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A Nonlinear Theory of Laser Noise and Coherence. I*

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With 2 Figures in the Text

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We consider the interaction of a set of atoms at random lattice sites with a decaying resonator mode. The optical transition is supposed to possess a homogeneously broadened Lorentzian line. The pumping is taken into account explicitly as a stochastic process. After elimination of the atomic coordinates a second order nonlinear differential equation for the light amplitude is found. In between excitation collisions this equation can be solved exactly if the resonator width is large as compared to all other frequency differences. In contrast to linear theories there exists a marked threshold. Below it the amplitude decreases after each excitation exponentially and the linewidth turns out to be identical with those of previous authors (for instance WAGNER and BIRNBAUM), if specialized to large cavity width. Above the threshold the light amplitude converges towards a stable value, whereas the phase undergoes some kind of undamped diffusion process. We then consider the general case with arbitrary cavity width. If the general equation of motion of the light amplitude is interpreted as that of a particle moving in two dimensions, it becomes clear that also in this case the amplitude oscillates above threshold around a stable value which is identical with that determined in previous papers by HAKEN and SAUERMANN neglecting laser noise. This stable value may, however, undergo shifts, if there are slow systematic changes of the cavity width, inversion etc. On the other hand the phase still fluctuates in an undamped way. After splitting off the phase factor the equations can be linearized and solved explicitly. With these solutions simple examples of correlation functions are calculated in a semiclassical way, thus yielding expressions for the line width above threshold. The results can also be used to evaluate from first principles correlation functions for different laser beams. As an example the complex degree of mutual coherence of two laser beams is determined. It vanishes if one of the lasers is still below threshold and its value is close to unity well above threshold for observation times small compared to the inverse laser linewidth.

§1. Introduction

There are nowadays two main groups of theoretical investigations of laser oscillation which go beyond the application of rate equations[1]. The one group[2], which mainly investigates laser noise, treats laser action by assuming that the noise of the spontaneous emission is amplified by

* Presented by H. SAUERMANN at Summerschool in Les Houches, 1964.

[1] STATZ, H., and G. A. DE MARS: Quantum Electronics, edit. by C. H. TOWNES. New York: Columbia University Press 1960. — TANG, C. L.: J. Appl. Phys. **34**, 2935 (1963).

[2] SCHAWLOW, A. L., and C. H. TOWNES: Phys. Rev. **112**, 1940 (1958). — WAGNER, W. G., and G. BIRNBAUM: J. Appl. Phys. **32**, 1185 (1961). — FLECK jr., J. A.: J. Appl. Phys. **34**, 2997 (1963). — POUND, R. V.: Ann. Phys. (N.Y.) **1**, 24 (1957). — WEBER, J.: Rev. Mod. Phys. **31**, 681 (1959). — STRANDBERG, M. P. W.: Phys. Rev. **106**, 617 (1957). — Fully quantum theoretical treatments are given by MCCUMBER, D. E.: Phys. Rev. **130**, 675 (1962). — WELLS, W. H.: Ann. Phys. (N.Y.) **12**, 1 (1961). — KEMENY, G.: Phys. Rev. **133** (1A), A 69 (1964). — SCHWABL, F., u. W. THIRRING: Ergeb. exakt. Naturw. **36**, 219 (1964).

stimulated emission. In these investigations a theory is used which is basically linear in the light oscillator coordinate. The other group of papers[3], however, neglects spontaneous emission completely and investigates the free oscillation of the system where just the nonlinearities play an essential role for the stabilisation of laser oscillation. Furthermore, only these nonlinearities make it possible to explain effects such as coexistence of laser-modes in a homogeneously broadened line[3,4] and hole-burning in an inhomogeneously broadened line leading to frequency pushing[3,5]. In this nonlinear treatment a definite threshold of the inversion occurs above and below which the behaviour of the system differs quite essentially. Below threshold there is no emission at all whereas above threshold there appears an infinitely sharp line. In the present paper an attempt is made to develop the basic features of a noise theory taking into account the nonlinearities from the very beginning. As we shall see these nonlinearities play also in this case a decisive role for the definition af a certain threshold. Below it we obtain essentially expressions which are well known from the linear theory of noise[2], however, with some corrections depending on the underlying model for the originally broadened line. Near threshold the expressions become very complicated so that at present one can tell only about the qualitative behaviour. Well above threshold the situation is easier again and will be treated for high photon densities.

Our present treatment allows also to make contact with a group of papers[6] in which coherence functions are investigated with adhoc assumptions about the amplitude and phase of laser light. As we will see below laser light can be described by a superposition of Glauber-states[6] with random phases, but fixed amplitude. By calculation of the complex degree of mutual coherence of two laser beams (well above threshold) we can substantiate from first principles recent considerations by Jordan and Ghielmetti[7], who conclude from experiments that the expectation value $\langle b \rangle$ of a single mode amplitude must be basically nonvanishing.

§2. The equations of motion

We assume a set of modes in the cavity which we describe by running waves as verified in the Gyroscope. By this assumption we avoid

[3] Haken, H.: Talk at the Conference on Optical Pumping. Heidelberg 1962. — Haken, H., and H. Sauermann: Z. Physik **173**, 261 (1963); **176**, 47 (1963). — Lamb jr., W. E.: Phys. Rev. **134**, A 1429 (1964). — The pumping process is introduced explicitly, however, in a different way by Bevensee, R. M.: J. Math. and Phys. **5**, 308 (1964).

[4] Tang, C. L., H. Statz, and G. A. de Mars: J. Appl. Phys. **34**, 2289 (1963).

[5] Bennett, W. R.: Phys. Rev. **128**, 1013 (1962).

[6] Mandel, L.: Phys. Rev. **134**, A 10 (1964). — Glauber, R. J.: Phys. Rev. **131**, 2766 (1963).

[7] Jordan, T. F., and G. Ghielmetti: Phys. Rev. Letters **12**, 607 (1964).

diffuculties which arise from an otherwise spatially inhomogeneous inversion. We assume further that these modes are strongly discriminated by different lifetimes within the cavity. We then may assume that only one mode is important for our investigation of the nonlinearities, since for sufficient high losses no essential wave amplitude and thus no essential nonlinearity can build up; note however, that these modes would contribute in the linear range to a linewidth. In the following we take this contribution into account by describing the broadening of the line by energy fluctuation (in the case of KUBO[8]) respectively by an imaginary part of the energy of the atoms.

If we neglect at first the pumping process the equations of motion are the same as given in a preceding paper[3]:

$$\dot{b}^{+}=i\omega b^{+}+i\sum_{\mu}h^{*}e^{i\underset{\sim}{k}\underset{\sim}{r}_{\mu}}\alpha_{\mu}^{+}, \tag{2.1}$$

$$\dot{\alpha}_{\mu}^{+}=i\varepsilon_{\mu}\alpha_{\mu}^{+}-i h e^{-i\underset{\sim}{k}\underset{\sim}{r}_{\mu}}\sigma_{\mu}b^{+} \tag{2.2}$$

with

$$\sigma_{\mu}=\alpha_{\mu}^{+}\alpha_{\mu}-\alpha_{\mu}\alpha_{\mu}^{+} \tag{2.3}$$

and

$$h=-\frac{e}{m}\sqrt{\frac{2\pi}{\hbar\omega V}}\int\varphi_{1}^{*}\underset{\sim}{n}\,\underset{\sim}{p}\,\varphi_{2}\,d\tau. \tag{2.4}$$

V is the volume of the cavity, $\underset{\sim}{p}$ is the momentum operator, φ_1 and φ_2 are the electronic states of one atom. b^+ is the creation operator of the light quantum of the mode under consideration or classically spoken its amplitude, whereas the α and α^+ are the transition operators for electrons in upward or downward direction. ω is complex in order to take into account a finite lifetime of the mode in the cavity.

The finite lifetime or linewidth of the atoms has to be described differently according to the physical situation for which we give two examples:

a) The atoms are in a fluctuating field (for instance by lattice vibrations) which does not induce transitions but which changes steadily the excitation energy. In this case we put $\varepsilon=\varepsilon_0+\eta(t)$ where ε_0 is the frequency of the atomic resonance and $\eta(t)$ a steadily fluctuating function ($\overline{\eta}=0$).

b) Due to their coupling to different modes the atoms may decay. According to SENITZKY[9] this decay can be described by providing ε with an imaginary part and adding to the equation of motion an operator F with certain properties.

In the following we will consider as a concrete example case a) since the representation becomes especially clear and one can also establish

[8] KUBO, R.: J. Phys. Soc. Japan **9**, 935 (1954).

