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Free Vibration of Rectangular Beams of Arbitrary Depth

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With 1 Figure

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Summary — Zusammenfassung

Free Vibration of Rectangular Beams of Arbitrary Depth. The state space approach is extended to the two dimensional elastodynamic problems. The formulation is in a form particularly amenable to consistent reduction to obtain approximate theories of any desired order. Free vibration of rectangular beams of arbitrary depth is investigated using this approach. The method does not involve the concept of the shear coefficient k. It takes into account the vertical normal stress and the transverse shear stress. The frequency values are calculated using the Timoshenko beam theory and the present analysis for different values of Poisson's ratio and they are in good agreement. Four cases of beams with different end conditions are considered.

Freie Schwingungen rechteckiger Balken beliebiger Höhe. Die Zustandsraum-Technik wird auf zweidimensionale elastodynamische Probleme ausgedehnt. Die Formulierung ist besonders geeignet für die Aufstellung von Näherungstheorien beliebigen Grades. Freie Schwingungen von Rechteckbalken beliebiger Höhe wurden mit Hilfe dieser Technik untersucht. Das Verfahren umgeht den Begriff des Schubbeiwerts k. Es berücksichtigt die senkrechte Normalbeanspruchung und die Querkraft. Die Frequenzwerte werden mit Hilfe der Balkentheorie von Timoshenko und der vorliegenden Analyse berechnet, und zwar für verschiedene Werte der Querdehnzahl. Die berechneten Werte befinden sich in guter Übereinstimmung. Vier Fälle von Balken mit verschiedenen Endbedingungen werden untersucht.

Notation

$2h$	depth of beam
k	Timoshenko shear constant
L	length of the beam
n	mode number
u, v	displacement in x, y directions
A	area of cross section
A_n	coefficient in series representation
E	modulus of elasticity
G	modulus of rigidity
I	moment of inertia about z-axis
ϱ	mass density
μ	Poisson's ratio
r	$\sqrt{I/A}/L$

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θ	$r \times n$
$\delta\sigma_x$, σ_y	direct stresses
τ_{xy}	shear stress
η	eigenvalue of square matrix
ω	frequency of harmonic vibration
λ	eigenvalue $= \sqrt{\frac{\varrho}{G}}\,\omega L$
Ω	frequency parameter $= \sqrt{\frac{\varrho A}{EI}}\,\frac{\omega L^2}{n^2}$
Ω^*	frequency parameter $= \Omega \times \theta$

1. Introduction

The Bernoulli-Euler equation for beams does not consider the effects of shear deformation, rotatory inertia and the vertical normal stress. Rayleigh [1] introduced the effect of rotatory inertia and Timoshenko [2] incorporated the effect of shear deformation. The shear stress and shear strain are not uniformly distributed over the cross section. Timoshenko introduced a dimensionless constant k to account for this. By taking k as the ratio of the average shear stress on a cross section to the product of the shear modulus and the angle of shear at the centroid, he arrived at a value of 0.67 for rectangular cross sections. Later he [3] suggested a value of 0.889 in order to bring the predictions of his equation into closer agreement with the three dimensional theory of small vibrations of elastic bodies. Cowper [4] derived the equations of the Timoshenko beam theory, by integration of the equations of the theory of elasticity. He obtained expressions for shear constant for different cross sections as functions of Poisson's ratio. For rectangular cross section its value is $10(1 + \mu)/(12 + 11\mu)$.

A different approach, using the two dimensional elasticity theory is given here to determine the frequencies of beams of arbitrary depth. This method does not resort to the selection of any shear constant.

Vlasov [5] developed the method of initial functions (MIF) for rectangular regions by expanding the unknowns in Maclaurin series in thickness coordinate. This method has been extended to twodimensional elastodynamic problems by Das and Setlur [6]. Bahar [7] used state space view point to approach Vlasov's formulation of elasticity. In the present paper the state space approach is extended to two dimensional elastodynamic problems. Using this method, the natural frequencies have been calculated for beams with various boundary conditions. The values by Timoshenko beam solution are also given for comparison.

