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The Lebesgue constant for Lagrange interpolation on equidistant nodes

T.M. Mills and S.J. Smith

Department of Mathematics, La Trobe University College of Northern Victoria, P.O. Box 199, Bendigo, Victoria 3550, Australia

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Summary. For $n=1, 2, 3, \ldots$, let Λ_n denote the Lebesgue constant for Lagrange interpolation based on the equidistant nodes $x_{k,n}=k$, $k=0, 1, 2, \ldots, n$. In this paper an asymptotic expansion for $\log \Lambda_n$ is obtained, thereby improving a result of A. Schönhage.

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1 Introduction

Consider the matrix X of points $x_{k,n}$, $k=0, 1, 2, \ldots, n$; $n=1, 2, 3, \ldots$, where

$$0 \leqq x_{0,n} < x_{1,n} < x_{2,n} < \ldots < x_{n,n} \leqq n,$$

and define

$$l_{k,n}(X, x) = \frac{\omega_n(X, x)}{\omega_n'(X, x_{k,n})(x - x_{k,n})},$$

$$\omega_n(X, x) = \prod_{k=0}^{n} (x - x_{k,n}).$$

As is well known, the magnitude of the Lebesgue constant

$$\Lambda_n(X) = \max_{0 \leqq x \leqq n} \sum_{k=0}^{n} |l_{k,n}(X, x)|$$

plays a crucial role in determining the convergence behaviour of Lagrange interpolation polynomials based on X (see, for instance, [2]). Now let X' denote the special case when X is the matrix of equally spaced points

$$X': \; x_{k,n} = k, \quad k = 0, 1, 2, \ldots, n; \quad n = 1, 2, 3, \ldots,$$

Offprint requests to: S.J. Smith

and put $\Lambda_n = \Lambda_n(X')$. As discussed in Trefethen and Weideman [3], various authors have developed estimates for Λ_n, although many of the efforts seem to have been duplicates of one another. The first good estimate for Λ_n was due to Turetskii [4], who obtained

$$\Lambda_n \sim \frac{2^{n+1}}{e\, n \log n}, \quad \text{as} \quad n \to \infty.$$

This result was improved slightly by Schönhage [1], whose result can be expressed in the form

$$\log \Lambda_n = (n+1)\log 2 - \log n - \log\log n - 1 - \frac{\gamma}{\log n} + O\left(\frac{1}{(\log n)^2}\right), \tag{1}$$

where $\gamma = 0.577\ldots$ is Euler's constant. In this paper we extend (1) by proving the following result.

Theorem. *There exists an asymptotic expansion of* $\log \Lambda_n$ *of the form*

$$\log \Lambda_n = (n+1)\log 2 - \log n - \log\log n - 1 + \sum_{k=1}^{m} \frac{A_k}{(\log n)^k} + O\left(\frac{1}{(\log n)^{m+1}}\right), \quad m = 1, 2, 3, \ldots, \tag{2}$$

where

$$A_1 = -\gamma, \quad A_2 = \gamma^2/2 - \pi^2/12, \quad A_3 = -\gamma^3/3 + \gamma\pi^2/6 - \zeta(3)/3, \quad A_4 = \gamma^4/4 + \gamma\zeta(3) - \gamma^2\pi^2/4 + \pi^4/90, \ldots. \tag{3}$$

$\left(\textit{Here } \zeta(3) = \sum_{r=1}^{\infty} r^{-3}.\right)$

We remark that it appears to be a difficult problem to obtain an explicit general formula for the A_k. We also point out that if Λ_n^* is defined by

$$\Lambda_n^* = \min_X \Lambda_n(X),$$

then

$$\log \Lambda_n^* = \log\log n + \log(2/\pi) + \frac{\gamma + \log(4/\pi)}{\log n} + o\left(\frac{1}{\log n}\right). \tag{4}$$

(See [2, Theorem 3.29].) A comparison of (2) and (4) shows that the equally spaced nodes X' can be regarded as very "bad" from the point of view of interpolation.

2 Proof of the Theorem

As shown by Schönhage [1],

$$\Lambda_n = \max_{0 \leqq x \leqq 1} \lambda_n(x), \tag{5}$$

where $\lambda_n(x)$ is the unique polynomial of degree n that satisfies

$$\lambda_n(0)=1; \quad \lambda_n(k)=(-1)^{k-1}, \quad k=1, 2, \ldots, n.$$

A straightforward induction argument shows that

$$\lambda_n(x)=\frac{2(1-x)_n}{n!}-1+2x+\sum_{k=2}^{n} g_k(x), \tag{6}$$

where

$$(1-x)_k=(1-x)(2-x)\ldots(k-x),$$

$$g_k(x)=\frac{2^k}{k!}\,x(1-x)_{k-1}=\frac{2^k}{k!}\,x\,\frac{\Gamma(k-x)}{\Gamma(1-x)}, \quad k=2, 3, \ldots. \tag{7}$$

Suppose x_k in (0, 1) is such that

$$\max_{0 \leqq x \leqq 1} g_k(x)=g_k(x_k). \tag{8}$$

From (5), (6) and (8) it follows that

$$\sum_{k=2}^{n}(g_k(x_n))+O(1) \leqq \Lambda_n \leqq \sum_{k=2}^{n}(g_k(x_k))+O(1). \tag{9}$$

