = (\operatorname{div} \tilde{D}v, w) = (\tilde{D}v, \tilde{D}w)$$

and consequently

$$(D\tilde{D}v, v) = (\tilde{D}v, \tilde{D}v) \geqq 0.$$

Analogously, for $\tau, \chi \in H_h$ we have

$$(\tilde{D}D\tau, \chi) = (D\tau, \operatorname{div} \chi) = (D\tau, D\chi)$$

and therefore

$$(\tilde{D}D\tau, \tau) = (D\tau, D\tau) \geqq 0.$$

Finally we observe that for $w \in V_h$, $\chi \in H_h$ there holds

$$(\tilde{D}w, \chi) = (w, \operatorname{div} \chi) = (w, D\chi). \tag{2.4}$$

Semidiscrete approximations

Using these discretization operators the problem (2.1) takes the following form: We seek (u_h, σ_h): $[0, T] \to V_h \times H_h$ such that

$$\begin{aligned} &\ddot{u}_h - D\sigma_h = f_h, \\ &\tilde{D}u_h + A\sigma_h = 0, \\ &u_h(0) = u_h^0, \quad \dot{u}_h(0) = u_h^1, \end{aligned} \tag{2.5}$$

where f_h is the L^2-projection of f in V_h. The operator $DA^{-1}\tilde{D}$ is symmetric and positive semidefinite on $(V_h, (\cdot, \cdot))$, and thus (2.5) has a unique solution.

We now prove the following result.

Theorem 2.1. *Let* (u, σ) *be the solution of problem* (1.2), *and* (u_h, σ_h) *the solution of* (2.1) *with initial conditions* $u_h^0 = P_1 u^0$ *and* $u_h^1 = P_1 u^1$. *Then*

$$\|u(t) - u_h(t)\| + \|\sigma(t) - \sigma_h(t)\| \leqq c h^r \left\{ \|u(t)\|_{r+1} + \left\{ \int_0^t \|\ddot{u}(s)\|_r^2 \, ds \right\}^{1/2} \right\}.$$

Proof. We shall compare the solution of the semidiscrete problem (2.5) with the pair $(P_1 u, \Pi_1 \sigma) \in V_h \times H_h$ defined in *(FE1)*. Let

$$P_1 u(t) = w(t), \qquad \Pi_1 \sigma(t) = \xi(t).$$

Then, denoting P: $L^2 \to V_h$ the L^2 projection operator onto V_h, using *(FE1)* we have

$$\ddot{w}(t) - D\xi(t) = \ddot{w}(t) - P\ddot{u}(t) + f_h(t)$$

and

$$\tilde{D}w(t) + A\xi(t) = 0. \tag{2.6}$$

From these relations and (2.5) we obtain the error equations

$$\ddot{e}(t) - D\eta(t) = P\ddot{u}(t) - \ddot{w}(t),$$
$$\tilde{D}e(t) + A\eta(t) = 0,$$

where $e = u_h - w$ and $\eta = \sigma_h - \xi$. Differentiating the second equation with respect to t we obtain

$$\ddot{e}(t) - D\eta(t) = P\ddot{u}(t) - \ddot{w}(t),$$
$$\tilde{D}\dot{e}(t) + A\dot{\eta}(t) = 0.$$

Taking the L^2-inner product of this system with $(\dot{e}, \eta) \in V_h \times H_h$ we have

$$(\ddot{e}, \dot{e}) - (D\eta, \dot{e}) + (\tilde{D}\dot{e}, \eta) + (A\dot{\eta}, \eta) = (P\ddot{u}(t) - \ddot{w}(t), \dot{e}).$$

Since $(D\eta, \dot{e}) = (\eta, \tilde{D}\dot{e})$, using the fact that A is symmetric we have

$$\frac{d}{dt}\{\|\dot{e}\|^2 + (A\eta, \eta)\} \leqq \|P\ddot{u}(t) - \ddot{w}(t)\|^2 + \|\dot{e}\|^2.$$

Hence we conclude, since A is positive definite on R_s

$$\frac{d}{dt}\{\|\dot{e}\|^2 + \|e\|^2 + (A\eta, \eta)\} \leqq \|P\ddot{u}(t) - \ddot{w}(t)\|^2 + 2\{\|\dot{e}\|^2 + \|e\|^2 + (A\eta, \eta)\}.$$

By Gronwall's lemma and *(FE2)* we therefore have

$$\|\dot{e}(t)\|^2 + \|e(t)\|^2 + \|\eta(t)\|^2 \leqq c(\|\dot{e}(0)\|^2 + \|e(0)\|^2 + \|\eta(0)\|^2) + c h^{2r} \int_0^t \|\ddot{u}(s)\|_r^2 \, ds.$$

The particular choice of the initial data gives that $\|\dot{e}(0)\|^2 + \|e(0)\|^2 + \|\eta(0)\|^2 = 0$; hence we obtain the desired result using once more the assumption *(FE2)*. □

Note that for given $v \in H^1$, $P_1 v$, $\Pi_1 \psi = \Pi_1 C\varepsilon(v)$, can be computed as the mixed finite element solution in $V_h \times H_h$ of a stationary problem. For the efficient solution of the resulting linear system we refer to [9] and the references therein.

Fully discrete approximations

For the time discretization of (2.5) we shall use *rational approximations to the cosine*. Specifically we shall consider rational functions of the form, cf. [8]

$$r(x) = \frac{1 + p_1 x^2 + p_2 x^4}{1 + q_1 x^2 + q_2 x^4}, \qquad q_1, q_2 > 0,$$

where we assume that $r(x)$ is a fourth-order accurate approximation to the cosine, i.e. $p_1 = q_1 - 1/2$, $p_2 = q_2 - q_1/2 + 1/24$, and that, for stability purposes, the pair (q_1, q_2) belongs to the stability region $\mathcal{R}$ of the $q_1, q_2 > 0$ quarterplane of Fig. 1 of [8].

Let now $k>0$ be the (constant) time step, $t_n=nk$, $n=0,1,\ldots,J$, with $t_J=T$. From the approximation property of $r(x)$ we have, for any smooth function $y=y(t)$,

$$(I-q_1 k^2 \partial_t^2+q_2 k^4 \partial_t^4)(y^{n+1}+y^{n-1})=2(I-p_1 k^2 \partial_t^2+p_2 k^4 \partial_t^4)\, y^n+O(k^6 y^{(6)}),$$

where $y^n=y(t_n)$.

Using (2.5) we have

$$\partial_t^2 u_h=D\sigma_h+f_h$$

and

$$\partial_t^4 u_h=\partial_t^2 D\sigma_h+\partial_t^2 f_h=-DA^{-1}\tilde{D}\ddot{u}_h+\partial_t^2 f_h=-DA^{-1}\tilde{D}D\sigma_h-DA^{-1}\tilde{D}f_h+\partial_t^2 f_h.$$

Motivated by these equations we may now discretize the first equation of (2.5) as follows: Denoting by $\begin{pmatrix}U^n\\ \Sigma^n\end{pmatrix}\in V_h\times H_h$ the fully discrete approximation of $\begin{pmatrix}u(t_n)\\ \sigma(t_n)\end{pmatrix}$ and omitting terms of $O(k^6)$, we are led to

$$\begin{aligned}
&U^{n+1}-2U^n+U^{n-1}-k^2 D(q_1\Sigma^{n+1}-2p_1\Sigma^n+q_1\Sigma^{n-1})\\
&\quad -k^4 DA^{-1}\tilde{D}D(q_2\Sigma^{n+1}-2p_2\Sigma^n+q_2\Sigma^{n-1})\\
&\quad =k^2(q_1 f^{n+1}-2p_1 f^n+q_1 f^{n-1})+k^4 DA^{-1}\tilde{D}(q_2 f^{n+1}-2p_2 f^n+q_2 f^{n-1})\\
&\quad -k^4(q_1-\tfrac{1}{12})\,\delta^2 f^n,
\end{aligned}$$

where $f^n:=f_h(t_n)$, $\delta^2 f^n=k^{-2}(f^{n+1}-2f^n+f^{n-1})$.