[9] SENITZKY, P.: Phys. Rev. **119**, 1807 (1960); **123**, 1525 (1961).

immediately the connection with the model of fluctuating dipoles as used by WAGNER and BIRNBAUM[2]. The calculations in case b), however, proceed quite similarly. By the transformation

$$\alpha_\mu^+ \to \alpha_\mu^+ e^{-i \underset{\sim}{k} \underset{\sim}{r}_\mu}$$

we can eliminate the dependence of the atomic position so that now we have to deal only with an equation of a mode with formally infinite wavelength. In order to exhibit the essential features more clearly we let the frequency of the cavity mode coincide with that of the atoms. After the transformation

$$b^+ \to b^+ e^{i \varepsilon_0 t}, \qquad \alpha^+ \to \alpha^+ e^{i \varepsilon_0 t}$$

we obtain

$$\dot{b}^+ = -\kappa b^+ + i h^* \sum_\mu \alpha_\mu^+, \tag{2.5}$$

$$\dot{\alpha}_\mu^+ = \eta_\mu(t) \alpha_\mu^+ - i h \sigma_\mu b^+. \tag{2.6}$$

In it $2\kappa = 1/t_0$. We assume that the distribution law of $\eta_\mu(t)$ is independent of μ. Eqs. (2.1) and (2.2) respectively (2.5) and (2.6) don't yet contain the pumping process. In our preceding papers we have taken it into account in the following way:

By the pumping process after a time T an average inversion σ_μ^0 will be established. The total change of σ_μ is then given by

$$\dot{\sigma}_\mu = \frac{\sigma_\mu^0 - \sigma_\mu}{T} + \textit{coherent change}.$$

Our earlier equations are thus to be interpreted in such a way that already one has averaged over the pumping process within the single equation for b^+, α^+ and σ. In order to take into account the statistical oscillation of the amplitude being brought about by the pumping process this averaging may be performed only at the end of the whole calculation. Therefore we consider now explicitly the single excitation collisions. Because by a collision an atom μ is brought again into the initial state (an excited state) the operator $\alpha_\mu^+(t)$ is to be replaced by the operator $\alpha_\mu^+(0)\, e^{i \varphi_{\mu,\nu}}$ after the collision at time $t_{\mu,\nu}$ where $\varphi_{\mu,\nu}$ is a random phase. Further by each excitation the inversion is changed and thus $\sigma_\mu(t)$ to be replaced by σ_μ^0 after each collision. The eqs. (2.1) and (2.2) respectively (2.5) and (2.6) are now valid only in between collisions. After each collision at atom μ at time $t_{\mu,\nu}$, however, new initial conditions are valid:

$$\alpha_\mu^+(t_{\mu,\nu}) = \alpha_\mu^+(0)\, e^{i \varphi_{\mu,\nu}}, \tag{2.7}$$

$$\sigma_\mu(t_{\mu,\nu}) = \sigma_\mu^0. \tag{2.8}$$

Since in the eq. (2.5) there occurs a sum over α_μ^+ we introduce it as new variable

$$S^+ = \sum_\mu \alpha_\mu^+ \tag{2.9}$$

and correspondingly

$$S_z = \tfrac{1}{2} \sum_\mu \sigma_\mu . \tag{2.10}$$

The equation of motion (2.5) thus reads

$$\dot{b}^+ = -\kappa b^+ + i h^* S^+ . \tag{2.5'}$$

By summing up over μ in (2.6) we obtain operators which refer to macroscopic quantities (for instance "total spin") and which we may treat with some caution in a classical way. Further we may note that summing over up over μ corresponds to an average over the random variable $\eta_\mu(t)$ where under some assumptions (compare KUBO [8]) one has

$$\left\langle \exp\left(i \int_0^t \eta(\tau)\, d\tau \right) \right\rangle_{\text{Av.}} = \exp(-t\gamma) .$$

Thus we get from (2.6)

$$\dot{S}^+ = -\gamma S^+ - i h 2 S_z b^+ . \tag{2.6'}$$

The equation for the pumping process (2.7) is transformed into

$$\begin{aligned} S^+(t_{\mu',\nu}+0) &= \sum_\mu \alpha_\mu^+(t_{\mu'\nu}+0) = \sum_{\mu \neq \mu'} \alpha_\mu^+(t_{\mu',\nu}-0) + \alpha_{\mu'}^+(0)\, e^{i\varphi_{\mu',\nu}} \\ &= \sum_\mu \alpha_\mu^+(t_{\mu'\nu}-0) + \alpha_{\mu'}^+(0)\, e^{i\varphi_{\mu',\nu}} - \alpha_{\mu'}^+(t_{\mu',\nu}-0) \end{aligned}$$

or

$$S^+(t_{\mu'\nu}+0) = S^+(t_{\mu',\nu}-0) + \delta\alpha_{\mu'} \tag{2.7'}$$

with

$$\delta\alpha_{\mu'} = \alpha_{\mu'}^+(0)\, e^{i\varphi_{\mu'\nu}} - \alpha_{\mu'}^+(t_{\mu',\nu}-0) . \tag{2.7''}$$

Correspondingly (2.8) goes over into

$$S_z(t_{\mu'\nu}+0) = S_z(t_{\mu'\nu}-0) + \delta\sigma_{\mu'} \tag{2.8'}$$

with

$$\delta\sigma_{\mu'} = \tfrac{1}{2}\left(\sigma_{\mu'}^0 - \sigma_{\mu'}(t_{\mu'\nu})\right) .$$

Provided $\kappa=\gamma=0$ a conservation law exists in between the collisions:

$$\dot{S}_z + (b^+ b)^\cdot = 0$$

which is to be changed for finite cavity width and $\gamma' \neq 0$ into

$$\dot{S}_z + (b^+ b)^\cdot = -2\gamma' S_z - 2\kappa b^+ b .$$

After integration and taking into account the jump condition (2.8′) one obtains finally (2.11)

$$S_z+b^+ b=D+\sum_{t_{\mu\,\nu}<t} \delta\sigma_\mu(t_{\mu,\nu})-2\gamma'\int S_z\,d\tau-2\kappa\int b^+ b\,d\tau \qquad (2.11)$$

where D is an integration constant.

The function described by $\sum\delta\sigma_\mu$ can be split into a continuous function $C_1+(t-t_0)\,F$ and a discontinuous function $g(t)$. The integrals $\int S_z\,d\tau$ and $\int b^+ b\,d\tau$ can be written in the form*

$$(t-t_0)\,\bar{S}_z+\vartheta_1(t) \qquad \text{respectively } (t-t_0)\,\overline{b^+ b}+\vartheta_2(t).$$

Because the integration smoothes fluctuations of S_z and $b^+ b$ to a large extent, we have $\vartheta_j(t)\ll g(t)$ so that ϑ_j can be neglected. Because in the steady state the time average of the total number of excited atoms and photons does not change one must have

$$F=2\gamma'\bar{S}_z+2\kappa\bar{n} \ (\textit{with } \bar{n}=\overline{b^+ b} \textit{ averaged number of photons}). \qquad (2.12\text{a})$$

From (2.11) we thus obtain finally

$$S_z+b^+ b=G_0+g(t) \qquad (2.12\text{b})$$

with

$$G_0=D+C_1.$$

By means of (2.12) we can express S_z in eq. (2.6′) by $b^+ b$. Further we can eliminate S^+ from eqs. (2.5′) and (2.6′) thus obtaining in between collisions

$$\ddot{b}^+=-(\kappa+\gamma)\,\dot{b}^+ +[2\,|h|^2(G_0+g(t)-b^+ b)-\kappa\gamma]\,b^+. \qquad (2.13)$$

This is the basic equation for our further considerations, where we will put $(\gamma+\kappa)=\alpha$. One can extend (2.13) to an equation valid for all times by taking into account the jump condition (2.8′)

$$\dot{S}^+=-\gamma S^+-i\,2h\,S_z\,b^+ +\sum_{t_{\mu,\,\nu}} \delta(t-t_{\mu,\nu})\,\delta\alpha_\mu^+(t_{\mu,\nu}). \qquad (2.14)$$

If one eliminates again S^+ from (2.14) and (2.5′) we get instead of (2.13)

$$\left.\begin{aligned}\ddot{b}^+ +(\kappa+\gamma)\,\dot{b}^+ -[2\,|h|^2(G_0+g(t)-b^+ b)-\kappa\gamma]\,b^+ \\ =i\sum_{t_{\mu,\,\nu}} h^*\,\delta(t-t_{\mu,\nu})\,\delta\alpha_\mu^+(t_{\mu,\nu}).\end{aligned}\right\} \qquad (2.15)$$

By means of this equation one can most explicitly explain the essential difference between the linear theory of laser noise and the treatment of laser action as free oscillation. In the linearized theory the nonlinear

* From here on we exclude from our analysis switching-on effects and spiking.

term b^+b is either completely neglected or replaced by $b^+b=\bar{n}$ and one puts $g(t)=0$.* One then has to solve a linear inhomogeneous differential equation that means to treat forced oscillation. On the other hand, as we will show in §3, the treatment of HAKEN and SAUERMANN[3] is equivalent to solving the homogeneous eq. (2.13) taking into account the nonlinear term.

In the present paper we wish to investigate the range of validity of both procedures by taking into account the nonlinearity as well as the inhomogeneous term.

§3. Discussion of the completely steady state

Before treating the noise problem we investigate as an idealized example the completely steady state. For this end we assume that the pumping process takes place continuously so that the discontinuous curve $g(t)$ (compare eq. (2.12)) vanishes completely. We further assume that we can neglect fully the fluctuation of the atomic amplitudes during the pumping process. In this case the equation of motion is given by

$$\ddot{b}^+ = -\alpha\dot{b}^+ + [2|h|^2(G_0 - b^+b) - \kappa\gamma]\,b^+ \tag{3.1}$$

with no additional conditions for b. In the steady state we have $\ddot{b}^+ = \dot{b}^+ = 0$ from which we obtain the condition

$$2|h|^2(G_0 - \bar{n}) - \kappa\gamma = 0. \tag{3.2}$$

For a given adjusted pumping rate F we can determine the unknown quantities S_z, n and G_0 by means of the eqs. (2.12a, b) and (3.2) thus obtaining

$$\bar{S}_z = \frac{\kappa\gamma}{2|h|^2}, \tag{3.3}$$

$$\bar{n} = \frac{1}{2\kappa}\left(F - \frac{\kappa\gamma\gamma'}{|h|^2}\right), \tag{3.4}$$

$$G_0 = S_z + \bar{n}.$$

Especially we obtain as lasing condition, e.g. a positive photon number,

$$F \geqq \frac{\kappa\gamma\gamma'}{|h|^2}. \tag{3.5}$$

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* A first step beyond this scheme was done by H. SAUERMANN and the present author (Talk given by H. SAUERMANN at Hochfrequenzausschuß, Karlsruhe, Spring 1964), who also put $b^+b=\bar{n}$, but took into account fluctuations of the inversion by means of a correlation function method. As we shall see below, this treatment belongs to the class of subthreshold theories which describe the line-narrowing but don't give rise to a stable light-amplitude.

