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On the coupled BEM and FEM for a nonlinear exterior Dirichlet problem in $\mathbf{R}^2$*

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Summary. In this paper we apply the coupling of boundary integral and finite element methods to solve a nonlinear exterior Dirichlet problem in the plane. Specifically, the boundary value problem consists of a nonlinear second order elliptic equation in divergence form in a bounded inner region, and the Laplace equation in the corresponding unbounded exterior region, in addition to appropriate boundary and transmission conditions. The main feature of the coupling method utilized here consists in the reduction of the nonlinear exterior boundary value problem to an equivalent monotone operator equation. We provide sufficient conditions for the coefficients of the nonlinear elliptic equation from which existence, uniqueness and approximation results are established. Then, we consider the case where the corresponding operator is strongly monotone and Lipschitz-continuous, and derive asymptotic error estimates for a boundary-finite element solution. We prove the unique solvability of the discrete operator equations, and based on a Strang type abstract error estimate, we show the strong convergence of the approximate solutions. Moreover, under additional regularity assumptions on the solution of the continuous operator equation, the asymptotic rate of convergence $O(h)$ is obtained.

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1 Introduction

At present, there are two different concepts for the combination of boundary element method (BEM) and finite element method (FEM). One, which we will not consider, uses BEM for the modeling of special finite element functions (see e.g.,

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Schnack (1987) and Hsiao et al. 1991). The second concept, which is the most popular, consists of subdividing the original domain into a finite number of subregions and using in each of them either FEM or BEM, where the latter lives on all boundaries of the subregions. This approach is particularly attractive for exterior problems or boundary value problems in domains extending to infinity. In this case, the general coupling procedure may be simply described as follows. First, one divides the domain into two subregions, a bounded inner and an unbounded outer region by introducing an auxiliary common boundary, if necessary. Next, the problem is reduced to an equivalent one in the bounded inner region. This reduction will be accomplished by deriving either a local natural boundary condition or a nonlocal boundary condition, which relates the Cauchy data of the solution, on the common boundary. Because of the necessity of deriving this boundary condition on the common boundary, one needs generally to apply boundary integral methods to the unbounded outer region. This reduction to an equivalent problem is by no means a unique process. The first significant result concerning the theoretical justification of a coupling procedure of this type based on the direct boundary integral method seems due to Brezzi and Johnson (1979) and Johnson and Nedelec (1980). This result has been generalized recently by Wendland (1986, 1988). Further theoretical developments with respect to various coupling procedures may be found in Costabel (1987), Feng (1983), Han (1987), Hsiao and Porter (1986), MacCamy and Marin (1980) and Porter (1986). For a complete survey of the coupling methods we refer to the recent papers Hsiao (1988) and Hsiao (1990).

It is worth remarking that in terms of general flexibility and applicability, the most suitable approaches are the ones given in Johnson and Nedelec (1980), Costabel (1987), Han (1987) and Wendland (1986, 1988), which are all based on an integral representation of the solution in the unbounded region from Green's Theorem, the so-called direct boundary integral method. The success of the coupling procedure described in Johnson and Nedelec (1980) and of its corresponding generalization given in Wendland (1986, 1988), hinges on the fact that the boundary integral operator of the double layer potential is compact. However, in many applications as e.g., in elasticity, this is not the case. Costabel (1987) and Han (1987) proposed modifications of the original method of Johnson and Nedelec in which the compactness does not play any role. Both Costabel's and Han's approaches are based on the addition of a boundary integral equation for the normal derivative (resp. traction in the case of elasticity). Costabel's method leads to a symmetric and non-coercive bilinear form, while Han's method, on the contrary, yields a coercive and non-symmetric bilinear form.

It has been shown recently that the coupling of boundary integral and finite element methods, originally designed for treating linear problems, works equally well for the case of an adequate combination of linear and nonlinear partial differential equations. To the best of the authors's knowledge, up to now only Costabel and Stephan (1988), Gatica (1989), Gatica and Hsiao (1989a, b, c, 1990), Berger (1989), and Berger et al. (1990) have applied the coupling method to nonlinear problems from the theoretical point of view. In Berger et al., for instance, the approach is based on the classical method of Johnson and Nedelec. On the other hand, Costabel and Stephan extended the symmetric method for linear problems (see Costabel (1987)) to the nonlinear case and obtained a variational formulation in which the weak solution constitutes a saddle point of the associated functional. The approach used in Gatica and Hsiao's papers is based on the theory of monotone operators and can be regarded as a generalization of Han's method to the nonlinear case.

In this paper we report on the results obtained in Gatica and Hsiao (1989a, b) for the two-dimensional case. In Sect. 2 we convert the nonlinear exterior boundary value problem to an equivalent nonlocal boundary problem in the inner region. Then, in Sect. 3 we give a weak formulation for the nonlocal boundary problem and reduce it to an equivalent operator equation form. Some known results on monotone operators are collected in Sect. 4. Existence and uniqueness of the solution of this operator equation are established in Sect. 5 by using the results from Sect. 4. In Sect. 6, we study the Galerkin approximations for the monotone operator equation and provide some abstract error estimates of the Céa and the Strang type. Finally, in Sect. 7 we derive asymptotic error estimates for a boundary-finite element solution of the operator equation for the case in which the corresponding operator is strongly monotone and Lipschitz-continuous.

2 The nonlocal boundary problem

We first specify the nonlinear exterior boundary value problem. Let Ω_0 be a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary Γ_0. Let Ω^- be the annular region bounded by Γ_0 and another smooth closed curve Γ. We denote by Ω^+ the complement of $\bar{\Omega}_0 \cup \bar{\Omega}^-$ (see Fig. 1 below). For any function v defined in $\Omega^- \cup \Omega^+$, we write $v^\pm$ for its limits on Γ from $\Omega^\pm$. Also, for $0 < r < 1$, $C^{1,r}(\bar{\Omega}^-) \cap C^{1,r}(\bar{\Omega}^+)$ denotes the space of functions v defined in $\bar{\Omega}^- \cup \bar{\Omega}^+$ such that $v|_{\bar{\Omega}^-} \in C^{1,r}(\bar{\Omega}^-)$ and $v|_{\bar{\Omega}^+} \in C^{1,r}(\bar{\Omega}^+)$, where for each nonnegative integer m, $C^{m,r}(\bar{\Omega}^\pm)$ is the space of those functions in $C^m(\bar{\Omega}^\pm)$ whose partial derivatives of order m satisfy a Holder continuity condition with exponent r in $\bar{\Omega}^\pm$. In addition, let a_i: $\bar{\Omega}^- \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$ be nonlinear mappings such that $a_i \in C^1(\bar{\Omega}^- \times \mathbb{R}^2)$ for all $i = 1, 2$. Then, given $f \in C(\bar{\Omega}^-)$, $g_0 \in C^{1,r}(\Gamma_0)$ and $b \in \mathbb{R}$, we consider the

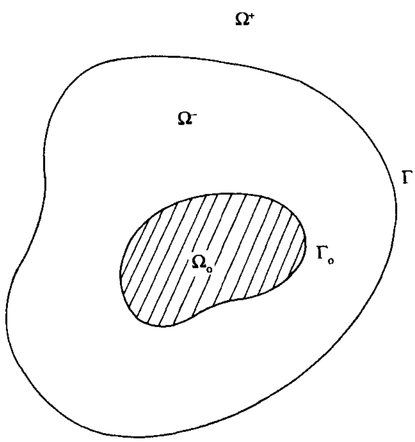

Fig. 1. Geometry of the exterior BVP

nonlinear exterior boundary value problem: *Find* $u \in C^2(\Omega^- \cup \Omega^+) \cap C^{1,r}(\bar{\Omega}^-) \cap C^{1,r}(\bar{\Omega}^+)$ *such that*

$$
\begin{aligned}
(2.1) \qquad & -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(\cdot, \nabla u(\cdot)) = f \quad \text{in } \Omega^- \\
& -\Delta u = 0 \quad \text{in } \Omega^+ \\
& u = g_0 \quad \text{on } \Gamma_0 \\
& u^- = u^+, \qquad \sum_{i=1}^{2} a_i(\cdot, (\nabla u)^-(\cdot)) \nu_i = \left(\frac{\partial u}{\partial \nu}\right)^+ \quad \text{on } \Gamma \\
& u(x) = \frac{b}{\pi} \log|x| + O(1) \quad \text{as } |x| \to +\infty ,
\end{aligned}
$$

where $\nu := (\nu_1, \nu_2)$ *denotes the unit outward normal to* Γ.

We remark that the regularity conditions specified above on the nonlinear coefficients a_i and the data f and g_0 will be relaxed when we consider the weak formulation of (2.1), which, as indicated before, is our main concern in this paper. In that case it will suffice to have $f \in L^2(\Omega^-)$ and $g_0 \in H^{1/2}(\Gamma_0)$, in addition to the Carathéodory and growth conditions for a_i (see Sect. 3). We comment further that if $f \in C(\bar{\Omega}^-)$ then clearly $f \in L^2(\Omega^-)$, and similarly if $g_0 \in C^{1,r}(\Gamma_0)$ then one can prove that $g_0 \in H^{1/2}(\Gamma_0)$. However, the converses are not necessarily true.

Explicit examples of nonlinearities a_i appear in physics, mechanics, etc. In the stationary heat conduction equation, for example, one has $a_i(x, \nabla u(x)) = k(x, \nabla u(x)) \frac{\partial u}{\partial x_i}$ where u is the temperature and k is the heat conductivity. Also, in some subsonic flow and fluid mechanics problems the nonlinear mappings a_i have similar expressions (see e.g., Feistauer 1986, 1987).

Our goal here is to solve the problem (2.1) by the coupling procedure as in Han (1987) for linear problems. We first reduce (2.1) to a nonlocal boundary problem. For this purpose we need some results from potential theory. Let $E(x, y) = -\frac{1}{2\pi} \log|x - y|$ be the fundamental solution for the two-dimensional Laplacian. Then, we assume that Γ is a closed Lyapunov curve and define the following continuous boundary integral operators (see Michlin (1970, 1978)):

$$
\begin{aligned}
(2.2) \qquad & \mathbf{V}: C^{0,r}(\Gamma) \to C^{1,r}(\Gamma) \\
& (\mathbf{V}\xi)(x) := \int_\Gamma E(x, y) \xi(y) \, ds_y ,
\end{aligned}
$$

$$
\begin{aligned}
(2.3) \qquad & \mathbf{K}: C^{0,r}(\Gamma) \to C^{1,r}(\Gamma) \\
& (\mathbf{K}\xi)(x) := \int_\Gamma \left\{ \frac{\partial}{\partial \nu(y)} E(x, y) \right\} \xi(y) \, ds_y ,
\end{aligned}
$$

$$
\begin{aligned}
(2.4) \qquad & \mathbf{K}': C^{0,r}(\Gamma) \to C^{1,r}(\Gamma) \\
& (\mathbf{K}'\xi)(x) := \int_\Gamma \left\{ \frac{\partial}{\partial \nu(x)} E(x, y) \right\} \xi(y) \, ds_y ,
\end{aligned}
$$

(2.5)
$$\mathbf{W}: C^{1,r}(\Gamma) \to C^{0,r}(\Gamma)$$

$$(\mathbf{W}\xi)(x) := -\frac{\partial}{\partial \nu(x)} \int_\Gamma \left\{ \frac{\partial}{\partial \nu(y)} E(x, y) \right\} \xi(y) \, ds_y .$$

Here **V**, **K**, **K**′ and **W** are, respectively, the boundary integral operators of the *simple, double, adjoint of the double and hypersingular layer potentials.* For further properties of these operators we refer to Michlin (1970) and Colton and Kress (1983). We remark that since our main interest is the studying of a weak formulation of (2.1), we will provide in Sect. 3 the continuity properties of the above operators in the Sobolev spaces $H^r(\Gamma)$.

The following lemma will also be needed (see Hsiao and Roach 1979; Hsiao 1986).

Lemma 2.1. *Let* $v \in C^2(\Omega^+) \cap C^{1,r}(\bar{\Omega}^+)$ *be the solution of the exterior Dirichlet problem*

$$-\Delta v = 0 \quad \text{in } \Omega^+$$

$$v \text{ prescribed on } \Gamma$$

$$v(x) = \frac{b}{\pi} \log|x| + O(1) \quad \text{as } |x| \to +\infty .$$

Then

$$\int_\Gamma \left(\frac{\partial v}{\partial \nu} \right)^+ ds = 2b .$$

Let us return now to our problem (2.1). From Green's theorem we have the representation

(2.6)
$$u(x) = \int_\Gamma \left\{ \frac{\partial}{\partial \nu(y)} E(x, y) \right\} u^+(y) \, ds_y - \int_\Gamma E(x, y) \left(\frac{\partial u}{\partial \nu} \right)^+ (y) \, ds_y - \lambda$$

for all $x \in \Omega^+$, where λ is a constant. Hence, in view of the jump conditions of the corresponding layer potentials, we arrive at the integral equations on Γ:

(2.7)
$$u^+ = (\tfrac{1}{2}\mathbf{I} + \mathbf{K}) u^+ - \mathbf{V} \left(\frac{\partial u}{\partial \nu} \right)^+ - \lambda ,$$

$$\left(\frac{\partial u}{\partial \nu} \right)^+ = -\mathbf{W} u^+ + (\tfrac{1}{2}\mathbf{I} - \mathbf{K}') \left(\frac{\partial u}{\partial \nu} \right)^+ .$$

Now we use the interface conditions from (2.1). We have $u^+ = u^-$, and if we put $\tilde{\sigma} = \sum_{i=1}^2 a_i(\cdot, (\nabla u)^-(\cdot)) \nu_i$, then $\left(\frac{\partial u}{\partial \nu} \right)^+ = \tilde{\sigma}$. We substitute into (2.7) to obtain, on Γ,

$$(\tfrac{1}{2}\mathbf{I} - \mathbf{K}) u^- + \mathbf{V} \tilde{\sigma} + \lambda = 0 ,$$

$$\tilde{\sigma} = \sum_{i=1}^{2} a_i(\cdot, (\nabla u)^-(\cdot)) \nu_i = -\mathbf{W} u^- + (\tfrac{1}{2}\mathbf{I} - \mathbf{K}') \tilde{\sigma} .$$

These formulae and Lemma 2.1 lead us to the *nonlocal boundary problem*: *Find* $u \in C^2(\Omega^-) \cap C^{1,r}(\bar{\Omega}^-)$, $\tilde{\sigma} \in C^{0,r}(\Gamma)$ *and* $\lambda \in \mathbb{R}$ *such that*

(2.8)
$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(\cdot, \nabla u(\cdot)) = f \quad \text{in } \Omega^-$$

$$u = g_0 \quad \text{on } \Gamma_0$$

$$\sum_{i=1}^{2} a_i(\cdot, (\nabla u)^-(\cdot))\nu_i = \tilde{\sigma} \qquad \text{and} \qquad \tilde{\sigma} = -\mathbf{W}u^- + (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\tilde{\sigma} \quad \text{on } \Gamma$$

$$(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u^- + \mathbf{V}\tilde{\sigma} + \lambda = 0 \quad \text{on } \Gamma$$

$$\int_\Gamma \tilde{\sigma}\, ds = 2b\,.$$

Note that in (2.8) the boundary conditions on Γ are nonlocal conditions, since the values u^- over the entire curve Γ are needed in order to compute $\tilde{\sigma}(x) = \sum_{i=1}^{2} a_i(x, (\nabla u)^-(x))\nu_i$ at a single point $x \in \Gamma$. Clearly, one may also consider other types of nonlocal conditions, and in principle, the nonlocal boundary problem (2.8) can be treated numerically by any conventional scheme, since it is a problem over the finite region $\bar{\Omega}^-$. We will adopt the Galerkin procedure and lead up to the coupling of the boundary element and finite element methods. Before doing so, we establish explicitly the equivalence between (2.1) and the nonlocal boundary problem (2.8).

Theorem 2.2. *The problems* (2.1) *and* (2.8) *are equivalent in the following sense:*

i) If v is a solution of (2.1), *then $(u, \tilde{\sigma}, \lambda)$ solves* (2.8), *where* $u := v|_{\bar{\Omega}^-}$, $\tilde{\sigma} := \left(\dfrac{\partial v}{\partial \nu}\right)^+$ *on Γ and the constant λ is obtained from the Green's representation for v in Ω^+. Conversely, if $(u, \tilde{\sigma}, \lambda)$ is a solution of* (2.8), *then v defined by*

(2.9)
$$v(x) = \begin{cases} u(x), \; x \in \bar{\Omega}^- \\ \displaystyle\int_\Gamma \left\{ \frac{\partial}{\partial \nu(y)} E(x,y) \right\} u^-(y)\, ds_y - \int_\Gamma E(x,y)\tilde{\sigma}(y)\, ds_y - \lambda, \; x \in \Omega^+ \end{cases}$$

is a solution of (2.1).

ii) The problem (2.1) *has a unique solution if and only if* (2.8) *has also a unique solution.*

Proof. Naturally, the first assertion of *i)* follows from our deduction of (2.8). Now, let $(u, \tilde{\sigma}, \lambda) \in [C^2(\Omega^-) \cap C^{1,r}(\bar{\Omega}^-)] \times C^{0,r}(\Gamma) \times \mathbb{R}$ be a solution of (2.8) and define v by (2.9). According to the regularity properties of u and $\tilde{\sigma}$, it follows that $v \in C^2(\Omega^- \cup \Omega^+) \cap C^{1,r}(\bar{\Omega}^-) \cap C^{1,r}(\bar{\Omega}^+)$. Also, since $v = u$ in $\bar{\Omega}^-$ and the double and single layer potentials are harmonic in Ω^+, we obtain that v clearly satisfies the first three equations in (2.1). On the other hand, taking limit in (2.9) as x approaches Γ from Ω^+ we get

(2.10)
$$v^+ = (\tfrac{1}{2}\mathbf{I} + \mathbf{K})u^- - \mathbf{V}\tilde{\sigma} - \lambda \quad \text{on } \Gamma\,.$$

Then, adding (2.10) to the fourth equation in (2.8) we deduce that $v^+ = u^-$ on Γ, and since $u^- = v^-$ we conclude that $v^+ = v^-$ on Γ. Analogously, taking limit in

(2.9) of the normal derivative as x approaches Γ from Ω^+ again, we obtain

$$\left(\frac{\partial v}{\partial \nu}\right)^+ = -\mathbf{W}u^- + (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\tilde{\sigma} ,$$

which together with the third equation in (2.8) imply $\left(\frac{\partial v}{\partial \nu}\right)^+ = \tilde{\sigma}$, and hence, since $v = u$ in $\bar{\Omega}^-$,

$$\sum_{i=1}^{2} a_i(\cdot, (\nabla u)^-(\cdot))\nu_i = \left(\frac{\partial v}{\partial \nu}\right)^+ \quad \text{on } \Gamma .$$

Finally, it is easy to see that the equality $\int_\Gamma \tilde{\sigma}\, ds = 2b$ and the formula (2.9) yield the asymptotic behavior $v(x) = (b/\pi)\log|x| + O(1)$ as $|x| \to +\infty$. This concludes the demonstration that v is a solution of (2.1).

We now prove *ii*). First, suppose that (2.1) has at most one solution and that $(u_j, \tilde{\sigma}_j, \lambda_j)$, $j = 1, 2$ are two solutions to (2.8). Then, according to our previous analysis we obtain that for $j = 1, 2$ the function v_j defined by (2.9) with $(u_j, \tilde{\sigma}_j, \lambda_j)$ instead of $(u, \tilde{\sigma}, \lambda)$, is a solution to (2.1). Thus, the uniqueness of (2.1) implies $v_1 = v_2$, which means $u_1 = u_2$ in $\bar{\Omega}^-$, and

$$\text{(2.11)} \qquad \int_\Gamma \left\{\frac{\partial}{\partial \nu(y)} E(x, y)\right\} u_1^-(y)\, ds_y - \int_\Gamma E(x, y)\tilde{\sigma}_1(y)\, ds_y - \lambda_1$$
$$= \int_\Gamma \left\{\frac{\partial}{\partial \nu(y)} E(x, y)\right\} u_2^-(y)\, ds_y - \int_\Gamma E(x, y)\tilde{\sigma}_2(y)\, ds_y - \lambda_2$$

for all $x \in \Omega^+$. Then, since by (2.8) $\tilde{\sigma}_j = \sum_{i=1}^2 a_i(\cdot, (\nabla u_j)^-(\cdot))\nu_i$ on Γ, we obtain that $\tilde{\sigma}_1 = \tilde{\sigma}_2$. Hence, the equality (2.11) gives $\lambda_1 = \lambda_2$. In this way we have shown that $(u_1, \tilde{\sigma}_1, \lambda_1) = (u_2, \tilde{\sigma}_2, \lambda_2)$, which proves that (2.8) has at most one solution.