For the discretization of the second equation of (2.5) we just obseve that it implies

$$(k^2 q_1+k^4 q_2\tilde{D}DA^{-1})(\tilde{D}u_h^{n+1}+A\sigma_h^{n+1}-\tilde{D}u_h^{n-1}-A\sigma_h^{n-1})=0.$$

Using these equations, we can now state the fully discrete scheme that defines the approximations $\begin{pmatrix}U^n\\ \Sigma^n\end{pmatrix}\in V_h\times H_h$, $\begin{pmatrix}U^n\\ \Sigma^n\end{pmatrix}\simeq\begin{pmatrix}u(t_n)\\ \sigma(t_n)\end{pmatrix}$. For some $1\leqq n\leqq J-1$, let $\begin{pmatrix}U^j\\ \Sigma^j\end{pmatrix}\in V_h\times H_h$, $0\leqq j\leqq n\leqq J-1$, be given approximations of $\begin{pmatrix}u(t_j)\\ \sigma(t_j)\end{pmatrix}$. Define then $\begin{pmatrix}U^{n+1}\\ \Sigma^{n+1}\end{pmatrix}$ as solution of the linear system

$$R_1\begin{pmatrix}U^{n+1}\\ \Sigma^{n+1}\end{pmatrix}-2S\begin{pmatrix}U^n\\ \Sigma^n\end{pmatrix}+R_2\begin{pmatrix}U^{n-1}\\ \Sigma^{n-1}\end{pmatrix}=F^n, \tag{2.7}$$

where R_1, R_2, S are linear operators on $V_h\times H_h$ defined by

$$R_1=\begin{pmatrix} I & -k^2 q_1 D-k^4 q_2 DA^{-1}\tilde{D}D\\ k^2 q_1\tilde{D}+k^4 q_2\tilde{D}DA^{-1}\tilde{D} & k^2 q_1 A+k^4 q_2\tilde{D}D\end{pmatrix},$$

$$S=\begin{pmatrix} I & -k^2 p_1 D-k^4 p_2 DA^{-1}\tilde{D}D\\ 0 & 0\end{pmatrix},$$

and

$$R_2=\begin{pmatrix} I & -k^2 q_1 D-k^4 q_2 DA^{-1}\tilde{D}D\\ -k^2 q_1\tilde{D}-k^4 q_2\tilde{D}DA^{-1}\tilde{D} & -k^2 q_1 A-k^4 q_2\tilde{D}D\end{pmatrix},$$

and where by F^n we denote

$$F^n = \begin{pmatrix} k^2(q_1 f^{n+1} - 2p_1 f^n + q_1 f^{n-1}) \\ 0 \end{pmatrix} + \begin{pmatrix} k^4 D A^{-1} \tilde{D}(q_2 f^{n+1} - 2p_2 f^n + q_2 f^{n-1}) - k^4 (q_1 - \frac{1}{12})\, \delta^2 f^n \\ 0 \end{pmatrix}.$$

We see that at each time step the calculation of $\begin{pmatrix} U^{n+1} \\ \Sigma^{n+1} \end{pmatrix}$ needs solving a linear system with a *positive definite matrix.* To see that R_1 is positive definite on $V_h \times H_h$ with respect to the inner product defined for $\begin{pmatrix} w_1 \\ \chi_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ \chi_2 \end{pmatrix} \in V \times H_{sd}$ by

$$\left(\left(\begin{pmatrix} w_1 \\ \chi_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ \chi_2 \end{pmatrix}\right)\right) = (w_1, w_2) + (\chi_1, \chi_2),$$

let $\begin{pmatrix} w \\ \chi \end{pmatrix} \in V_h \times H_h$, and observe that

$$\begin{aligned} \left(\left(R_1 \begin{pmatrix} w \\ \chi \end{pmatrix}, \begin{pmatrix} w \\ \chi \end{pmatrix}\right)\right) = {} & (w, w) - k^2 q_1 (D\chi, w) - k^4 q_2 (D A^{-1} \tilde{D} D \chi, w) \\ & + k^2 q_1 (\tilde{D} w, \chi) + k^4 q_2 (\tilde{D} D A^{-1} \tilde{D} w, \chi) \\ & + k^2 q_1 (A\chi, \chi) + k^4 q_2 (\tilde{D} D \chi, \chi). \end{aligned}$$

Since A^{-1} is symmetric we have, using (2.4),

$$(D A^{-1} \tilde{D} D \chi, w) = (\chi, \tilde{D} D A^{-1} \tilde{D} w).$$

Hence, since A is a positive definite,

$$\left(\left(R_1 \begin{pmatrix} w \\ \chi \end{pmatrix}, \begin{pmatrix} w \\ \chi \end{pmatrix}\right)\right) = (w, w) + k^2 q_1 (A\chi, \chi) + k^4 q_2 (D\chi, D\chi) > 0, \qquad \begin{pmatrix} w \\ \chi \end{pmatrix} \neq 0.$$

Note on implementation

Regarding the question of solving the linear system

$$R_1 \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.7a}$$

on $V_h \times H_h$ we make the following observations. Letting

$$k^2 q_1 A Y + k^4 q_2 \tilde{D} D Y =: \tilde{Y}, \tag{2.7b}$$

we see that

$$X = D A^{-1} \tilde{Y} + a \tag{2.7c}$$

and

$$\tilde{Y}+k^2 q_1 \tilde{D}DA^{-1}\tilde{Y}+k^4 q_2 \tilde{D}DA^{-1}\tilde{D}DA^{-1}\tilde{Y}=\tilde{b},$$

where $\tilde{b}=b-(k^2 q_1 \tilde{D}+k^4 q_2 \tilde{D}DA^{-1}\tilde{D})a$. If $L_D:=\tilde{D}DA^{-1}$, then the above linear system becomes

$$(2.7\text{d}) \qquad (I+k^2 q_1 L_D+k^4 q_2 L_D^2)\tilde{Y}=\tilde{b}.$$

Hence the computation of (X, Y) requires the solution of the linear systems (2.7d), (2.7b) and computing X by (2.7c).

In the special case where $1+q_1 x^2+q_2 x^4=(1+\alpha x^2)^2$, $\alpha>0$ – these are the schemes constructed in [6] – the system (2.7d) becomes

$$(I+\alpha k^2 L_D)^2\tilde{Y}=\tilde{b}.$$

Then, if we assume that *(FE3)* holds, we may conclude that to compute $\tilde{Y}$ from the above we have to solve two systems, with symmetric positive definite matrix, of the form: Find $Y_1\in H_h$ such that

$$(AY_1,\chi)+\alpha k^2(\operatorname{div} Y_1, \operatorname{div}\chi)=(b_1,\chi) \quad \forall\chi\in H_h.$$

Analogously, the system (2.7b) has the form: Find $Y\in H_h$ such that

$$2\alpha k^2(AY,\chi)+\alpha^2 k^4(\operatorname{div} Y, \operatorname{div}\chi)=(\tilde{Y},\chi) \quad \forall\chi\in H_h.$$

Consistency of the Scheme (2.7)

In the convergence proof that will follow, we shall compare the approximation $\begin{pmatrix}U^n\\ \Sigma^n\end{pmatrix}$ to the "elliptic projection" $\begin{pmatrix}P_1 u(t_n)\\ \Pi_1 \sigma(t_n)\end{pmatrix}=:\begin{pmatrix}w^n\\ \xi^n\end{pmatrix}$. For this purpose we estimate first $\begin{pmatrix}a^n\\ \gamma^n\end{pmatrix}$, defined by

$$(2.8) \qquad R_1\begin{pmatrix}w^{n+1}\\ \xi^{n+1}\end{pmatrix}-2S\begin{pmatrix}w^n\\ \xi^n\end{pmatrix}+R_2\begin{pmatrix}w^{n-1}\\ \xi^{n-1}\end{pmatrix}=:\begin{pmatrix}a^n\\ \gamma^n\end{pmatrix}.$$

Using *(FE1)* we see that for every n (2.6) implies that $\gamma^n=0$. On the other hand,

$$(2.9) \qquad \begin{aligned} a^n=&(w^{n+1}-2w^n+w^{n-1})-k^2 D(q_1\xi^{n+1}-2p_1\xi^n+q_1\xi^{n-1})\\ &-k^4 DA^{-1}\tilde{D}D(q_2\xi^{n+1}-2p_2\xi^n+q_2\xi^{n-1}).\end{aligned}$$