Since we treat in the present paper the pumping process in a formally different way than in our preceding papers[3] we investigate the connection between both descriptions more closely. The adjusted pumping rate F is according to its definition on page 101 given by

$$F=\frac{1}{t}\sum^{t_{\mu\nu}<t}\frac{1}{2}\left(\sigma_\mu(0)-\sigma_\mu(t_{\mu\nu})\right). \tag{3.6}$$

If T_p is the mean time between two pumping processes at the same atom we can transform the right hand side into

$$\frac{N\sigma(0)}{2T_p}-\frac{\bar{S}_z}{T_p}. \tag{3.7}$$

Using eq. (2.12a) we get

$$2\kappa\bar{n}=\frac{N\sigma(0)}{2T_p}-S_z\left(\frac{1}{T_p}+2\gamma'\right). \tag{3.8}$$

On the other hand we have found in our preceding investigations

$$2\kappa n=\frac{N d_0}{2T}-S_z\frac{1}{T} \tag{3.9}$$

and a relation identical with (3.3) which therefore will not be discussed here. From a comparison of (3.7) and (3.8) follows that both descriptions give in case of vanishing pumping fluctuations the same result where one has for the effective pumping time T

$$\frac{1}{T}=\frac{1}{T_p}+2\gamma'$$

and further

$$\frac{d_0}{T}=\frac{\sigma(0)}{T_p}.$$

§4. The limiting case of a large cavity linewidth

Eq. (2.15) cannot be solved in closed form on account of $g(t)$ and especially on account of the nonlinear term b^+bb^+. However, we want to show that there are interesting limiting cases in which one can solve (2.15) exactly or at least in a good approximation explicitly. For this case we consider the characteristic frequencies of the system. These are (compare (2.14), (2.3′) and (2.15)) the resonator width $\sim\kappa$, the reciprocal phase memory time of the atoms $\sim\gamma$ and the frequencies in $g(t)$ and on the right hand side of eq. (2.15). In general the frequencies in $g(t)$ will be of the order of the pumping time itself. In the following we will disregard this term since its contribution to the linewidth above threshold is small as will be shown below. It should be noted, however, that our treatment is fully capable of taking into account that time-dependence. We consider now

$$\sum_{t_{\mu,\nu}} \delta(t-t_{\mu,\nu})\,\delta\alpha_\mu^+ .$$

$\delta\alpha_\mu^+$ still contains random phases. In this function again frequencies of the order $1/T_p$ are essential as can be shown by a more detailed investigation*.

We now treat the limiting case that the cavity width κ is large as compared to the other frequencies of the system. In this case we obtain as equation between collisions

$$\kappa \dot{b}^+ - [2|h|^2 (G_0 + g(t) - b^+ b) - \kappa\gamma]\, b^+ = 0 . \tag{4.1}$$

If we denote the solution of eq. (4.1) between two collisions by $b_l^+(t)$ the following recursive relation must hold on account of the pumping condition

$$\kappa\, b_{l+1}^+(t_{\mu,\nu}) = \kappa\, b_l^+(t_{\mu,\nu}) + i\, h^*\, \delta\alpha_\mu^+(t_{\mu,\nu}) . \tag{4.2}$$

For the complete solution of the problem we have to perform the following steps:

1. The explicit solution between two collisions must be found.
2. The integration constants are to be determined using (4.2).
3. One has to construct correlation functions using a certain statistical average over the collisions.

We start solving the first problem:

For $g(t)=0$ one could take the solution $b^+ = constant$ as discussed in §3 by which we cannot fulfill, however, the jump condition (4.2). Thus we have to look for the general solution valid also for $g(t) \neq 0$ which reads

$$b^+ = \sqrt{\frac{\kappa'}{2}}\, C \left\{ e^{-\int_{t_l}^{t} H\, d\tau} \left(1 + |C|^2 \int_{t_l}^{t} e^{\int_{t_l}^{\tau} H(\sigma)\, d\sigma}\, d\tau \right) \right\}^{-\frac{1}{2}} . \tag{4.3}$$

In it

$$\kappa' = \frac{\kappa}{2|h|^2} \tag{4.4a}$$

and

$$H(t) = \frac{2}{\kappa'} \left(G_0 + g(t) - \frac{\kappa\gamma}{2|h|^2} \right) . \tag{4.4b}$$

C is an integration constant which may be in general complex and which is to be determined from eq. (4.2) by recursion. Before solving this task we discuss special cases of the general solution for low, middle and high

* I am grateful to Mr. CH. SCHMID for this detailed investigation.

pumping. In order to make the calculation as simple as possible we assume that the function $g(t)$ which describes the random fluctuations of the inversion vanishes between two collisions. In this case we obtain generally

$$b^+ = C\sqrt{\frac{\kappa'}{2}}\left\{e^{-2G(t-t_l)}\left(1-\frac{|C|^2}{2G}\right)+\frac{|C|^2}{2G}\right\}^{-\frac{1}{2}} \tag{4.5}$$

where

$$G=\frac{2|h|^2 G_0-\kappa\gamma}{\kappa}.$$

Having the steady state solution of §3 in mind we may assume that for

A. small pumping G_0 is maller than $\kappa\cdot\gamma/2|h|^2$. Further in this case the light amplitude and thus C will be small. Thus we can approximate the eq. (4.5) by

$$b^+ \approx \sqrt{\frac{\kappa'}{2}}\, C\, e^{+G(t-t_l)}\left\{1+\frac{|C|^2}{4G}\left(1-e^{+2G(t-t_l)}\right)\right\}. \tag{4.6}$$

For very small light amplitudes one can neglect also the quadratic term in C thus obtaining finally

$$b^+ \simeq \sqrt{\frac{\kappa'}{2}}\, C\, e^{+G(t-t_l)}. \tag{4.7}$$

B. Is $G=0$ we obtain

$$b^+ = \sqrt{\frac{\kappa'}{2}}\, C\{1+|C|^2(t-t_l)\}^{-\frac{1}{2}}. \tag{4.8}$$

In this case we are on the average just at threshold.

C. The pumping is supposed to be still higher. In this case G_0 is shurely bigger than $\kappa\cdot\gamma/2|h|^2$.

We assume further that the light amplitude ($\sim C$) has become so great that

$$\left|1-\frac{|C|^2}{2G}\right| \ll \frac{|C|^2}{2G} \tag{4.9}$$

holds.

Then we can approximate (4.5) by

$$b^+ = \frac{C}{|C|}\sqrt{\kappa' G}\left\{1+e^{-2G(t-t_l)}\,\frac{1}{2}\left(1-\frac{2G}{|C|^2}\right)\right\}. \tag{4.10}$$

The solution (4.5) and its special cases are completely different as to their time dependence depending on

$$G\equiv\frac{2|h|^2 G_0-\kappa\gamma}{\kappa}\gtrless 0$$

(compare also (3.2)).

For $G<0$ the light amplitude decreases exponentially after each excitation collision. Its decay constant which for spontaneous emission would be equal γ is decreased on account of the inversion $\sim G_0$. As we will show below explicity the linewidth becomes smaller by this effect.

For $G>0$ one would obtain for a linear equation an exponential increase of the light amplitude. On account of the nonlinearity according to eq. (4.5) the following behaviour results however. Let the amplitude $b^+=b_0^+$ be given at collision time $t_l \cdot b^+$ approaches for large t the value (compare (4.10))

$$b^+=\frac{C}{|C|}\sqrt{G_0-\kappa\gamma/2|h|^2}$$

or the number of photons the value

$$\bar{n}=G_0-\kappa\gamma/2|h|^2$$

which agrees completely with that of the stationary state as discussed in §3. After each collision the light amplitudes thus approach that of the free oscillation.

We now consider how one can fulfill the recursion condition in the limiting case A. We have immediately

$$C_{l+1}=C_l e^{G(t_{l+1}-t_l)}+\delta_l$$

where

$$\delta_l=\frac{i h^*}{\kappa}\sqrt{\frac{2}{\kappa'}}\,\delta\alpha_\mu^+(t_\mu)$$

and thus for the general solution at time t

$$b^+=\sqrt{\frac{\kappa'}{2}}\,e^{Gt}\left\{\sum_{t_l<t}\delta_l e^{-Gt_l}+C_0 e^{-Gt_0}\right\}. \tag{4.11}$$

§5. Calculation of the line-width below threshold

In order to calculate the linewidth we have to determine the correlation function $\langle b^+(t+\tau)b(t)\rangle$ where $\langle\cdots\rangle$ means the quantum mechanical average as well as that over the collision process. By means of (4.11) we have

$$\left.\begin{aligned}&\langle b^+(t+\tau)\,b(t)\rangle\\&=\frac{|h|^2}{\kappa^2}\sum_{\mu,\nu}\sum_{\mu',\nu'}\overline{e^{G(t+\tau-t_{\mu,\nu})+G(t-t_{\mu',\nu'})}\langle\delta\alpha_\mu^+(t_{\mu,\nu})\,\delta\alpha_\mu(t_{\mu',\nu'})\rangle}.\end{aligned}\right\} \tag{5.1}$$

The sums over the atoms μ respectively μ' and the collision times $t_{\mu,\nu}$ respectively $t_{\mu'\nu'}$ run to the last collision. The bar means average over the collision times. Because in the present case there is no phase relation between different atoms we have $\mu'=\mu$.