Conversely, suppose that (2.8) has no more than one solution and that v_1, v_2 are two solutions to (2.1). Then, by *i*) and the uniqueness of (2.8) we deduce that $v_1 = v_2$ in $\bar{\Omega}^-$, $\left(\frac{\partial v_1}{\partial \nu}\right)^+ = \left(\frac{\partial v_2}{\partial \nu}\right)^+$ on Γ, and $\lambda_1 = \lambda_2$. Now, since by (2.1) $v_1^+ = v_1^-$ and $v_2^+ = v_2^-$ we deduce that $v_1^+ = v_2^+$, and then, from the Green's representation for v_j in Ω^+ we conclude that $v_1 = v_2$ in Ω^+. This completes the proof. □

Having proved the above equivalence, we now focus our attention on the nonlocal boundary problem (2.8). Indeed, the derivation of a convenient weak formulation for (2.8) is the main purpose of the next section.

3 The weak formulation

In order to provide a weak formulation of (2.8), we need first to introduce a few notations and collect some results.

3.1 Preliminaries

In what follows, for m integer, r real, let $H^m(\Omega^-)$ and $H^r(\Gamma)$ denote the usual Sobolev spaces equipped with norms $\|\cdot\|_{H^m(\Omega^-)}$ and $\|\cdot\|_{H^r(\Gamma)}$ respectively (see e.g.

Kufner et al. (1977)). Furthermore, for $m > 0$, let $|\cdot|_{H^m(\Omega^-)}$ denote the seminorm,

$$|v|_{H^m(\Omega^-)} := \left\{ \sum_{|\alpha|=m} \int_{\Omega^-} |D^\alpha v|^2 \, dx \right\}^{1/2} ,$$

and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H^r(\Gamma)$ and $H^{-r}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product,

$$\langle \xi, \delta \rangle := \int_\Gamma \xi(s) \delta(s) \, ds \quad \forall (\xi, \delta) \in H^r(\Gamma) \times H^{-r}(\Gamma) .$$

We now introduce the subspace $H^1_{\Gamma_0}(\Omega^-)$ of $H^1(\Omega^-)$ defined by

$$H^1_{\Gamma_0}(\Omega^-) := \{ v \in H^1(\Omega^-) : v|_{\Gamma_0} = 0 \} . \tag{3.1}$$

Then, it is easy to prove that $\|\cdot\|_{H^1(\Omega^-)}$ and $|\cdot|_{H^1(\Omega^-)}$ are equivalent on $H^1_{\Gamma_0}(\Omega^-)$, i.e., there exists a positive constant C such that

$$\| v \|^2_{H^1(\Omega^-)} \leqq C |v|^2_{H^1(\Omega^-)} \quad \forall v \in H^1_{\Gamma_0}(\Omega^-) . \tag{3.2}$$

Let us now consider the boundary integral operators of the layer potentials acting on the Sobolev spaces $H^r(\Gamma)$. The following results are well known (see e.g. Hsiao 1989; Hsiao and Wendland 1977, 1981).

Lemma 3.1. *For C^∞ boundary Γ, the operators defined by* (2.2)–(2.5)

$$\begin{aligned} &\mathbf{V}: H^{r-1/2}(\Gamma) \to H^{r+1/2}(\Gamma); \mathbf{K}: H^{r+1/2}(\Gamma) \to H^{r+3/2}(\Gamma) \\ &\mathbf{K}': H^{r-1/2}(\Gamma) \to H^{r+1/2}(\Gamma); \mathbf{W}: H^{r+1/2}(\Gamma) \to H^{r-1/2}(\Gamma) \end{aligned} \tag{3.3}$$

are continuous for any $r \in \mathbb{R}$. In addition, if diameter of Γ is less than 1, *then $\mathbf{V}$ is a bijective mapping from $H^{r-1/2}(\Gamma)$ onto $H^{r+1/2}(\Gamma)$ for all $r \in \mathbb{R}$, and there exists a positive constant C such that*

$$\langle \mathbf{V}\xi, \xi \rangle \geqq C \| \xi \|^2_{H^{-1/2}(\Gamma)} \quad \forall \xi \in H^{-1/2}(\Gamma) . \tag{3.4}$$

The first part of Lemma 3.1 clearly indicates that $\mathbf{V}$, $\mathbf{K}$, and $\mathbf{K}'$ are pseudodifferential operators of order -1, whereas $\mathbf{W}$ is a pseudodifferential operator of order $+1$ (see Kohn and Nirenberg 1965). We comment that for the purposes of the weak formulation of (2.8), we shall be particularly interested in $r = 0$. In this case, as shown in Hsiao and Wendland (1977), it suffices to require Γ to be of class C^2 in order to obtain the above continuity properties. Now, we also observe that the assumption on the diameter of Γ insures that $\mathbf{V}$ has a positive kernel, from which one proves that (3.4) holds (Hsiao and Wendland 1977, Corollary 1). However, this restriction on the size of Γ can be removed if we consider a convenient subspace of $H^{-1/2}(\Gamma)$. In fact, let D be twice the diameter of Γ, and let us rewrite the operator $\mathbf{V}$ as follows,

$$(\mathbf{V}\xi)(x) = \frac{1}{2\pi} \int_\Gamma \log \left\{ \frac{D}{|x-y|} \right\} \xi(y) \, ds_y - \frac{1}{2\pi} \log D \langle 1, \xi \rangle$$

for all $\xi \in H^{-1/2}(\Gamma)$. Then, if $H_0^{-1/2}(\Gamma)$ denotes the subspace of $H^{-1/2}(\Gamma)$ defined by

$$H_0^{-1/2}(\Gamma) := \{ \xi \in H^{-1/2}(\Gamma) : \langle 1, \xi \rangle = 0 \} , \tag{3.5}$$

we obtain clearly that

$$(3.6)\qquad (\mathbf{V}\xi)(x) = \frac{1}{2\pi}\int_\Gamma \log\left\{\frac{D}{|x-y|}\right\}\xi(y)\,ds_y \quad \forall \xi \in H_0^{-1/2}(\Gamma).$$

This shows that $\mathbf{V}$ has a strictly positive kernel on $H_0^{-1/2}(\Gamma)$, and consequently, there exists $C > 0$ such that

$$(3.7)\qquad \langle \mathbf{V}\xi, \xi\rangle \geqq C\|\xi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \xi \in H_0^{-1/2}(\Gamma),$$

independently of the diameter of Γ. It is worth remarking that the subspaces $H^1_{\Gamma_0}(\Omega^-)$ and $H_0^{-1/2}(\Gamma)$ introduced here (cf. (3.1), (3.5)) will become the correct spaces for the weak formulation of (2.8).

On the other hand, if Γ is a Lipschitz continuous boundary, then for all $r \in [-1/2, 1/2]$ the mapping properties of $\mathbf{V}$ and $\mathbf{W}$ remain the same as in (3.3), but $\mathbf{K}$ and $\mathbf{K}'$ become now pseudodifferential operators of order 0, only. More precisely, we have the following result (see Costabel 1988, Theorems 1, 2).

Lemma 3.2. *For $C^{0,1}$ boundary Γ, the operators defined by* (2.2)–(2.5)

$$\mathbf{V}\colon H^{r-1/2}(\Gamma) \to H^{r+1/2}(\Gamma);\ \mathbf{K}\colon H^{r+1/2}(\Gamma) \to H^{r+1/2}(\Gamma)$$

$$\mathbf{K}'\colon H^{r-1/2}(\Gamma) \to H^{r-1/2}(\Gamma);\ \mathbf{W}\colon H^{r+1/2}(\Gamma) \to H^{r-1/2}(\Gamma)$$

are continuous for all $r \in [-1/2, 1/2]$. Furthermore, there exists $C > 0$ such that

$$(3.8)\qquad \langle \mathbf{V}\xi, \xi\rangle \geqq C\|\xi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \xi \in H_0^{-1/2}(\Gamma).$$

Now, it is not difficult to see that for $\xi \in H^{1/2}(\Gamma)$,

$$(\mathbf{W}\xi)(x) := -\frac{\partial}{\partial\nu(x)}\int_\Gamma\left\{\frac{\partial}{\partial\nu(y)}E(x,y)\right\}\xi(y)\,ds_y$$

$$= -\frac{d}{ds_x}\int_\Gamma E(x,y)\frac{d\xi(y)}{ds_y}\,ds_y.$$

Hence, integrating by parts on Γ, we get

$$(3.9)\qquad \langle \delta, \mathbf{W}\xi\rangle = \left\langle \mathbf{V}\left(\frac{d\xi}{ds}\right), \left(\frac{d\delta}{ds}\right)\right\rangle \quad \forall \xi, \delta \in H^{1/2}(\Gamma),$$

where $\frac{d\xi}{ds}$ stands for the derivative of ξ with respect to the arc length s. We remark that the above expression is well defined because the map $H^{1/2}(\Gamma) \ni \xi \to \frac{d\xi}{ds} \in H^{-1/2}(\Gamma)$ is continuous (see Nečas 1967), and from Lemma 3.2 with $r = 0$, $\mathbf{V}: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is also continuous. In particular, for v, $z \in H^1(\Omega^-)$ we have by the trace theorem v^-, $z^- \in H^{1/2}(\Gamma)$, and hence,

$$(3.10)\qquad \langle z^-, \mathbf{W}v^-\rangle = \left\langle \mathbf{V}\left(\frac{dv^-}{ds}\right), \left(\frac{dz^-}{ds}\right)\right\rangle.$$

Moreover, since $\frac{dv^-}{ds} \in H_0^{-1/2}(\Gamma)$, a direct application of (3.8) and (3.10) gives

$$\langle v^-, \mathbf{W}v^- \rangle = \left\langle \mathbf{V}\left(\frac{dv^-}{ds}\right), \left(\frac{dv^-}{ds}\right) \right\rangle \geqq C \left\| \frac{dv^-}{ds} \right\|^2_{H^{-1/2}(\Gamma)} \geqq 0 \tag{3.11}$$

for all $v \in H^1(\Omega^-)$.

The results given in this subsection will be needed later in the paper.

3.2 The weak formulation and the operator equation

We first reformulate the nonlocal boundary problem (2.8) in a weak sense. To this end, we assume that $\partial\Omega^- \in C^{0,1}$, $f \in L^2(\Omega^-)$ and $g_0 \in H^{1/2}(\Gamma_0)$. Then the problem (2.8) reads: Find $(u, \tilde{\sigma}, \lambda) \in H^1(\Omega^-) \times H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$- \sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(\cdot, \nabla u(\cdot)) = f \quad \text{in } \Omega^- \tag{3.12}$$

$$u = g_0 \quad \text{on } \Gamma_0$$

$$\sum_{i=1}^{2} a_i(\cdot, (\nabla u)^-(\cdot))\nu_i = \tilde{\sigma} \qquad \text{and} \qquad \tilde{\sigma} = -\mathbf{W}u^- + (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\tilde{\sigma} \quad \text{on } \Gamma$$

$$(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u^- + \mathbf{V}\tilde{\sigma} + \lambda = 0 \quad \text{on } \Gamma$$

$$\langle 1, \tilde{\sigma} \rangle = 2b ,$$

where the nonlinear partial differential equation in (3.12) must be understood in the distributional sense. Note that in view of the trace theorem we now have $u^- \in H^{1/2}(\Gamma)$, and $u|_{\Gamma_0} \in H^{1/2}(\Gamma_0)$. Also, the conormal $\sum_{i=1}^2 a_i(\cdot, (\nabla u)^-(\cdot))\nu_i$ is interpreted as a distribution in $H^{-1/2}(\Gamma)$ via the divergence theorem. In other words, for any $v \in H^1(\Omega^-)$ with $\sum_{i=1}^2 \frac{\partial}{\partial x_i} a_i(\cdot, \nabla v(\cdot)) \in L^2(\Omega^-)$, the formal expression $\sum_{i=1}^2 a_i(\cdot, (\nabla v)^-(\cdot))\nu_i$ denotes the distribution in $H^{-1/2}(\Gamma)$ which is defined as follows,

$$\left\langle z^-, \sum_{i=1}^{2} a_i(\cdot, (\nabla v)^-(\cdot))\nu_i \right\rangle := \sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla v(x)) \frac{\partial z}{\partial x_i}\, dx + \int_{\Omega^-} \left\{ \sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(\cdot, \nabla v(\cdot)) \right\} z\, dx \tag{3.13}$$

for all $z \in H^1_{\Gamma_0}(\Omega^-)$. Here, we have assumed explicitly that the nonlinear coefficients a_i are such that $a_i(\cdot, \nabla v(\cdot)) \in L^2(\Omega^-)$ for all $v \in H^1(\Omega^-)$. This assumption will make sense later on when we specify the Carathéodory and growth conditions for a_i.

Now for the weak formulation, we multiply the partial differential equation in (3.12) by any function $z \in H^1_{\Gamma_0}(\Omega^-)$ and apply the divergence theorem (cf. (3.13)) and the third equation in (3.12) to yield

$$\sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla u(x)) \frac{\partial z}{\partial x_i}\, dx + \langle z^-, \mathbf{W}u^- - (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\tilde{\sigma} \rangle = \int_{\Omega^-} f z\, dx . \tag{3.14}$$

Similarly, we multiply the fourth equation in (3.12) by any test function $\delta \in H^{-1/2}(\Gamma)$ and integrate over Γ to obtain

$$\langle \mathbf{V}\tilde{\sigma}, \delta \rangle + \langle (\tfrac{1}{2}\mathbf{I} - \mathbf{K})u^-, \delta \rangle + \lambda \langle 1, \delta \rangle = 0 . \tag{3.15}$$

Equations (3.14) and (3.15) then lead us to the weak formulation: *Given* $(f, g_0, b) \in L^2(\Omega^-) \times H^{1/2}(\Gamma_0) \times \mathbb{R}$, *find* $(u, \tilde{\sigma}, \lambda) \in H^1(\Omega^-) \times H^{-1/2}(\Gamma) \times \mathbb{R}$ *such that* $(u - g) \in H^1_{\Gamma_0}(\Omega^-)$, $\langle 1, \tilde{\sigma} \rangle = 2b$, *and* $(u, \tilde{\sigma}, \lambda)$ *satisfies*

$$\sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla u(x)) \frac{\partial z}{\partial x_i}\, dx + \mathbf{B}((u, \tilde{\sigma}), (z, \delta)) + \lambda \langle 1, \delta \rangle = \int_{\Omega^-} f z\, dx \tag{3.16}$$

for all $(z, \delta) \in H^1_{\Gamma_0}(\Omega^-) \times H^{-1/2}(\Gamma)$, *where* $g \in H^1(\Omega^-)$ *is an extension of* g_0 *with* $g|_{\Gamma_0} = g_0$, *and* $\mathbf{B}$ *is the bilinear form defined by*

$$\mathbf{B}((v, \xi), (z, \delta)) := \langle z^-, \mathbf{W}v^- \rangle - \langle z^-, (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\xi \rangle + \langle \mathbf{V}\xi, \delta \rangle + \langle (\tfrac{1}{2}\mathbf{I} - \mathbf{K})v^-, \delta \rangle \tag{3.17}$$

for all $(v, \xi), (z, \delta) \in H^1(\Omega^-) \times H^{-1/2}(\Gamma)$.

We remark that (3.16) and (3.12) are equivalent. Moreover, if $(u, \tilde{\sigma}, \lambda)$ is a sufficiently smooth solution of (3.16), then it is also a classical solution of the nonlocal boundary problem (2.8). Obviously, the regularity properties of a solution of (3.16) will depend on the smoothness of the coefficients a_i, the data g_0, f, and the boundary $\partial\Omega^-$. The question of regularity will not be addressed here.

Now, in order to deduce a more suitable formulation, we set $w = u - g$ in Ω^-, and $\sigma = \tilde{\sigma} - \frac{2b}{|\Gamma|}$ on Γ. It is easily seen that $(w, \sigma) \in H^1_{\Gamma_0}(\Omega^-) \times H_0^{-1/2}(\Gamma)$. Consequently, we reformulate (3.16) as follows: *Given* $(f, g, b) \in L^2(\Omega^-) \times H^1(\Omega^-) \times \mathbb{R}$, *find* $(w, \sigma, \lambda) \in H^1_{\Gamma_0}(\Omega^-) \times H_0^{-1/2}(\Gamma) \times \mathbb{R}$ *such that*

$$\sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla(w + g)(x)) \frac{\partial z}{\partial x_i}\, dx + \mathbf{B}((w, \sigma), (z, \delta)) + \lambda \langle 1, \delta \rangle = \mathcal{F}(z, \delta) \tag{3.18}$$

for all $(z, \delta) \in H^1_{\Gamma_0}(\Omega^-) \times H^{-1/2}(\Gamma)$, *where* $\mathcal{F}$ *is the linear functional on* $H^1(\Omega^-) \times H^{-1/2}(\Gamma)$ *defined by*

$$\mathcal{F}(z, \delta) := \int_{\Omega^-} f z\, dx - \mathbf{B}\left(\left(g, \frac{2b}{|\Gamma|}\right), (z, \delta)\right) \quad \forall (z, \delta) \in H^1(\Omega^-) \times H^{-1/2}(\Gamma) . \tag{3.19}$$

Furthermore, we can still reduce (3.16) to a simpler problem by considering the following formulation which is a simplification of (3.18): *Given* $(f, g, b) \in L^2(\Omega^-) \times H^1(\Omega^-) \times \mathbb{R}$, *find* $(w, \sigma) \in H^1_{\Gamma_0}(\Omega^-) \times H_0^{-1/2}(\Gamma)$ *such that*

$$\sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla(w + g)(x)) \frac{\partial z}{\partial x_i}\, dx + \mathbf{B}((w, \sigma), (z, \delta)) = \mathbf{F}(z, \delta) \tag{3.20}$$

for all $(z,\delta)\in H^1_{\Gamma_0}(\Omega^-)\times H_0^{-1/2}(\Gamma)$, *where* $\mathbf{F}$ *is the restriction of the functional* $\mathscr{F}$ *on the subspace* $H^1_{\Gamma_0}(\Omega^-)\times H_0^{-1/2}(\Gamma)$, *i.e.*,

$$\text{(3.21)}\qquad \mathbf{F}(z,\delta):=\int_{\Omega^-} fz\,dx-\mathbf{B}\left(\left(g,\frac{2b}{|\Gamma|}\right),(z,\delta)\right)\quad \forall (z,\delta)\in H^1_{\Gamma_0}(\Omega^-)\times H_0^{-1/2}(\Gamma)\,.$$

In fact, we have the following result.

Theorem 3.3. *The weak formulations* (3.18) *and* (3.20) *are equivalent in the following sense:*

i) If (w,σ,λ) *is a solution of* (3.18) *then* (w,σ) *solves* (3.20). *Conversely, if* (w,σ) *is a solution of* (3.20), *then* (w,σ,λ) *is a solution of* (3.18), *where the constant* λ *is given by*

$$\text{(3.22)}\qquad \lambda=-\frac{1}{|\Gamma|}\mathbf{B}\left(\left(w+g,\sigma+\frac{2b}{|\Gamma|}\right),(0,1)\right).$$

ii) The problem (3.18) *has a unique solution if and only if* (3.20) *has also a unique solution.*

Proof. The fact that every solution (w,σ,λ) of (3.18) provides a solution (w,σ) of (3.20) is clear. Now, let $(w,\sigma)\in H^1_{\Gamma_0}(\Omega^-)\times H_0^{-1/2}(\Gamma)$ be a solution of (3.20). Given $\delta\in H^{-1/2}(\Gamma)$ we define

$$\delta_0=\delta-\frac{\langle 1,\delta\rangle}{|\Gamma|}\,.$$

Since $\delta_0\in H_0^{-1/2}(\Gamma)$ we can write from (3.20)

$$\sum_{i=1}^{2}\int_{\Omega^-} a_i(x,\nabla(w+g)(x))\frac{\partial z}{\partial x_i}\,dx+\mathbf{B}((w,\sigma),(z,\delta_0))=\mathbf{F}(z,\delta_0)$$

for all $z\in H^1_{\Gamma_0}(\Omega^-)$. But, it is clear that

$$\mathbf{B}((w,\sigma),(z,\delta_0))=\mathbf{B}((w,\sigma),(z,\delta))-\frac{1}{|\Gamma|}\mathbf{B}((w,\sigma),(0,1))\langle 1,\delta\rangle\,,$$

and also

$$\mathbf{F}(z,\delta_0)=\mathscr{F}(z,\delta)-\frac{1}{|\Gamma|}\mathscr{F}(0,1)\langle 1,\delta\rangle\,.$$

It follows from the above equations that

$$\sum_{i=1}^{2}\int_{\Omega^-} a_i(x,\nabla(w+g)(x))\frac{\partial z}{\partial x_i}\,dx+\mathbf{B}((w,\sigma),(z,\delta))+\lambda\langle 1,\delta\rangle=\mathscr{F}(z,\delta)$$

for all $(z,\delta)\in H^1_{\Gamma_0}(\Omega^-)\times H^{-1/2}(\Gamma)$, where

$$\lambda=\frac{1}{|\Gamma|}\{\mathscr{F}(0,1)-\mathbf{B}((w,\sigma),(0,1))\}\,.$$

This proves that (w,σ,λ) is a solution of (3.18). Moreover, from (3.19) we have $\mathscr{F}(0,1)=-\mathbf{B}\left(\left(g,\frac{2b}{|\Gamma|}\right),(0,1)\right)$, which together with the above equality yields (3.22).