To estimate the first term of the right-hand side of (2.9), we write

$$(2.10\text{a}) \quad \begin{aligned} w^{n+1}-2w^n+w^{n-1}&=P(w^{n+1}-u^{n+1})-2P(w^n-u^n)+P(w^{n-1}-u^{n-1})\\ &\quad+P(u^{n+1}-2u^n+u^{n-1})\\ &=:a_1^n+P(u^{n+1}-2u^n+u^{n-1}),\end{aligned}$$

where *(FE2)* gives that

$$\|a_1^n\| \leqq C k_2 h^r \sup_{s\in[t^{n-1},t^{n+1}]} \|\partial_t^2 u(s)\|_r. \tag{2.10b}$$

For the next term of the right-hand side (2.9), we see, using *(FE1)* and the definition of D that for $w\in V_h$

$$(D\xi^n, w) = (\operatorname{div}\xi^n, w) = (\operatorname{div}\sigma^n, w) = (D\sigma^n, w),$$

i.e. $D\xi^n = D\sigma^n$. Hence using (1.3)

$$\begin{aligned}(2.10c)\quad -k^2 D(q_1\xi^{n+1} - 2p_1\xi^n + q_1\xi^{n-1}) &= -k^2 D(q_1\sigma^{n+1} - 2p_1\sigma^n + q_1\sigma^{n-1})\\ &= -k^2 P\partial_t^2(q_1 u^{n+1} - 2p_1 u^n + q_1 u^{n-1})\\ &\quad + k^2(q_1 f^{n+1} - 2p_1 f^n + q_1 f^{n-1}).\end{aligned}$$

Finally, for the estimation of the last term of (2.9) using again *(FE1)*, the first equation of (1.3) and the definition of (w^n, ξ^n), we first have

$$\begin{aligned}(2.11a)\quad DA^{-1}\tilde{D}D\xi^n &= DA^{-1}\tilde{D}D\sigma^n = DA^{-1}\tilde{D}(\partial_t^2 Pu^n - f^n)\\ &= DA^{-1}\tilde{D}(P-P_1)\partial_t^2 u^n + DA^{-1}\tilde{D}\partial_t^2 w^n - DA^{-1}\tilde{D}f^n\\ &= DA^{-1}\tilde{D}(P-P_1)\partial_t^2 u^n - D\partial_t^2\xi^n - DA^{-1}\tilde{D}f^n\\ &= DA^{-1}\tilde{D}(P-P_1)\partial_t^2 u^n - P\partial_t^4 u^n + f^{(2)n} - DA^{-1}\tilde{D}f^n.\end{aligned}$$

Now using the definition of D, $\tilde{D}$ and *(FE2)* we obtain, for $\omega\in V_h$,

$$\begin{aligned}(2.11b)\quad (DA^{-1}\tilde{D}(P-P_1)\partial_t^2 u^n, \omega) &= ((P-P_1)\partial_t^2 u^n, DA^{-1}\tilde{D}\omega)\\ &\leqq Ch^r\|\partial_t^2 u^n\|_r \|DA^{-1}\tilde{D}\omega\|.\end{aligned}$$

Next, for the terms containing second derivatives of f we have in view of (2.9), using that $q_2 - p_2 = (q_1 - 1/12)/2$,

$$\begin{aligned}(2.11c)\quad (q_2 f^{(2)n+1} - 2p_2 f^{(2)n} + q_2 f^{(2)n-1}) &= q_2(f^{(2)n+1} - 2f^{(2)n} + f^{(2)n-1})\\ &\quad + (q_1 - 1/12)(f^{(2)n} - \delta^2 f^n) + (q_1 - 1/12)\delta^2 f^n,\end{aligned}$$

where

$$k^4\|f^{(2)n+1} - 2f^{(2)n} + f^{(2)n-1}\| \leqq ck^6 \sup_{s\in[t^{n-1},t^{n+1}]} \|\partial_t^4 f(s)\|, \tag{2.11d}$$

and

$$k^4\|f^{(2)n} - \delta^2 f^n\| \leqq ck^6 \sup_{s\in[t^{n-1},t^{n+1}]} \|\partial_t^4 f(s)\|. \tag{2.11e}$$

Finally, since the cosine scheme that we have in mind is fourth-order accurate, we have

$$\begin{aligned}(2.12)\quad \|P(I - q_1 k^2\partial_t^2 + q_2 k^4\partial_t^4)(u^{n+1} + u^{n-1}) - 2P(I - p_1 k^2\partial_t^2 + p_2 k^4\partial_t^4)u^n\|\\ \leqq ck^6 \sup_{s\in[t^{n-1},t^{n+1}]} \|\partial_t^6 u(s)\|.\end{aligned}$$

Combining (2.9), (2.10a–c), (2.11a–e) and (2.12) we conclude

$$(2.13)\quad a^n = k^2(q_1 f^{n+1} - 2p_1 f^n + q_1 f^{n-1}) + k^4 D A^{-1} \tilde{D}(q_2 f^{n+1} - 2p_2 f^n + q_2 f^{n-1}) - k^4(q_1 - \tfrac{1}{12})\,\delta^2 f^n - d^n,$$

where

$$(2.14)\quad |(d^n, \omega)| \leqq c_1(u,f)\, k^2(k^4 + h^r)(\|\omega\| + k^2\|DA^{-1}\tilde{D}\omega\|), \qquad \forall \omega \in V_h,$$

with $c_1(u,f) = c \sup_{s\in[0,T]} (\|\partial_t^2 u(s)\|_r + \|\partial_t^6 u(s)\| + \|\partial_t^4 f(s)\|)$. The Eqs. (2.8), (2.13) and (2.14) contain the consistency property of our scheme that will be required later.

Convergence.

Let $E^n = U^n - w^n$, $Z^n = \Sigma^n - \xi^n$ and, for $1 \leqq n \leqq J$,

$$\mathscr{E}^n = \|E^n - E^{n-1}\|^2 + k^2\|Z^n + Z^{n-1}\|^2 + k^2\|Z^n - Z^{n-1}\|^2 + k^4\|D(Z^n + Z^{n-1})\|^2 + k^4\|D(Z^n - Z^{n-1})\|^2.$$

We are now ready to state and prove the following optimal-order convergence result for our scheme.

Theorem 2.2. *Assume that* $\begin{pmatrix} U^0 \\ \Sigma^0 \end{pmatrix}, \begin{pmatrix} U^1 \\ \Sigma^1 \end{pmatrix}$ *are given elements of* $V_h \times H_h$, *chosen so that*

$$(2.15)\qquad \tilde{D}U^j + A\Sigma^j = 0, \qquad j = 0, 1$$

and

$$(2.16)\qquad \mathscr{E}^1 + k^2\|E^0\|^2 \leqq c k^2 (k^4 + h^r)^2.$$

Then, for every n, $2 \leqq n \leqq J$, $\begin{pmatrix} U^n \\ \Sigma^n \end{pmatrix} \in V_h \times H_h$ *exists uniquely as the solution of* (2.7). *If* (q_1, q_2) *belongs to the stability region* $\mathscr{R}$ *of Fig. 1 of* [8], *and in addition* $q_1 > 1/4$, *and k is sufficiently small, there exists a positive constant c, independent of k and h such that*:

$$\max_{0 \leqq n \leqq J} (\|E^n\| + \|Z^n\|) \leqq c(u,f)(k^4 + h^r)$$

and

$$\max_{0 \leqq n \leqq J} (\|u^n - U^n\| + \|\sigma^n - \Sigma^n\|) \leqq c(u,f)(k^4 + h^r).$$

Proof. We have already seen that given $\begin{pmatrix} U^n \\ \Sigma^n \end{pmatrix}, \begin{pmatrix} U^{n-1} \\ \Sigma^{n-1} \end{pmatrix}$ we may compute $\begin{pmatrix} U^{n+1} \\ \Sigma^{n+1} \end{pmatrix}$ as the unique solution of the linear system (2.7).