Since all atoms behave in the same manner we can neglect the index μ and can replace the sum over μ by N. Writing $\langle\cdots\rangle$ by means of the definition of $\delta\alpha_\mu$ explicitly we obtain

$$\frac{|h|^2}{\kappa^2} N \sum_{\nu,\nu'} e^{G(t+\tau-t_\nu)+G(t-t_{\nu'})}\{A\} \tag{5.2}$$

where

$$\left.\begin{aligned}\{A\} = &\langle\alpha^+(0)\,e^{i\varphi_\nu}\alpha(0)\,e^{-i\varphi_{\nu'}}\rangle - \langle\alpha^+(t_\nu-0)\,\alpha(0)\,e^{-i\varphi_{\nu'}}\rangle - \\ &-\langle\alpha^+(0)\,e^{i\varphi_\nu}\alpha(t_{\nu'}-0)\rangle + \langle\alpha^+(t_\nu-0)\,\alpha(t_{\nu'}-0)\rangle\,.\end{aligned}\right\} \tag{5.3}$$

Since there are no phase relations between the single collision the first bracket is only non vanishing if $\nu=\nu'$. If the α's in the last bracket stem from different collisions also this vanishes. On the other hand the α's occurring in the second and third bracket may refer to a common collision. If in the second bracket a collision has appeared at time $t_{\nu'}$ which defines an initial phase $\varphi_{\nu'}$, α^+ retains this phase* until the next collision at time $t_\nu = t_{\nu'+1}$.

This bracket is therefore non vanishing if $\nu=\nu'+1$. Correspondingly we have for the third term $\nu=\nu'-1$. Thus we obtain for (5.2)

$$\left.\begin{aligned}\frac{|h|^2}{\kappa^2} N \sum_{\nu'}^{\nu_0} e^{G(t+\tau-t_{\nu'})+G(t-t_{\nu'})}\{&\langle\alpha^+\alpha\rangle_0 - e^{G(t_{\nu'}-t_{\nu'+1})}\times \\ &\times\langle\alpha^+(t_{\nu'+1}-0)\,\alpha(0)\,e^{-i\varphi_{\nu'}}\rangle - e^{G(t_{\nu'}-t_{\nu'-1})}\times \\ &\times\langle\alpha^+(0)\,e^{i\varphi_{\nu'}}(t_{\nu'}-0)\rangle + \langle\alpha^+(t_{\nu'}-0)\,\alpha(t_{\nu'}-0)\rangle\}\,.\end{aligned}\right\} \tag{5.4}$$

For the evaluation of the expectation value $\langle\cdots\rangle$ we have to use the equations of motion (2.6) in which again the light amplitude occurs. Because the lightfield is relatively small we may use in (5.4) for α^+ the zero'th approximation. This reads for the model of a fluctuating field

$$\alpha^+(t) = e^{-i\int_0^t \eta(\tau)\,d\tau}\,\alpha^+(0)\,.$$

Thus we obtain

$$\langle\alpha^+(t_{\nu'+1}-0)\,\alpha(0)\,e^{-i\varphi_{\nu'}}\rangle = \langle\alpha^+\alpha\rangle_0\left\langle e^{-i\int_{t_{\nu'}}^{t_{\nu'+1}}\eta(\tau)\,d\tau}\right\rangle = \langle\alpha^+\alpha\rangle_0\,e^{-\gamma(t_{\nu'+1}-t_{\nu'})}$$

and

$$\langle\alpha^+(t_{\nu'}-0)\,\alpha(t_{\nu'}-0)\rangle = \langle\alpha^+\alpha\rangle_0\,.$$

* There is of course an additional change of phase given by $\exp\left\{i\int^t \eta(\tau)\,d\tau\right\}$ which we take into account below.

Using $N\langle\alpha^+\alpha\rangle_0 = N_2 =$ *number of the excited atoms* we have now to determine

$$\left.\begin{array}{l}\frac{|h|^2}{\kappa^2} N_2 \sum_{\nu'} e^{G(t+\tau-t_{\nu'})+G(t-t_{\nu'})} \times \\ \quad \times \{2 - e^{-\gamma(t_{\nu'+1}-t_{\nu'})+|G|(t_{\nu'+1}-t_{\nu'})} - e^{-\gamma(t_{\nu'}-t_{\nu'-1})-|G|(t_{\nu'}-t_{\nu'-1})}\}.\end{array}\right\} \tag{5.5}$$

In order to average over t_ν we use a Poisson-distribution. We treat first the sums which stem from the first and third term in $\{\cdots\}$ in (5.5):

$$\left.\begin{array}{l}\frac{|h|^2}{\kappa^2} N_2 \sum_{\nu=0}^{\infty}\left[\frac{1}{T_p}\int_0^\infty e^{-2|G|\sigma-\frac{\sigma}{T_p}}\left(\frac{\sigma}{T_p}\right)^\nu \frac{1}{\nu!}\right] d\sigma \times \\ \qquad \times\left\{2-\frac{1}{T_p}\int_0^\infty e^{-(|G|+\gamma)\sigma'-\frac{\sigma'}{T_p}} d\sigma'\right\}.\end{array}\right\} \tag{5.5a}$$

The sum over $[\cdots]$ gives $\frac{1}{2|G|T_p}$ and thus (5.5a) reads

$$\frac{|h|^2}{\kappa^2} N_2 e^{-|G||\tau|} \frac{1}{2|G|T_p}\left\{2-\frac{1}{T_p(|G|+\gamma)+1}\right\}. \tag{5.6}$$

Because the second term of $\{\cdots\}$ in (5.5) contains also a collision which lies between t and $t+\tau$, we transform the sum as follows

$$\sum_{\nu'}^{\nu_0} e^{G(t+\tau-t_{\nu'})+G(t-t_{\nu'})} e^{G(t_{\nu'}-t_{\nu'-1})} + e^{G(t+\tau-t_{\nu_0+1})+G(t-t_{\nu_0})} \tag{5.5b}$$

with the condition $t \leqq t_{\nu_0+1} \leqq t+\tau$ and $t_{\nu_0} \leqq t$.

Taking into account the factor $e^{-\gamma(t_{\nu'}-t_{\nu'-1})}$ the sum yields

$$-\frac{|h|^2 N_2}{\kappa^2} e^{-|G||\tau|} \frac{1}{2|G|T_p} \frac{1}{T_p(|G|+\gamma)+1}.$$

The average over the second term in (5.5b) is given by

$$\frac{1}{T_p}\int_t^{t+\tau} e^{G(t+\tau+t')+Gt} e^{-\frac{(t'-t)}{T_p}} dt' \, e^{-\gamma t'} \frac{1}{T_p}\int_{-\infty}^{t} e^{-Gt''} e^{\gamma t''-\frac{t-t''}{T_p}} dt''$$

for which one obtains immediately

$$\frac{1}{(T_p(\gamma-|G|)+1)(T_p(\gamma+|G|)+1)}\left(e^{G\tau}-e^{-\frac{\tau}{T_p}-\gamma\tau}\right).$$

For $\frac{1}{T_p} \gg |G|$ we obtain finally $e^{G\tau} \frac{1}{(T_p(\gamma+|G|)+1)(T_p(\gamma-|G|)+1)}$.
If we collect all terms coming from (5.5) we have:

$$\frac{|h|^2}{\kappa^2} N_2 e^{-|G|\,|\tau|} \left\{ \frac{1}{2|G|\,T_p} \left(2 - \frac{2}{T_p(|G|+\gamma)+1} - \frac{1}{(T_p(\gamma+|G|)+1)(T_p(\gamma-|G|)+1)} \right) \right\}.$$

For $|G|T_p \ll 1$ this simplifies to

$$\frac{|h|^2}{\kappa^2} N_2 e^{-|G|\,|\tau|} \frac{\gamma}{|G|}.$$

The spectral distribution is given by

$$g(\omega) = \int \langle b^+(t+\tau)\, b(t) \rangle e^{i\omega\tau} d\tau$$

or

$$g(\omega) = N_2 \frac{2\gamma |h|^2}{|\kappa\gamma - |h|^2 (N_2 - N_1)|^2 + \kappa^2 \omega^2}$$

where we have used that $2G_0 = N_2 - N_1 =$ *inversion of the atomic levels.* Eq. (5.5) gives the same linewidth as the paper of WAGNER and BIRNBAUM[2] if we specialize the latter to the case: κ larger as compared to the other frequencies.

If we use the model of fluctuating dipoles we also get the same intensity. Using other models there may occur, however, other intensities which shall be treated in later papers. For slow pumping ($T_p \gamma \gg 1$) one finds a lowering of the intensity. As final formula we obtain now

$$g(\omega) = \frac{2N_2}{T_p} \frac{|h|^2}{(\kappa\gamma - |h|^2 (N_2 - N_1))^2 + \kappa^2 \omega^2}$$

with the same linewidth as above.