2. Formulation of the Problem

The governing equations of the plane stress case, without body forces, of the theory of elasticity are (Fig. 1)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \frac{\varrho\,\partial^2 u}{\partial t^2}$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \frac{\varrho\,\partial^2 v}{\partial t^2}. \tag{1}$$

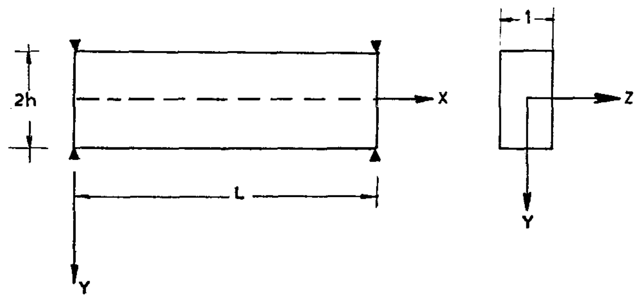

Fig. 1. Coordinate system

Using Hooke's law the stresses can be written as

$$\begin{aligned}
\sigma_x &= \frac{2G}{1-\mu}\left(\frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y}\right) \\
\sigma_y &= \frac{2G}{1-\mu}\left(\frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x}\right) \qquad (2) \\
\tau_{xy} &= G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).
\end{aligned}$$

Let

$$\begin{aligned}
&U = Gu, \qquad V = Gv, \qquad X = \tau_{xy}, \qquad Y = \sigma_y \\
&\frac{\partial}{\partial x} = \alpha, \qquad \frac{\partial}{\partial y} = \beta, \qquad \frac{\varrho}{G}\frac{\partial^2}{\partial t^2} = \zeta^2.
\end{aligned} \qquad (3)$$

By eliminating σ_x between (1) and (2) the following basic equation is obtained

$$\beta \begin{bmatrix} U \\ V \\ Y \\ X \end{bmatrix} = \begin{bmatrix} 0 & -\alpha & 0 & 1 \\ -\mu\alpha & 0 & \frac{1-\mu}{2} & 0 \\ 0 & \zeta^2 & 0 & -\alpha \\ -2(1+\mu)\,\alpha^2 + \zeta^2 & 0 & -\mu\alpha & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ Y \\ X \end{bmatrix} \qquad (4)$$

Let Z denote the state vector

$$[U, V, Y, X]^\tau.$$

Eq. (4) can be written as

$$\frac{d}{dy}[Z] = [A]\,[Z]. \qquad (5)$$

The integration of the vector matrix differential equation yields

$$[Z] = \exp\,[y \cdot A]\,Z(0), \qquad (6)$$

where $Z(0)$ correspond to Z at $y = 0$. The exponential matrix is the transfer matrix that maps the initial state vector into the field. The characteristic equation of the determinant associated with the matrix $[A]$ is

$$[\eta^2 + (\alpha^2 - \zeta^2)]\left[\eta^2 + \left(\alpha^2 - \frac{1-\mu}{2}\zeta^2\right)\right] = 0. \tag{7}$$

The roots are

$$\eta = \pm i\delta_1, \quad \pm i\delta_2, \tag{8}$$

where

$$\delta_1 = \sqrt{\alpha^2 - \zeta^2}; \qquad \delta_2 = \sqrt{\alpha^2 - \frac{1-\mu}{2}\zeta^2}. \tag{9}$$

According to the Cayley-Hamilton theorem the square matrix sarisfies its own characteristic equation. Hence the exponential matrix can be written as

$$\exp[yA] = a_0 I + a_1 A + a_2 A^2 + a_3 A^3. \tag{10}$$

The expression (10) must also be satisfied if the matrix A is replaced by its own eigenvalues. Hence

$$\exp(y\eta) = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3. \tag{11}$$

Substituting the roots (8) into (11) and solving the system of equations the following values are obtained for a_0, a_1, a_2 and a_3.

$$\begin{aligned} a_0 &= 2(\delta_2^2 \cos y\delta_1 - \delta_1^2 \cos y\delta_2)/(1+\mu)\,\zeta^2 \\ a_1 &= 2\left(\delta_2^2 \frac{\sin y\delta_1}{\delta_1} - \delta_1^2 \frac{\sin y\delta_2}{\delta_2}\right)\Big/(1+\mu)\,\zeta^2 \\ a_2 &= 2(\cos y\delta_1 - \cos y\delta_2)/(1+\mu)\,\zeta^2 \\ a_3 &= 2\left(\frac{\sin y\delta_1}{\delta_1} - \frac{\sin y\delta_2}{\delta_2}\right)\Big/(1+\mu)\,\zeta^2. \end{aligned} \tag{12}$$