Subtracting (2.8) from (2.7) and using (2.13) we obtain the error equation:

$$R_1\begin{pmatrix}E^{n+1}\\Z^{n+1}\end{pmatrix}-2S\begin{pmatrix}E^{n}\\Z^{n}\end{pmatrix}+R_2\begin{pmatrix}E^{n-1}\\Z^{n-1}\end{pmatrix}=\begin{pmatrix}d^{n}\\0\end{pmatrix}, \tag{2.17}$$

where $d^n \in V_h$ satisfies the estimate (2.10). Next, take the $((\cdot,\cdot))$ inner product of both sides of (2.13) with the vector

$$\begin{pmatrix}E^{n+1}-E^{n-1}\\Z^{n+1}-2(p_2/q_2)\,Z^n+Z^{n-1}\end{pmatrix},$$

and obtain

$$\begin{aligned}(2.18)\quad &(E^{n+1}-2E^n+E^{n-1}, E^{n+1}-E^{n-1})\\ &-k^2(D(q_1 Z^{n+1}-2p_1 Z^n+q_1 Z^{n-1}), E^{n+1}-E^{n-1})\\ &-k^4(DA^{-1}\tilde{D}D(q_2 Z^{n+1}-2p_2 Z^n+q_2 Z^{n-1}), E^{n+1}-E^{n-1})\\ &+k^2(q_1\tilde{D}(E^{n+1}-E^{n-1}), Z^{n+1}-2(p_2/q_2)\,Z^n+Z^{n-1})\\ &+k^4(q_2\tilde{D}DA^{-1}\tilde{D}(E^{n+1}-E^{n-1}), Z^{n+1}-2(p_2/q_2)\,Z^n+Z^{n-1})\\ &+k^2(q_1 A(Z^{n+1}-Z^{n-1}), Z^{n+1}-2(p_2/q_2)\,Z^n+Z^{n-1})\\ &+k^4(q_2\tilde{D}D(Z^{n+1}-Z^{n-1}), Z^{n+1}-2(p_2/q_2)\,Z^n+Z^{n-1})\\ &=(d^n, E^{n+1}-E^{n-1}).\end{aligned}$$

Put now $\mu=2(p_1-(p_2\,q_1)/q_2)$ and observe that

$$\begin{aligned}&(q_1\tilde{D}(E^{n+1}-E^{n-1}), Z^{n+1}-2(p_2/q_2)\,Z^n+Z^{n-1})\\ &\quad=(E^{n+1}-E^{n-1}, D(q_1 Z^{n+1}-2p_1 Z^n+q_1 Z^{n-1}))+\mu(E^{n+1}-E^{n-1}, DZ^n)\end{aligned}$$

and

$$\begin{aligned}&(q_1 A(Z^{n+1}-Z^{n-1}), Z^{n+1}-2(p_2/q_2)\,Z^n+Z^{n-1})\\ &\quad=(Z^{n+1}-Z^{n-1}, A(q_1 Z^{n+1}-2p_1 Z^n+q_1 Z^{n-1}))+\mu(Z^{n+1}-Z^{n-1}, AZ^n).\end{aligned}$$

Using (2.4) we have

$$\begin{aligned}&(q_2\tilde{D}DA^{-1}\tilde{D}(E^{n+1}-E^{n-1}), Z^{n+1}-2(p_2/q_2)\,Z^n+Z^{n-1})\\ &\quad=(E^{n+1}-E^{n-1}, DA^{-1}\tilde{D}D(q_2 Z^{n+1}-2p_2 Z^n+q_2 Z^{n-1})).\end{aligned}$$

We now observe that the second component of the vector equation (2.17) gives for $1\leqq n\leqq J-1$,

$$(k^2 q_1+k^4 q_2\tilde{D}DA^{-1})(\tilde{D}(E^{n+1}-E^{n-1})+A(Z^{n+1}-Z^{n-1}))=0.$$

In view of (2.15) this implies that for $0\leqq n\leqq J$

$$(k^2 q_1 A+k^4 q_2\tilde{D}D)\,A^{-1}(\tilde{D}E^n+AZ^n)=0,$$

from which since the operator $\tilde{D}D$ is positive semidefinite on H_h, q_1, $q_2>0$ and A is positive define on R_s, we have for every n, $0\leqq n\leqq J$, that

$$\tilde{D}E^n+AZ^n=0. \tag{2.19}$$

Consequently,

$$\mu(DZ^n, E^{n+1}-E^{n-1})=\mu(Z^n, \tilde{D}(E^{n+1}-E^{n-1}))=-\mu(Z^n, A(Z^{n+1}-Z^{n-2})).$$

Combining the above relations, we see that (2.18) becomes, for $1 \leqq n \leqq J-1$

$$(2.20)\qquad \begin{aligned}&(E^{n+1}-2E^n+E^{n-1}, E^{n+1}-E^{n-1})\\ &+k^2(A(Z^{n+1}-Z^{n-1}), q_1 Z^{n+1}-2p_1 Z^n+q_1 Z^{n-1})\\ &+k^4(\tilde{D}D(Z^{n+1}-Z^{n-1}), q_2 Z^{n+1}-2p_2 Z^n+q_2 Z^{n-1})\\ &=(d^n, E^{n+1}-E^{n-1}).\end{aligned}$$

Applying now to (2.20) standard summation techniques valid for such symmetric, two-step schemes, cf. Sect. 2 of [8], we see that for $1 \leqq M \leqq J-1$ there holds

$$(2.21)\qquad \begin{aligned}&\|E^{M+1}-E^M\|+\tfrac{1}{2}\{k^2(q_1-p_1)(A(Z^{M+1}+Z^M), Z^{M+1}+Z^M)\\ &+k^2(q_1+p_1)(A(Z^{M+1}-Z^M), Z^{M+1}-Z^M)\\ &+k^4(q_2-p_2)\|D(Z^{M+1}+Z^M)\|^2+k^4(q_2+p_2)\|D(Z^{M+1}-Z^M)\|^2\}\\ &\leqq c\mathscr{E}^1+\sum_{n=1}^{M}(d^n, E^{n+1}-E^{n-1}).\end{aligned}$$

Now (2.10), (2.19) give the estimtate

$$\begin{aligned}&|(d^n, E^{n+1}-E^{n-1})|\\ &\leqq ck^2(k^4+h^r)(\|E^{n+1}-E^{n-1}\|+k^2\|DA^{-1}\tilde{D}(E^{n+1}-E^{n-1})\|)\\ &\leqq ck^2(k^4+h^r)(\|E^{n+1}-E^{n-1}\|+k^2\|D(Z^{n+1}-Z^{n-1})\|)\\ &\leqq ck^3(k^4+h^r)^2+ck(\|E^{n+1}-E^{n-1}\|^2+k^4\|D(Z^{n+1}-Z^{n-1})\|^2).\end{aligned}$$

Since $(q_1, q_2) \in \mathscr{R}$ and $q_1 > 1/4$ we have, cf. [8], that the coefficients of all terms in the left-hand side of (2.21) are positive. Hence

$$\mathscr{E}^{M+1} \leqq c\mathscr{E}^1+c(Mk)k^2(k^3+h^r)^2+ck\sum_{n=1}^{M+1}\mathscr{E}^n \qquad 0 \leqq M \leqq J-1.$$

Hence for k sufficiently small, (2.16) and the above give

$$\mathscr{E}^{M+1} \leqq c(Mk)k^2(k^4+h^r)^2+ck\sum_{n=1}^{M}\mathscr{E}^n, \qquad 0 \leqq M \leqq J-1.$$

By Gronwall's lemma we finally obtain the desired results, using

$$\|E^{M+1}\| \leqq \sum_{j=0}^{M}\|E^{j+1}-E^j\|+\|E^0\|,$$

(2.16) once more and *(FE2)*. □

We describe now one choice of the initial values of the scheme for which (2.15) and (2.16) are valid. We compute $\hat{u}^1$, $\hat{\sigma}^1$ using derivatives of the given initial data by the formulas

$$\hat{u}^1 = u^0 + k u^1 + \frac{k^2}{2!}\partial_t^2 u(0) + \frac{k^3}{3!}\partial_t^3 u(0) + \frac{k^4}{4!}\partial_t^4 u(0),$$

and

$$\hat{\sigma}^1 = \sigma(0) + k\partial_t \sigma(0) + \frac{k^2}{2!}\partial_t^2 \sigma(0) + \frac{k^3}{3!}\partial_t^3 \sigma(0),$$

and put $U^0 = P_1 u^0$, $\Sigma^0 = \Pi_1 \sigma(0)$ and $U^1 = P_1 \hat{u}^1$, $\Sigma^1 = \Pi_1 \hat{\sigma}^1$. Then $E^0 = 0$, $Z^0 = 0$ and, using (2.6), since time differentiation commutes with the operators $\tilde{D}P_1$, $A\Pi_1$,

$$\tilde{D}U^j + A\Sigma^j = 0, \qquad j = 0, 1.$$

We have at once that $\|E^1\| \leqq \|u^1 - \hat{u}^1\| + \|(I - P_1)(u^1 - \hat{u}^1)\| \leqq c k(h^r + k^4)$, and $\|Z^1\| \leqq \|\sigma(k) - \hat{\sigma}^1\| + \|(I - \Pi_1)(\sigma(k) - \hat{\sigma}^1)\| \leqq c(h^r + k^4)$.