We now prove *ii*). It is clear from *i*) that the uniqueness of (3.20) follows from that of (3.18). Conversely, suppose that (3.20) has at most one solution, and let $(w_j, \sigma_j, \lambda_j)$, $j = 1, 2$ be two solutions of (3.18). Since for $j = 1, 2$, (w_j, σ_j) is a solution of (3.20), the uniqueness of (3.20) implies $(w_1, \sigma_1) = (w_2, \sigma_2)$. Therefore, from (3.18) we deduce that $(\lambda_1 - \lambda_2)\langle 1, \delta\rangle = 0 \ \forall \delta \in H^{-1/2}(\Gamma)$, which gives $\lambda_1 = \lambda_2$. This completes the proof. □

From the above theorem we conclude that in order to study the existence and uniqueness of solution to (3.16) (or (3.18)), it suffices to consider the equivalent weak formulation given by (3.20) which involves only the two unknowns w and σ. Consequently, from now on we shall direct our efforts to study the solvability of this problem. As indicated in the beginning of this paper, the approach we adopt here consists in the reduction of (3.20) to an equivalent operator equation. For this purpose, we need now to specify some conditions on the nonlinear coefficients a_i. So, let $a_i : \Omega^- \times \mathbb{R}^2 \to \mathbb{R}$ be such that the following conditions are fulfilled:

(H.1) Carathéodory conditions. The function $a_i(\cdot, \alpha)$ is measurable in Ω^- for all $\alpha \in \mathbb{R}^2$ and $a_i(x, \cdot)$ is continuous in $\mathbb{R}^2$ for almost all $x \in \Omega^-$.

(H.2) Growth condition. There exist functions $\phi_i \in L^2(\Omega^-)$, $i = 1, 2$ such that

$$|a_i(x, \alpha)| \leqq C\{1 + |\alpha|\} + |\phi_i(x)|$$

for all $\alpha \in \mathbb{R}^2$ and for almost all $x \in \Omega^-$.

Here and in the sequel, C is a generic constant.

As a consequence of (H.1) and (H.2) one can prove the following important result (for details of the proof see Gatica 1989, Chapter 2; and also Vainberg 1964).

Theorem 3.4. *Suppose that* (H.1) *and* (H.2) *are satisfied. Let A_i be the Nemytsky operator defined by*

$$(A_i v)(x) := a_i(x, \nabla(v + g)(x)) \quad \forall v \in H^1(\Omega^-) \forall x \in \Omega^- . \tag{3.23}$$

Then A_i is a continuous map from $H^1(\Omega^-)$ into $L^2(\Omega^-)$ and the inequality

$$\| A_i v \|^2_{L^2(\Omega^-)} \leqq C\{Area(\Omega^-) + |v + g|^2_{H^1(\Omega^-)} + \| \phi_i \|^2_{L^2(\Omega^-)}\} \tag{3.24}$$

holds for all $v \in H^1(\Omega^-)$ and for all $i = 1, 2$.

It is important to remark that in view of the previous theorem, all of the integrals over Ω^- on the left hand side of the formulations (3.16), (3.18) and (3.20) make sense. Hence, we are now in a position to reduce (3.20) to an equivalent operator equation form. We define the Hilbert space

$$\mathscr{H} := H^1(\Omega^-) \times H^{-1/2}(\Gamma) ,$$

and the subspace

$$\mathbf{H} := H^1_{\Gamma_0}(\Omega^-) \times H_0^{-1/2}(\Gamma) ,$$

with the product norms

$$\|(v, \xi)\|_{\mathscr{H}} := \{\|v\|^2_{H^1(\Omega^-)} + \|\xi\|^2_{H^{-1/2}(\Gamma)}\}^{1/2} \quad \forall (v, \xi) \in \mathscr{H} ,$$

and

$$\|(v, \xi)\|_{\mathbf{H}} := \|(v, \xi)\|_{\mathscr{H}} \quad \forall (v, \xi) \in \mathbf{H} .$$

Further, let $\mathscr{H}^*$ and $\mathbf{H}^*$ be the duals of $\mathscr{H}$ and $\mathbf{H}$, respectively, with the norms defined by

$$\|l\|_{\mathscr{H}^*} := \sup_{\substack{(z,\delta)\in\mathscr{H}\\(z,\delta)\neq 0}} \frac{|[l,(z,\delta)]|}{\|(z,\delta)\|_{\mathscr{H}}} \quad \forall l \in \mathscr{H}^* , \tag{3.25}$$

and

$$\|l\|_{\mathbf{H}^*} := \sup_{\substack{(z,\delta)\in\mathbf{H}\\(z,\delta)\neq 0}} \frac{|[l,(z,\delta)]|}{\|(z,\delta)\|_{\mathbf{H}}} \quad \forall l \in \mathbf{H}^* , \tag{3.26}$$

where $[\cdot,\cdot]$ denotes the duality pairing on both $\mathscr{H}^* \times \mathscr{H}$ and $\mathbf{H}^* \times \mathbf{H}$.

We remark that the bilinear form $\mathbf{B}$ (see (3.17)) is bounded on $\mathscr{H} \times \mathscr{H}$ as a consequence of Lemma 3.1 (or Lemma 3.2), and the trace theorem. That is, there exists $M > 0$ such that

$$|\mathbf{B}((v,\xi),(z,\delta))| \leqq M\|(v,\xi)\|_{\mathscr{H}}\|(z,\delta)\|_{\mathscr{H}} \quad \forall (v,\xi),(z,\delta) \in \mathscr{H} . \tag{3.27}$$

This clearly implies that the linear functionals $\mathscr{F}$ (see (3.19)) and $\mathbf{F}$ (see (3.21)) are also bounded, i.e., $\mathscr{F} \in \mathscr{H}^*$ and $\mathbf{F} \in \mathbf{H}^*$.

Now, we observe that for fixed (w, σ), the expression on the left hand side of the equality (3.20) is a linear functional in (z, δ). Then, in virtue of Theorem 3.4 and the boundedness of $\mathbf{B}$, we can introduce the nonlinear operators $\mathscr{T} : \mathscr{H} \to \mathscr{H}^*$ and $\mathbf{T} : \mathbf{H} \to \mathbf{H}^*$, where $\mathscr{T}$ is defined by

$$[\mathscr{T}(v,\xi),(z,\delta)] := \sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla(v+g)(x)) \frac{\partial z}{\partial x_i}\, dx + \mathbf{B}((v,\xi),(z,\delta)) \tag{3.28}$$

for all (v,ξ), $(z,\delta) \in \mathscr{H}$, and $\mathbf{T}$ is the corresponding restriction of $\mathscr{T}$ on $\mathbf{H}$, i.e.,

$$[\mathbf{T}(v,\xi),(z,\delta)] := [\mathscr{T}(v,\xi),(z,\delta)] \tag{3.29}$$

for all (v,ξ), $(z,\delta) \in \mathbf{H}$. Note that $\mathbf{T}$ can be defined, equivalently, as $\mathbf{T} := \mathbf{i}^* \cdot \mathscr{T} \cdot \mathbf{i}$, where $\mathbf{i} : \mathbf{H} \subsetneq \mathscr{H}$ and $\mathbf{i}^* : \mathscr{H}^* \subsetneq \mathbf{H}^*$ are the usual continuous injections.

Consequently, as the main point of this section, the weak formulation (3.20) can be written in the form of an operator equation: *Find* $(w, \sigma) \in \mathbf{H}$ *such that*

$$\mathbf{T}(w,\sigma) = \mathbf{F} , \tag{3.30}$$

or, equivalently, such that

$$[\mathbf{T}(w,\sigma),(z,\delta)] = [\mathbf{F},(z,\delta)] \quad \forall (z,\delta) \in \mathbf{H} . \tag{3.31}$$

The operator equation form (3.30) then allows us to study the solvability of the weak formulation (3.20) by the theory of monotone operators. This analysis will be carried out in Sect. 5. For that purpose, some known results on monotone operators will be provided in the next section.

We remark that at this point it may seem that the introduction of the space $\mathscr{H}$, the functional $\mathscr{F}$, and the operator $\mathscr{T}$, is unnecessary. However, the usefulness of this setting will become transparent in Sects. 6 and 7 when we study the Galerkin approximations of the operator equation (3.30).

We end this section with the following corollary of Theorem 3.4.

Theorem 3.5. *Under the assumptions* (H.1) *and* (H.2), *the operators* $A_i: H^1(\Omega^-) \to L^2(\Omega^-)$ *and* $\mathbf{T}: \mathbf{H} \to \mathbf{H}^*$ *have the following properties:*

i) There exists $M_0 > 0$ *such that*

$$\| A_i v \|_{L^2(\Omega^-)} \leqq C \| (v, \xi) \|_{\mathbf{H}} \tag{3.32}$$

for all $(v, \xi) \in \mathbf{H}$ *with* $\| (v, \xi) \|_{\mathbf{H}} \geqq M_0$.

ii) $\mathbf{T}$ *is bounded and continuous.*

Proof. It follows easily from (3.24) that

$$\| A_i v \|_{L^2(\Omega^-)} \leqq C \{ M_0 + \| (v, \xi) \|_{\mathbf{H}} \} \quad \forall (v, \xi) \in \mathbf{H} , \tag{3.33}$$

where the constant M_0 is given by

$$M_0 := \{ Area(\Omega^-) + |g|^2_{H^1(\Omega^-)} + \sum_{i=1}^{2} \| \phi_i \|^2_{L^2(\Omega^-)} \}^{1/2} .$$

Hence, (3.32) is a simple consequence of (3.33).

Now, by using (3.27), (3.33), and Schwarz's inequality, we obtain from the definition of $\mathbf{T}$ in (3.29),

$$\begin{aligned} |[\mathbf{T}(v, \xi), (z, \delta)]| &\leqq \sum_{i=1}^{2} \| A_i v \|_{L^2(\Omega^-)} \left\| \frac{\partial z}{\partial x_i} \right\|_{L^2(\Omega^-)} + M \| (v, \xi) \|_{\mathbf{H}} \| (z, \delta) \|_{\mathbf{H}} \\ &\leqq C \{ M_0 + \| (v, \xi) \|_{\mathbf{H}} \} |z|_{H^1(\Omega^-)} + M \| (v, \xi) \|_{\mathbf{H}} \| (z, \delta) \|_{\mathbf{H}} \\ &\leqq C \{ M_0 + \| (v, \xi) \|_{\mathbf{H}} \} \| (z, \delta) \|_{\mathbf{H}} \quad \forall (v, \xi), (z, \delta) \in \mathbf{H} . \end{aligned}$$

The above inequality and (3.26) imply,

$$\| \mathbf{T}(v, \xi) \|_{\mathbf{H}^*} \leqq C \{ M_0 + \| (v, \xi) \|_{\mathbf{H}} \} \quad \forall (v, \xi) \in \mathbf{H} ,$$

which clearly shows that $\mathbf{T}$ is bounded, i.e., it maps bounded sets of $\mathbf{H}$ into bounded sets of $\mathbf{H}^*$.

To prove the continuity of $\mathbf{T}$, we let $\{ (v^{(k)}, \xi^{(k)}) \}$ be a sequence in $\mathbf{H}$ which converges to $(v, \xi) \in \mathbf{H}$. Then, for all $(z, \delta) \in \mathbf{H}$ we have from (3.29),

$$\begin{aligned} [\mathbf{T}(v^{(k)}, \xi^{(k)}) - \mathbf{T}(v, \xi), (z, \delta)] &= \sum_{i=1}^{2} \int_{\Omega^-} \{ (A_i v^{(k)})(x) - (A_i v)(x) \} \frac{\partial z}{\partial x_i} \, dx \\ &\quad + \mathbf{B}((v^{(k)}, \xi^{(k)}) - (v, \xi), (z, \delta)) . \end{aligned}$$

Thus, Schwarz's inequality, (3.27), and (3.26), lead to the estimate

$$\begin{aligned} \| \mathbf{T}(v^{(k)}, \xi^{(k)}) - \mathbf{T}(v, \xi) \|_{\mathbf{H}^*} \leqq C \Bigg\{ &\sum_{i=1}^{2} \| A_i v^{(k)} - A_i v \|_{L^2(\Omega^-)} \\ &+ \| (v^{(k)}, \xi^{(k)}) - (v, \xi) \|_{\mathbf{H}} \Bigg\} , \end{aligned}$$

which, by the continuity of the Nemytsky operators, shows that

$$\| \mathbf{T}(v^{(k)}, \xi^{(k)}) - \mathbf{T}(v, \xi) \|_{\mathbf{H}^*} \to 0 .$$

This proves the continuity of $\mathbf{T}$. □

The above theorem will be needed in Sect. 5.

4 Some known results on monotone operators

The results to be presented here have been taken mainly from the books Nečas (1986) and Oden (1986). Throughout this section we always assume that **T** is a mapping from a reflexive and separable Banach space **X** into its dual **X***. Also, $[\cdot,\cdot]$ denotes the duality pairing between **X*** and **X**, and $\|\cdot\|$ is the norm on **X**. Hereafter, the symbols "$\rightharpoonup$" and "$\to$" mean convergence in the weak and strong sense, respectively. To begin with, we introduce the following definitions:

Definition 1. **T** is said to satisfy the property (M) if whenever

$$u_n \rightharpoonup u, \qquad \mathbf{T}u_n \rightharpoonup f \qquad \text{and} \qquad \limsup_{n\to+\infty}\,[\mathbf{T}u_n, u_n] \leqq [f, u]\,,$$

then $\mathbf{T}u = f$.

Definition 2. **T** is called *coercive* on **X** if

$$\lim_{\|u\|\to+\infty} \frac{[\mathbf{T}u, u]}{\|u\|} = +\infty\,.$$

Definition 3. **T** is called *demicontinuous* if $u_n \to u \Rightarrow \mathbf{T}u_n \rightharpoonup \mathbf{T}u$.

We now state the first result regarding the existence of solutions to the operator equation $\mathbf{T}u = f$, with $f \in X^*$.

Theorem 4.1. *Suppose that* **T** *is coercive, demicontinuous, bounded, and satisfies the property* (M). *Then* $\mathbf{T}(\mathbf{X}) = \mathbf{X}^*$, *and* $\mathbf{T}^{-1}$ *is a multivalued bounded mapping.*

Proof. See Nečas (1986, Theorem 3.3.6). □

The above theorem can be simplified by using the following result.

Theorem 4.2. *Suppose that* **T** *is bounded and satisfies the property* (M). *Then* **T** *is demicontinuous, and is therefore continuous when restricted to any finite dimensional subspace of* **X**.

Proof. See Oden (1986, Theorem 28.1). □

The fact that restrictions of demicontinuous operators to finite dimensional spaces are continuous follows from the well known property that weakly convergent sequences are also strongly convergent in this case. Then, as a consequence of Theorems 4.1 and 4.2, we obtain

Theorem 4.3. *Suppose that* **T** *is coercive, bounded, and satisfies the property* (M). *Then* $\mathbf{T}(\mathbf{X}) = \mathbf{X}^*$, *and* $\mathbf{T}^{-1}$ *is a multivalued bounded mapping.*

We remark that a direct proof of this result, which makes no use of Theorem 4.2, is given in Oden (1986, Theorem 28.2). Also, we comment that the notion of operators of type (M), which was introduced in Brezis (1968), is sometimes too general to be of great value in applications. For our purposes in this paper, it will suffice to consider instead the following class of monotone operators.

Definition 4. The operator $\mathbf{T}$ is called

i) monotone if

$$[\mathbf{T}u - \mathbf{T}v, u - v] \geqq 0 \quad \forall u, v \in \mathbf{X} ,$$

ii) strictly monotone if

$$[\mathbf{T}u - \mathbf{T}v, u - v] > 0 \quad \forall u, v \in \mathbf{X}, u \neq v ,$$

iii) strongly monotone if there exists $C > 0$ such that

$$[\mathbf{T}u - \mathbf{T}v, u - v] \geqq C\|u - v\|^2 \quad \forall u, v \in \mathbf{X} .$$

Definition 5. The operator $\mathbf{T}$ is called *hemicontinuous* if the mapping $\mathbb{R} \ni t \to [\mathbf{T}(u + tv), w] \in \mathbb{R}$ is continuous for all $u, v, w \in \mathbf{X}$.

The relationship between monotone and type (M) operators is made clear in the following statement.

Theorem 4.4. *If* $\mathbf{T}$ *is monotone and hemicontinuous, then* $\mathbf{T}$ *satisfies the property* (M).

Proof. See Nečas (1986, Theorem 3.3.14). □

Of particular interest to us here are the following theorems for the finite dimensional problems.

Theorem 4.5. *Let* $\mathbf{X}_h$ *be a finite dimensional subspace of* $\mathbf{X}$. *Suppose that* $\mathbf{T}$ *is coercive and demicontinuous. Then, given* $f \in \mathbf{X}^*$, *there exists at least one* $u_h \in \mathbf{X}_h$ *such that*

$$[\mathbf{T}u_h, v_h] = [f, v_h] \quad \forall v_h \in \mathbf{X}_h . \tag{4.1}$$

Proof. The main part of the proof consists in the reduction of (4.1) to an equivalent fixed point equation in $\mathbb{R}^m$, where m is the dimension of $\mathbf{X}_h$. Hence, a direct application of Brouwer's fixed point theorem yields the desired conclusion. For details, see Oden (1986, Theorem 27.2). □

Theorem 4.6. *Let* $\mathbf{X}_h$ *be finite dimensional, and let* $\mathbf{T}_h : \mathbf{X}_h \to \mathbf{X}_h^*$ *be monotone. Then* $\mathbf{T}_h$ *is bounded.*

Proof. See Oden (1986, Theorem 27.3). □

Now, we are in a position to deduce an important existence-uniqueness result.

Theorem 4.7. *Suppose that* $\mathbf{T}$ *is monotone, hemicontinuous, bounded, and coercive. Then* $\mathbf{T}(\mathbf{X}) = \mathbf{X}^*$. *Moreover, if* $\mathbf{T}$ *is strictly monotone, then* $\mathbf{T}$ *is one-to-one.*

Proof (main ideas). It is clear that the surjectivity of the operator $\mathbf{T}$ is a consequence of Theorems 4.3 and 4.4. However, it is worth mentioning that a constructive proof which makes use of the Galerkin approximations (4.1) can also be employed. In fact, since $\mathbf{X}$ is separable, there is a family $\{\mathbf{X}_m\}_{m\in\mathbb{N}}$ of finite dimensional subspaces such that $\mathbf{X}_1 \subset \mathbf{X}_2 \subset \cdots \subset \mathbf{X}$, and $\bigcup\{\mathbf{X}_m : m\in\mathbb{N}\}$ is dense in $\mathbf{X}$. It can be proved (see e.g., Oden (1986, Lemma 27.1)) that if $\mathbf{T}$ is monotone, bounded, and hemicontinuous, then $\mathbf{T}$ is demicontinuous. Therefore, given $f\in\mathbf{X}^*$, a direct application of Theorem 4.5 allows us to construct a sequence of Galerkin approximations $\{u_m\}_{m\in\mathbb{N}}$ satisfying

$$u_m\in\mathbf{X}_m, \qquad [\mathbf{T}u_m, v_m] = [f, v_m] \quad \forall v_m\in\mathbf{X}_m\,.$$

Hence, the coerciveness of $\mathbf{T}$ implies that $\{u_m\}_{m\in\mathbb{N}}$ is bounded, and since $\mathbf{X}$ is reflexive, there exists a subsequence $\{w_m\}_{m\in\mathbb{N}} \subseteq \{u_m\}_{m\in\mathbb{N}}$ which converges weakly to an element $w\in\mathbf{X}$. The rest of the proof reduces to show that $\mathbf{T}w = f$. For further details, see Oden (1986, Theorem 27.1).

Finally, if $\mathbf{T}$ is strictly monotone and u, u' are two different solutions to $\mathbf{T}u = f$, then

$$0 < [\mathbf{T}u - \mathbf{T}u', u - u'] = [f - f, u - u'] = 0$$

which is a contradiction. So, necessarily $u = u'$. □

We end this brief collection of results on monotone operators with two simple corollaries of Theorem 4.7.

Theorem 4.8. *Suppose that* $\mathbf{T}$ *is monotone, continuous, bounded, and coercive. Then* $\mathbf{T}(\mathbf{X}) = \mathbf{X}^*$. *Moreover, if* $\mathbf{T}$ *is strictly monotone, then* $\mathbf{T}$ *is one-to-one.*

Proof. It follows easily from Theorem 4.7 and the obvious fact that continuity implies demicontinuity. □

Theorem 4.9. *Suppose that* $\mathbf{T}$ *is strongly monotone, continuous, and bounded. Then* $\mathbf{T}(\mathbf{X}) = \mathbf{X}^*$, *and* $\mathbf{T}$ *is one-to-one. In other words, for any* $f\in\mathbf{X}^*$ *there exists a unique* $u\in\mathbf{X}$ *such that* $\mathbf{T}u = f$.