§ 6. Graphical discussion of light amplitude and linewidth above threshold

Whereas the light amplitude between collisions is already known from eq. (4.5) we have now to determine the integration constant C by the recursion formula (4.2)

$$C_{l+1} = C_l \left\{ e^{-2G(t_{l+1}-t_l)} + \frac{|C_l|^2}{2G} (1 - e^{-2G(t_{l+1}-t_l)}) \right\}^{-\frac{1}{2}} + \delta_{l+1}, \quad (6.1)$$

$$= C_l \frac{d_l e^{G(t_{l+1}-t_l)}}{\sqrt{|C_l|^2 + d_l^2}} + \delta_{l+1} \quad (6.2)$$

where

$$d_l = \sqrt{2G/C\, e^{2G(t_{l+1}-t_l)} - 1}\,.$$

For a preliminary discussion we assume that the intervals $t_{l+1}-t_l$ are independent of l putting $e^{G(t_{l+1}-t_l)} = f > 1$. It follows

$$C_{l+1} = C_l \frac{df}{\sqrt{|C_l|^2 + d^2}} + \delta_{l+1}\,. \tag{6.3}$$

With $D_l = \frac{C_l}{d}$; $\vartheta_{l+1} = \frac{\delta_{l+1}}{d}$ we get

$$D_{l+1} = D_l \frac{f}{\sqrt{|D_l|^2 + 1}} + \vartheta_{l+1}\,. \tag{6.4}$$

Neglecting for the moment ϑ_{l+1} we consider the establishment of the stationary state. For it $D_l = D$ is determined by the equation $1 = \frac{f}{\sqrt{D^2+1}}$ with $D^2 = f^2 - 1$ which is consistent with

$$\bar{n} = |b|^2 = G_0 - \kappa\gamma/2\,|h|^2.$$

The dynamical approach to the equilibrium can be visualized in a simple way by a graphical plot (Fig. 1). According to it we can obtain D_{l+1} from D_l in the following manner. First plot D_l along the x-axis and go from there parallel to the y-axis. Because $\cos\alpha = \frac{D_l}{\sqrt{|D_l|^2+1}}$ one obtains D_{l+1} by plotting f along the line OB and projecting it on the x-axis. If D_l lies on the right hand side from the equilibrium value D, D_{l+1} is coming now to the left hand side of D_l, thus coming closer to the equilibrium value. The corresponding situation appears if D_l lies in the beginning on the left hand side of the equilibrium position. Thus one may recognize immediately the approach to equilibrium. This approach is of course disturbed by δ_l respectively ϑ_l. Let us assume that D_{l+1} is real. We investigate first for which maximal D_l the system cannot be further pushed from its equilibrium value by collisions. This implies the condition

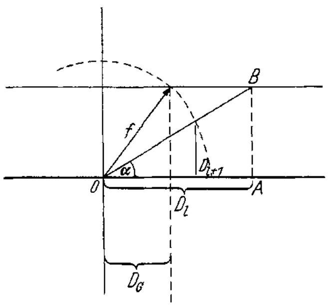

Fig. 1. Construction of the constant D_{l+1} from D_l

$$D_{l\,\max} - D^0_{l+1} \geqq |\vartheta_{l+1}| \qquad (D^0_{l+1} \leqq D_{l\,\max} + |\vartheta_{l+1}|)$$

or

$$D_{l\,\max} - \frac{D_{l\,\max} f}{\sqrt{D^2_{l\,\max}+1}} \geqq \vartheta_{l+1}\,.$$

As follows from Fig. 1 such a value can always be found. The question if also such value for $D_{l\,\min}$ exists is of more importance:

$$D^0_{l+1} - D_{l\,\min} \geqq |\vartheta_{l+1}|$$

or

$$\frac{D_{l\,\min} f}{\sqrt{D^2_{l\,\min}+1}} - D_{l\,\min} \geqq |\vartheta_{l+1}|\,.$$

As one sees immediately from Fig. 1 this limiting value certainly does not exist if one starts with $D_l=0$. If on the other hand a certain value D_l is crossed over to the right hand side the system can get trapped. In this case it further approaches its stable amplitude. One should bear in mind that our present analysis was based on the assumption of constant intervals Δt_ν. In realistic cases, they will of course undergo fluctuations and our discussion then refers to the most probable collision time. As we will see below by a different analysis our conclusions are still valid for such a statistical behaviour. If we insert realistic numbers we can further show that for sufficient high inversion the minimum condition can be fulfilled.

From our analysis we can understand the build up of the coherent light amplitude: Assume that an inversion G is already present. First we have $D \sim C \sim b^+ = 0$. Then according to (6.3) the light amplitude fluctuates around Zero. By a favourable combination of the randomly fluctuating δ's the amplitude can reach a stable area, where it fluctuates now around its equilibrium value. With a very low probability (depending on the pumping rate) the amplitude can jump out of the stable region again if $\Delta t \ll \Delta T$ and the phases of the jumps are unfavourable.

In the next chapter we will discuss this question quantitatively. So far the occurring quantities were assumed real. Because the δ_l's are complex, however, the D's and thus also the C's will become also complex quantities. All our above considerations can be, nowever, easily extended to the complex D-plane. What has been said about the real D is now valid for its absolute value. However, we have now still collisions in tangential direction, which cannot be stabilized and give rise to a finite linewidth. For the determination of this linewidth $\Delta\nu$ on account of phase jumps we treat these jumps as diffusion process along the circle with length $U = 2\pi\sqrt{f^2-1}$. If the length of the single step in tangential direction is given by ϑ_T the probability that the length x is reached at time t is given by

$$W(x,t) = \frac{1}{(2\pi\langle\vartheta^2\rangle + |\Delta T|)^{\frac{1}{2}}}\, e^{-\frac{x^2}{2T/\Delta T\,\langle\vartheta_T^2\rangle}}.$$

At the time $t_0=\frac{2\pi(f^2-1)\Delta T}{\langle\vartheta^2 T\rangle}$ the diffusion process has reached the phase 2π. For $\langle\vartheta^2\rangle$ we have

$$\langle\vartheta^2\rangle=\frac{1}{N'}\left\langle\left|\sum_{\substack{\mu\nu\\ N\text{ collisions}}}\delta\alpha_\mu^+(t_{\mu\nu})_T\, i h^*\right|^2\right\rangle\frac{1}{d^2}\,2\frac{|h|^2}{\kappa^2};\qquad \delta^2\approx\frac{1}{\Delta T}$$

$$=\frac{N}{N'}\sum_{\substack{\nu\nu'\\ N\text{ collisions}}}\langle\delta\alpha^+(t_\nu)_T\,\delta\alpha(t_{\nu'})_T\rangle=\sum_{\nu=1,0,+1}\langle\delta\alpha^+(t_0)_T\,\delta\alpha^+(t_\nu)_T\rangle.$$

For the calculation of $\langle\cdots\rangle$ one has to write $\delta\alpha$ again as difference $\alpha^+(0)\,e^{i\varphi}-\alpha^+(t-0)$ where $\alpha^+(t-0)$ stems from the last collision.

In order to determine $\alpha^+(t-0)$ we have to take into account to action of the lightfield on the atoms, where we can consider the lightfield as a fixed quantity. If $hb^+>\sqrt{|\eta|^2}$ we can neglect in first approximation the fluctuations of the energy of atoms.

Further it is useful to use the Schrödinger picture:

$$\underbrace{(h^*\alpha^+ b+h\alpha b^+)}_{\tilde H}\varphi=i\dot\varphi$$

with the general solution

$$\left\{(a\sin\omega t+c\cos\omega t)(\downarrow)+\frac{i h^* b}{|h|\,|b|}(a\cos\omega t-c\sin\omega t)(\uparrow)\right\}$$

and the normalization $|a|^2+|c|^2=1$ and $\omega=|hb|$.

In order to determine the direction of the collision we calculate

$$\langle\alpha^+(t)\rangle=-\frac{i h b^+}{|h|\,|b|}\left\{a^* c\cos^2\omega t-a c^*\sin^2\omega t+\frac{|a|^2-|c|^2}{2}\sin 2\omega t\right\}.$$

Due to the collision condition (4.2) one has

$$\delta\alpha^+=\alpha^+(0)\,e^{i\varphi_t}-|h|\frac{b^+}{|b|}\left(|a|^2-\frac{1}{2}\right)\sin 2\omega t\qquad(\text{where } a\gg c).$$

Whereas the first term gives rise to a collision into an arbitrary direction the second one gives rise to collisions parallel to b^+. Its sign depends on the collision time $t\equiv T_p\gtrless 1/2\omega$. If the collision time is large as compared to $1/2\omega$ which holds for high fieldstrength b^+, we have $\sin 2\omega t<0$, the atom is mostly in the groundstate. From there the electron falls down into a lower level or a further change of sign occurs. This means that the collision now occur in the direction of smaller b-values.

If $T_p \gg 1/2\omega$ one can expect a strong cancellation of expressions $\alpha^+(t_{\mu,\nu}-0)$. The decisive contribution to the diffusion process therefore stems in general from the first term $\alpha^+(0)\, e^{i\varphi_\nu}$ with random phases. The calculation of the diffusion constant D^2 is now straight forward and is given in §7 (compare (7.12)).

We now return to the discussion of the motion of the amplitude in radial direction. As we have seen above the system makes small oscillations around the stable position $D_G = \sqrt{f^2-1}$. This suggests to extend D_l around D_G:

$$D_l = \sqrt{f^2-1} + \xi_l .$$

Inserting this into (6.4) and treating ξ_l as small quantity yields $\xi_{l+1} = \frac{\xi_l}{f^2} + \vartheta_l$ or on account of $f = e^{G\Delta T}$

$$\Delta\xi = -\xi\, 2G\,\Delta T + \vartheta .$$

From this one obtains the differential equation

$$\dot{\xi} = -2\xi G + \tilde{\vartheta} \quad \text{with } \tilde{\vartheta} = \frac{\vartheta}{\Delta T}$$

which describes a diffusion process with a restoring force. Postponing the explicit solution of the equation to the next paragraph we summarize the results of the present paragraph: For a sufficiently high inversion a stable value for $|D|^2$ and thus for b^+b occurs around which the system makes small fluctuations. On the other hand there are no restoring forces for the phase of b. It undergoes a diffusion process in the complex b-plane. We thus must conclude that for treating a differential equation for b one is usually not allowed to linearize. One has first to split off the phase and must investigate its fluctuation separately.