Substituting these values in (10) the transfer matrix $[L]$ is obtained. From (6) we get

$$\begin{bmatrix} U \\ V \\ Y \\ X \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} U_0 \\ V_0 \\ Y_0 \\ X_0 \end{bmatrix} \tag{13}$$

where U_0, V_0, Y_0, X_0 are all initial unknown functions in the plane $y = 0$. The coefficients L_{11}, L_{12} etc., of the transfer matrix are all differential operators and have the following values

$$
\begin{aligned}
L_{11} &= L_{44} = \frac{(-2\alpha^2 + \zeta^2)}{\zeta^2} \cos y\delta_1 + \frac{2\alpha^2}{\zeta^2} \cos y\delta_2 \\
L_{12} &= L_{34} = \frac{2\alpha(\alpha^2 - \zeta^2)}{\zeta^2\delta_1} \sin y\delta_1 - \frac{\alpha(2\alpha^2 - \zeta^2)}{\zeta^2\delta_2} \sin y\delta_2 \\
L_{13} &= L_{24} = -\frac{\alpha}{\zeta^2} \cos y\delta_1 + \frac{\alpha}{\zeta^2} \cos y\delta_2 \\
L_{14} &= -\frac{(\alpha^2 - \zeta^2)}{\zeta^2\delta_1} \sin y\delta_1 + \frac{\alpha^2}{\zeta^2\delta_2} \sin y\delta_2 \\
L_{21} &= L_{43} = \frac{\alpha(2\alpha^2 - \zeta^2)}{\zeta^2\delta_1} \sin y\delta_1 - \frac{\alpha[2\alpha^2 - (1-\mu)\,\zeta^2]}{\zeta^2\delta_2} \sin y\delta_2 \\
L_{22} &= L_{33} = \frac{2\alpha^2}{\zeta^2} \cos y\delta_1 - \frac{(2\alpha^2 - \zeta^2)}{\zeta^2} \cos y\delta_2 \\
L_{23} &= \frac{\alpha^2}{\zeta^2\delta_1} \sin y\delta_1 - \frac{(2\alpha^2 - (1-\mu)\,\zeta^2)}{2\zeta^2\delta_2} \sin y\delta_2 \\
L_{31} &= L_{42} = \frac{2\alpha(2\alpha^2 - \zeta^2)}{\zeta^2} \cos y\delta_1 - \frac{2\alpha(2\alpha^2 - \zeta^2)}{\zeta^2} \cos y\delta_2 \\
L_{32} &= \frac{-4\alpha^2(\alpha^2 - \zeta^2)}{\zeta^2\delta_1} \sin y\delta_1 + \frac{(2\alpha^2 - \zeta^2)^2}{\zeta^2\delta_2} \sin y\delta_2 \\
L_{41} &= \frac{(2\alpha^2 - \zeta^2)^2}{\zeta^2\delta_1} \sin y\delta_1 - \frac{2\alpha^2[2\alpha^2 - (1-\mu)\,\zeta^2]}{\zeta^2\delta_2} \sin y\delta_2 .
\end{aligned}
\tag{14}
$$

The above coefficients of the transfer matrix are symmetric with respect to the secondary diagonal, due to the isotropy of the elastic body. Thrse expressions are in agreement with those given in Ref. [6], where they have been obtained by assuming the solution of (4) in the form of Maclaurin series in the y direction.

3. Application of the Method

3.1. Beam Subjected to Symmetrical Loading

Taking $y = 0$ as the reference plane, because of symmetry in loading we obtain

$$V_0 = X_0 = 0. \tag{15}$$

Hence Eqs. (13) simplify to

$$
\begin{aligned}
U &= L_{11}U_0 + L_{13}Y_0 \\
V &= L_{21}U_0 + L_{23}Y_0 \\
Y &= L_{31}U_0 + L_{33}Y_0 \\
X &= L_{41}U_0 + L_{43}Y_0.
\end{aligned}
\tag{16}
$$

On the plane $y = \pm h$

$$Y = -p(x, t), \qquad X = 0. \tag{17}$$

Using these values of Y and X, the Eqs. (16) reduce to the following two equations

$$\begin{bmatrix} L_{31} & L_{33} \\ L_{41} & L_{43} \end{bmatrix}_{y=h} \begin{bmatrix} U_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} -p \\ 0 \end{bmatrix}. \tag{18}$$