Proof. Let $|||\cdot|||$ be the norm on $\mathbf{X}^*$. Then by the strong monotonicity of $\mathbf{T}$ we deduce

$$\frac{[\mathbf{T}u, u]}{\|u\|} \geqq C\|u\| + \frac{[\mathbf{T}0, u]}{\|u\|} \geqq C\|u\| - |||\mathbf{T}0||| \quad \forall u\in\mathbf{X}\,,$$

which clearly shows that $\mathbf{T}$ is coercive. (Note that $\mathbf{T}0$ is not necessarily equal to zero.) Hence, Theorem 4.8 yields the desired result. □

5 Existence and uniqueness

In this section we provide sufficient conditions for the solvability of the operator equation (3.30), which as shown in Sect. 3, is equivalent to the weak formulation (3.20). The approach used here is the same as in Gatica and Hsiao (1989c).

For purposes of clarity, we recall from (3.28)–(3.29) the definition of the nonlinear operator $\mathbf{T}: \mathbf{H} \to \mathbf{H}^*$,

$$\text{(5.1)} \qquad [\mathbf{T}(v, \xi), (z, \delta)] = \sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla(v + g)(x)) \frac{\partial z}{\partial x_i}\, dx + \mathbf{B}((v, \xi), (z, \delta))$$

for all $(v, \xi), (z, \delta) \in \mathbf{H} := H^1_{\Gamma_0}(\Omega^-) \times H_0^{-1/2}(\Gamma)$, where $\mathbf{B}$ is the bounded bilinear form (cf. (3.17)),

$$\mathbf{B}((v, \xi), (z, \delta)) := \langle z^-, \mathbf{W}v^- \rangle - \langle z^-, (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\xi \rangle + \langle \mathbf{V}\xi, \delta \rangle + \langle (\tfrac{1}{2}\mathbf{I} - \mathbf{K})v^-, \delta \rangle \tag{5.2}$$

for all $(v, \xi), (z, \delta) \in \mathscr{H} := H^1(\Omega^-) \times H^{-1/2}(\Gamma)$.

We are interested in the operator equation for the unknown (w, σ) (cf. (3.30)),

$$\mathbf{T}(w, \sigma) = \mathbf{F}\ , \tag{5.3}$$

where $\mathbf{F} \in \mathbf{H}^*$ is the bounded linear functional defined by (3.21).

In order to apply the results on monotone operators given in Sect. 4, we need to make further assumptions on the nonlinear coefficients a_i in (5.1). We consider the following conditions:

(H.3) Coerciveness condition. There exist a constant $C_1 > 0$ and a function $C_2 \in L^1(\Omega^-)$ such that

$$\sum_{i=1}^{2} a_i(x, \boldsymbol{\alpha})\alpha_i \geqq C_1 |\boldsymbol{\alpha}|^2 - C_2(x)$$

for all $\boldsymbol{\alpha} := (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and for almost all $x \in \Omega^-$.

(H.4) Monotonicity condition.

$$\sum_{i=1}^{2} (a_i(x, \boldsymbol{\alpha}) - a_i(x, \boldsymbol{\alpha}'))(\alpha_i - \alpha_i') \geqq 0$$

for all $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbb{R}^2$ and for almost all $x \in \Omega^-$.

(H.5) Strict monotonicity condition.

$$\sum_{i=1}^{2} (a_i(x, \boldsymbol{\alpha}) - a_i(x, \boldsymbol{\alpha}'))(\alpha_i - \alpha_i') > 0$$

for all $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbb{R}^2$, $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$, and for almost all $x \in \Omega^-$.

(H.6) Strong monotonicity condition. The nonlinear coefficients $a_i(x, \cdot)$ have continuous first order partial derivatives in $\mathbb{R}^2$ for almost all $x \in \Omega^-$. In addition, there exists $C > 0$ such that

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial \alpha_j} a_i(x, \boldsymbol{\alpha})\beta_i\beta_j \geqq C \sum_{i=1}^{2} \beta_i^2$$

for all $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} := (\beta_1, \beta_2) \in \mathbb{R}^2$ and for almost all $x \in \Omega^-$.

Clearly, these conditions are not mutually exclusive. Hence, our main results can be summarized in the following theorems depending on the assumptions on a_i.

Theorem 5.1. *Suppose that the coefficients a_i satisfy the assumptions* (H.1)–(H.4). *Let $\mathbf{T}: \mathbf{H} \to \mathbf{H}^*$ be the operator defined by* (5.1) *and let $\mathbf{F} \in \mathbf{H}^*$ be the bounded linear functional defined by* (3.21). *Then there exists a solution $(w, \sigma) \in \mathbf{H}$ of the equation* (5.3), *and the solution is unique, if* (H.5) *is satisfied.*

Theorem 5.2. *Suppose that the coefficients a_i satisfy the assumptions* (H.1), (H.2) *and* (H.6). *Let* $\mathbf{T}: \mathbf{H} \to \mathbf{H}^*$ *be the operator defined by* (5.1) *and let* $\mathbf{F} \in \mathbf{H}^*$ *be the bounded linear functional defined by* (3.21). *Then there exists a unique solution* $(w, \sigma) \in \mathbf{H}$ *of the equation* (5.3).

The proofs of these theorems are lengthy, but the arguments here are straight forward if one applies the results on monotone operators given in the previous section. To facilitate the proofs, let us first make some observations.

We note that the bilinear form $\mathbf{B}$, in addition to being bounded (see (3.27)), satisfies also a sort of positiveness condition. In fact, since

$$\langle v^-, (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\xi \rangle = \langle (\tfrac{1}{2}\mathbf{I} - \mathbf{K})v^-, \xi \rangle ,$$

we obtain from (5.2),

$$\mathbf{B}((v, \xi), (v, \xi)) = \langle v^-, \mathbf{W}v^- \rangle + \langle \mathbf{V}\xi, \xi \rangle$$

for all $(v, \xi) \in \mathbf{H}$, and hence, making use of (3.8) (cf. Lemma 3.2) and (3.11), we conclude that

$$\mathbf{B}((v, \xi), (v, \xi)) \geqq C \| \xi \|^2_{H^{-1/2}(\Gamma)} \quad \forall (v, \xi) \in \mathbf{H} . \tag{5.4}$$

The above inequality indicates that the operator $\mathbf{T}$ is completely dominated by the nonlinear term in $\mathbf{T}$, i.e., the integral over Ω^- involving the Nemytsky operators A_i, $i = 1, 2$ (see (5.1)). This is precisely why all of our hypotheses are imposed on the nonlinear mappings a_i. On the other hand, we recall from Theorem (3.5) that under the assumptions (H.1) and (H.2), the operator $\mathbf{T}$ defined by (5.1) is bounded and continuous on $\mathbf{H}$. This result will be used in the proof of both Theorem 5.1 and Theorem 5.2, below.

Proof of Theorem 5.1. The conclusion of this theorem is a direct application of Theorem 4.8. The assumption (H.3) naturally gives the coerciveness property of $\mathbf{T}$ (see Definition 2 in Sect. 4) as can be seen in the following. We may write from the definition of the Nemytsky operators A_i (3.23),

$$\sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla(v + g)(x)) \frac{\partial v}{\partial x_i} dx = \sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla(v + g)(x)) \frac{\partial}{\partial x_i}(v + g) dx - \sum_{i=1}^{2} \int_{\Omega^-} (A_i v)(x) \frac{\partial g}{\partial x_i} dx ,$$

and hence, from (H.3) together with $i)$ in Theorem 3.5, we get

$$\sum_{i=1}^{2} \int_{\Omega^-} a_i(x, \nabla(v + g)(x)) \frac{\partial v}{\partial x_i} dx \geqq C_1 |v + g|^2_{H^1(\Omega^-)} - \| C_2 \|_{L^1(\Omega^-)} - C \| (v, \xi) \|_{\mathbf{H}} |g|_{H^1(\Omega^-)} \tag{5.5}$$

for all $(v, \xi) \in \mathbf{H}$ with $\| (v, \xi) \|_{\mathbf{H}} \geqq M_0$. Now, according to (3.2) we have $|v|^2_{H^1(\Omega^-)} \geqq C \| v \|^2_{H^1(\Omega^-)} \forall v \in H^1_{\Gamma_0}(\Omega^-)$. Hence, we easily obtain,

$$|v + g|^2_{H^1(\Omega^-)} \geqq |v|^2_{H^1(\Omega^-)} - 2|v|_{H^1(\Omega^-)} |g|_{H^1(\Omega^-)} \geqq C \| v \|^2_{H^1(\Omega^-)} - 2|g|_{H^1(\Omega^-)} \| (v, \xi) \|_{\mathbf{H}} \tag{5.6}$$

for all $(v, \xi) \in \mathbf{H}$. Consequently, we deduce from (5.1), (5.4), (5.5) and (5.6),

$$\frac{[\mathbf{T}(v, \xi), (v, \xi)]}{\|(v, \xi)\|_{\mathbf{H}}} \geqq \tilde{C}_1 \|(v, \xi)\|_{\mathbf{H}} - C|g|_{H^1(\Omega^-)} - \|C_2\|_{L^1(\Omega^-)} \|(v, \xi)\|_{\mathbf{H}}^{-1}$$

for all $(v, \xi) \in \mathbf{H}$ with $\|(v, \xi)\|_{\mathbf{H}} \geqq M_0$. The above inequality proves that $\mathbf{T}$ is coercive on $\mathbf{H}$.

Similarly, in view of (5.1) and (5.4), we obtain

$$(5.7) \qquad [\mathbf{T}(v, \xi) - \mathbf{T}(z, \delta), (v, \xi) - (z, \delta)] \geqq C \|\xi - \delta\|_{H^{-1/2}(\Gamma)}^2 + \sum_{i=1}^{2} \int_{\Omega^-} \{a_i(x, \nabla\tilde{v}(x)) - a_i(x, \nabla\tilde{z}(x))\} \left\{ \frac{\partial \tilde{v}}{\partial x_i} - \frac{\partial \tilde{z}}{\partial x_i} \right\} dx$$

for all $(v, \xi), (z, \delta) \in \mathbf{H}$, with $\tilde{v} := v + g$ and $\tilde{z} := z + g$. Then, since (H.4) holds, we deduce from (5.7),

$$[\mathbf{T}(v, \xi) - \mathbf{T}(z, \delta), (v, \xi) - (z, \delta)] \geqq C \|\xi - \delta\|_{H^{-1/2}(\Gamma)}^2 \geqq 0$$

for all $(v, \xi), (z, \delta) \in \mathbf{H}$, which proves that $\mathbf{T}$ is monotone.

In addition, suppose that (H.5) is satisfied, and let $(v, \xi), (z, \delta) \in \mathbf{H}$ such that $(v, \xi) \neq (z, \delta)$. If $\xi \neq \delta$, then (5.7) gives

$$(5.8) \qquad [\mathbf{T}(v, \xi) - \mathbf{T}(z, \delta), (v, \xi) - (z, \delta)] > 0 .$$

Now, if $v \neq z$, then making use of (3.2) again, we get

$$|\tilde{v} - \tilde{z}|_{H^1(\Omega^-)} = |v - z|_{H^1(\Omega^-)} \geqq C \|v - z\|_{H^1(\Omega^-)} > 0 .$$

Thus, there must exist a subset D of Ω^- with $Area(D) > 0$ such that $\nabla\tilde{v}(x) \neq \nabla\tilde{z}(x)$ a.e. in D. Hence, the assumption (H.5) and (5.7) imply that the inequality (5.8) also holds in this case. This shows that $\mathbf{T}$ is strictly monotone. Finally, snce by (H.1) and (H.2) $\mathbf{T}$ is bounded and continuous, an application of Theorem 4.8 completes this proof. □

Proof of Theorem 5.2. The existence and uniqueness results given here follow directly from Theorem 4.9. It remains only to show that under (H.6), $\mathbf{T}$ is strongly monotone on $\mathbf{H}$. In fact, for $v, z \in H^1_{\Gamma_0}(\Omega^-)$, let us put again $\tilde{v} := v + g$, $\tilde{z} := z + g$, and define the real valued function $h_i : [0, 1] \to \mathbb{R}$ by

$$h_i(t) := a_i(x, \boldsymbol{\alpha}(x, t)) \quad \forall t \in [0, 1] ,$$

where $\boldsymbol{\alpha}(x, t) := \nabla\tilde{z}(x) + t(\nabla\tilde{v}(x) - \nabla\tilde{z}(x))$. It follows that

$$h_i'(t) = \sum_{j=1}^{2} \frac{\partial}{\partial \alpha_j} a_i(x, \boldsymbol{\alpha}(x, t)) \left\{ \frac{\partial \tilde{v}}{\partial x_j} - \frac{\partial \tilde{z}}{\partial x_j} \right\} ,$$

and since $a_i(x, \nabla\tilde{v}(x)) - a_i(x, \nabla\tilde{z}(x)) = h_i(1) - h_i(0)$, we deduce that

$$a_i(x, \nabla\tilde{v}(x)) - a_i(x, \nabla\tilde{z}(x)) = \int_0^1 \sum_{j=1}^{2} \frac{\partial}{\partial \alpha_j} a_i(x, \boldsymbol{\alpha}(x, t)) \left\{ \frac{\partial \tilde{v}}{\partial x_j} - \frac{\partial \tilde{z}}{\partial x_j} \right\} dt .$$

Consequently, (H.6) and (3.2) yield

$$\sum_{i=1}^{2} \int_{\Omega^-} \{a_i(x, \nabla\tilde{v}(x)) - a_i(x, \nabla\tilde{z}(x))\} \left\{\frac{\partial\tilde{v}}{\partial x_i} - \frac{\partial\tilde{z}}{\partial x_i}\right\} dx$$

$$\geqq C \sum_{i=1}^{2} \int_{\Omega^-} \left\{\frac{\partial\tilde{v}}{\partial x_i} - \frac{\partial\tilde{z}}{\partial x_i}\right\}^2 dx = C|\tilde{v} - \tilde{z}|^2_{H^1(\Omega^-)}$$

$$= C|v - z|^2_{H^1(\Omega^-)} \geqq C\|v - z\|^2_{H^1(\Omega^-)} .$$

Then, by substituting this inequality into (5.7), we conclude

$$[\mathbf{T}(v, \xi) - \mathbf{T}(z, \delta), (v, \xi) - (z, \delta)] \geqq C\|(v, \xi) - (z, \delta)\|^2_{\mathbf{H}}$$

for all (v, ξ), $(z, \delta) \in \mathbf{H}$, which proves the strong monotonicity of $\mathbf{T}$. □

Having proved these existence-uniqueness results for the operator equation (5.3) (or (3.30)), we now direct our attention toward the Galerkin approximations of the corresponding solution (w, σ). This is precisely the aim of the next section.

6 Galerkin approximations of the operator equation

To formulate the Galerkin approximations of the solution of (5.3), we let $\{\mathbf{H}_h\}_{h\in S}$ be a family of finite-dimensional subspaces of $\mathbf{H}$ such that $\bigcup\{\mathbf{H}_h, h \in S\}$ is dense in $\mathbf{H}$, where the parameter h is in an index set S and represents, without loss of generality, a measure of the size of the corresponding finite elements. Then, *the Galerkin approximation of the solution* (w, σ) *of* (5.3) *is defined as an element* $(w_h, \sigma_h) \in \mathbf{H}_h$ *satisfying the Galerkin equations*

$$[\mathbf{T}(w_h, \sigma_h), (z_h, \delta_h)] = [\mathbf{F}, (z_h, \delta_h)] \tag{6.1}$$

for all $(z_h, \delta_h) \in \mathbf{H}_h$.

We remark that, as for the operator equation (5.3), the existence, uniqueness as well as the convergence of the Galerkin approximations (w_h, σ_h) of (6.1) depend strongly on the assumptions of the nonlinear coefficients a_i in $\mathbf{T}$. From Theorem 5.1 we know that there exists a unique solution $(w, \sigma) \in \mathbf{H}$ of (5.3), provided (H.5) is fulfilled in addition to the assumptions (H.1)–(H.3). In this case, we will obtain the following expected results for the Galerkin approximations.

Theorem 6.1. *Suppose that the coefficients* a_i *satisfy the conditions* (H.1)–(H.3), *and* (H.5). *Then there exists a unique solution* $(w_h, \sigma_h) \in \mathbf{H}_h$ *of the equation* (6.1). *Moreover, there is a subsequence* $\{(w_{\bar{h}}, \sigma_{\bar{h}})\}$ *of* $\{(w_h, \sigma_h)\}_{h\in S}$ *such that*

$$(w_{\bar{h}}, \sigma_{\bar{h}}) \rightharpoonup (w, \sigma) \quad as\ \bar{h} \to 0 .$$

Proof. Let $\mathbf{i}_h : \mathbf{H}_h \to \mathbf{H}$ and $\mathbf{i}_h^* : \mathbf{H}^* \to \mathbf{H}_h^*$ be the canonical continuous injections. Then (6.1) can be written, equivalently, as the "discrete" operator equation

$$\mathbf{T}_h(w_h, \sigma_h) = \mathbf{F}_h , \tag{6.2}$$

where $\mathbf{T}_h$: $\mathbf{H}_h \to \mathbf{H}_h^*$ and $\mathbf{F}_h \in \mathbf{H}_h^*$ are defined by $\mathbf{T}_h := \mathbf{i}_h^* \cdot \mathbf{T} \cdot \mathbf{i}_h$ and $\mathbf{F}_h := \mathbf{F} \cdot \mathbf{i}_h$, respectively. Since $\mathbf{T}$ is bounded, continuous, monotone and coercive on $\mathbf{H}$, it is easy to prove that $\mathbf{T}_h$ holds the same properties on $\mathbf{H}_h$. Hence, an application of Theorem 4.8 implies that $\mathbf{T}_h$ is one-to-one and onto. The existence of a subsequence of $\{(w_h, \sigma_h)\}_{h\in S}$ converging weakly to (w, σ) is deduced easily from the proof of Theorem 4.7. □

We remark that the weak convergence of the Galerkin approximations can be improved if the condition (H.6) is satisfied. In fact, we have the following result.

Theorem 6.2. *Suppose that the coefficients a_i satisfy the conditions* (H.1), (H.2) *and* (H.6). *Then there exists a constant $C > 0$ independent of h such that*

$$\|(w, \sigma) - (w_h, \sigma_h)\|_{\mathbf{H}} \leqq C \inf_{(v_h, \xi_h)\in \mathbf{H}_h} \{ \|(w, \sigma) - (v_h, \xi_h)\|_{\mathbf{H}} + \|\mathbf{T}(w, \sigma) - \mathbf{T}(v_h, \xi_h)\|_{\mathbf{H}^*}\}, \tag{6.3}$$

where $(w, \sigma)\in \mathbf{H}$ and $(w_h, \sigma_h)\in \mathbf{H}_h$ are the unique solutions of (5.3) *and* (6.1), *respectively.*

Proof. It follows easily from the triangle inequality and the strong monotonicity property of $\mathbf{T}$. □

Now, in order to obtain some kind of rate of convergence, an additional condition will be needed in contrast to the linear problems (see Michlin 1962). For this purpose, one will introduce a Lipschitz condition below. We comment that if $\mathbf{T}$ were a bounded linear operator, then (6.3) would become the usual Céa's lemma (see Ciarlet 1978).

We recall that $\mathbf{T}: \mathbf{H} \to \mathbf{H}^*$ is *Lipschitz continuous*, if there exists $C > 0$ such that

$$\|\mathbf{T}(v, \xi) - \mathbf{T}(z, \delta)\|_{\mathbf{H}^*} \leqq C\|(v, \xi) - (z, \delta)\|_{\mathbf{H}} \tag{6.4}$$

for all $(v, \xi), (z, \delta)\in \mathbf{H}$. We shall show that a sufficient condition on the nonlinear coefficients a_i to ensure (6.4) is the following condition:

(H.7) Lipschitz condition. The nonlinear functions $a_i(x, \cdot)$ have continuous first order partial derivatives in $\mathbb{R}^2$ for almost all $x\in\Omega^-$. Also, there exists $C_0 > 0$ such that for each $i, j\in\{1, 2\}$, $\frac{\partial}{\partial\alpha_j} a_i(x, \boldsymbol{\alpha})$ satisfies the Carathéodory conditions (H.1), and

$$\left|\frac{\partial}{\partial\alpha_j} a_i(x, \boldsymbol{\alpha})\right| \leqq C_0$$

for all $\boldsymbol{\alpha}\in\mathbb{R}^2$ and for almost all $x\in\Omega^-$.

Since the bilinear form $\mathbf{B}$ in (5.1) is bounded (cf. (3.27)), to establish (6.4) it suffices to show that the Nemytsky operators A_i defined by (3.23) are Lipschitz continuous on $H^1(\Omega^-)$. In fact, similarly as in the proof of Theorem 5.2, we have for $v, z \in H^1(\Omega^-)$,

$$\|A_i v - A_i z\|^2_{L^2(\Omega^-)} = \int_{\Omega^-} |a_i(x, \nabla\tilde{v}(x)) - a_i(x, \nabla\tilde{z}(x))|^2\, dx$$

$$= \int_{\Omega^-} \left| \sum_{j=1}^{2} \int_0^1 \frac{\partial}{\partial\alpha_j} a_i(x, \boldsymbol{\alpha}(x,t)) \left\{ \frac{\partial\tilde{v}}{\partial x_j} - \frac{\partial\tilde{z}}{\partial x_j} \right\} dt \right|^2 dx\,,$$

where $\tilde{v} := v + g$, $\tilde{z} := z + g$ and $\boldsymbol{\alpha}(x,t) := \nabla\tilde{z}(x) + t(\nabla\tilde{v}(x) - \nabla\tilde{z}(x))$. It follows from (H.7) that

$$\|A_i v - A_i z\|^2_{L^2(\Omega^-)} \leqq 2C_0^2 |\tilde{v} - \tilde{z}|^2_{H^1(\Omega^-)} \leqq C \|v - z\|^2_{H^1(\Omega^-)}$$

which proves that $A_i : H^1(\Omega^-) \to L^2(\Omega^-)$ is Lipschitz continuous. Therefore, we conclude that if the condition (H.7) is satisfied, then both operators $\mathbf{T}$ and $\mathscr{T}$ are Lipschitz continuous on $\mathbf{H}$ and $\mathscr{H}$, respectively.