§7. Lightamplitude and linewidth above threshold at high inversion

We now consider the general eq. (2.15). Our consideration of §6 suggests to put b^+ in the form $r e^{i\varphi}$.* Inserting into (2.15) and separating the real and imaginary part gives

$$\left.\begin{aligned} &\ddot{r} + (\kappa+\gamma)\dot{r} - [\underbrace{2|h|^2 G_0 - \kappa\gamma}_{\tilde{G}} - 2|h|^2 \cdot r^2 - \dot{\varphi}^2]\, r \\ &\quad = \mathrm{Re}\{e^{-i\varphi}\, i h^* \sum_{t_{\mu\nu}} \delta(t-t_{\mu\nu})\,\delta\alpha_\mu^+(t_{\mu,\nu})\} \end{aligned}\right\} \qquad (7.1)$$

* Because b and therefore also r and φ are operators the Ansatz $b = r e^{i\varphi}$ requires some precaution. Its exact definition were $b = Tr \exp\{\int(\tau)\, d\tau\}$ where T is the time order operator. Since all calculations, however, go through the same way we disregard of this sophisticated procedure here.

and

$$2\dot{r}\,\dot{\varphi}+r\,\ddot{\varphi}+(\kappa+\gamma)\,\dot{\varphi}\,r=\mathrm{Im}\,\{\cdots\}\,. \tag{7.2}$$

In analogy to §6 we expect that r will make only small oscillations around an equilibrium value r_0. Following up a remark by H. KOPPE* this can be most easily visualized if one considers eq. (2.15) as that of a particle moving in two dimensions and having the potential energy $-\frac{\tilde{G}}{2}|x|^2+$ $+|h|^2\frac{|x|^4}{2}=V(|x|)$. The plot of $V(x)$ in Fig. 2 shows in fact that one has for positive inversion ($G>0$) an equilibrium value $x_0 \neq 0$. It is interesting to see what happens below threshold. In this case the minimum exists only for $x=0$, that means after each collision process the light amplitude decreases as has been discussed in §5. Making according to these considerations for r the ansatz $r=r_0+\rho$** where $r_0^2=\frac{\tilde{G}}{2|h|^2}$ we obtain the following equations

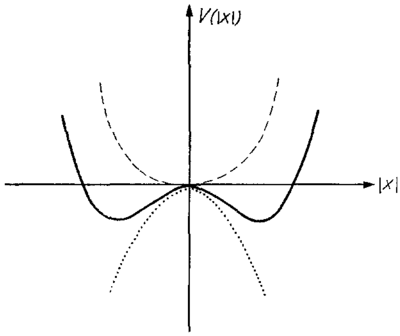

Fig. 2. Plot of "potential energy" versus light amplitude. — — — below threshold (*linear and nonlinear* theory); ······· above threshold, *linear* theory leads to instability; ——— above threshold, *nonlinear* theory

$$\left.\begin{aligned}&\ddot{\rho}+(\kappa+\gamma)\,\dot{\rho}-(r_0+\rho)\,\tilde{G}+2\,|h|^2\,r_0^3+\\&\qquad+3\cdot 2\,|h|^2\,r_0^2\,\rho-\dot{\varphi}^2(r_0+\rho)=\mathrm{Re}\,\{\cdots\}\end{aligned}\right\} \tag{7.3}$$

or

$$\ddot{\rho}+(\kappa+\gamma)\,\dot{\rho}+2\rho\,\tilde{G}-\dot{\rho}^2\,r_0=\mathrm{Re}\,\{\cdots\} \tag{7.4}$$

and

$$\ddot{\varphi}+(\kappa+\gamma)\,\dot{\varphi}=\frac{1}{r_0}\,\mathrm{Im}\,\{\cdots\} \tag{7.5}$$

where we have kept only linear terms of ρ.

The solution of eq. (7.4) and (7-5) seems still difficult because the term $e^{-i\varphi}$ represents a strong nonlinearity. In §6, however, we have seen that $\delta\alpha_\mu^+$ possesses random phases***. Thus we may take $e^{-i\varphi}$ into $\delta\alpha_\mu^+$ which can be treated again as random variable $\delta\tilde{\alpha}_\mu^+$. By means of the

* H. KOPPE, private communication.

** This ansatz where r_0 is a C-number corresponds to a unitary transformation, which preserves for instance commutation relations.

*** The non-stochastic part of δa^+ has a phase factor which cancels out in (7.1) and (7.2) completely.

method of variation of the constant we can solve eq. (7.5) immediately:

$$\varphi-\varphi_0=\frac{1}{\alpha r_0}\sum_{t_{\mu\nu}>t_0}^{t} I_{\mu\nu}(1-e^{-\alpha(t-t_{\mu\nu})}) \quad \text{with } I_{\mu\nu}=\operatorname{Im} i h^* \delta\tilde{\alpha}_\mu^+(t_{\mu\nu}). \quad (7.6)$$

With this solution we can enter into eq. (7.6) obtaining an inhomogeneous linear equation for ρ. Its solution reads

$$\rho(t)=\int^{t} F(\sigma)\,K(t,\sigma)\,d\sigma \tag{7.7}$$

with the abbreviations:

$$K(t,\sigma)=\frac{1}{2i\sqrt{L}}\{e^{i\omega_1(t-\sigma)}-e^{+i\omega_2(t-\sigma)}\},$$

$$\omega_{1,2}=\frac{i\alpha}{2}\pm\sqrt{-\frac{\alpha^2}{4}+2\tilde{G}}=\frac{i\alpha}{2}\pm\sqrt{L},$$

$$F(\sigma)=\Big\{\sum_{t_{\mu,\nu}}\delta(\sigma-t_{\mu,\nu})\,R_{\mu,\nu}\Big\}+r_0\,\dot{\varphi}^2,$$

and

$$R_{\mu\nu}=\operatorname{Re}\{e^{-i\varphi}\, i h^*\,\delta\alpha_\mu^+(t_{\mu,\nu})\}.$$

In the following we neglect the term $\dot{\varphi}^2 r_0$ due to its smallness. On account of the δ-function occurring in $F(\sigma)$ (7.7) simplifies to

$$\rho=\sum_{t_{\mu\nu}}^{t} R_{\mu\nu}\,K(t,t_{\mu\nu}). \tag{7.8}$$

Since we are only interested in the stationary solution we need not add a (damped) solution of the homogeneous equation. By means of (7.8) and (7.6) one can calculate in principle all correlation functions. We do this here for several simple examples:

1. Mean square of the phases as a function of time $\langle(\varphi(t)-\varphi_0)^2\rangle$

By means of (7.6) we obtain immediately

$$\langle\cdots\rangle=\frac{1}{r_0^2}\sum_{t_{\mu\nu}\geqq t_0}^{t}\frac{I_{\mu\nu}^2}{\alpha^2}(1-e^{-\alpha(t-t_{\mu\nu})})^2 \tag{7.9}$$

where we suppose that the phases of $I_{\mu\nu}$ are uncorrelated. From (7.7) follows

$$\langle\cdots\rangle=\frac{1}{r_0^2}N\frac{\overline{I_{\mu\nu}^2}}{\alpha^2}\sum_{\nu=\nu_A}^{\nu_E}(1-e^{-\alpha(t-t_\nu)})^2$$

where we have only to average over collisions between t_0 and t. Let $T_p M = (t - t_0) = T$ then we have

$$\sum_{\nu=\nu_A}^{\nu_E} 1 = \frac{T}{T_p}; \qquad \sum_{\nu=\nu_A}^{\nu_E} e^{-\alpha(t-t_\nu)} \approx \frac{1}{T_p \alpha} (1 - e^{-\alpha T}). \tag{7.10}$$

Thus we obtain

$$\langle \cdots \rangle = \frac{1}{r_0^2} N \frac{\overline{I^2}}{\alpha^2} \frac{1}{T_p} \left\{ T - \frac{2}{\alpha} (1 - e^{-\alpha T}) + \frac{1}{2\alpha} (1 - e^{-2\alpha T}) \right\} \tag{7.11}$$

which tends to Zero for $\alpha T \ll 1$ proportional to T^3, whereas it gives for $T\alpha \gg 1$

$$\langle \cdots \rangle \approx \frac{1}{r_0^2} N \frac{\overline{I^2}}{\alpha^2} \frac{T}{T_p}. \tag{7.12}$$

Introducing the photon flux $P = 2\kappa \bar{n}$ we obtain

$$\langle \cdots \rangle = \frac{2\kappa N \overline{I^2}}{\alpha^2 P} \frac{T}{T_p} = \frac{1}{P} \frac{T}{2T_p} \frac{g^2 \kappa}{\alpha^2} \rho. \quad \text{where} \quad g^2 = V |h|^2 \tag{7.13}$$

The linewidth is thus given by

$$\Delta \nu \approx \frac{1}{P} \frac{\kappa}{2T_p} \frac{g^2 \rho}{\alpha^2}. \tag{7.14}$$

It is interesting to compare this formula with that obtained below threshold. In the limit of large κ it reads

$$P = 2\kappa \bar{n} = \frac{2|h|^2}{\kappa} N_2 \frac{\gamma}{\Delta \nu}$$

or

$$\Delta \nu = \frac{2|g|^2 \rho_2 \gamma}{P \cdot \kappa}$$

where

$$\rho_2 = \frac{N_2}{V}. \tag{7.15}$$

One thus obtains a very pronounced analogy where $2T_p$ corresponds to $1/\gamma$. This analogy becomes still stronger, when we specialize below threshold to large pumping time T_p, that means if we assume strong decrease of the atomic amplitude between collisions. The only difference consists in the factor $2\rho_2$, respectively ρ. The factor 2 is of no importance because the linewidth comes in both cases from different relations.