Also from the definitions of D and Π_1 we see that

$$\begin{aligned} \|DZ^1\|^2 &= (D\Pi_1(\sigma(t_1) - \hat{\sigma}^1), D\Pi_1(\sigma(t_1) - \hat{\sigma}^1)) \\ &= (\operatorname{div} \Pi_1(\sigma(t_1) - \hat{\sigma}^1), D\Pi_1(\sigma(t_1) - \hat{\sigma}^1)) \\ &= (\operatorname{div}(\sigma(t_1) - \hat{\sigma}^1), D\Pi_1(\sigma(t_1) - \hat{\sigma}^1)) \\ &\leqq c k^3 \|DZ^1\|. \end{aligned}$$

Hence *(FE2)* gives (2.16).

Second order in time fully discrete schemes

To construct such schemes we consider a rational approximation to the cosine of the form

$$r(x) = \frac{1 + p_1 x^2}{1 + q_1 x^2}, \qquad q_1 > 0.$$

where $p_1 = q_1 - 1/2$ and assume for stability purposes that $q_1 > 1/4$. Then, for every smooth function $y = y(t)$ there holds

$$(I - q_1 k^2 \partial_t^2)(y^{n+1} + y^{n-1}) = 2(I - p_1 k^2 \partial_t^2) y^n + O(k^4 y^{(4)}).$$

In a similar way as in the case of the fourth order schemes we led to the following fully discrete scheme for finding approximations of $\begin{pmatrix} U^n \\ \Sigma^n \end{pmatrix}$ of $\begin{pmatrix} u(t_n) \\ \sigma(t_n) \end{pmatrix}$

in $V_h \times H_h$: let $\begin{pmatrix} U^j \\ \Sigma^j \end{pmatrix} \in V_h \times H_h$, $0 \leqq j \leqq J-1$ be given approximations of $\begin{pmatrix} u(t_j) \\ \sigma(t_j) \end{pmatrix}$. Then define $\begin{pmatrix} U^{n+1} \\ \Sigma^{n+1} \end{pmatrix}$ as the solution of the linear system

$$R_1' \begin{pmatrix} U^{n+1} \\ \Sigma^{n+1} \end{pmatrix} - 2S' \begin{pmatrix} U^n \\ \Sigma^n \end{pmatrix} + R_2' \begin{pmatrix} U^{n-1} \\ \Sigma^{n-1} \end{pmatrix} = F_n', \tag{2.19}$$

where R_1', R_2', S' are linear operators on $V_h \times H_h$ having the following form

$$R_1' = \begin{pmatrix} I & -k^2 q_1 D \\ k^2 q_1 \tilde{D} & k^2 q_1 A \end{pmatrix},$$

$$S' = \begin{pmatrix} I & -k^2 p_1 D \\ 0 & 0 \end{pmatrix},$$

$$R_2' = \begin{pmatrix} I & -k^2 q_1 D \\ -k^2 q_1 \tilde{D} & -k^1 q_1 A \end{pmatrix},$$

and F_n' is the element of $V_h \times H_h$ given by

$$F_n' = \begin{pmatrix} k^2 (q_1 f^{n+1} - 2p_1 f^n + q_1 f^{n-1}) \\ 0 \end{pmatrix}.$$

It is easily seen that since $q_1 > 0$ R_1' is positive definite on $(V_h \times H_h, ((\cdot, \cdot)))$ and therefore invertible. If we now let $E^n = U^n - P_1 u(t_n)$, $Z^n = \Sigma^n - \Pi_1 \sigma(t_n)$ and, for $1 \leqq n \leqq J$, put

$$\mathscr{E}^n = \|E^n - E^{n-1}\|^2 + k^2 \|Z^n + Z^{n-1}\|^2 + k^2 \|Z^n - Z^{n-1}\|^2,$$

we may prove the following result along the same lines as the proof of Theorem 2.2.

Theorem 2.3. *Assume that* $\begin{pmatrix} U^0 \\ \Sigma^0 \end{pmatrix}$, $\begin{pmatrix} U^1 \\ \Sigma^1 \end{pmatrix}$ *are given elements of* $V_h \times H_h$ *chosen so that*

$$\mathscr{E}^1 \leqq c k^2 (k^2 + h^r)^2.$$

Then, for every n, $2 \leqq n \leqq J$, $\begin{pmatrix} U^n \\ \Sigma^n \end{pmatrix}$ *exists uniquely in* $V_h \times H_h$ *as the solution of* (2.19). *If* $q_1 > 1/4$ *then, for* k *sufficiently small, there exists a positive constant* c, *independent of* k, h *such that*:

$$\max_{0 \leqq n \leqq J} (\|E^n\| + \|Z^n\|) \leqq c(k^2 + h^r)$$

and

$$\max_{0 \leqq n \leqq J} (\|u^n - U^n\| + \|\sigma^n - \Sigma^n\|) \leqq c(k^2 + h^r).$$

3 Velocity-stress discretization

Semidiscretization

In this section we shall study fully discrete mixed finite element approximations of the solution of problem (1.4), the "velocity-stress" formulation of the elastodynamics problem (1.2). We first discuss the semidiscrete approximation of the problem (1.4). We seek (v_h, σ_h): $[0, T] \to V_h \times H_h$ such that

$$\begin{aligned} (\dot{v}_h, w) - (\operatorname{div} \sigma_h, w) &= (f, w), \qquad \forall w \in V_h \quad 0 \leqq t \leqq T, \\ a(\dot{\sigma}_h, \chi) + (v_h, \operatorname{div} \chi) &= 0, \qquad \forall \chi \in H_h \quad 0 \leqq t \leqq T, \\ v_h(0) = v_h^0, &\qquad \sigma_h(0) = \sigma_h^0, \end{aligned}$$

where (v_h^0, σ_h^0) is a given approximation of $(v^0, \sigma(0))$ on $V_h \times H_h$. Using the discretization operators D and $\tilde{D}$ this is equivalent to seeking (v_h, σ_h): $[0, T] \to V_h \times H_h$, such that

$$(3.1) \qquad \begin{aligned} \dot{v}_h - D\sigma_h &= f_h, \qquad 0 \leqq t \leqq T, \\ A\dot{\sigma}_h + \tilde{D} v_h &= 0, \qquad 0 \leqq t \leqq T, \\ v_h(0) = v_h^0, &\qquad \sigma_h(0) = \sigma_h^0, \end{aligned}$$

where f_h is the L^2-projection of f in V_h. We now define the operator $\mathscr{D}_h$ on $V_h \times H_h$ as

$$\mathscr{D}_h = \begin{pmatrix} 0 & D \\ -A^{-1}\tilde{D} & 0 \end{pmatrix}.$$

Then the two differential equations in (3.1) can be written in the form

$$(3.2) \qquad \partial_t \begin{pmatrix} v_h \\ \sigma_h \end{pmatrix} = \mathscr{D}_h \begin{pmatrix} v_h \\ \sigma_h \end{pmatrix} + \begin{pmatrix} f_h \\ 0 \end{pmatrix}, \qquad 0 \leqq t \leqq T.$$