As a corollary of Theorem 6.2 we can now state the following result concerning the error estimates of the Galerkin approximations.

Theorem 6.3. *Suppose that the coefficients a_i satisfy the conditions* (H.1), (H.2), (H.6) *and* (H.7). *Then there exists a constant $C > 0$ independent of h such that*

$$\text{(6.5)} \qquad \|(w,\sigma) - (w_h, \sigma_h)\|_{\mathbf{H}} \leqq C \inf_{(v_h, \xi_h) \in \mathbf{H}_h} \|(w,\sigma) - (v_h, \xi_h)\|_{\mathbf{H}}$$

where $(w,\sigma) \in \mathbf{H}$ and $(w_h, \sigma_h) \in \mathbf{H}_h$ are the unique solutions of (5.3) *and* (6.1), *respectively.*

We observe that as in the linear case, these simple, yet crucial estimates (6.3) and (6.5), show that the problem of estimating the error between the solution (w, σ) and the Galerkin approximations (w_h, σ_h) is reduced to a question in the approximation theory. Furthermore, for the numerical implementations, one does not solve (6.1) exactly. This leads us to consider a modification of that formulation. More precisely, for reasons that will become evident in Sect. 7 we now extend our concept of Galerkin approximations to a more general setting in which $\mathscr{H}$, $\mathscr{F}$ and $\mathscr{T}$ come into play (see Remark before Theorem 3.5). To this end, we now let $\{\mathbf{H}_h\}_{h \in S}$ be a family of finite-dimensional subspaces of $\mathscr{H}$ (not necessarily subspaces of $\mathbf{H}$!). In addition, let $\mathbf{T}_h : \mathbf{H}_h \to \mathbf{H}_h^*$ be an operator that approximates $\mathscr{T}$ on $\mathbf{H}_h$, and let $\mathbf{F}_h \in \mathbf{H}_h^*$ be an approximation of $\mathscr{F}$ on $\mathbf{H}_h$. *Then we redefine a Galerkin approximation of the solution (w, σ) of* (5.3) *as an element $(w_h, \sigma_h) \in \mathbf{H}_h$ (if it exists) such that*

$$\text{(6.6)} \qquad [\mathbf{T}_h(w_h, \sigma_h), (z_h, \delta_h)]_h = [\mathbf{F}_h, (z_h, \delta_h)]_h$$

for all $(z_h, \delta_h) \in \mathbf{H}_h$, where $[\cdot, \cdot]_h$ denotes the duality pairing on $\mathbf{H}_h^* \times \mathbf{H}_h$. As far as the existence of a solution of (6.6) is concerned, it suffices from Theorem 4.6 and Theorem 4.8, to assume that $\mathbf{T}_h$ is a continuous, monotone and coercive operator on $\mathbf{H}_h$. Note that from Theorem 4.6, since $\mathbf{H}_h$ is finite-dimensional, the monotonicity of $\mathbf{T}_h$ implies that $\mathbf{T}_h$ is bounded. Similarly, from Theorem 4.9, we see that there exists a unique solution of (6.6) if $\mathbf{T}_h$ is continuous and strongly monotone on $\mathbf{H}_h$. Moreover, we can establish the following important theorem.

Theorem 6.4. *Suppose that the coefficients a_i satisfy the conditions* (H.1), (H.2), (H.6) *and* (H.7). *Let* $\mathbf{F}_h \in \mathbf{H}_h^*$ *be an approximation of* $\mathscr{F}$ *on* $\mathbf{H}_h$, *and let* $\mathbf{T}_h : \mathbf{H}_h \to \mathbf{H}_h^*$ *be an operator that approximates* $\mathscr{T}$ *on* $\mathbf{H}_h$, *with the properties*:

i) $\mathbf{T}_h$ *is continuous*

ii) $\mathbf{T}_h$ *is uniformly strongly monotone, i.e., there exist constants* $h_0 > 0$ *and* $\tilde{C} > 0$, *independent of* h, *such that*

$$[\mathbf{T}_h(v_h, \xi_h) - \mathbf{T}_h(z_h, \delta_h), (v_h, \xi_h) - (z_h, \delta_h)]_h \geqq \tilde{C} \| (v_h, \xi_h) - (z_h, \delta_h) \|_{\mathscr{H}}^2$$

for all (v_h, ξ_h), $(z_h, \delta_h) \in \mathbf{H}_h$ *and for all* $h \in (0, h_0)$. *Then* (5.3) *has a unique solution* $(w, \sigma) \in \mathbf{H}$ *and* (6.6) *has a unique solution* $(w_h, \sigma_h) \in \mathbf{H}_h$. *Moreover, the following Strang type estimates hold for all* $h \in (0, h_0)$:

$$(6.7) \quad \| (w, \sigma) - (w_h, \sigma_h) \|_{\mathscr{H}} \leqq C \left\{ \sup_{\substack{(z_h, \delta_h) \in \mathbf{H}_h \\ (z_h, \delta_h) \neq 0}} \frac{|[\mathbf{F}_h, (z_h, \delta_h)]_h - [\mathscr{T}(w, \sigma), (z_h, \delta_h)]|}{\| (z_h, \delta_h) \|_{\mathscr{H}}} \right.$$
$$+ \inf_{(v_h, \xi_h) \in \mathbf{H}_h} \left(\| (w, \sigma) - (v_h, \xi_h) \|_{\mathscr{H}} \right.$$
$$\left.\left. + \sup_{\substack{(z_h, \delta_h) \in \mathbf{H}_h \\ (z_h, \delta_h) \neq 0}} \frac{|[\mathscr{T}(v_h, \xi_h), (z_h, \delta_h)] - [\mathbf{T}_h(v_h, \xi_h), (z_h, \delta_h)]_h|}{\| (z_h, \delta_h) \|_{\mathscr{H}}} \right) \right\}$$

where $C > 0$ *is a constant independent of* h.

Proof. From Theorem 5.2 and the previous remark, it remains only to prove the estimate (6.7). We have again by the triangle inequality,

$$(6.8) \quad \| (w, \sigma) - (w_h, \sigma_h) \|_{\mathscr{H}} \leqq \| (w, \sigma) - (v_h, \xi_h) \|_{\mathscr{H}} + \| (w_h, \sigma_h) - (v_h, \xi_h) \|_{\mathscr{H}}$$

for all $(v_h, \xi_h) \in \mathbf{H}_h$. Now, since by *ii)* $\mathbf{T}_h$ is uniformly strongly monotone and $(w_h, \sigma_h) \in \mathbf{H}_h$ is the solution of (6.6), we obtain, with $(z_h, \delta_h) := (w_h, \sigma_h) - (v_h, \xi_h)$,

$$\| (z_h, \delta_h) \|_{\mathscr{H}}^2 \leqq C \{ [\mathbf{F}_h, (z_h, \delta_h)]_h - [\mathbf{T}_h(v_h, \xi_h), (z_h, \delta_h)]_h \}$$

for all $(v_h, \xi_h) \in \mathbf{H}_h$. It follows easily that

$$(6.9) \quad \| (z_h, \delta_h) \|_{\mathscr{H}}^2 \leqq C \{ |[\mathbf{F}_h, (z_h, \delta_h)]_h - [\mathscr{T}(w, \sigma), (z_h, \delta_h)]|$$
$$+ |[\mathscr{T}(w, \sigma), (z_h, \delta_h)] - [\mathscr{T}(v_h, \xi_h), (z_h, \delta_h)]|$$
$$+ |[\mathscr{T}(v_h, \xi_h), (z_h, \delta_h)] - [\mathbf{T}_h(v_h, \xi_h), (z_h, \delta_h)]_h| \} .$$

Note that by (H.7) $\mathscr{T}$ is Lipschitz continuous on $\mathscr{H}$. Hence, dividing (6.9) by $\| (z_h, \delta_h) \|_{\mathscr{H}}$, and then taking the supremum with respect to $(z_h, \delta_h) \in \mathbf{H}_h$ on each term of the right hand side, we deduce that

$$\| (w_h, \sigma_h) - (v_h, \xi_h) \|_{\mathscr{H}} \leqq C \left\{ \sup_{\substack{(z_h, \delta_h) \in \mathbf{H}_h \\ (z_h, \delta_h) \neq 0}} \frac{|[\mathbf{F}_h, (z_h, \delta_h)]_h - [\mathscr{T}(w, \sigma), (z_h, \delta_h)]|}{\| (z_h, \delta_h) \|_{\mathscr{H}}} \right.$$
$$+ \| (w, \sigma) - (v_h, \xi_h) \|_{\mathscr{H}}$$
$$\left. + \sup_{\substack{(z_h, \delta_h) \in \mathbf{H}_h \\ (z_h, \delta_h) \neq 0}} \frac{|[\mathscr{T}(v_h, \xi_h), (z_h, \delta_h)] - [\mathbf{T}_h(v_h, \xi_h), (z_h, \delta_h)]_h|}{\| (z_h, \delta_h) \|_{\mathscr{H}}} \right\}$$

for all $(v_h, \xi_h) \in \mathbf{H}_h$. Together with (6.8) this completes the proof. □

It is important to note that the estimate (6.7) can be considered as a slight variant of the well known first and second Strang's lemmas for treating linear problems with variational crimes (see e.g. Ciarlet 1978, Chap. 4). To conclude this section, we remark that the Strang type estimate (6.7) plays a fundamental role in the asymptotic error estimates. In Sect. 7 an explicit operator $\mathbf{T}_h$ is given to possess the properties *i)* and *ii)* in Theorem 6.4, and specific error estimates based on (6.7) are also obtained for a family of finite element subspaces $\{\mathbf{H}_h\}_{h\in(0,h_0)}$.

7 Asymptotic error estimates

The main goal of this section is to derive asymptotic error estimates for a boundary-finite element solution of Eq. (5.3) for the case in which the corresponding operator is strongly monotone and Lipschitz-continuous. The problem is discretized by using linear conforming triangular elements on a polygonal domain $\bar{\Omega}_h$ approximating $\bar{\Omega}^-$, and using piecewise constants functions on the corresponding polygonal boundary Γ_h that approximates Γ. Two discrete problems are defined, one on $\bar{\Omega}_h$ and the other on $\bar{\Omega}^-$. With the hypotheses that the conditions (H.1), (H.2), (H.6) and (H.7) are satisfied on a domain containing $\bar{\Omega}^- \cup \bar{\Omega}_h$, we conclude that both discrete problems are unique solvable. The solution of the discrete problem on $\bar{\Omega}^-$ yields the Galerkin solution (w_h, σ_h) defined by (6.6). Similarly, the solution of the discrete problem on $\bar{\Omega}_h$ induces, by means of a convenient transformation, another approximation which we denote by $(\bar{w}_h, \bar{\sigma}_h)$ and call the Quasi-Galerkin solution. Then, based on the Strang type estimate given in Theorem 6.4, we prove the strong convergence of (w_h, σ_h) and $(\bar{w}_h, \bar{\sigma}_h)$ to the solution (w, σ) of (5.3). Moreover, under additional regularity assumptions on (w, σ), we show that the approximate solutions converge to (w, σ) with the asymptotic rate of convergence $O(h)$. The present section can be regarded as the first piece of work concerned with the asymptotic error analysis of the coupled boundary and finite element methods for a *nonlinear problem.* For the linear case we refer to Wendland (1986, 1988).

7.1 Preliminaries

Throughout the rest of this section we write Ω instead of Ω^-. Let $\tilde{\Omega} \subset \mathbb{R}^2$ be a bounded domain with smooth boundary such that $\bar{\Omega}$ is strictly contained in $\tilde{\Omega}$. We assume that the data f and g are such that $f \in L^2(\tilde{\Omega})$, and $g \in H^{2+\varepsilon}(\tilde{\Omega})$ for some $\varepsilon > 0$. In addition, the nonlinear coefficients a_i are supposed to satisfy the following assumptions:

(H.1)′ Carathéodory conditions. The function $a_i(\cdot, \boldsymbol{\alpha})$ is measurable in $\tilde{\Omega}$ for all $\boldsymbol{\alpha} \in \mathbb{R}^2$ and $a_i(x, \cdot)$ is continuous in $\mathbb{R}^2$ for almost all $x \in \tilde{\Omega}$.

(H.2)′ Growth condition. There exist functions $\phi_i \in H^1(\tilde{\Omega})$, $i = 1, 2$ such that

$$|a_i(x, \boldsymbol{\alpha})| \leqq C\{1 + |\boldsymbol{\alpha}|\} + |\phi_i(x)|$$

for all $\boldsymbol{\alpha} \in \mathbb{R}^2$ and for almost all $x \in \tilde{\Omega}$.

(H.6)′ Strong monotonicity condition. The nonlinear coefficients $a_i(x, \cdot)$ have continuous first order partial derivatives in $\mathbb{R}^2$ for almost all $x \in \tilde{\Omega}$. In addition, there exists $C > 0$ such that

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial \alpha_j} a_i(x, \boldsymbol{\alpha}) \beta_i \beta_j \geqq C \sum_{i=1}^{2} \beta_i^2$$

for all $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} := (\beta_1, \beta_2) \in \mathbb{R}^2$ and for almost all $x \in \tilde{\Omega}$.

(H.7)′ Lipschitz condition. The nonlinear functions $a_i(x, \cdot)$ have continuous first order partial derivatives in $\mathbb{R}^2$ for almost all $x \in \tilde{\Omega}$. Also, there exists $C_0 > 0$ such that for each $i, j \in \{1, 2\}$, $\frac{\partial}{\partial \alpha_j} a_i(x, \boldsymbol{\alpha})$ satisfies the Carathéodory conditions (H.1)′, and

$$\left| \frac{\partial}{\partial \alpha_j} a_i(x, \boldsymbol{\alpha}) \right| \leqq C_0$$

for all $\boldsymbol{\alpha} \in \mathbb{R}^2$ and for almost all $x \in \tilde{\Omega}$.

Having established our main hypotheses, we now rewrite the operator equation (5.3), equivalently, as: *Find* $(w, \sigma) \in \mathbf{H}$ *such that*

$$\textbf{(P)} \qquad [\mathbf{T}(w, \sigma), (z, \delta)] = [\mathbf{F}, (z, \delta)] \quad \forall (z, \delta) \in \mathbf{H},$$

where $\mathbf{F}$ *is the bounded linear functional defined by* (3.21). We will refer to **(P)** as the *continuous problem.*

We remark that from the point of view of the formulation of problem **(P)**, the assumptions on f and g, and the conditions (H.1)′ and (H.2)′ can be weakened. In fact, as proved in the previous sections, $f \in L^2(\Omega)$, $g \in H^1(\Omega)$, and the hypotheses (H.1) and (H.2) suffice to prove that $\mathbf{T}$ and $\mathbf{F}$ are well defined. Similarly, from the point of view of the solvability of problem **(P)**, the assumption (H.6)′ can also be simplified. Indeed, according to Theorem 5.2, (H.1), (H.2), and (H.6) guarantee the existence of a unique solution of **(P)**.

For the purposes of this section, we find it convenient to redefine the bilinear form $\mathbf{B}$. We note that since

$$\frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial \nu(y)} \log \left\{ \frac{1}{|x - y|} \right\} ds_y = -\tfrac{1}{2} \quad \forall x \in \Gamma,$$

the boundary integral operator $\mathbf{K}$ of the double layer potential (see (2.3) and Lemma 3.2), satisfies the identity,

$$\langle \mathbf{K} v^-, \delta \rangle = \mathbf{d}(v^-, \delta) - \tfrac{1}{2} \langle v^-, \delta \rangle \quad \forall (v, \delta) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$$

where $\mathbf{d}\colon H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}$ is the bounded bilinear form defined by

$$\mathbf{d}(\xi, \delta) := \frac{1}{2\pi} \int_\Gamma \int_\Gamma \frac{(x - y) \cdot \nu(y)}{|x - y|^2} \{\xi(y) - \xi(x)\} \delta(x) \, ds_x ds_y \tag{7.1}$$

for all $(\xi, \delta) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. Similarly, we introduce the bounded bilinear form $\mathbf{b}\colon H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}$ defined by

$$\mathbf{b}(\xi, \delta) := \langle \mathbf{V} \xi, \delta \rangle \quad \forall \xi, \delta \in H^{-1/2}(\Gamma), \tag{7.2}$$

where **V** is the boundary integral operator of the simple layer potential (see (2.2) and Lemma 3.2). Observe, in view of (3.6), that

$$\mathbf{b}(\xi, \delta) = \frac{1}{2\pi} \int_\Gamma \int_\Gamma \log \left\{ \frac{D}{|x-y|} \right\} \xi(y)\delta(x)\, ds_y\, ds_x \tag{7.3}$$

for all $\xi \in H_0^{-1/2}(\Gamma)$ and for all $\delta \in H^{-1/2}(\Gamma)$, where D denotes twice the diameter of Γ. Moreover, according to (3.8) (cf. Lemma 3.2), we have

$$\mathbf{b}(\xi, \xi) \geqq C \| \xi \|^2_{H^{-1/2}(\Gamma)} \quad \forall \xi \in H_0^{-1/2}(\Gamma) . \tag{7.4}$$

Also, from (3.10), we clearly have

$$\langle z^-, \mathbf{W} v^- \rangle = \mathbf{b}\left(\frac{dv^-}{ds}, \frac{dz^-}{ds} \right) \quad \forall v, z \in H^1(\Omega) .$$

Now, if we let $\mathbf{A}(\cdot, \cdot)$ denote the semilinear form

$$\mathbf{A}(v, z) := \sum_{i=1}^{2} \int_\Omega (A_i v)(x) \frac{\partial z}{\partial x_i} dx \quad \forall v, z \in H^1(\Omega) ,$$

with $(A_i v)(x) := a_i(x, \nabla(v+g)(x))$ being the Nemytsky operator, then the operator $\mathscr{T}$ and the bilinear form **B** may be redefined:

$$[\mathscr{T}(v, \xi), (z, \delta)] := \mathbf{A}(v, z) + \mathbf{B}((v, \xi), (z, \delta)) ,$$

and

$$\mathbf{B}((v, \xi), (z, \delta)) := \mathbf{b}\left(\frac{dv^-}{ds}, \frac{dz^-}{ds} \right) + \mathbf{b}(\xi, \delta) + \mathbf{d}(z^-, \xi) - \mathbf{d}(v^-, \delta) + \langle v^-, \delta \rangle - \langle z^-, \xi \rangle \tag{7.5}$$

for all $(v, \xi), (z, \delta) \in \mathscr{H}$.

This setting will be used in Subsection 7.3 to define the finite element discretizations associated to the continuous problem (**P**). To that end, we now introduce the corresponding triangulations of the domain.

7.2 Triangulations of the domain and their properties

Let $\{\Omega_h\}_{h \in (0, h_0)}$ be the set of polygonal approximations of Ω obtained by approximating Γ and Γ_0 by two simple closed piecewise linear curves Γ_h and $\Gamma_{o,h}$, respectively, with all their vertices lying on $\partial\Omega$. We assume that the bounded domain $\tilde{\Omega}$ introduced in Sect. 7.1 is such that

$$\bar{\Omega}_h \subset \tilde{\Omega} \quad \forall h \in (0, h_0) . \tag{7.6}$$

Observe that $\partial\Omega_h = \Gamma_h \cup \Gamma_{0,h}$. Now, let Π_h be a triangulation of Ω_h, i.e., a set $\Pi_h := \{\tau_1, \ldots, \tau_m\}$ consisting of a finite number of closed triangles, which has the following properties:

i) $\bar{\Omega}_h = \cup_{j=1}^m \tau_j$.

ii) If $\tau_i, \tau_j \in \Pi_h$, $\tau_i \neq \tau_j$, then either $\tau_i \cap \tau_j = \emptyset$ or $\tau_i \cap \tau_j$ is a common vertex or $\tau_i \cap \tau_j$ is a common side of τ_i and τ_j.

We denote by $\mathscr{V}_h := \{P_1, \ldots, P_M\}$ the set of all vertices of Π_h, and assume:

iii) $\mathscr{V}_h \subseteq \bar{\Omega}$, $\mathscr{V}_h \cap \partial\Omega_h \subseteq \partial\Omega$ for all $h \in (0, h_0)$.
iv) At most two vertices of each $\tau_j \in \Pi_h$ lie on $\partial\Omega_h := \Gamma_h \cup \Gamma_{0,h}$.