2. Correlation function of the phase factors

This correlation function reads

$$\langle \exp(i\,\varphi(t_2) - i\,\varphi(t_1))\rangle . \tag{7.16}$$

According to (7.6) we split φ into a sum over the single atomic contribution obtaining thus for (7.16) a product

$$\prod_{\mu} \langle \exp(\varphi_\mu(t_2) - \varphi_\mu(t_1))\rangle \tag{7.17}$$

$$\varphi_\mu = \text{part of the sum over } \mu .$$

In it φ_μ denotes just a partial sum with fixed atomic index but still going over all collisions. We now assume that the contributions of a single atom for the phase jumps are small. Thus we can expand the exponential into a power series which we brake off with the second term:

$$\prod_{\mu} \{1 + i\,\langle \varphi_\mu(t_2) - \varphi_\mu(t_1)\rangle - \tfrac{1}{2}\langle(\varphi_\mu(t_2) - \varphi_\mu(t_1))^2\rangle\} . \tag{7.18}$$

Putting the brackets into the exponent again we find

$$e^{i \sum_{\mu} \langle \Delta\varphi_\mu\rangle - \frac{1}{2} \sum_{\mu} \langle(\Delta\varphi_\mu)^2\rangle} . \tag{7.19}$$

Whereas $\langle \Delta\,\varphi_\mu\rangle$ vanishes, $\frac{1}{2}\langle(\Delta\,\varphi_\mu)^2\rangle$ has already been determined in eq. (7.9). We thus find finally

$$\langle \cdots \rangle = e^{-\Delta\nu t/2} \tag{7.20}$$

where $\Delta\nu$ is given by (7.14).

3. Calculation of the correlation function of the amplitude

In this part we will show, that already slightly above threshold the fluctuations are not capable to throw the system out of its stable value $|b| = r_0$.

a) Influence of the fluctuation of atomic phases. The quantity to be determined reads

$$\langle r(t+\tau)\,r(t)\rangle .$$

Using $r = r_0 + \rho$ this can be split into

$$r_0^2 + r_0\,\langle \rho(t+\tau)\rangle + r_0\,\langle \rho(t)\rangle + \langle \rho(t+\tau)\,\rho(t)\rangle .$$

Because $\langle \rho\rangle$ vanishes if averaged over the phases of the α's we need only to determine $\langle \rho(t+\tau)\,\rho(t)\rangle$.

Taking again the limit that the coherent parts of $\delta\alpha^+$ vanish between collisions we have immediately

$$\langle\rho(t+\tau)\,\rho(t)\rangle = N\,\overline{R_\mu^2}\sum_{\nu}^{\nu_0} K^*(t+\tau, t_\nu)\,K(t, t_\nu)\,. \tag{7.21}$$

The average over the collision t_ν can be performed as in §5. We confine ourselves at first to the limiting case of large κ. Because in this case

$$K(t, t_\nu) = \frac{1}{\kappa}\,e^{-2G(t-t_\nu)} \quad (\text{for } t > t_\nu)$$

the calculation proceeds like in §5 with the result

$$\frac{|\hbar|^2}{\kappa^2}\,\frac{N}{4}\,e^{-2G|\tau|}\,\frac{1}{4|G|\,T_p} \tag{7.22}$$

On account of

$$G = \frac{2|h|^2 G_0 - \kappa\gamma}{\kappa} = \frac{2|h|^2\,\bar{n}}{\kappa}$$

eq. (7.22) also can be written

$$\frac{1}{32}\,\frac{N}{\kappa\,T_p\,\bar{n}}\,e^{-2|G|\tau}\,.$$

From this expression is it evident that the contribution of $\langle\rho^2\rangle$ is by many orders of magnitude smaller than $\bar{n}$ when $\bar{n}$ is somewhat above threshold. It should be noted, however, that these expressions as well as the other neglected nonlinear terms of ρ might be important just at threshold.

b) Influence of the fluctuation of the atomic inversion. In all our above considerations we have neglected the influence on the amplitude by the fluctuation of inversion. We want to show that this contribution also is vanishingly small if laser action is considered somewhat above threshold. In order to take into account the fluctuation of inversion we have to replace on the left hand side of eq. (7.4) G_0 by $G_0 + g(t)$, so that we have

$$\ddot{\rho} + (\kappa+\gamma)\,\dot{\rho} + 2\rho\,\tilde{G} - r_0\,2|h|^2\,g(t) \tag{7.23}$$

where

$$g(t) = \left(\sum_{t_{\mu\nu}}^{t} \Theta(t - t_{\mu\nu}) - \frac{N}{T_p}\,t\right) d_0\,.$$

For sake of simplicity we again consider only the limiting case of large κ. The differential equation then reads

$$\kappa\,\dot{\rho} + 2\rho\,\tilde{G} = 2|h|^2\,r_0\,g(t)$$

with the solution

$$\rho=e^{-Gt}2|h|^2\frac{r_0}{\kappa}\int_0^t e^{G\tau}\left(\sum\Theta(t-t_{\mu\nu})-\frac{N}{T_p}t\right)dt\,d_0\,,\qquad G=\frac{2\tilde{G}}{\kappa}.\tag{7.24}$$

Carrying out the integration yields

$$2|h|^2\frac{r_0}{\kappa}d_0\left(\sum\frac{1-e^{-G(t-t_{\mu\nu})}}{G}-\frac{N}{T_p}\frac{t}{G}+\frac{N}{T_pG^2}(1-e^{-Gt})\right).\tag{7.25}$$

The first term of the sum cancels against the third term. The last term vanishes, because it represents only a swiching-on effect. In order to get an estimate of the contribution we consider

$$\left.\begin{aligned}\langle\rho^2\rangle=\frac{r_0^2d_0^2}{\tilde{G}^2}|h|^4\Bigg(&\frac{N^2}{(T_pG)^2}-\frac{2N}{T_pG}\sum_{\mu\nu}e^{-G(t-t_{\mu\nu})}+\\&+\sum_{\mu\nu}\sum_{\mu'\nu'}e^{-2Gt+G(t_{\mu\nu}+t_{\mu'\nu'})}\Bigg).\end{aligned}\right\}\tag{7.26}$$

Using

$$\overline{\sum_{\mu\nu}}=N\overline{\sum_{\nu}e^{-G(t-t_\nu)}}=\frac{N}{GT_p}\tag{7.27}$$

and splitting the average into that over different and equal molecules we obtain

$$\rho^2=\frac{r_0^2d_0^2}{\tilde{G}^2}|h|^4N\left\{-\frac{1}{(GT_p)^2}+\sum_{\nu,\nu'}\overline{e^{-2Gt+Gt_\nu+Gt_{\nu'}}}\right\}.\tag{7.28}$$

After performing the average over the last sum we get

$$\rho^2=|h|^4\frac{r_0^2d_0^2N}{\tilde{G}^2}\frac{1}{2GT_p}\left(1-\frac{2}{T_pG+2}\right).\tag{7.29}$$

Thus we find in any case that

$$\rho^2\ll\frac{r_0^2d_0^2N}{\tilde{G}^2}|h|^2\approx\frac{\bar{n}N}{\bar{n}^2}=\frac{N}{\bar{n}}\tag{7.30}$$

which is of the order of $N/\bar{n}$ and therefore somewhat above threshold again many orders of magnitude smaller than $r_0^2=\bar{n}$ itself.

c) Correlation function for the amplitudes for the general eq. (7.4). Our above calculation of the correlation function were done for the limiting case of large κ. We want now to demonstrate by means of an example that these results are not changed qualitatively when we consider the general equation. The determination of $\langle\rho(t+\tau)\rho(t)\rangle$ using the general solution (7.7) yields

$$e^{-\frac{\alpha}{2}\tau}\left\{e^{-i\sqrt{L}\tau}A+e^{i\sqrt{L}\tau}A^*\right\}$$

where

$$A=\frac{N\bar{R}^2}{(8\tilde{G}-\alpha^2)T_p}\left\{\frac{1}{\alpha}-\frac{1}{(\alpha+2i\sqrt{L})}\right\}.$$

Because $\frac{\tilde{G}}{\bar{R}^2}\sim\bar{n}$ we have $A\sim\frac{N}{T_p}\Big/(\kappa\bar{n})$ from which we see that this is of exactly the same order of magnitude as discussed in a). It remains limited for high inversion (for $T_p\to 0$ or $N\to\infty$) and is again orders of magnitude smaller than $\bar{n}$ itself. Note, however, that this expression might become important near threshold. As is obvious from the time dependence of this correlation function there occur two satellits around the frequency ω with a dip in the middle. This represents just a kind of hole burning within a homogeneously broadened line due to the build up of the stationary amplitude of oscillation.

§8. Connection with Glauber states

All our considerations above were based on the Heisenberg picture. It is interesting, however, to see how the corresponding wave function in the Schrödinger picture would look like. For this purpose we realize that the connection between Heisenberg and Schrödinger picture requires

$$\langle\psi_0^*\,\Omega(t)\,\psi_0\rangle=(\psi(t)\,\Omega(0)\,\psi(t)) \tag{8.1}$$

where $\Omega(t)$ is the time dependent Heisenberg operator and ψ_0 the wave function at initial time, whereas $\Omega(0)$ is the time dependent operator in the Schrödinger picture. As we have found above b can be written above threshold in the form $b_0\cdot e^{i\varphi(t)}$. We thus require

$$b_0\,e^{i\varphi(t)}\,\psi_0=b\,\psi(t)\,. \tag{8.2}$$

The solution of this equation reads for $t=t_0$

$$\psi_0=\text{Norm}\,e^{+b^+\,b_0\,e^{i\varphi_0}}\,\Phi_0\,. \tag{8.3}$$

This solution can be immediately extended to all times by putting

$$\psi(t)=N\cdots e^{+b^+\,b_0\,e^{i\varphi(t)}}\,\phi_0\,.$$

(8.3) can be put into connection with Glauber states by taking $\psi(t)$ as

$$\psi(t)=\int\delta(\varphi-\varphi(t))\,e^{b^+\,b_0\,e^{i\varphi}}\,\Phi_0\,d\varphi \tag{8.4}$$

where the integral represents a superposition of Glauber-states with fixed amplitude but variable φ. Although the calculations above have been performed as if $\varphi(t)$ is a C-number all the considerations can also well be done if $\varphi(t)$ is an operator.