The second equation is made an identity by introducing an auxiliary function φ such that

$$U_0 = L_{43}^{(h)} \varphi, \qquad Y_0 = -L_{41}^{(h)} \varphi. \tag{19}$$

The first equation of (18) becomes

$$(L_{31} L_{43} - L_{33} L_{41})_h \, \varphi = -p. \tag{20}$$

Substituting the values of the operators from Eq. (14), the following exact partial differential equation is obtained for the free vibration symmetric with respect to the middle plane

$$\left[\frac{2\alpha^2[2\alpha^2 - (1-\mu)\,\zeta^2]}{\zeta^2} \cos h\delta_1 \, \frac{\sin h\delta_2}{\delta_2} - \frac{(2\alpha^2 - \zeta^2)^2}{\zeta^2} \cos h\delta_2 (\sin h\delta_1)/\delta_1\right] \varphi = 0 \tag{21}$$

Expanding the trignometric expressions and retaining terms up to h^3, we get a fourth order theory. Taking terms of higher powers of h, higher order theories are obtained. When φ is known U_0 and Y_0 can be obtained from Eq. (19) and the stresses can be obtained from Eq. (16).

3.2 Beam Subjected to Anti Symmetrical Loading

Taking $y = 0$ as the reference plane, because of antisymmetry in loading we obtain

$$U_0 = Y_0 = 0. \tag{22}$$

On the plane $y = \pm h$

$$Y = \pm p(x, t) \qquad X = 0. \tag{23}$$

Using these values of Y and X, the Eqs. (13) reduce to the following two equations

$$\begin{bmatrix} L_{32} & L_{34} \\ L_{42} & L_{44} \end{bmatrix}_{y=h} \begin{bmatrix} V_0 \\ X_0 \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix}. \tag{24}$$

Introducing an auxiliary function F such that

$$V_0 = L_{44}^{(h)} F, \qquad X_0 = -L_{42}^{(h)} F \tag{25}$$

the second of Eq. (24) is identically satisfied and the first of Eq. (24) leads to the following differential equation for the free vibration

$$\left[\frac{(2\alpha^2 - \zeta^2)^2}{\zeta^2 \delta_2} \cos h\delta_1 \sin h\delta_2 - \frac{4\alpha^2(\alpha^2 - \zeta^2)}{\zeta^2 \delta_1} \sin h\delta_1 [\cos h\delta_2]\right] F = 0. \tag{26}$$

Eq. (26) is the exact transcendental partial differential equation for the free vibration anti-symmetric with respect to the middle plane. Expanding the trignometric expressions and retaining a finite number of terms, solution of a desired order can be obtained. When F is known, the values of X_0 and V_0 can be obtained from Eq. (25) and the stresses can be obtained from Eq. (13). Since the present formulation has no restriction on the depth the theory could be used for both shallow (long) and deep (short) beams. Sundara Raja Iyengar et al. [8] have given detailed numerical work for higher order theories for static case.

3.3 Boundary Conditions

The boundary conditions can be expressed in terms of the auxiliary function φ or F. As an illustration, the boundary conditions are given for a sixth order theory of the antisymmetric case.

(i) Hinged End ($v = 0$, $u = 0$)

$$v = 0 \quad \text{gives} \quad \left[1 - \frac{2+\mu}{2} h^2\alpha^2\right] F = 0$$

$$u = 0 \quad \text{gives} \quad \alpha^2 F = \alpha^4 F = 0.$$

Combining them we get the necessary conditions as

$$F = \alpha^2 F = \alpha^4 F = 0. \tag{27}$$

(ii) Clamped End ($u = 0$, $v = 0$)

$$\alpha F = \alpha^3 F = 0$$

$$\left[1 - \frac{2+\mu}{2} h^2\alpha^2\right] F = 0. \tag{28}$$

(iii) Free End ($\sigma_x = 0$, $\tau_{xy} = 0$)

Of the two conditions any one can be satisfied exactly and the other approximately. Assuming that σ_x is satisfied exactly then

$$\alpha^2 F = \alpha^4 F = 0. \tag{29a}$$

The remaining condition is obtained by satisfying $\tau_{xy} = 0$ approximately as

$$\int_{-h}^{h} \tau_{xy}\, dy = 0. \tag{29b}$$

3.4 Solution of the Differential Equation

As the methods of solution of the differential equation for the symmetric and the anti-symmetric case are identical, the method is explained for the anti-symmetric case.