For the needs of the error analysis we define the following bilinear form on $V \times H_{sd}$:

$$\left(\left(\begin{pmatrix} w_1 \\ \chi_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ \chi_2 \end{pmatrix}\right)\right)_A = (w_1, w_2) + (A\chi_1, \chi_2), \qquad \begin{pmatrix} w_1 \\ \chi_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ \chi_2 \end{pmatrix} \in V \times H_{sd}.$$

Since A is symmetric and positive definite, $((\cdot, \cdot))_A$ is a inner product on $V \times H_{sd}$ equivalent to the inner product $((\cdot, \cdot))$ defined in Sect. 2. Furthermore observe that

$$\left(\left(\mathscr{D}_h \begin{pmatrix} w_1 \\ \chi_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ \chi_2 \end{pmatrix}\right)\right)_A = (D\chi_1, w_2) - (AA^{-1}\tilde{D}w_1, \chi_2) = (D\chi_1, w_2) - (\tilde{D}w_1, \chi_2)$$

and

$$\left(\left(\begin{pmatrix} w_1 \\ \chi_1 \end{pmatrix}, \mathscr{D}_h \begin{pmatrix} w_2 \\ \chi_2 \end{pmatrix}\right)\right)_A = (D\chi_2, w_1) - (AA^{-1}\tilde{D}w_2, \chi_1) = (D\chi_2, w_1) - (\tilde{D}w_2, \chi_1).$$

Hence, using (2.8), we have

$$((\mathscr{D}_h X, Y))_A = -((X, \mathscr{D}_h Y))_A, \qquad \forall X, Y \in V_h \times H_h,$$

i.e. $\mathscr{D}_h$ is antisymmetric on $V_h \times H_h$. As a consequence,

$$((\mathscr{D}_h X, X))_A = 0, \qquad \forall X \in V_h \times H_h.$$

If (u, σ) is the solution of (1.2) let $w(t) := P_1\, u(t)$ and $\xi(t) := \Pi_1\, \sigma(t)$. Since time differentiation commutes with the operator P_1, letting $z(t) := \dot{w}(t)$, we have

$$z(t) = P_1\, \dot{u}(t) = P_1\, v(t).$$

Differentiating the second equation of *(FE1)* with respect to t we obtain

$$\begin{aligned} &(\operatorname{div} \xi(t), q) = (\operatorname{div} \sigma(t), q), && \forall q \in V_h,\ t \in [0, T], \\ &(A\, \dot{\xi}(t), \chi) + (z(t), \operatorname{div} \chi) = 0, && \forall \chi \in H_h,\ t \in [0, T]. \end{aligned} \tag{3.3}$$

We shall compare the solution of the semidiscrete problem (3.1) with the element $(z(t), \xi(t)) := (P_1\, v(t), \Pi_1\, \sigma(t))$ of $V_h \times H_h$, and prove the following result, which is the analog in the case of the elastodynamics equations of the result of Geveci [15].

Theorem 3.1. *Let (v, σ) be the solution of* (1.4), *where $v = \dot{u}$. If (v_h, σ_h) is the solution of* (3.1) *then*

$$\begin{aligned} \|v(t) - v_h(t)\| + \|\sigma(t) - \sigma_h(t)\| &\leqq c(\|v^0 - v_h(0)\| + \|\sigma^0 - \sigma_h(0)\|) \\ &+ c h^r (\|v^0\|_r + \|\sigma^0\|_r + \left\{ \int_0^t (\|\sigma(s)\|^2 + \|\dot{v}(s)\|^2)\, ds \right\}^{1/2}. \end{aligned}$$

Proof. Let $(z(t), \xi(t)) \in V_h \times H_h$ be as above. Then *(FE2)* gives

$$\left|\left|\left| \begin{pmatrix} z \\ \xi \end{pmatrix} - \begin{pmatrix} v \\ \sigma \end{pmatrix} \right|\right|\right|_A \leqq c h^r (\|v\|_r + \|\sigma\|_r). \tag{3.4}$$

Using now the definitions of z, ξ and Eqs. (3.3) we have

$$\partial_t \begin{pmatrix} z \\ \xi \end{pmatrix} - \mathscr{D}_h \begin{pmatrix} z \\ \xi \end{pmatrix} = \partial_t \begin{pmatrix} z \\ \xi \end{pmatrix} - \begin{pmatrix} D\xi \\ -A^{-1} \tilde{D} z \end{pmatrix},$$

where $D\xi = D\sigma$ and

$$(\tilde{D} z, \chi) = -(A\, \dot{\xi}, \chi), \qquad \forall \chi \in H_h,$$

i.e. that

$$-A^{-1} \tilde{D} z = \dot{\xi}. \tag{3.5}$$

Using (1.4) we get the relation

$$\partial_t \begin{pmatrix} z \\ \xi \end{pmatrix} - \mathscr{D}_h \begin{pmatrix} z \\ \xi \end{pmatrix} = \partial_t \begin{pmatrix} z - P v \\ 0 \end{pmatrix} + \begin{pmatrix} f_h \\ 0 \end{pmatrix} =: \boldsymbol{P} R_t + \begin{pmatrix} f_h \\ 0 \end{pmatrix},$$

where $\boldsymbol{P}$ is the L^2-projection in $V_h \times H_h$. Putting now $\theta = \begin{pmatrix} v_h - z \\ \sigma_h - \xi \end{pmatrix}$, we have from the above and (3.2) that

$$\partial_t \theta - \mathscr{D}_h \theta = -\boldsymbol{P} R_t, \qquad t \in [0, T]. \tag{3.6}$$

Let E_t be the solution operator of the corresponding to (3.6) homogeneous problem on $V_h \times H_h$, i.e. denote by $W_h(t) = E_t W^0 \in V_h \times H_h$ the solution of the problem

$$\partial_t W_h - \mathscr{D}_h W_h = 0, \qquad W_h(0) = W^0, \qquad W^0 \in V_h \times H_h.$$

Since $\mathscr{D}_h$ is antisymmetric on $(V_h \times H_h, ((\cdot, \cdot))_A)$ we easily obtain

$$|||W_h||| = |||E_t W^0|||_A = |||W^0|||_A.$$

Duhamel's principle gives now that the solution of (3.6) is

$$\theta(t) = E_t \theta(0) - \int_0^t E_{t-s} P R_t(s)\, ds.$$

Hence

$$|||\Theta(t)|||_A \leqq |||E_t \Theta(0)|||_A + \int_0^t |||E_{t-s} P R_t(s)|||_A\, ds$$

$$\leqq |||\Theta(0)|||_A + \int_0^t |||R_t(s)|||_A\, ds.$$

Since now

$$|||R_t(s)|||_A = \|\dot{v}(t) - \dot{z}(t)\| \leqq c h^r \|\dot{v}\|_r,$$

we complete the proof using (3.4) and that the norms $|||\cdot|||_A$ and $|||\cdot|||$ are equivalent. □

Fully discrete approximations

Since the operator $\mathscr{D}_h$ has purely imaginary eigenvalues it is reasonable to discretize (3.2) in time using rational approximations of e^{ix}, $x \in \mathbb{R}$ [5, 7].

To this end consider a rational function $\tilde{r}(z)$ which is an up to fourth order accurate approximation of $\exp(z)$. Let $\tilde{r}(z) = P(z)/Q(z)$ where P and Q are relatively prime polynomials of degree up to two, with the following properties:

$$Ri \quad \exists \nu, 1 \leqq \nu \leqq 4: \ |\tilde{r}(z) - e^z| \leqq c|z|^{\nu+1}, \qquad z \in \mathbb{C},$$
$$Rii \qquad |\tilde{r}(z)| \leqq 1, \quad \text{for every } z \in i\mathbb{R}.$$

It is a straightforward consequence of Rii that there holds

$$Riii \quad Q(z) \neq 0, \qquad \text{for every } z \in i\mathbb{R}.$$

Examples of methods that satisfy these assumptions are given in the following table [7], in which $P(z) = 1 + p_1 z + p_2 z^2$, $Q(z) = 1 + q_1 z + q_2 z^2$.