For definiteness, from now on we assume that $\partial\Omega := \Gamma \cup \Gamma_0$ is piecewise of class C^2 and add the condition:

v) The points of $\partial\Omega$ where the condition of C^2-smoothness is not satisfied are elements of $\mathscr{V}_h$.

We let h_j and θ_j denote the length of the maximum side and the magnitude of the minimum angle of $\tau_j \in \Pi_h$, respectively, and set

$$h := \max\{h_j : j = 1, \ldots, m\} \qquad \text{and} \qquad \theta_h := \min\{\theta_j : j = 1, \ldots, m\} .$$

We assume that there exists a positive constant θ_0 such that

(vi) $\theta_h \geqq \theta_0 > 0 \quad \forall h \in (0, h_0)$.

The set of triangulations $\{\Pi_h\}_{h \in (0, h_0)}$ with all the properties i)–vi) is called a *system of regular triangulations.*

We now introduce further notations. Let τ_j be a *boundary triangle*, i.e., a triangle with two vertices on Γ, or on Γ_0, and let P_1^j, P_2^j, P_3^j be its vertices (in a local notation) with $P_2^j, P_3^j \in \Gamma$, or $P_2^j, P_3^j \in \Gamma_0$. Let $\mathscr{C}_j$ be the part of Γ (or Γ_0) which is approximated by the segment $P_2^j P_3^j$. Then the closed triangle τ_j^{id} with two straight sides $P_1^j P_2^j$, $P_1^j P_3^j$ and one curved side $\mathscr{C}_j$ is called the *ideal triangle* associated with the triangle τ_j. In other words, τ_j is the approximation of τ_j^{id}.

From now on, for simplicity, we will assume that the region bounded by Γ is convex. Then, we adopt the following notation:

$$\Omega - \bar{\Omega}_h = \cup_{r=1}^{N_1} \tau_{r,\Gamma}^+ \cup \cup_{r=1}^{N_2} \tau_{r,\Gamma_0}^+$$

and

$$\Omega_h - \bar{\Omega} = \cup_{r=1}^{N_3} \tau_{r,\Gamma_0}^- ,$$

where $\tau_{r,\Gamma}^+, \tau_{r,\Gamma_0}^+, \tau_{r,\Gamma_0}^-$ are domains with boundaries

$$\partial\tau_{r,\Gamma}^+ = \mathscr{C}_{r,\Gamma}^+ \cup l_{r,\Gamma}^+, \qquad \partial\tau_{r,\Gamma_0}^+ = \mathscr{C}_{r,\Gamma_0}^+ \cup l_{r,\Gamma_0}^+, \qquad \partial\tau_{r,\Gamma_0}^- = \mathscr{C}_{r,\Gamma_0}^- \cup l_{r,\Gamma_0}^- .$$

Here, $\mathscr{C}_{r,\Gamma}^+ \subseteq \Gamma$ (resp. $\mathscr{C}_{r,\Gamma_0}^+ \subseteq \Gamma_0$, $\mathscr{C}_{r,\Gamma_0}^- \subseteq \Gamma_0$) is the curved side of an ideal triangle τ_j^{id}, and $l_{r,\Gamma}^+ \subseteq \Gamma_h$ (resp. $l_{r,\Gamma_0}^+ \subseteq \Gamma_{0,h}$, $l_{r,\Gamma_0}^- \subseteq \Gamma_{0,h}$) is the corresponding side of the triangle $\tau_j \in \Pi_h$ that approximates τ_j^{id}. We assume that h is sufficiently small so that $\mathscr{C}_{r,\Gamma}^+$ and $l_{r,\Gamma}^+$ (resp. $\mathscr{C}_{r,\Gamma_0}^+$ and l_{r,Γ_0}^+, $\mathscr{C}_{r,\Gamma_0}^-$ and l_{r,Γ_0}^-) intersect only at the end points of the line segment $l_{r,\Gamma}^+$ (resp. l_{r,Γ_0}^+, l_{r,Γ_0}^-). It is clear from the above definitions that

$$\Gamma = \cup_{r=1}^{N_1} \mathscr{C}_{r,\Gamma}^+, \qquad \Gamma_h = \cup_{r=1}^{N_1} l_{r,\Gamma}^+ ,$$

$$\Gamma_0 = \cup_{r=1}^{N_2} \mathscr{C}_{r,\Gamma_0}^+ \cup \cup_{r=1}^{N_3} \mathscr{C}_{r,\Gamma_0}^- ,$$

and

$$\Gamma_{0,h} = \cup_{r=1}^{N_2} l_{r,\Gamma_0}^+ \cup \cup_{r=1}^{N_3} l_{r,\Gamma_0}^- .$$

Throughout the rest of the section, in our estimates we shall work with various constants. For simplicity, the same symbol C will be used to denote generic constants which are *independent of h.* Also, if necessary, the constant h_0 will be replaced by a convenient smaller one denoted again by h_0. We use $|\cdot|$ to mean the

Lebesgue measure defined in $\mathbb{R}^2$, as well as the one dimensional measure defined on $\partial\Omega$ or $\partial\Omega_h$. We now state a useful lemma. We omit its proof since it is too technical. However, we refer the interested reader to Gatica and Hsiao (1989b).

Lemma 7.1. *There exists a positive constant C such that*

$$\sum_{r=1}^{N_1} |\tau_{r,\Gamma}^+|, \qquad \sum_{r=1}^{N_2} |\tau_{r,\Gamma_0}^+|, \qquad \sum_{r=1}^{N_3} |\tau_{r,\Gamma_0}^-| \leqq Ch^2, \quad \textit{and}$$
$$\sum_{r=1}^{N_1} \|v\|^2_{L^2(\tau_{l_{r,\Gamma}^+})} \leqq Ch^2 \|v\|^2_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$

hold for all $h \in (0, h_0)$.

Proof. See Gatica and Hsiao (1989b, Lemma 3.1, Lemma 3.3). □

7.3 The finite element discretizations

In this subsection we introduce the two discrete problems mentioned in the beginning of the section, which lead to the Galerkin and Quasi-Galerkin approximations of the solution (w, σ) of the continuous problem (**P**). We mainly follow the approach given in Johnson and Nedelec (1980) for linear problems.

7.3.1 The discrete problem on $\bar{\Omega}_h$

We first define, similarly as in (3.1) and (3.5), the following Sobolev spaces:

$$H^1_{\Gamma_{0,h}}(\Omega_h) := \{\tilde{v} \in H^1(\Omega_h) : \tilde{v}|_{\Gamma_{0,h}} = 0\},$$

$$H_0^{-1/2}(\Gamma_h) := \{\tilde{\xi} \in H^{-1/2}(\Gamma_h) : \int_{\Gamma_h} \tilde{\xi}(s)\, ds = 0\},$$

and the associated product space,

$$\tilde{\mathbf{H}} := H^1_{\Gamma_{0,h}}(\Omega_h) \times H_0^{-1/2}(\Gamma_h).$$

Also, we introduce the finite element spaces,

$$\tilde{H}_h^1 := \{\tilde{v} \in C(\bar{\Omega}_h) : \tilde{v}|_{\tau_j} \text{ is affine } \forall j = 1, \ldots, m\},$$

$$\tilde{H}_{h,0}^1 := \{\tilde{v} \in \tilde{H}_h^1 : \tilde{v}(P_j) = 0 \,\forall P_j \in \mathscr{V}_h \cap \Gamma_{0,h}\}, \tag{7.7}$$

$$\tilde{H}_h^{-1/2} := \{\tilde{\xi} \in L^2(\Gamma_h) : \tilde{\xi}|_{l_{r,\Gamma}^+} \text{ is constant } \forall r = 1, \ldots, N_1\},$$

$$\tilde{H}_{h,0}^{-1/2} := \{\tilde{\xi} \in \tilde{H}_h^{-1/2} : \int_{\Gamma_h} \tilde{\xi}(s)\, ds = 0\}, \tag{7.8}$$

and the product space,

$$\tilde{\mathbf{H}}_h := \tilde{H}_{h,0}^1 \times \tilde{H}_{h,0}^{-1/2}. \tag{7.9}$$

It is clear that $\tilde{\mathbf{H}}_h$ is a finite element subspace of $\tilde{\mathbf{H}}$. We now define the usual interpolation operator $\tilde{I}^h: H^2(\Omega_h) \ni \tilde{v} \to \tilde{I}^h \tilde{v} \in \tilde{H}_h^1$ by the relation

$$(\tilde{I}^h \tilde{v})(P_j) = \tilde{v}(P_j) \quad \forall P_j \in \mathscr{V}_h .$$

We adopt the notation $\tilde{v}^h := \tilde{I}^h \tilde{v}$ for all $\tilde{v} \in H^2(\Omega_h)$.

Since by hypothesis $g \in H^{2+\varepsilon}(\tilde{\Omega})$, and by (7.6) $\bar{\Omega}_h \subseteq \tilde{\Omega}$, we have clearly that $g \in H^2(\Omega_h)$. Thus, according to (H.1)′ and (H.2)′, we can define the semilinear form

$$\tilde{\mathbf{A}}_h(v, z) = \sum_{i=1}^{2} \int_{\Omega_h} a_i(x, \nabla(v + \tilde{g}^h)(x)) \frac{\partial z}{\partial x_i} dx \tag{7.10}$$

for all $v, z \in H^1(\Omega_h)$. Analogously, following (7.1) and (7.3), we introduce the discrete bilinear forms

$$\tilde{\mathbf{b}}_h(\xi, \delta) := \frac{1}{2\pi} \int_{\Gamma_h} \int_{\Gamma_h} \log \left\{ \frac{D}{|x - y|} \right\} \xi(y) \delta(x) \, ds_y \, ds_x \tag{7.11}$$

for all $\xi, \delta \in H^{-1/2}(\Gamma_h)$,

$$\tilde{\mathbf{d}}_h(\xi, \delta) := \frac{1}{2\pi} \int_{\Gamma_h} \int_{\Gamma_h} \frac{(x - y) \cdot \nu_h(y)}{|x - y|^2} \{\xi(y) - \xi(x)\} \delta(x) \, ds_x \, ds_y \tag{7.12}$$

for all $(\xi, \delta) \in H^{1/2}(\Gamma_h) \times H^{-1/2}(\Gamma_h)$, and

$$\langle v^-, \delta \rangle_h := \int_{\Gamma_h} v^-(x) \delta(x) \, ds_x \tag{7.13}$$

for all $v \in H^1(\Omega_h)$, $\delta \in H^{-1/2}(\Gamma_h)$. We remark that in (7.13) v^- denotes now the trace of $v \in H^1(\Omega_h)$ on Γ_h, i.e., $v^- \in H^{1/2}(\Gamma_h)$. Thus, $\langle \cdot, \cdot \rangle_h$ stands for the duality pairing between $H^{1/2}(\Gamma_h)$ and $H^{-1/2}(\Gamma_h)$ with respect to the $L^2(\Gamma_h)$-inner product. Also, in (7.12), we denote by $\nu_h(y)$ the outward normal to Γ at the point $\psi_h(y) \in \Gamma$, $y \in \Gamma_h$, where $\psi_h: \Gamma_h \to \Gamma$ is the mapping that associates to each $y \in \Gamma_h$, $y \notin \mathscr{V}_h$, the intersection point of Γ with the line passing through y perpendicular to Γ_h, and $\psi_h(y) = y$ for all $y \in \mathscr{V}_h \cap \Gamma_h$. It is clear that for h sufficiently small, ψ_h is a bijection.

Now, similarly as in (7.5), we define the discrete bilinear form corresponding to $\mathbf{B}$,

$$\tilde{\mathbf{B}}_h((v, \xi), (z, \delta)) := \tilde{\mathbf{b}}_h\left(\frac{dv^-}{ds_h}, \frac{dz^-}{ds_h} \right) + \tilde{\mathbf{b}}_h(\xi, \delta) + \tilde{\mathbf{d}}_h(z^-, \xi) - \tilde{\mathbf{d}}_h(v^-, \delta) + \langle v^-, \delta \rangle_h - \langle z^-, \xi \rangle_h \tag{7.14}$$

for all $(v, \xi), (z, \delta) \in H^1(\Omega_h) \times H^{-1/2}(\Gamma_h)$, where d/ds_h indicates tangential derivative along Γ_h. We remark that $\tilde{\mathbf{B}}_h$ is bounded as a consequence of the trace theorem on $H^1(\Omega_h)$ and the continuity properties of the boundary integral operators on the Lipschitzian boundary Γ_h (see Lemma 3.2). By (H.1)′ and (H.2)′ we can define the operator $\tilde{\mathbf{T}}: \tilde{\mathbf{H}} \to \tilde{\mathbf{H}}^*$, with

$$[\tilde{\mathbf{T}}(v, \xi), (z, \delta)]_{\tilde{\mathbf{H}}} := \tilde{\mathbf{A}}_h(v, z) + \tilde{\mathbf{B}}_h((v, \xi), (z, \delta)) \tag{7.15}$$

for all $(v, \xi), (z, \delta) \in \tilde{\mathbf{H}}$, where $[\cdot, \cdot]_{\tilde{\mathbf{H}}}$ denotes the corresponding duality pairing. Also, due to the boundedness of $\tilde{\mathbf{B}}_h$ we can introduce the bounded linear functional $\tilde{\mathbf{F}}: \tilde{\mathbf{H}} \to \mathbf{R}$, with

$$[\tilde{\mathbf{F}}, (z, \delta)]_{\tilde{\mathbf{H}}} := \int_{\Omega_h} f z \, dx - \tilde{\mathbf{B}}_h\left(\left(\tilde{g}^h, \frac{2b}{|\Gamma|} \right), (z, \delta)\right) \tag{7.16}$$

for all $(z, \delta) \in \tilde{\mathbf{H}}$. The discrete analogue of (P) can now be formulated: *Find* $(\tilde{w}_h, \tilde{\sigma}_h) \in \tilde{\mathbf{H}}_h$ *such that*

$$(\tilde{\mathbf{P}}_h) \qquad [\tilde{\mathbf{T}}(\tilde{w}_h, \tilde{\sigma}_h), (\tilde{v}_h, \tilde{\xi}_h)]_{\tilde{h}} = [\tilde{\mathbf{F}}, (\tilde{v}_h, \tilde{\xi}_h)]_{\tilde{h}}$$

for all $(\tilde{v}_h, \tilde{\xi}_h) \in \tilde{\mathbf{H}}_h$, where $\tilde{\mathbf{H}}_h$ is the finite element subspace of $\tilde{\mathbf{H}}$ defined in (7.9).

By following the same arguments of the proof of Theorem 3.4 (see Gatica 1989, Chapter 2), we easily deduce that under (H.1)′ and (H.2)′, the *discrete Nemytsky operator* $\tilde{A}_i^h$, defined by

$$(\tilde{A}_i^h v)(x) := a_i(x, \nabla(v + \tilde{g}^h)(x)) ,$$

is a continuous map from $H^1(\Omega_h)$ into $L^2(\Omega_h)$, and hence, the operator $\tilde{\mathbf{T}}$ defined by (7.15) is bounded and continuous. By using Lemma 3.2 and (3.11), since Γ_h is Lipschitzian, one can easily prove that $\tilde{\mathbf{B}}_h$ satisfies the inequality

$$(7.17) \qquad \tilde{\mathbf{B}}_h((v, \xi), (v, \xi)) \geqq C \| \xi \|^2_{H^{-1/2}(\Gamma_h)}$$

for all $(v, \xi) \in \tilde{\mathbf{H}}$, which is the discrete analogue of (5.4). Then, following the same procedure as in Sect. 5 (see proof of Theorem 5.2) and using (7.17) and (H.6)′, we deduce that $\tilde{\mathbf{T}}$ is a strongly monotone operator on the whole space $\tilde{\mathbf{H}}$.

As a consequence, we can state the following result concerning the solvability of $(\tilde{\mathbf{P}}_h)$.

Theorem 7.2. *Under the assumptions* (H.1)′, (H.2)′ *and* (H.6)′, *there exists a unique solution pair* $(\tilde{w}_h, \tilde{\sigma}_h) \in \tilde{\mathbf{H}}_h$ *to the problem* $(\tilde{\mathbf{P}}_h)$.

Proof. The proof follows directly from Theorem 4.9. □

We comment that the problem $(\tilde{\mathbf{P}}_h)$ leads to a nonlinear system of algebraic equations for the unknowns $\tilde{w}_h(P_j)$ per each vertex $P_j \in \mathscr{V}_h$, $P_j \notin \Gamma_{0,h}$, and the unknowns $\tilde{\sigma}_h|_{l^+_{r,\Gamma}}$ per each side $l^+_{r,\Gamma}$ of the polygonal boundary Γ_h. The algorithms for solving the nonlinear system determined by $(\tilde{\mathbf{P}}_h)$ as well as their numerical implementations will not be discussed here. The solution $(\tilde{w}_h, \tilde{\sigma}_h)$ of the discrete problem $(\tilde{\mathbf{P}}_h)$, an element of the space $C(\bar{\Omega}_h) \times L^2(\Gamma_h)$, cannot be directly compared with the solution (w, σ) of the continuous problem (**P**), since they are defined on different domains. However, we shall see in the next subsection that a suitable transformation $\mathscr{M}$: $C(\bar{\Omega}_h) \times L^2(\Gamma_h) \to C(\bar{\Omega}) \times L^2(\Gamma)$ can be introduced so that $\mathscr{M}(\tilde{w}_h, \tilde{\sigma}_h)$ constitutes what we call the Quasi-Galerkin approximation of (w, σ). Moreover, by using this mapping $\mathscr{M}$, and based on $(\tilde{\mathbf{P}}_h)$, we show that a discrete problem $(\mathbf{P}_h)$ of the form (6.6) can be formulated for which the assumptions *i*) and *ii*) of Theorem 6.4 are fulfilled.

7.3.2 The discrete problem on $\bar{\Omega}$

We first convert the subspace $\tilde{\mathbf{H}}_h$ of $\tilde{\mathbf{H}}$ into a subspace of $\mathscr{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$. For this purpose, we shall use the properties of the mapping ψ_h: $\Gamma_h \to \Gamma$ defined previously. Using ψ_h^{-1} to transform integrals along Γ_h to integrals along Γ, we have for any $\tilde{\xi} \in L^2(\Gamma_h)$

$$(7.18) \qquad \int_{\Gamma_h} \tilde{\xi}(s)\, ds = \int_{\Gamma} (\tilde{\xi} \circ \psi_h^{-1})(s) J_h(s)\, ds ,$$

where $J_h(s) = (d/ds)\,\psi_h^{-1}(s)$ denotes the derivative of ψ_h^{-1} in the tangential direction to Γ. The identity (7.18) suggests the definition of the mapping $\mathcal{M}_\Gamma : L^2(\Gamma_h) \to L^2(\Gamma)$ with

$$\mathcal{M}_\Gamma(\tilde{\xi}) := (\tilde{\xi} \circ \psi_h^{-1}) J_h \quad \forall \tilde{\xi} \in L^2(\Gamma_h) .$$

Note that if $\tilde{\xi} \in \tilde{H}_{h,0}^{-1/2}$, where $\tilde{H}_{h,0}^{-1/2}$ is the finite element subspace defined in (7.8), and if $\xi = \mathcal{M}_\Gamma(\tilde{\xi})$, then

$$0 = \int_{\Gamma_h} \tilde{\xi}(s)\, ds = \int_{\Gamma} (\tilde{\xi} \circ \psi_h^{-1})(s) J_h(s)\, ds = \int_{\Gamma} \xi(s)\, ds ,$$

which proves that $\xi \in H_0^{-1/2}(\Gamma)$. Therefore, if we define

$$H_{h,0}^{-1/2} = \mathcal{M}_\Gamma(\tilde{H}_{h,0}^{-1/2}) ,$$

then clearly $H_{h,0}^{-1/2}$ is a subspace of $H_0^{-1/2}(\Gamma)$.