Now, if $0 \leqq x \leqq 1$, then $g_k(x) \leqq 2^k k^{-1}$, and so

$$\begin{aligned}\sum_{k=2}^{n-[\log_2 n]} g_k(x_k) &\leqq \sum_{k=2}^{[n/2]}(2^k)+2n^{-1}\sum_{k=[n/2+1]}^{n-[\log_2 n]}(2^k)\\ &=O(2^{n/2})+O(2^n n^{-2})\\ &=O(2^n n^{-2}).\end{aligned}$$

Thus (9) yields

$$\sum_{k=n-[\log_2 n]}^{n}(g_k(x_n))+O(1) \leqq \Lambda_n \leqq \sum_{k=n-[\log_2 n]}^{n}(g_k(x_k))+O(2^n n^{-2}). \tag{10}$$

To make use of (10) we need to estimate x_k for $n-[\log_2 n]\leqq k\leqq n$. Now, x_k is given by the solution in (0, 1) to the equation

$$\frac{d}{dx}\log[x(1-x)(2-x)\dots(k-1-x)]=0.$$

This gives

$$\frac{1}{x}-\sum_{r=1}^{k-1}\frac{1}{r-x}=0,$$

or

$$\sum_{j=1}^{\infty}\left(\sum_{r=1}^{k-1}r^{-j}\right)x^j=1.$$

Define

$$S_1=\gamma;\qquad S_j=\sum_{r=1}^{\infty}r^{-j},\qquad j=2,3,\dots.$$

Since

$$\sum_{r=1}^{k-1}r^{-1}=\log k+S_1+O(k^{-1}),$$

$$\sum_{r=1}^{k-1}r^{-j}=S_j+O(k^{1-j}),\qquad j=2,3,\dots,$$

it follows that x_k satisfies

$$(11)\qquad (\log k+S_1+O(k^{-1}))\,x_k+\sum_{j=2}^{\infty}(S_j+O(k^{1-j}))\,x_k^j=1.$$

Now consider the equation

$$(12)\qquad (\log n+S_1)\,x+\sum_{j=2}^{\infty}S_jx^j=1,$$

which has a unique solution $x=x_n^*$ in (0, 1). Upon substituting $x=x_n^*$ in (12), then subtracting (11), it follows from $x_k=O((\log k)^{-1})$, $x_n^*=O((\log n)^{-1})$, that

$$x_n^*-x_k=O((n\log n)^{-1}),\qquad n-[\log_2 n]\leqq k\leqq n,$$

where the $O((n\log n)^{-1})$ term can be made independent of k. Consequently, if $n-[\log_2 n]\leqq k\leqq n$, x_k is given to within $O((n\log n)^{-1})$ terms as the solution in (0, 1) to (12), which can be written more succinctly as

$$\psi(1-x)+x^{-1}=\log n.$$

(Here $\psi(\cdot)$ denotes the logarithmic derivative of the gamma function.) Note that (12) determines an asymptotic expansion for $x=x_k$, $n-[\log_2 n]\leqq k\leqq n$, of the form

$$(13)\qquad x=\sum_{j=1}^{m}\frac{B_j}{(\log n)^j}+O\left(\frac{1}{(\log n)^{m+1}}\right),\qquad m=1,2,3,\dots,$$

where the first few of the coefficients B_j are given by

$$(14)\qquad B_1=1,\quad B_2=-\gamma,\quad B_3=\gamma^2-\pi^2/6,$$
$$B_4=-\gamma^3+\gamma\pi^2/2-\zeta(3),\quad B_5=\gamma^4-\gamma^2\pi^2+4\gamma\zeta(3)+2\pi^4/45.$$

We next consider $g_k(x)$, as defined by (7). From the well-known asymptotic expansion for the gamma function,

$$\Gamma(x)=\sqrt{2\pi}\,e^{-x+(x-1/2)\log x}(1+O(x^{-1})),$$

it follows that if x is given by (13), and if $n-[\log_2 n]\leqq k\leqq n$, then

$$\frac{g_k(x)}{2^k}=\frac{x}{\Gamma(1-x)}\frac{\Gamma(k-x)}{\Gamma(k+1)}=\frac{x}{\Gamma(1-x)}\frac{e^{-x\log n}}{n}(1+O(n^{-1})).$$

Thus both $\sum_{k=n-[\log_2 n]}^{n}(g_k(x_n))$ and $\sum_{k=n-[\log_2 n]}^{n}(g_k(x_k))$ are of the form

$$\left(\sum_{k=n-[\log_2 n]}^{n}2^k\right)\frac{x}{\Gamma(1-x)}\frac{e^{-x\log n}}{n}(1+O(n^{-1}))=\frac{2^{n+1}}{n}\frac{x\,e^{-x\log n}}{\Gamma(1-x)}(1+O(n^{-1})),$$

where x is given by (13). From (10) we conclude that

$$\Lambda_n=\frac{2^{n+1}}{n}\frac{x\,e^{-x\log n}}{\Gamma(1-x)}+O(2^n n^{-2}),$$

and thus

$$(15)\qquad \log\Lambda_n=(n+1)\log 2-(1+x)\log n+\log x-\log\Gamma(1-x)+O(n^{-1}\log n)$$
$$=(n+1)\log 2-(1+x)\log n+\log x-\sum_{k=1}^{\infty}S_k\frac{x^k}{k}+O(n^{-1}\log n).$$

Upon substituting (13) and (14) in (15), the asymptotic expansion (2) and coefficients (3) are obtained, and hence the theorem is established.

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