Approximations of e^z	ν	q_1	q_2	p_1	p_2
Euler	1	-1	0	0	0
Crank-Nicolson	2	$-1/2$	0	1/2	0
Calahan[a]	3	-2λ	λ^2	$1-2\lambda$	$\lambda^2 - 2\lambda + 1/2$
Padé	2	-1	1/2	0	0
Padé	3	$-2/3$	1/6	1/3	0
Padé	4	$-1/2$	1/12	1/2	1/12

[a] $\lambda = (1/2)(1 + 1/\sqrt{3})$

A consequence of Ri is that for every smooth function $y=y(t)$ and $k>0$, there holds

$$y(t+k)+q_1\,k\,y'(t+k)+q_2\,k^2\,y''(t+k) = y(t)+p_1\,k\,y'(t)+p_2\,k^2\,y''(t)+O(k^{\nu+1}\,y^{(\nu+1)}). \tag{3.7}$$

Using (3.2), observe that

$$\partial_{tt}\begin{pmatrix}v_h\\ \sigma_h\end{pmatrix}=\mathscr{D}_h\begin{pmatrix}\dot v_h\\ \dot\sigma_h\end{pmatrix}+\partial_t\begin{pmatrix}f_h\\ 0\end{pmatrix}=\begin{pmatrix}-DA^{-1}\tilde D v_h\\ -A^{-1}\tilde D D\sigma_h\end{pmatrix}+\begin{pmatrix}\dot f_h\\ -A^{-1}\tilde D f_h\end{pmatrix}.$$

Motivated now by (3.7) we obtain the following fully discrete scheme: Let $\begin{pmatrix}V^j\\ \Sigma^j\end{pmatrix}\in V_h\times H_h$ $0\leqq j\leqq n\leqq J-1$ be given approximations of $\begin{pmatrix}v(t_j)\\ \sigma(t_j)\end{pmatrix}$. We define $\begin{pmatrix}V^{n+1}\\ \Sigma^{n+1}\end{pmatrix}\in V_h\times H_h$ as the solution of the linear system

$$\mathscr{A}\begin{pmatrix}V^{n+1}\\ \Sigma^{n+1}\end{pmatrix}=\mathscr{B}\begin{pmatrix}V^n\\ \Sigma^n\end{pmatrix}+\mathscr{F}^n, \tag{3.8}$$

where $\mathscr{A}$, $\mathscr{B}$ are linear operators on $V_h\times H_h$ defined by

$$\mathscr{A}=Q(k\mathscr{D}_h)=\begin{pmatrix}I-k^2q_2\,DA^{-1}\tilde D & kq_1D\\ -kq_1\,A^{-1}\tilde D & I-k^2q_2\,A^{-1}\tilde D D\end{pmatrix}$$

and

$$\mathscr{B}=P(k\mathscr{D}_h)=\begin{pmatrix}I-k^2p_2\,DA^{-1}\tilde D & kp_1D\\ -kp_1\,A^{-1}\tilde D & I-k^2p_2\,A^{-1}\tilde D D\end{pmatrix},$$

and where by $\mathscr{F}^n$ we denote

$$\mathscr{F}^n=-\begin{pmatrix}k(q_1f^{n+1}-p_1f^n)+k^2(q_2f^{(1)n+1}-p_2f^{(1)n})\\ -k^2A^{-1}\tilde D(q_2f^{n+1}-p_2f^n)\end{pmatrix}.$$

Since $\mathscr{A}=Q(k\mathscr{D}_h)$, if $\{\lambda_j\}_{j=1}^{M_h}$ are the eigenvalues of $\mathscr{D}_h$, $\lambda_j\in i\mathbb{R}$, $j=1,2,\ldots,M_h$, then the eigenvalues of $\mathscr{A}$ are $Q(k\lambda_j)$, $j=1,2,\ldots,M_h$. Therefore, $Riii$ gives that zero is not an eigenvalue of $\mathscr{A}$. Hence, the operator $\mathscr{A}$ is invertible and we can compute $\begin{pmatrix}V^{n+1}\\ \Sigma^{n+1}\end{pmatrix}$ as the unique solution of (3.8).

Note on implementation

For the efficient solution of the system

$$\mathscr{A}\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}a\\ b\end{pmatrix}, \tag{3.8a}$$

we note first that if $q_2=0$, i.e. $\mathscr{A}=I+q_1 k\mathscr{D}_h$, (3.8a) can be solved directly using block elimination:

$$AY+q_1^2 k^2 \tilde{D} D Y = A b + q_1 k \tilde{D} a \qquad (3.8\text{b})$$
$$X = -q_1 k D Y + a.$$

For $q_2 \neq 0$, write $\mathscr{A}=(I-\alpha_1 k\mathscr{D}_h)(I-\alpha_2 k\mathscr{D}_h)$, where α_1, α_2 are complex in general – the roots of $Q(z)$. In this case in order to compute (X, Y) we have to solve two systems of the form (3.8b).

Another alternative for solving (3.8a) is a preconditioned iterative method, similar to the one analyzed by Bales in [7] in the case of Galerkin approximations for the wave equation. The basic steps of this method are as follows: First, consider an equivalent form of the system (3.8a),

$$\mathscr{A}^* \mathscr{A} \begin{pmatrix} X \\ Y \end{pmatrix} = \mathscr{A}^* \begin{pmatrix} a \\ b \end{pmatrix}, \qquad (3.8\text{c})$$

where $\mathscr{A}^*$ is the adjoint of $\mathscr{A}$. The matrix of the system (3.8c) is symmetric and positive definite. Solve (3.8c) with an iterative method in which the calculation of an approximation (X^{p+1}, Y^{p+1}) to (X, Y), given (X^j, Y^j), $0 \leqq j \leqq p$, requires solving systems involving only a matrix A_0 – the preconditioner – with certain properties, cf. [7], and such that $A_0^{-1}\Phi$, $\Phi \in V_h \times H_h$, can be computed *easily*. In our case an appropriate choice of A_0 is $A_0 = \mathscr{A}_0^* \mathscr{A}_0$, where $\mathscr{A}_0 = I - \alpha^2 k^2 \mathscr{D}_h^2$ for some real number α, i.e. where

$$\mathscr{A}_0 = \begin{pmatrix} I + k^2\alpha^2 D A^{-1}\tilde{D} & 0 \\ 0 & I + k^2 \alpha^2 A^{-1} \tilde{D} D \end{pmatrix}.$$

$\mathscr{A}_0$ is symmetric and positive definite on $(V_h \times H_h, ((\cdot,\cdot))_A)$ and thus $A_0 = \mathscr{A}_0^2$. We conclude therefore that the computation of $A_0^{-1}\Phi$ requires solving two systems of the form: Find $(\tilde{X}, \tilde{Y}) \in V_h \times H_h$ such that

$$\tilde{X} + k^2 \alpha^2 D A^{-1} \tilde{D} \tilde{X} = a_1$$
$$A\tilde{Y} + k^2 \alpha^2 \tilde{D} D \tilde{Y} = b_1.$$

Stability

Since $\mathscr{B} = P(k\mathscr{D}_h)$, the eigenvalues of $\mathscr{B}$ are the numbers $P(k\lambda_j)$, $j=1,2,\ldots,M_h$. Let $\{\Phi_j\}_{j=1}^{M_h}$ be an orthonormal system of eigenvectors of $\mathscr{D}_h$ with respect to $((\cdot,\cdot))_A$ extended for complex elements as

$$\left(\left(\begin{pmatrix} w_1 \\ \chi_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ \chi_2 \end{pmatrix}\right)\right)_A = (w_1, \bar{w}_2) + (A\chi_1, \bar{\chi}_2).$$

If $\Psi \in V_h \times H_h$ with $\Psi = \sum_j c_j \Phi_j$, then $|||\mathscr{A}\Psi|||_A = \left\{ \sum_{j=1}^{M_h} |Q(k\lambda_j)|^2 |c_j|^2 \right\}^{1/2}$ and $|||\mathscr{B}\Psi|||_A = \left\{ \sum_{j=1}^{M_h} |P(k\lambda_j)|^2 |c_j|^2 \right\}^{1/2}$. Hence, using Rii, we have

$$|||\mathscr{A}\Psi|||_A \leqq |||\mathscr{B}\Psi|||_A, \quad \forall \Psi \in V_h \times H_h. \tag{3.9}$$