Similarly, let us define the mapping $\mathcal{M}_\Omega : C(\bar{\Omega}_h) \to C(\bar{\Omega})$ by

$$\mathcal{M}_\Omega \tilde{v} := v \quad \forall \tilde{v} \in C(\bar{\Omega}_h) ,$$

where $v(x) = \tilde{v}(x)$ for all $x \in \bar{\Omega} \cap \bar{\Omega}_h$; $v(x) = v(y)$ for all x on the line segment joining $y \in \Gamma_h$ and $\psi_h(y) \in \Gamma$; and $v(x) = v(y)$ for all x on the line segment joining $y \in l_{r,\Gamma_0}^+$ and $\psi_{h,0}(y) \in \mathscr{C}_{r,\Gamma_0}^+$ for all $r = 1, \ldots, N_2$. Here $\psi_{h,0}$ is the mapping that associates to each $y \in l_{r,\Gamma_0}^+$, $y \notin \mathscr{V}_h$ the intersection point of $\mathscr{C}_{r,\Gamma_0}^+$ with the line passing through y perpendicular to l_{r,Γ_0}^+, and $\psi_{h,0}(y) = y \,\forall\, y \in \mathscr{V}_h \cap l_{r,\Gamma_0}^+$. Then we introduce the following subspaces of $H^1(\Omega)$:

$$H_h^1 := \mathcal{M}_\Omega(\tilde{H}_h^1) ,$$

and

$$H_{h,0}^1 := \mathcal{M}_\Omega(\tilde{H}_{h,0}^1) .$$

According to the definitions of $\mathcal{M}_\Omega$ and $\tilde{H}_{h,0}^1$ (see (7.7)), it is clear that $v(x) = 0$ for all $x \in \cup_{r=1}^{N_2} \tau_{r,\Gamma_0}^+$ and for all $v \in H_{h,0}^1$. However, $H_{h,0}^1$ is not a subspace of $H_{\Gamma_0}^1(\Omega)$. In fact, although every $v \in H_{h,0}^1$ vanishes at the end points of the curve sides $\mathscr{C}_{r,\Gamma_0}^-$, $r = 1, \ldots, N_3$, its values on the rest of the points of $\mathscr{C}_{r,\Gamma_0}^-$ are not necessarily zero. In other words, $H_{h,0}^1$ is not a subspace of $H_{\Gamma_0}^1(\Omega)$ because the boundary condition at Γ_0 is not fully satisfied. We then define the subspace of $\mathscr{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$,

$$\mathbf{H}_h := H_{h,0}^1 \times H_{h,0}^{-1/2} .$$

It follows from the previous comment that $\mathbf{H}_h$ is not a subspace of $\mathbf{H} := H_{\Gamma_0}^1(\Omega) \times H_0^{-1/2}(\Gamma)$. Hence, our next goal is to use the subspace $\mathbf{H}_h$ to define a Galerkin approximation of (w, σ) in the form of (6.6).

Let us introduce the mapping $H^2(\Omega) \ni v \to I^h v := v^h \in H_h^1 = \mathcal{M}_\Omega(\tilde{H}_h^1)$, where v^h is the unique element of H_h^1 such that $v^h(P_j) = v(P_j) \,\forall\, P_j \in \mathscr{V}_h$. Then, given $g^h := I^h g$, by (H.1)′ and (H.2)′, we can define the semilinear form,

$$\mathbf{A}_h(v, z) := \sum_{i=1}^{2} \int_{\Omega^h} a_i(x, \nabla(v + g^h)(x)) \frac{\partial z}{\partial x_i}\, dx \tag{7.19}$$

for all $v, z \in H_{h,0}^1 = \mathcal{M}_\Omega(\tilde{H}_{h,0}^1)$. Here Ω^h is the domain bounded by Γ_h and Γ_0.

Now, by changing integrations from Γ_h to Γ, the discrete bilinear forms $\tilde{\mathbf{b}}_h$ and $\tilde{\mathbf{d}}_h$ from (7.11) and (7.12) then satisfy the following identities:

$$\tilde{\mathbf{b}}_h(\tilde{\xi}, \tilde{\delta}) = \mathbf{b}_h(\mathscr{M}_\Gamma \tilde{\xi}, \mathscr{M}_\Gamma \tilde{\delta}) \quad \forall \tilde{\xi}, \tilde{\delta} \in \tilde{H}_{h,0}^{-1/2}, \tag{7.20}$$

and

$$\tilde{\mathbf{d}}_h(\tilde{v}^-, \tilde{\delta}) = \mathbf{d}_h((\mathscr{M}_\Omega \tilde{v})^-, \mathscr{M}_\Gamma \tilde{\delta}) \quad \forall \tilde{v} \in \tilde{H}_{h,0}^1 \ \forall \tilde{\delta} \in \tilde{H}_{h,0}^{-1/2}, \tag{7.21}$$

where $\mathbf{b}_h$ and $\mathbf{d}_h$ are the bilinear forms defined by

$$\mathbf{b}_h(\xi, \delta) := \frac{1}{2\pi} \int_\Gamma \int_\Gamma \log \left\{ \frac{D}{|\psi_h^{-1}(x) - \psi_h^{-1}(y)|} \right\} \xi(y) \delta(x) \, ds_y \, ds_x \tag{7.22}$$

for all $\xi, \delta \in H_{h,0}^{-1/2}$, and

$$\mathbf{d}_h(v^-, \delta) := \tag{7.23}$$

$$\frac{1}{2\pi} \int_\Gamma \int_\Gamma \frac{(\psi_h^{-1}(x) - \psi_h^{-1}(y)) \cdot \nu(y)}{|\psi_h^{-1}(x) - \psi_h^{-1}(y)|^2} (v^-(y) - v^-(x)) \delta(x) J_h(y) \, ds_y \, ds_x$$

for all $v \in H_h^1$ and for all $\delta \in H_{h,0}^{-1/2}$. Note that in the definition of $\mathbf{d}_h$ we have used also the property that $v(\psi_h^{-1}(x)) = v(x)$ for all $x \in \Gamma$ and for all $v \in H_h^1$.

Similarly, given $v = \mathscr{M}_\Omega \tilde{v}$ with $\tilde{v} \in \tilde{H}_{h,0}^1$, and $\delta = \mathscr{M}_\Gamma \tilde{\delta}$ with $\tilde{\delta} \in \tilde{H}_{h,0}^{-1/2}$, we obtain

$$\langle v^-, \delta \rangle = \int_\Gamma v^-(s) \delta(s) \, ds = \int_\Gamma v^-(\psi_h^{-1}(s)) (\tilde{\delta} \circ \psi_h^{-1})(s) J_h(s) \, ds$$

$$= \int_{\Gamma_h} \tilde{v}^-(s) \tilde{\delta}(s) \, ds = \langle \tilde{v}^-, \tilde{\delta} \rangle_h .$$

Hence, we deduce that

$$\langle \tilde{v}^-, \tilde{\delta} \rangle_h = \langle (\mathscr{M}_\Omega \tilde{v})^-, \mathscr{M}_\Gamma \tilde{\delta} \rangle \quad \forall \tilde{v} \in \tilde{H}_{h,0}^1 \ \forall \tilde{\delta} \in \tilde{H}_{h,0}^{-1/2} . \tag{7.24}$$

Furthermore, given $\tilde{v} \in \tilde{H}_{h,0}^1$ and $v = \mathscr{M}_\Omega \tilde{v} \in H_{h,0}^1$, we clearly have $v^-(s) = \tilde{v}^-(\psi_h^{-1}(s))$, from which it follows that

$$\frac{dv^-}{ds}(s) = \frac{d\tilde{v}^-}{ds_h}(\psi_h^{-1}(s)) \frac{d}{ds} \psi_h^{-1}(s) .$$

Hence, we can write

$$\frac{dv^-}{ds} = \mathscr{M}_\Gamma \left(\frac{d\tilde{v}^-}{ds_h} \right) . \tag{7.25}$$

Also, since $\dfrac{d\tilde{v}^-}{ds_h} \in \tilde{H}_{h,0}^{-1/2}$, we deduce that $\dfrac{dv^-}{ds} \in H_{h,0}^{-1/2}$.

Collecting (7.20), (7.21), (7.24) and (7.25), we see that the discrete bilinear form $\tilde{\mathbf{B}}_h$ (cf. (7.14)) satisfies the identity

$$\tilde{\mathbf{B}}_h((\tilde{v}, \tilde{\xi}), (\tilde{z}, \tilde{\delta})) = \mathbf{B}_h(\mathscr{M}(\tilde{v}, \tilde{\xi}), \mathscr{M}(\tilde{z}, \tilde{\delta})) \tag{7.26}$$

for all $(\tilde{v}, \tilde{\xi}), (\tilde{z}, \tilde{\delta}) \in \tilde{\mathbf{H}}_h$, where $\mathbf{B}_h : \mathbf{H}_h \times \mathbf{H}_h \to \mathbb{R}$ is the bilinear form defined by

$$\mathbf{B}_h((v, \xi), (z, \delta)) := \mathbf{b}_h \left(\frac{dv^-}{ds}, \frac{dz^-}{ds} \right) + \mathbf{b}_h(\xi, \delta) + \mathbf{d}_h(z^-, \xi) \tag{7.27}$$

$$- \mathbf{d}_h(v^-, \delta) + \langle v^-, \delta \rangle - \langle z^-, \xi \rangle$$

for all $(v, \xi), (z, \delta) \in \mathbf{H}_h$, and $\mathcal{M}: C(\bar{\Omega}_h) \times L^2(\Gamma_h) \to C(\bar{\Omega}) \times L^2(\Gamma)$ is defined naturally by

$$\mathcal{M}(\tilde{v}, \tilde{\xi}) := (\mathcal{M}_\Omega \tilde{v}, \mathcal{M}_\Gamma \tilde{\xi}) \quad \forall (\tilde{v}, \tilde{\xi}) \in C(\bar{\Omega}_h) \times L^2(\Gamma_h) .$$

Now, in order to formulate the discrete problem on $\bar{\Omega}$ in terms of $\mathbf{A}_h$, $\mathbf{B}_h$ and $\mathbf{H}_h$, we need the following lemma (see Johnson and Nedelec 1980; Leroux 1977).

Lemma 7.3. *Let* $\mathbf{d}$, $\mathbf{b}$, $\mathbf{b}_h$, $\mathbf{d}_h$ *be the bilinear forms defined in* (7.1), (7.2), (7.22) *and* (7.23), *respectively. Then there exists a positive constant C such that for all* $h \in (0, h_0)$,

$$|\mathbf{b}(\xi, \delta) - \mathbf{b}_h(\xi, \delta)| \leqq Ch \| \xi \|_{H^{-1/2}(\Gamma)} \| \delta \|_{H^{-1/2}(\Gamma)} \quad \forall \xi, \delta \in H_{h,0}^{-1/2} ,$$

and

$$|\mathbf{d}(v^-, \delta) - \mathbf{d}_h(v^-, \delta)| \leqq Ch^{3/2} \| v \|_{H^1(\Omega)} \| \delta \|_{H^{-1/2}(\Gamma)} \quad \forall (v, \delta) \in H_h^1 \times H_{h,0}^{-1/2} .$$

As a by-product of this lemma one deduces from its proof (see e.g. Johnson and Nedelec 1980) that for all $\xi, \delta \in H_{h,0}^{-1/2}$, $v \in H_h^1$,

$$(7.28) \qquad |\mathbf{b}(\xi, \delta) - \mathbf{b}_h(\xi, \delta)| \leqq Ch^2 \| \xi \|_{L^2(\Gamma)} \| \delta \|_{L^2(\Gamma)} .$$

$$(7.29) \qquad |\mathbf{d}(v^-, \delta) - \mathbf{d}_h(v^-, \delta)| \leqq Ch^2 \| \delta \|_{L^2(\Gamma)} \| v \|_{H^1(\Omega)} .$$

As a consequence of Lemma 7.3, the continuity of $\mathbf{b}$ and $\mathbf{d}$ (cf. (7.1), (7.2)) and the trace theorem, we deduce that the bilinear form $\mathbf{B}_h$ defined by (7.27) is bounded on $\mathbf{H}_h \times \mathbf{H}_h$. Moreover, in virtue of the inequalities (7.28) and (7.29), we observe that the bilinear forms $\mathbf{b}_h$ and $\mathbf{d}_h$ are also well defined and bounded on $L^2(\Gamma) \times L^2(\Gamma)$ and $H^1(\Omega) \times L^2(\Gamma)$, respectively. Hence, the bilinear form $\mathbf{B}_h$ is well defined and bounded on $(H^1(\Omega) \times L^2(\Gamma)) \times (H^1(\Omega) \times L^2(\Gamma))$, as well.

We are now in a position to introduce the discrete operator equation in $\mathbf{H}_h$. First, we define the operator $\mathbf{T}_h: \mathbf{H}_h \to \mathbf{H}_h^*$, with

$$(7.30) \qquad [\mathbf{T}_h(v, \xi), (z, \delta)]_h := \mathbf{A}_h(v, z) + \mathbf{B}_h((v, \xi), (z, \delta))$$

for all (v, ξ), $(z, \delta) \in \mathbf{H}_h$, where $[\cdot, \cdot]_h$ denotes the duality pairing on $\mathbf{H}_h^* \times \mathbf{H}_h$. Further, similarly as in (7.16) and (3.21), since $\left(g^h, \frac{2b}{|\Gamma|} \right) \in H^1(\Omega) \times L^2(\Gamma)$, we can define the bounded linear functional $\mathbf{F}_h: \mathbf{H}_h \to \mathbf{R}$,

$$(7.31) \qquad [\mathbf{F}_h, (z, \delta)]_h := \int_{\Omega^h} fz\,dx - \mathbf{B}_h\left(\left(g^h, \frac{2b}{|\Gamma|} \right), (z, \delta)\right)$$

for all $(z, \delta) \in \mathbf{H}_h$. Therefore, we now formulate the following problem: *Find* $(w_h, \sigma_h) \in \mathbf{H}_h$ *such that*

$$(\mathbf{P}_h) \qquad [\mathbf{T}_h(w_h, \sigma_h), (z_h, \delta_h)]_h = [\mathbf{F}_h, (z_h, \delta_h)]_h$$

for all $(z_h, \delta_h) \in \mathbf{H}_h$. The solution of $(\mathbf{P}_h)$ (if it exists) will be called the *Galerkin approximation* of the solution (w, σ) of the continuous problem $(\mathbf{P})$.

At this point, in view of (7.26), one may wonder whether the functional $\mathbf{F}_h$ in (7.31) and the semilinear form $\mathbf{A}_h$ in (7.19) can be redefined so that the identities

$$(7.32) \qquad [\tilde{\mathbf{F}}, (\tilde{z}, \tilde{\delta})]_{\tilde{h}} = [\mathbf{F}_h, \mathcal{M}(\tilde{z}, \tilde{\delta})]_h \quad \forall (\tilde{z}, \tilde{\delta}) \in \tilde{\mathbf{H}}_h ,$$

and

$$(7.33) \qquad \tilde{\mathbf{A}}_h(\tilde{v}, \tilde{z}) = \mathbf{A}_h(\mathcal{M}_\Omega \tilde{v}, \mathcal{M}_\Omega \tilde{z}) \quad \forall \tilde{v}, \tilde{z} \in \tilde{H}_{h,0}^1$$

hold. If this were the case, then (7.32) and (7.33) together with (7.26) would imply that $(\mathbf{P}_h)$ is just a reformulation of $(\tilde{\mathbf{P}}_h)$, and hence, the solution of $(\mathbf{P}_h)$ would be given by $\mathcal{M}(\tilde{w}_h, \tilde{\sigma}_h)$, since $\mathbf{H}_h = \mathcal{M}(\tilde{\mathbf{H}}_h)$. Unfortunately, this is not possible. In fact, if $(z, \delta) = \mathcal{M}(\tilde{z}, \tilde{\delta})$, and $v = \mathcal{M}_\Omega \tilde{v}$, then from (7.10) and (7.16) we easily obtain that

$$\tilde{\mathbf{A}}_h(\tilde{v}, \tilde{z}) = \mathbf{A}_h(v, z) + \sum_{i=1}^{2} \sum_{r=1}^{N_3} \int_{\tau^-_{r,\Gamma_0}} a_i(x, \nabla(\tilde{v} + \tilde{g}^h)(x)) \frac{\partial \tilde{z}}{\partial x_i}\, dx ,$$

and

$$[\tilde{\mathbf{F}}, (\tilde{z}, \tilde{\delta})]_{\tilde{h}} = \int_{\Omega^h} fz\, dx - \mathbf{B}_h\left(\left(g^h, \mathcal{M}_\Gamma\left(\frac{2b}{|\Gamma|}\right)\right), (z, \delta)\right) + \sum_{r=1}^{N_3} \int_{\tau^-_{r,\Gamma_0}} f\tilde{z}\, dx .$$

Because of the extra terms involving the integration on $\bigcup_{r=1}^{N_3} \tau^-_{r,\Gamma_0}$, the conjecture is false. Nevertheless, one can prove that these extra terms are sufficiently small so that the solution (w_h, σ_h) of $(\mathbf{P}_h)$ can be approximated by the solution $(\bar{w}_h, \bar{\sigma}_h)$ of $(\tilde{\mathbf{P}}_h)$ in the sense that

$$\|(w_h, \sigma_h) - (\bar{w}_h, \bar{\sigma}_h)\|_{\mathscr{H}} = O(h) ,$$

where

$$(\bar{w}_h, \bar{\sigma}_h) := \mathcal{M}(\tilde{w}_h, \tilde{\sigma}_h) . \tag{7.34}$$

Because of this property, we refer to $(\bar{w}_h, \bar{\sigma}_h)$ as the *Quasi-Galerkin approximation* of (w, σ).

Now, the fact that the assumptions of the Theorem 6.4 are satisfied for the approximating operator $\mathbf{T}_h$ in (7.30) will follow from (H.1)′, (H.2)′, (H.6)′, Lemma 7.3, the coerciveness property of $\mathbf{b}$ (see (7.4)), and a discrete Friedrichs inequality (see Gatica and Hsiao 1989b, Lemma 4.6). Here, the condition that $\partial\Omega$ is piecewise C^3 must be added to our hypotheses. We omit the proof, but again we refer the interested reader to Gatica and Hsiao (1989b) for details.

Therefore, we deduce that there exists a unique $(w_h, \sigma_h) \in \mathbf{H}_h$ solution of $(\mathbf{P}_h)$. Moreover, if $(w, \sigma) \in \mathbf{H}$ is the unique solution of the continuous problem $(\mathbf{P})$, then we have the following error estimate:

$$\begin{aligned} \|(w, \sigma) - (w_h, \sigma_h)\|_{\mathscr{H}} \leqq C \Bigg\{ & \sup_{\substack{(z_h, \delta_h) \in \mathbf{H}_h \\ (z_h, \delta_h) \neq 0}} \frac{|[\mathbf{F}_h, (z_h, \delta_h)]_h - [\mathscr{T}(w, \sigma), (z_h, \delta_h)]|}{\|(z_h, \delta_h)\|_{\mathscr{H}}} \\ & + \inf_{(v_h, \xi_h) \in \mathbf{H}_h} \Bigg(\|(w, \sigma) - (v_h, \xi_h)\|_{\mathscr{H}} \\ & + \sup_{\substack{(z_h, \delta_h) \in \mathbf{H}_h \\ (z_h, \delta_h) \neq 0}} \frac{|[\mathscr{T}(v_h, \xi_h), (z_h, \delta_h)] - [\mathbf{T}_h(v_h, \xi_h), (z_h, \delta_h)]_h|}{\|(z_h, \delta_h)\|_{\mathscr{H}}} \Bigg) \Bigg\} \end{aligned} \tag{7.35}$$

This abstract error estimate will be needed for the convergence proof as well as the asymptotic rate of convergence in the next section.

7.4 Convergence analysis and error estimates

The main result concerning the convergence analysis of $(\mathbf{P}_h)$ and $(\tilde{\mathbf{P}}_h)$ can be summarized in the following theorem.

Theorem 7.4. *Let $(w, \sigma) \in \mathbf{H}$ and $(w_h, \sigma_h) \in \mathbf{H}_h$ be the unique solutions of the problems* (**P**) *and* ($\mathbf{P}_h$), *respectively. In addition, let $(\bar{w}_h, \bar{\sigma}_h) \in \mathbf{H}_h$ be the Quasi-Galerkin approximation defined by* (7.34). *Then,*

$$\lim_{h \to 0} \|(w, \sigma) - (w_h, \sigma_h)\|_{\mathscr{H}} = \lim_{h \to 0} \|(w, \sigma) - (\bar{w}_h, \bar{\sigma}_h)\|_{\mathscr{H}} = 0 .$$

Furthermore, if $w \in H^{2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, and if $\sigma \in H^{1/2}(\Gamma)$, then there exist positive constants C, h_0 independent of h, such that

$$\|(w, \sigma) - (w_h, \sigma_h)\|_{\mathscr{H}} \leqq Ch, \qquad and \qquad \|(w, \sigma) - (\bar{w}_h, \bar{\sigma}_h)\|_{\mathscr{H}} \leqq Ch$$

for all $h \in (0, h_0)$.

The proof for the error $\|(w, \sigma) - (w_h, \sigma_h)\|_{\mathscr{H}}$ is based on the estimate (7.35), and is given in this Subsection. Similarly, it is not difficult to prove (see e.g. Gatica and Hsiao (1989b, Sect. 5.2)) that $\|(w_h, \sigma_h) - (\bar{w}_h, \bar{\sigma}_h)\|_{\mathscr{H}} = O(h)$. Clearly, this estimate will complete the proof of Theorem 7.4.

In what follows we shall estimate the terms appearing on the right hand side of (7.35). We will need the following theorem, which is a consequence of the usual interpolation result for linear triangular elements (see e.g. Ciarlet 1978), the Calderón extension Theorem (see Gilbarg and Trudinger 1983, Theorem 7.25), and the Sobolev imbedding Theorem (see Kufner et al. 1977).

Theorem 7.5. *Given $\varepsilon > 0$, there exists a positive constant C independent of $h \in (0, h_0)$, such that*

$$\|v - v^h\|_{H^1(\Omega)} \leqq Ch \|v\|_{H^{2+\varepsilon}(\Omega)}$$

for all $v \in H^{2+\varepsilon}(\Omega)$ and for all $h \in (0, h_0)$, where $v^h := I^h v \in H_h^1$.

Proof. See Gatica and Hsiao (1989b, Theorem 5.3). □

The following result will be required to estimate the third term on the right hand side of (7.35).