Expanding the rignometric terms in Eq. (26) and retaining terms upto h^3 the following differential equation is obtained

$$\left[\frac{2(1+\mu)}{3} h^3\alpha^4 - \frac{2}{3}(2+\mu) h^3\alpha^2\zeta^2 + \frac{7-\mu}{12} h^3\zeta^4 + h\zeta^2\right] F = 0. \tag{30}$$

In the expanded form, this will be

$$\left[\frac{2(1+\mu)}{3} h^3 \frac{\partial^4}{\partial x^4} - \frac{2(2+\mu)\varrho}{3G} h^3 \frac{\partial^4}{\partial x^2\,\partial t^2} + \frac{7-\mu}{12} h^3 \left(\frac{\varrho}{G}\right)^2 \frac{\partial^4}{\partial t^4} + \frac{h\varrho}{G}\frac{\partial^2}{\partial t^2}\right] F = 0. \tag{31}$$

For free vibration one can assume

$$F(x, t) = F_1(x) \cos \omega t, \tag{32}$$

where ω is the circular frequency and $F_1(x)$ is a function of x only. For a beam with hinged ends, the required boundary conditions will be satisfied if F_1 is taken as

$$F_1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{33}$$

Substituting Eqs. (32) and (33) into Eq. (31) we get a fourth degree polynomial in ω from which ω can be obtained. The frequency is expressed in the nondimensional form as eigenvalue λ where

$$\lambda = \sqrt{\frac{\varrho}{G}}\, \omega L. \tag{34}$$

In the case of higher order theories, the polynomial in λ will be correspondingly of higher degree. From Eq. (25) as a first approximation one obtains

$$V_0 = F. \tag{35}$$

If in Eq. (31) the first and last term only are retained one has

$$\left[\frac{2(1+\mu)}{3} h^3 \frac{\partial^4}{\partial x^4} + h \frac{\varrho}{G} \frac{\partial^2}{\partial t^2}\right] F = 0. \tag{36}$$

Substituting Eq. (35) in Eq. (36) and simplifying we obtain the following familiar equation of elementary beam theory.

$$\left[\frac{2}{3} E h^3 \frac{\partial^4}{\partial x^4} + 2h\varrho \frac{\partial^2}{\partial t^2}\right] \bar{v} = 0 \tag{37}$$

where $\bar{v}$ is the transverse deflection of the middle plane.

The numerical values of λ have been computed by the fourth, sixth and eighth order M.I.F. theories. Some of the frequency values are given in Table 1 for Poisson's ratio of 0.3. In Table 1 θ is defined as

$$\theta = \frac{n\sqrt{I/A}}{L}. \tag{38}$$

The frequency parameter Ω is defined as

$$\Omega = \sqrt{\frac{\varrho A}{EI}}\, \frac{\omega L^2}{n^2}. \tag{39}$$

The value of Ω by the elementary theory is π^2 and is independent of the depth-span ratio. The values of Ω are also computed by the Timoshenko beam theory using the shear constant k given by Cowper [4]. Table 2 gives the comparison between the frequency values computed using the eighth order M.I.F. theory and Timoshenko beam theory for different values of Poisson's ratio.

For beams with other boundary conditions, assuming trial values of λ, the general solution is written for the ordinary differential equation in F_1. Substituting this solution in the expressions for boundary conditions, we get a set of homogeneous simultaneous equations. For non-trivial solution the coefficient determinant should be zero. The value of λ which satisfies this condition is found. In Tables 3 to 5 the first five eigenvalues λ computed using the fourth and sixth order M.I.F. theories are given for beams with other boundary conditions. In these three tables, the values of λ are given for two values of r where r is defined as

$$r = \frac{\sqrt{I/A}}{L}.$$

The value of the Poisson's ratio is 0.3 in all the cases. The eigenvalues computed using the Timoshenko beam theory and elementary beam theory are also given for comparison.

In the case of vibration symmetric with respect to the middle plane, for beam with hinged ends, the frequency parameter Ω is π/θ for a given value of θ. In otherwords the parameter $\Omega^* = \Omega \times \theta$ is a constant and is π.

4. Discussion of Results and Conclusion

In the case of hinged beams the nth harmonic of a particular beam of span L, correspond to the fundamental frequency of a shorter beam of span L/n. Hence the depth span ratio and the mode number are combined in the single parameter θ and the non-dimensional frequency parameter are given for different values of θ. From the values given in Table 1 it can be seen that the frequencies calculated using the elementary beam theory are always higher and the difference increases for higher values of θ. From the same table it can be seen that the frequencies computed using the present approach show convergence as the order is increased.