Consistency

We shall compare the approximations $\begin{pmatrix} V^n \\ \Sigma^n \end{pmatrix}$ with the element $\begin{pmatrix} P_1 v^n \\ \Pi_1 \sigma^n \end{pmatrix} =: \begin{pmatrix} z^n \\ \xi^n \end{pmatrix}$ of $V_h \times H_h$. For this we need an estimate for $\begin{pmatrix} a_1^n \\ \gamma_1^n \end{pmatrix} \in V_h \times H_h$, defined by

$$\mathscr{A}\begin{pmatrix} z^{n+1} \\ \xi^{n+1} \end{pmatrix} - \mathscr{B}\begin{pmatrix} z^n \\ \xi^n \end{pmatrix} =: \begin{pmatrix} a_1^n \\ \gamma_1^n \end{pmatrix}. \tag{3.10}$$

We have

$$\begin{pmatrix} a_1^n \\ \gamma_1^n \end{pmatrix} = \begin{pmatrix} z^{n+1} - z^n + kD(q_1 \xi^{n+1} - p_1 \xi^n) \\ \xi^{n+1} - \xi^n - kA^{-1}\tilde{D}(q_1 z^{n+1} - z^n) \end{pmatrix} - \begin{pmatrix} k^2 DA^{-1}\tilde{D}(q_2 z^{n+1} - p_2 z^n) \\ k^2 A^{-1}\tilde{D}D(q_2 \xi^{n+1} - p_2 \xi^n) \end{pmatrix}. \tag{3.11}$$

As we have observed before, from the definition of z^n, ξ^n we have that $D\xi^n = D\sigma^n$ and the relation (3.5). Hence using (1.4),

$$\begin{pmatrix} D\xi^n \\ -A^{-1}\tilde{D}z^n \end{pmatrix} = \partial_t \begin{pmatrix} Pv^n \\ \xi^n \end{pmatrix} - \begin{pmatrix} f^n \\ 0 \end{pmatrix}. \tag{3.12a}$$

Observe also that using again (3.3), (3.5) and (1.4) yields

$$-DA^{-1}\tilde{D}z^n = D\dot{\xi}^n = D\dot{\sigma}^n$$

and

$$P\partial_{tt} v^n = -DA^{-1}\tilde{D}z^n + \dot{f}^n. \tag{3.12b}$$

Now note that using (1.4) and *(FE3)* gives

$$(\dot{v}, \operatorname{div}\chi) - (\operatorname{div}\sigma, \operatorname{div}\chi) = (f, \operatorname{div}\chi), \quad \forall \chi \in H_h,$$

and

$$(\dot{v}, \operatorname{div}\chi) - (\operatorname{div}\sigma, \operatorname{div}\chi) = (\dot{v}, \operatorname{div}\chi) - (D\xi, D\chi) = (\tilde{D}\dot{v}, \chi) - (\tilde{D}D\xi, \chi), \quad \forall \chi \in H_h.$$

Hence

$$\tilde{D}\dot{v}^n = \tilde{D}D\xi^n + \tilde{D}f^n.$$

Also, using (1.4) we see that $\tilde{D}\dot{v}^n = -\tilde{P}A\partial_t^2 \sigma^n$, where $\tilde{P}$: $H_{sd} \to H_h$ is the L^2-projection. Note that for $\tau \in H_{sd}$ we have

$$(A^{-1}\tilde{P}A\tau, \chi) = (\tilde{P}A\tau, A^{-1}\chi) = (\tau, \chi) = (\tilde{P}\tau, \chi) \quad \forall \chi \in H_h,$$

i.e. that $A^{-1}\tilde{P}A=\tilde{P}$ on H_{sd}. We finally obtain that

$$-A^{-1}\tilde{D}D\xi^n - A^{-1}\tilde{D}f^n = \tilde{P}\partial_t^2\sigma^n. \tag{3.12c}$$

The relations (3.11) and (3.12a–c) give

$$\begin{aligned}\binom{a_1^n}{\gamma_1^n} &= \boldsymbol{P}(I+kq_1\,\partial_t+k^2q_2\,\partial_t^2)\binom{v^{n+1}}{\sigma^{n+1}} - \boldsymbol{P}(I+kp_1\,\partial_t+k^2p_2\,\partial_t^2)\binom{v^n}{\sigma^n}\\ &\quad + \mathscr{F}^n + \boldsymbol{P}\begin{pmatrix}(P_1-I)(v^{n+1}-v^n)\\ (\Pi_1-I)(\sigma^{n+1}-\sigma^n+kq_1\,\partial_t\sigma^{n+1}-kp_1\,\partial_t\sigma^n)\end{pmatrix}\\ &=: \Gamma_1^n + \mathscr{F}^n + \Gamma_2^n,\end{aligned} \tag{3.13}$$

where, from (3.8) we have,

$$|||\Gamma_1^n|||_A \leqq c k^{\nu+1}\Big(\sup_{s\in[t^n,t^{n+1}]}\|\partial_t^{\nu+1}u(s)\| + \sup_{s\in[t^n,t^{n+1}]}\|\partial_t^{\nu+1}\sigma(s)\|\Big) \tag{3.14}$$

and from *(FE2)*

$$|||\Gamma_2^n|||_A \leqq c k h^r\Big(\sup_{s\in[t^n,t^{n+1}]}\|\dot{u}(s)\|_r + \sup_{s\in[t^n,t^{n+1}]}\|\dot{\sigma}(s)\|_r\Big). \tag{3.15}$$

Convergence

We have now the following result:

Theorem 3.2. *Let that* $(V^0,\Sigma^0)=(z^0,\xi^0)$. *Then for every* n, $1\leqq n\leqq J$, $\binom{V^n}{\Sigma^n}\in V_h\times H_h$ *exists uniquely as the solution of the linear system* (3.8). *Asuming that the rational function* $\tilde{r}(x)$ *satisfies Ri, Rii and putting* $\Theta^n=\binom{V^n-z^n}{\Sigma^n-\xi^n}$, *we have that if* $q_2=0$ *for* $\nu=2$, *there exists a positive constant c, independent of k, h such that:*

$$\max_{0\leqq n\leqq J}|||\Theta^n|||_A \leqq c(k^\nu+h^r),$$

and consequently

$$\max_{0\leqq n\leqq J}(\|v^n-V^n\|+\|\sigma^n-\Sigma^n\|)\leqq c(k^\nu+h^r).$$

Proof. Combining (3.8), (3.10–13) we have that Θ^n satisfies

$$\mathscr{A}\Theta^{n+1}=\mathscr{B}\Theta^n-\Gamma_1^n-\Gamma_2^n.$$

From (3.9) and (3.14), (3.15) we have

$$\begin{aligned}|||\mathscr{A}\Theta^{n+1}|||_A &\leqq |||\mathscr{B}\Theta^n|||_A + |||\Gamma_1^n|||_A + |||\Gamma_2^n|||_A\\ &\leqq |||\mathscr{A}\Theta^n|||_A + ck(k^\nu+h^r), \qquad 0\leqq n\leqq J-1.\end{aligned}$$

Hence, summing, we obtain

$$|||\mathscr{A}\Theta^n|||_A \leqq |||\mathscr{A}\Theta^0|||_A + cnk(k^v + h^r), \qquad 1 \leqq n \leqq J,$$

and since $\Theta^0 = 0$ we conclude that

$$|||\mathscr{A}\Theta^n|||_A \leqq c(k^v + h^r), \qquad 0 \leqq n \leqq J.$$

Since $\mathscr{D}_h$ is antisymmetric on $(V_h \times H_h, ((\cdot, \cdot))_A)$ we have

$$|||\mathscr{A}\Psi|||_A^2 = ((\Psi, \Psi))_A + k^2(q_1^2 - 2q_2)((\mathscr{D}_h\Psi, \mathscr{D}_h\Psi))_A + k^4 q_2^4((\mathscr{D}_h^2\Psi, \mathscr{D}_h^2\Psi))_A.$$

We see easily that $q_1^2 - 2q_2 \geqq 0$, so that for every $\Psi \in V_h \times H_h$ there holds

$$|||\Psi|||_A \leqq |||\mathscr{A}\Psi|||_A.$$

Hence we have

$$|||\Theta^n|||_A \leqq c(k^v + h^r), \qquad 0 \leqq n \leqq J. \quad \square$$

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