Theorem 7.6. *There exists a positive constant C independent of $h \in (0, h_0)$, such that*

$$|\mathbf{A}(v, z) - \mathbf{A}_h(v, z)| \leqq Ch(1 + \|v\|_{H^1(\Omega)}) \|z\|_{H^1(\Omega)}$$

for all $v, z \in H_{h,0}^1$ and for all $h \in (0, h_0)$.

Proof. Let $v, z \in H_{h,0}^1$, and define the semilinear form

$$\hat{\mathbf{A}}_h(v, z) := \sum_{i=1}^{2} \int_{\Omega^h} a_i(x, \nabla(v + g)(x)) \frac{\partial z}{\partial x_i} dx .$$

Then, by the triangle inequality we have

$$(7.36) \qquad |\mathbf{A}(v, z) - \mathbf{A}_h(v, z)| \leqq |\mathbf{A}(v, z) - \hat{\mathbf{A}}_h(v, z)| + |\hat{\mathbf{A}}_h(v, z) - \mathbf{A}_h(v, z)| .$$

Now, according to the definition of **A** (see Subsec. 7.1), we can write

$$\mathbf{A}(v, z) - \hat{\mathbf{A}}_h(v, z) = \sum_{i=1}^{2} \sum_{r=1}^{N_1} \int_{\tau_{r,r}^+} a_i(x, \nabla(v + g)(x)) \frac{\partial z}{\partial x_i} dx .$$

It follows from (H.2)′ and Schwarz's inequality that

$$(7.37) \qquad |\mathbf{A}(v, z) - \hat{\mathbf{A}}_h(v, z)|$$

$$\leqq C\left\{ \sum_{r=1}^{N_1} (|\tau_{r,r}^+| + |g|_{H^1(\tau_{r,r}^+)}^2 + |v|_{H^1(\tau_{r,r}^+)}^2 + \sum_{i=1}^{2} \|\phi_i\|_{L^2(\tau_{r,r}^+)}^2) \right\}^{1/2}$$

$$\cdot \left\{ \sum_{r=1}^{N_1} |z|_{H^1(\tau_{r,r}^+)}^2 \right\}^{1/2} .$$

Since $g \in H^{2+\varepsilon}(\tilde{\Omega})$ and $\phi_i \in H^1(\tilde{\Omega})$, we obtain from Lemma 7.1 that

$$\sum_{r=1}^{N_1} |g|_{H^1(\tau_{r,r}^+)}^2 \leqq Ch^2 \|g\|_{H^2(\Omega)}^2 ,$$

and

$$\sum_{r=1}^{N_1} \|\phi_i\|_{L^2(\tau_{r,r}^+)}^2 \leqq Ch^2 \|\phi_i\|_{H^1(\Omega)}^2 .$$

On the other hand, it is not difficult to prove (see Gatica and Hsiao 1989b, Lemma 4.5) that

$$\sum_{r=1}^{N_1} |v|_{H^1(\tau_{r,r}^+)}^2 \leqq Ch|v|_{H^1(\Omega)}^2 ,$$

for all $v \in H_{h,0}^1$. Therefore, Lemma 7.1 and applying the above inequality to v and z imply that

$$|\mathbf{A}(v, z) - \hat{\mathbf{A}}_h(v, z)| \leqq C\{h^2 + h|v|_{H^1(\Omega)}^2\}^{1/2} h^{1/2} |z|_{H^1(\Omega)}$$

from (7.37), and hence

$$(7.38) \qquad |\mathbf{A}(v, z) - \hat{\mathbf{A}}_h(v, z)| \leqq Ch\{h^{1/2} + \|v\|_{H^1(\Omega)}\} \|z\|_{H^1(\Omega)} .$$

We note that here the constant C depends also on $\|g\|_{H^2(\Omega)}$ and $\|\phi_i\|_{H^1(\Omega)}$, $i = 1, 2$. As for the second term in the right side of (7.36), we have

$$\hat{\mathbf{A}}_h(v, z) - \mathbf{A}_h(v, z) = \sum_{i=1}^{2} \int_{\Omega^h} \{A_i(v)(x) - A_i(v + g^h - g)(x)\} \frac{\partial z}{\partial x_i} dx ,$$

where A_i denotes the Nemytsky operator which maps $H^1(\Omega)$ continuously into $L^2(\Omega)$ (see Sect. 3). Thus, since by (H.7)′ A_i is Lipschitz continuous (see Sect. 6), we obtain that

$$|\hat{\mathbf{A}}_h(v, z) - \mathbf{A}_h(v, z)| \leqq \sum_{i=1}^{2} \|A_i(v) - A_i(v + g^h - g)\|_{L^2(\Omega)} \|z\|_{H^1(\Omega)}$$

$$\leqq C\|g - g^h\|_{H^1(\Omega)} \|z\|_{H^1(\Omega)} .$$

Hence, by applying Theorem 7.5 to $g \in H^{2+\varepsilon}(\Omega)$, we conclude,

$$|\hat{\mathbf{A}}_h(v, z) - \mathbf{A}_h(v, z)| \leqq Ch \|g\|_{H^{2+\varepsilon}(\Omega)} \|z\|_{H^1(\Omega)} . \tag{7.39}$$

The inequalities (7.38), (7.39) and (7.36) yield the desired result. □

We are ready now to estimate the third term on the right hand side of (7.35). In fact, as a consequence of Lemma 7.3 and Theorem 7.6 we can prove the following result.

Theorem 7.7. *There exists a positive constant C independent of $h \in (0, h_0)$, such that*

$$\sup_{\substack{(z_h, \delta_h) \in \mathbf{H}_h \\ (z_h, \delta_h) \neq 0}} \frac{|[\mathcal{T}(v_h, \xi_h), (z_h, \delta_h)] - [\mathbf{T}_h(v_h, \xi_h), (z_h, \delta_h)]_h|}{\|(z_h, \delta_h)\|_{\mathcal{H}}} \leqq Ch\{1 + \|(v_h, \xi_h)\|_{\mathcal{H}}\}$$

for all $(v_h, \xi_h) \in \mathbf{H}_h$ and for all $h \in (0, h_0)$.

Proof. See Gatica and Hsiao (1989b, Theorem 5.5). □

In order to estimate the first term on the right hand side of (7.35) we make use of some results involving curved finite elements (see Feistauer and Zenišek 1987; Zlamal 1973) and deduce that for each $z_h \in H^1_{h,0}$ there exists $\hat{z}_h \in H^1_{\Gamma_0}(\Omega)$ such that

$$\|z_h - \hat{z}_h\|_{H^1(\Omega)} \leqq Ch \|z_h\|_{H^1(\Omega)} . \tag{7.40}$$

With this auxiliary result, we can now prove the following two lemmas.

Lemma 7.8. *There exists a positive constant C independent of $h \in (0, h_0)$ such that*

$$\| [\mathcal{F}, (z_h, \delta_h)] - [\mathcal{T}(w, \sigma), (z_h, \delta_h)] | \leqq Ch \|(z_h, \delta_h)\|_{\mathcal{H}}$$

for all $(z_h, \delta_h) \in \mathbf{H}_h$ and for all $h \in (0, h_0)$.

Proof. Let $(z_h, \delta_h) \in \mathbf{H}_h$. Then, we can write

$$[\mathcal{F}, (z_h, \delta_h)] = [\mathcal{F}, (\hat{z}_h, \delta_h)] + [\mathcal{F}, (z_h - \hat{z}_h, 0)] , \tag{7.41}$$

and, similarly,

$$[\mathcal{T}(w, \sigma), (z_h, \delta_h)] = [\mathcal{T}(w, \sigma), (\hat{z}_h, \delta_h)] + [\mathcal{T}(w, \sigma), (z_h - \hat{z}_h, 0)] . \tag{7.42}$$

Since $\hat{z}_h \in H^1_{\Gamma_0}(\Omega)$ and $\delta_h \in H^{-1/2}_{h,0} \subseteq H_0^{-1/2}(\Gamma)$, we see that $(\hat{z}_h, \delta_h) \in \mathbf{H}$. Then, since (w, σ) is the unique solution of the problem (**P**), we easily obtain that

$$[\mathcal{T}(w, \sigma), (\hat{z}_h, \delta_h)] = [\mathcal{F}, (\hat{z}_h, \delta_h)] .$$

Consequently, by subtracting (7.42) from (7.41), and using (7.40), we deduce

$$\begin{aligned} |[\mathcal{F},(z_h, \delta_h)] - [\mathcal{T}(w, \sigma), (z_h, \delta_h)]| &= |[\mathcal{F}, (z_h - \hat{z}_h, 0)] - [\mathcal{T}(w, \sigma), (z_h - \hat{z}_h, 0)]| \\ &\leqq \{ \|\mathcal{F}\|_{\mathcal{H}^*} + \|\mathcal{T}(w, \sigma)\|_{\mathcal{H}^*}\} \|z_h - \hat{z}_h\|_{H^1(\Omega)} \\ &\leqq Ch\{ \|\mathcal{F}\|_{\mathcal{H}^*} + \|\mathcal{T}(w, \sigma)\|_{\mathcal{H}^*}\} \|z_h\|_{H^1(\Omega)} \\ &\leqq Ch \|(z_h, \delta_h)\|_{\mathcal{H}} . \end{aligned}$$

This completes the proof. □

Lemma 7.9. *There exists a positive constant C independent of* $h \in (0, h_0)$ *such that*

$$|[\mathbf{F}_h, (z_h, \delta_h)]_h - [\mathscr{F}, (z_h, \delta_h)]| \leqq Ch \|(z_h, \delta_h)\|_{\mathscr{H}} \tag{7.43}$$

for all $(z_h, \delta_h) \in \mathbf{H}_h$ *and for all* $h \in (0, h_0)$.

Proof. We easily obtain from the definitions of $\mathscr{F}$ and $\mathbf{F}_h$ (see (3.19) and (7.31)),

$$\begin{aligned} |[\mathbf{F}_h, (z_h, \delta_h)]_h - [\mathscr{F}, (z_h, \delta_h)]| &\leqq \left| \int_{\Omega} f z_h \, dx - \int_{\Omega^h} f z_h \, dx \right| \\ &+ |\mathbf{B}((g - g^h, 0), (z_h, \delta_h))| + \left| \mathbf{B}\left(\left(g^h, \frac{2b}{|\Gamma|} \right), (z_h, \delta_h) \right) \right. \\ &\left. - \mathbf{B}_h\left(\left(g^h, \frac{2b}{|\Gamma|} \right), (z_h, \delta_h) \right) \right| . \end{aligned} \tag{7.44}$$

Now, since $\Omega - \bar{\Omega}^h = \cup_{r=1}^{N_1} \tau_{r,\Gamma}^+$, we get by Schwarz's inequality,

$$\begin{aligned} \left| \int_{\Omega} f z_h \, dx - \int_{\Omega^h} f z_h \, dx \right| &= \left| \sum_{r=1}^{N_1} \int_{\tau_{r,\Gamma}^+} f z_h \, dx \right| \\ &\leqq \sum_{r=1}^{N_1} \|f\|_{L^2(\tau_{r,\Gamma}^+)} \|z_h\|_{L^2(\tau_{r,\Gamma}^+)} \\ &\leqq \left\{ \sum_{r=1}^{N_1} \|f\|_{L^2(\tau_{r,\Gamma}^+)}^2 \right\}^{1/2} \left\{ \sum_{r=1}^{N_1} \|z_h\|_{L^2(\tau_{r,\Gamma}^+)}^2 \right\}^{1/2} . \end{aligned}$$

Hence, by applying Lemma 7.1 to $z_h \in H^1(\Omega)$, for given $f \in L^2(\Omega)$, we deduce

$$\left| \int_{\Omega} f z_h \, dx - \int_{\Omega^h} f z_h \, dx \right| \leqq C \|f\|_{L^2(\Omega)} h \|z_h\|_{H^1(\Omega)} . \tag{7.45}$$

Now, from the boundedness of $\mathbf{B}$ and applying Theorem 7.5 to g, it follows that

$$|\mathbf{B}((g - g^h, 0), (z_h, \delta_h))| \leqq C \|g\|_{H^{2+\varepsilon}(\Omega)} h \|(z_h, \delta_h)\|_{\mathscr{H}} . \tag{7.46}$$

Further, from the definitions of $\mathbf{B}$ and $\mathbf{B}_h$ (see (7.5) and (7.27)), we obtain easily

$$\begin{aligned} &\left| \mathbf{B}\left(\left(g^h, \frac{2b}{|\Gamma|} \right), (z_h, \delta_h) \right) - \mathbf{B}_h\left(\left(g^h, \frac{2b}{|\Gamma|} \right), (z_h, \delta_h) \right) \right| \\ &\leqq \left| \mathbf{b}\left(\frac{dg^h}{ds}, \frac{dz_h}{ds} \right) - \mathbf{b}_h\left(\frac{dg^h}{ds}, \frac{dz_h}{ds} \right) \right| + \frac{2b}{|\Gamma|} |\mathbf{b}(1, \delta_h) - \mathbf{b}_h(1, \delta_h)| \\ &+ \frac{2b}{|\Gamma|} |\mathbf{d}(z_h^-, 1) - \mathbf{d}_h(z_h^-, 1)| + |\mathbf{d}((g^h)^-, \delta_h) - \mathbf{d}_h((g^h)^-, \delta_h)| . \end{aligned} \tag{7.47}$$

Then it follows from Lemma 7.3 and the approximation property of g^h (see Theorem 7.5) that the first and fourth term on the right hand side of (7.47) are bounded, respectively, by

$$C \|g\|_{H^{2+\varepsilon}(\Omega)} h \|z_h\|_{H^1(\Omega)}, \text{ and } C \|g\|_{H^{2+\varepsilon}(\Omega)} h^{3/2} \|\delta_h\|_{H^{-1/2}(\Gamma)} .$$

Moreover, from (7.28) and (7.29), we have, respectively,

$$|\mathbf{b}(1, \delta_h) - \mathbf{b}_h(1, \delta_h)| \leqq Ch^2 \|\delta_h\|_{L^2(\Gamma)} ,$$

and

$$|\mathbf{d}(z_h^-, 1) - \mathbf{d}_h(z_h^-, 1)| \leqq Ch^2 \|z_h\|_{H^1(\Omega)} .$$

Therefore, since $H_{h,0}^{-1/2}$ satisfies the inverse assumption (see Leroux 1977; Hsiao and Wendland 1977), it follows that the second term on the right hand side of (7.47) is bounded by $Ch^{3/2} \|\delta_h\|_{H^{-1/2}(\Gamma)}$. Thus, we obtain the estimate

$$(7.48) \quad \left|\mathbf{B}\left(\left(g^h, \frac{2b}{|\Gamma|}\right), (z_h, \delta_h)\right) - \mathbf{B}_h\left(\left(g^h, \frac{2b}{|\Gamma|}\right), (z_h, \delta_h)\right)\right| \leqq Ch \|(z_h, \delta_h)\|_{\mathscr{H}} .$$

Consequently, (7.44), (7.45), (7.46) and (7.48) yield (7.43). □

We are now in a position to estimate the first term on the right hand side of (7.35). In fact, we have the following main result.

Theorem 7.10. *There exists a positive constant C independent of $h \in (0, h_0)$, such that*

$$\sup_{\substack{(z_h, \delta_h) \in \mathbf{H}_h \\ (z_h, \delta_h) \neq 0}} \frac{|[\mathbf{F}_h, (z_h, \delta_h)]_h - [\mathscr{T}(w, \sigma), (z_h, \delta_h)]|}{\|(z_h, \delta_h)\|_{\mathscr{H}}} \leqq Ch$$

for all $h \in (0, h_0)$.

Proof. It follows easily from triangle inequality and Lemmas 7.8 and 7.9. □

As a consequence of Theorem 7.7 and Theorem 7.10, the abstract error estimate (7.35) can be rewritten as follows,

$$(7.49) \quad \|(w, \sigma) - (w_h, \sigma_h)\|_{\mathscr{H}} \leqq C\{h + e_h(w, \sigma)\} ,$$

where

$$(7.50) \quad e_h(w, \sigma) = \inf_{(v_h, \xi_h) \in \mathbf{H}_h} \{\|(w, \sigma) - (v_h, \xi_h)\|_{\mathscr{H}} + (1 + \|(v_h, \xi_h)\|_{\mathscr{H}})h\} .$$

Now, we recall from Leroux (1977) (see also Johnson and Nedelec 1980) the following *approximation property* of the subspace $H_{h,0}^{-1/2}$: For any $\xi \in H_0^{s-1/2}(\Gamma)$, $0 \leqq s \leqq 1$, there exists $\xi^h \in H_{h,0}^{-1/2}$ such that

$$(7.51) \quad \|\xi - \xi^h\|_{H^{-1/2}(\Gamma)} \leqq Ch^s \|\xi\|_{H^{s-1/2}(\Gamma)} \quad \forall h \in (0, h_0) ,$$

where the constant C is independent of h and ξ. We are ready now to complete the proof of Theorem 7.4.

Proof of Theorem 7.4. As expected, the proof reduces to estimate the term $e_h(w, \sigma)$ on the right hand side of (7.49). First of all, we note that the spaces $H_{\Gamma_0}^1(\Omega) \cap C^\infty(\bar{\Omega})$, and $H_0^{-1/2}(\Gamma) \cap C^\infty(\Gamma)$ are dense in $H_{\Gamma_0}^1(\Omega)$ and $H_0^{-1/2}(\Gamma)$, respectively (see e.g., Gilbarg and Trudinger 1983, Theorem 7.25). We then define the product space

$$\mathbf{H}^\infty := [H_{\Gamma_0}^1(\Omega) \cap C^\infty(\bar{\Omega})] \times [H_0^{-1/2}(\Gamma) \cap C^\infty(\Gamma)] ,$$

which is dense in $\mathbf{H}$. Thus, given $\eta > 0$, there exists $(w_\eta, \sigma_\eta) \in \mathbf{H}^\infty$ such that

$$\|(w, \sigma) - (w_\eta, \sigma_\eta)\|_{\mathscr{H}} < \frac{\eta}{2} .$$

Since $w_\eta \in C^\infty(\bar{\Omega})$ and $\sigma_\eta \in C^\infty(\Gamma)$, from Theorem 7.5 and the approximation property (7.51), we deduce that there exists $0 < \bar{h}(\eta) < h_0$ such that

$$\|(w_\eta, \sigma_\eta) - (w_\eta^h, \sigma_\eta^h)\|_{\mathscr{H}} < \frac{\eta}{2}$$

for all $h < \bar{h}(\eta)$.

Therefore, given $\eta > 0$, there exists $0 < \bar{h}(\eta) < h_0$ such that

$$\|(w, \sigma) - (w_\eta^h, \sigma_\eta^h)\|_{\mathscr{H}} < \eta \tag{7.52}$$

for all $h < \bar{h}(\eta)$. Since $(w_\eta^h, \sigma_\eta^h) \in \mathbf{H}_h$, we obtain from (7.50) and (7.52) that

$$\begin{aligned} e_h(w, \sigma) &\leqq \|(w, \sigma) - (w_\eta^h, \sigma_\eta^h)\|_{\mathscr{H}} + (1 + \|(w_\eta^h, \sigma_\eta^h)\|_{\mathscr{H}})h \\ &\leqq \eta + (1 + \eta + \|(w, \sigma)\|_{\mathscr{H}})h \quad \forall h \in (0, \bar{h}(\eta)) . \end{aligned} \tag{7.53}$$

In summary, it follows from (7.49) and (7.53) that given $\eta > 0$, there exists $h(\eta) := \min\{\eta, \bar{h}(\eta)\}$ such that

$$\|(w, \sigma) - (w_h, \sigma_h)\|_{\mathscr{H}} \leqq C\{h + \eta + (1 + \eta + \|(w, \sigma)\|_{\mathscr{H}})h\} \leqq C\eta$$

for all $h < h(\eta)$. This proves the convergence result.

In addition, if $w \in H^{2+\varepsilon}(\Omega)$ and $\sigma \in H^{1/2}(\Gamma)$, then by Theorem 7.5 and the approximation property (7.51) (with $s = 1$), we get for all $h \in (0, h_0)$

$$\|(w, \sigma) - (w^h, \sigma^h)\|_{\mathscr{H}} \leqq C\{\|w\|_{H^{2+\varepsilon}(\Omega)} + \|\sigma\|_{H^{1/2}(\Gamma)}\}h , \tag{7.54}$$

where $w^h := I^h w \in H^1_{h,0}$, and σ^h is given by (7.51) with σ instead of ξ. It follows from (7.50) and (7.54) that

$$e_h(w, \sigma) \leqq \|(w, \sigma) - (w^h, \sigma^h)\|_{\mathscr{H}} + (1 + \|(w^h, \sigma^h)\|_{\mathscr{H}})h \leqq Ch . \tag{7.55}$$

Hence, substituting (7.55) into (7.49) we obtain the convergence rate $O(h)$.

We emphasize that $(\mathbf{P}_h)$ has been introduced only to facilitate the proof of the convergence result. For numerical purposes, the actual computations must be carried on the discrete problem $(\tilde{\mathbf{P}}_h)$, and then transferred to $\bar{\Omega}$ using the maping $\mathscr{M}$.

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