Table 1. *Frequency parameter Ω for beam with hinged ends, Poisson's ratio = 0.3*

θ	MIF IV	MIF VI	MIF VIII
0.01	9.8476	9.8502	9.8502
0.05	9.3633	9.4229	9.4233
0.1	8.2703	8.4202	8.4237
0.5	3.1882	3.3132	3.3336
1.0	1.6957	1.7509	1.7666
2.0	0.8635	0.8863	0.8983
3.0	0.5777	0.5920	0.5950

From Table 2 it is clear that the frequencies depend on the value of the Poisson's ratio and the frequency values are in agreement with the Timoshenko beam theory with the values of k given by Cowper.

From Tables 3 to 5 it is clear that the results by the present theory are in agreement with Timoshenko beam theory. In the case of beams with both ends clamped, and one end clamped and the other end hinged, the eigenvalues obtained by the sixth order M.I.F. theory are slightly higher than those obtained using Timoshenko beam theory. In the case of beam with one end clamped and the other end free, the eigenvalues obtained using the sixth order M.I.F. theory are lower than those obtained using Timoshenko beam theory. Similar results are also obtained in the case of beams with one end hinged and the other end free.

In the case of vibration symmetric with respect to the middle plane, the results are in agreement with the usual theory of extensional vibration of rods.

Table 2. *Frequency parameter Ω for beam with hinged ends for different values of Poisson's ratio*

	$\mu = 0$		$\mu = 0.3$		$\mu = 0.50$	
	MIF VIII	Timoshenko	MIF VIII	Timoshenko	MIF VIII	Timoshenko
0.01	9.8531	9.8531	9.8502	9.8499	9.8482	9.8478
0.05	9.4859	9.4859	9.4233	9.4169	9.3822	9.3716
0.1	8.5976	8.5975	8.4237	8.4047	8.3125	8.2820
0.5	3.6066	3.6233	3.3336	3.3029	3.1767	2.1265
1.0	1.9230	1.9612	1.7666	1.7546	1.6703	1.6467
2.0	0.9701	1.0050	0.8983	0.8927	0.8432	0.8354
3.0	0.6571	0.6733	0.5950	0.5971	0.5627	0.5595

Table 3. *Frequency parameter λ for beam with clamped ends. Poisson's ratio = 0.3*

r	Mode	Elementary theory	Timoshenko	MIF IV	MIF VI
	1	1.0823	1.0109	0.9852	1.0096
	2	2.9833	2.5847	2.5000	2.5897
0.03	3	5.8485	4.6498	4.6282	4.6706
	4	9.6679	7.0408	6.9568	7.1129
	5	14.4421	9.6461	9.5416	9.7798
0.06	1	2.1645	1.7229	1.7153	1.7277
	2	5.9666	3.9203	3.8533	3.9325
	3	11.6970	6.5127	6.4924	6.7129
	4	19.3358	9.2847	9.2817	9.5583
	5	28.8843	12.1740	12.1145	12.5460

Table 4. *Frequency parameter λ for beam with one end clamped and the other end hinged. Poisson's ratio = 0.3*

r	Mode	Elemantary theory	Timoshenko	MIF IV	MIF VI
0.03	1	0.7458	0.7161	0.7160	0.7162
	2	2.4170	2.1795	2.1792	2.1798
	3	5.0428	4.2064	4.2019	4.2148
	4	8.6235	6.6099	6.5291	6.6750
	5	13.1591	9.2576	9.1895	9.3648
0.06	1	1.4917	1.2906	1.2901	1.2913
	2	4.8340	3.5204	3.5198	3.5235
	3	10.0856	6.1979	6.1783	6.3246
	4	17.2471	9.0796	9.0854	9.3608
	5	26.3181	12.0541	12.0766	12.4062

Table 5. *Frequency parameter λ for beam with one end clamped and the other end free. Poisson's ratio = 0.3*

r	Mode	Elementary theory	Timoshenko	MIF IV	MIF VI
0.03	1	0.1701	0.1687	0.1661	0.1635
	2	1.0659	1.0081	0.9902	0.9762
	3	2.9845	2.6430	2.6298	2.5847
	4	5.8484	4.7783	4.6753	4.5361
	5	9.6679	7.2526	7.1902	6.9965
0.06	1	0.3402	0.3293	0.3232	0.3190
	2	2.1318	1.7596	1.7437	1.7178
	3	5.9690	4.1658	4.1468	4.0228
	4	11.6969	6.8873	6.8654	6.7428
	5	19.3358	9.7659	9.7999	9.5847

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