§9. The complex degree of mutual coherence of two laser beams

The coherence function under consideration reads

$$\langle b_1^+(t+\tau)\, b_2(t)\rangle . \tag{9.1}$$

Because the amplitudes b_1 and b_2 stem from two different kinds of atoms we can (9.1) factorize

$$\langle b_1^+(t)\rangle\, \langle b_2(t)\rangle . \tag{9.2}$$

Above we have seen that

$$\langle b^+(t)\rangle = \left\{ \begin{array}{ll} 0 & \text{below threshold} \\ b_0 \langle e^{i\varphi(t)}\rangle & \text{above threshold}. \end{array} \right\} \tag{9.3}$$

This shows again that the amplitude of the laser light is stable in accordance with recent conclusions by JORDAN and GHIELMETTI [7] from experiments [10, 11], whereas the visibility of interference fringes is limited by phase fluctuations. If measurement is performed over a time T (which means averaging over t for a time interval T) one obtains for the complex degree of mutual coherence

$$\left.\begin{array}{l} \dfrac{1}{\sqrt{\langle b_1^+ b_1\rangle \langle b_2^+ b_2\rangle}}\, \dfrac{1}{T} \displaystyle\int_0^T \langle b_1^+(t+\tau)\rangle\, \langle b_2(t)\rangle\, dt \\ \qquad \approx \dfrac{1}{T} \displaystyle\int_0^T \langle e^{-i\varphi_1(t+\tau)}\rangle\, \langle e^{i\varphi_2(t)}\rangle\, dt \\ \qquad = e^{i(\varphi_{20}-\varphi_{10})}\, e^{-\Delta\nu_1\tau}\, \dfrac{1}{T(\Delta\nu_1+\Delta\nu_2)}\left(1-e^{-(\Delta\nu_1+\Delta\nu_2)T}\right). \end{array}\right\} \tag{9.4}$$

For times smaller than the inverse linewidth we obtain thus a fringe visibility close to unity. It were interesting to compare these results with that from two modes coming from the same laser. In this example the factorization of (9.1) is no more valid and therefore one may obtain other formulas for the fringe visibility. This will be investigated in forth coming papers.

[10] MAGYAR, G., and L. MANDEL: Nature **198**, 255 (1963).
[11] LIPSETT, M. S., and L. MANDEL: Nature **198**, 553 (1963).

§ 10. Concluding Remarks

The main objective of our paper was to bridge the gap between linear and nonlinear theories of laser action. As we have shown linear theories represent a very good approximation at small inversion. On the other hand there is a marked threshold beyond which the system behaves qualitatively very differently from below threshold, its amplitude oscillating around a stable value. For the explicit evaluation of the line-width by phase "diffusion" above threshold we have used a special model in which it was assumed, that the pumping time T_p is bigger than $h \cdot b$ and that the lower optical level is emptied to a ground level before the next excitation. The treatment of other situations is straight forward and will be published elsewhere. In our above treatment we have further assumed complete resonance between the atomic system and the cavity mode. This limitation can easily be released. One then finds linear, power independent mode pulling as given by TOWNES[12].

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Appendix

Extension of eq. (2.15) to standing waves and several modes

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We want to show, how one can derive equations corresponding to eq. (2.15) for this more general case. For this end we start with the basic equations

$$\dot{b}_\lambda^+ = i\,\omega_\lambda\, b_\lambda^+ + i \sum_\mu h_{\mu\lambda}^*\, \alpha_\mu^+, \tag{A.1}$$

$$\dot{\alpha}_\mu^+ = i\,\varepsilon\,\alpha_\mu^+ - i \sum_\lambda h_{\mu\lambda}\, b_\lambda^+\, \sigma_\mu + \sum_\nu \delta(t - t_{\mu\nu})\, \delta\alpha_\mu^+(t_{\mu\nu}), \tag{A.2}$$

$$\dot{\sigma}_\mu = -2\gamma\,\sigma_\mu + 2i\,\alpha_\mu \sum_\lambda h_{\mu\lambda}\, b_\lambda^+ - 2i\,\alpha_\mu^+ \sum_\lambda h_{\mu\lambda}^*\, b_\lambda + \sum_\nu \delta(t - t_{\mu\nu})\, \delta\sigma_\mu(t_{\mu\nu}) \tag{A.3}$$

which are the same as those of our preceding papers[3], being supplemented, however, by the pumping terms. We take $h_{\mu\lambda}$ in the form

$$h_{\mu\lambda} = h\sqrt{2}\,\sin k_\lambda x_\mu \tag{A.4}$$

and assume the following properties:

$$\sum_\mu h_{\mu\lambda}\, h_{\mu\lambda'} = N h^2\, \delta_{\lambda\lambda'}, \tag{A.5}$$

$$\sum_\lambda h_{\mu\lambda}\, h_{\mu'\lambda} = N h^2\, \delta_{\mu\mu'}. \tag{A.6}$$

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[12] TOWNES, C. H.: Quantum Electronics, edit. by J. R. SINGER. New York: Columbia University Press 1961.

We construct now

$$S_\lambda^+ = \frac{1}{h} \sum_\mu h_{\mu\lambda} \alpha_\mu^+, \tag{A.7}$$

$$S_{\lambda,\lambda'}^z = \frac{1}{2h^2} \sum_\mu h_{\mu\lambda} h_{\mu\lambda'} \sigma_\mu . \tag{A.8}$$

Note, that on account of (A.6):

$$\alpha_\mu^+ = \frac{1}{Nh} \sum_\lambda h_{\mu\lambda} S_\lambda^+ . \tag{A.9}$$

Eq. (A.1) goes immediately over into

$$\dot{b}_\lambda^+ = i\,\omega_\lambda b_\lambda^+ + i\,h\,S_\lambda^+ \tag{A.1'}$$

Multiplying eq. (A.2) with $\frac{1}{h} h_{\mu\lambda}$ and summing up over μ yields

$$\dot{S}_\lambda^+ = i\,\varepsilon S_\lambda^+ - 2\,i\,h \sum_{\lambda'} b_{\lambda'}^+ S_{\lambda,\lambda'}^z + \underbrace{\sum_\mu \sum_\nu h_{\mu\lambda}\,\delta(t-t_{\mu\nu})\,\delta\alpha_\mu^+(t_{\mu\nu})}_{\Delta_\lambda^+} . \tag{A.2'}$$

Finally we multiply (A.3) with $\frac{1}{h^2} h_{\mu\lambda} h_{\mu\lambda'}$ and sum up over μ. With help of (A.9) we obtain:

$$\left.\begin{aligned} \dot{S}_{\lambda,\lambda'}^z = {} & i\Big(\sum_{\lambda''}\sum_{\lambda'''} b_{\lambda''}^+ S_{\lambda'''}^- h\,A(\lambda,\lambda',\lambda'',\lambda''') - \text{con. compl.}\Big) - 2\gamma S_{\lambda,\lambda'}^z + \\ & \underbrace{+\frac{1}{2h^2}\sum_\mu\sum_\nu \delta(t-t_{\mu\nu})\,h_{\mu\lambda}h_{\mu\lambda'}\,\delta\sigma_\mu(t_{\mu\nu})}_{\Delta_{\lambda,\lambda'}^z} \end{aligned}\right\} \tag{A.3'}$$

where

$$A = \frac{1}{h^4 N} \sum_\mu h_{\mu\lambda} h_{\mu\lambda'} h_{\mu\lambda''} h_{\mu\lambda'''} .$$

According to eq. (A.1') we express S_λ^+ by b_λ^+:

$$\left.\begin{aligned} \dot{S}_{\lambda,\lambda'}^z = - \Big\{ & \sum_{\lambda''} b_{\lambda''}^+ \sum_{\lambda'''} (\dot{b}_{\lambda'''} + i\,\omega_{\lambda'''}^* b_{\lambda'''})\,A(\lambda,\lambda',\lambda'',\lambda''') + \\ & + \text{con. compl.}\Big\} - 2\gamma S_{\lambda,\lambda'}^z + \Delta_{\lambda,\lambda'}^z . \end{aligned}\right\} \tag{A.10}$$

We confine ourselves now to a single mode:

$$\lambda = \lambda' = \lambda'' = \lambda''' = \lambda_0 , \qquad A = \tfrac{3}{2}.$$

(A.10) then reads:

$$\dot{S}_{\lambda_0,\lambda_0}^z = -2(b^+ b)\cdot\tfrac{3}{2} - 2\kappa\, b^+ b \cdot \tfrac{3}{2} - 2\gamma S_{\lambda_0,\lambda_0}^z + \Delta_{\lambda_0,\lambda_0}^z . \tag{A.11}$$

The further treatment is identical with that on page 101 leading to

$$\left.\begin{aligned}&\ddot{b}^{+}+\{\kappa+\gamma-i(\omega_0+\varepsilon_0)\}\,\dot{b}^{+}-\\&-\{2h^2(G_0-\tfrac{3}{2}b^{+}b)-(i\varepsilon_0-\gamma)(i\omega_0-\kappa)\}\,b^{+}=i\,h\,\Delta_{\lambda_0}^{+}\end{aligned}\right\}\qquad(\text{A.12})$$

In order to determine b^+ for the *completely steady* state we put

$$b^{+}=b_0^{+}\,e^{i\Omega t}\qquad(b_0^{+}:\ \text{constant}).$$

Splitting then (A. 12) into real and imaginary part yields for Ω the power independent mode-pulling as mentioned in § 10, and for n_0 (photon number) an expression which coincides for moderate pumping power with that determined in a previous paper[3].

The simultaneous action of several modes can be treated similarly. Again the results for the completely steady state coincide with those of a previous paper[3] for moderate pumping power and if population pulsations are neglected.