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Multiple Wiener-Ito Integrals Possessing a Continuous Extension

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Summary. Let $F(W)$ be a Wiener functional defined by $F(W)=I_n(f)$ where $I_n(f)$ denotes the multiple Wiener-Ito integral of order n of the symmetric $L^2([0,1]^n)$ kernel f. We show that a necessary and sufficient condition for the existence of a continuous extension of F, i.e. the existence of a function $\phi(\cdot)$ from the continuous functions on $[0,1]$ which are zero at zero to $\mathbb{R}$ which is continuous in the supremum norms and for which $\phi(W)=F(W)$ *a.s*, is that there exists a multimeasure $\mu(dt_1,\dots,dt_n)$ on $[0,1]^n$ such that $f(t_1,\dots,t_n)=\mu((t_1,1],(t_2,1],\dots,(t_n,1])$ a.e. Lebesgue on $[0,1]^n$. Recall that a multimeasure $\mu(A_1,\dots,A_n)$ is for every fixed i *and every* fixed $A_i,\dots,A_{i-1},A_{i+1},\dots,A_n$ a signed measure in A_i and there exists multimeasures which are not measures. It is, furthermore, shown that if $f(t_1, t_2,\dots,t_n)=\mu((t_1,1],\dots,(t_n,1])$ then all the traces $f^{(k)}$, $k\le\left[\frac{n}{2}\right]$ of f exist, each $f^{(k)}$ induces an $n-2k$ multimeasure denoted by $\mu^{(k)}$, the following relation holds

$$I_n(f)=\sum_{k=0}^{[n/2]}\left(-\frac{1}{2}\right)^k\frac{n!}{k!(n-2k)!}\int_{[0,1]^{n-2k}}W_{t_1}\cdots W_{t_{n-2k}}\cdot\mu^{(k)}(dt_1,\dots,dt_{n-2k})$$

and each of the integrals in the above expression equals the multiple Stratonovich or Ogawa type integral of the trace $f^{(k)}$, namely

$$\int_{[0,1]^{n-2k}}W_{t_1}\cdots W_{t_{n-2k}}\mu^{(k)}(dt_1,\dots,dt_{n-2k})=I_{n-2k}{}^\circ(f^{(k)})\,.$$

1. Introduction

A. Statement of the Problem

Let $K(t_1,\dots,t_n)$ $t_i\in[0,1]$ be a symmetric n-kernel and assume that $K\in L^2([0,1]^n)$. $\{W_t,\ 0\le t\le 1\}$ is the standard Brownian motion on $C_o([0,1])$

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where $C_o([0, 1])$ denotes the class of real valued continuous functions on $[0, 1]$ which are zero at zero. Let $F(W)$ be the Brownian functional defined by

$$F(W) = I_n(K) = \int_{[0,1]^n} K(t_1, \ldots, t_n) W(dt_1) \cdots W(dt_n) .$$

Then $F(W)$ is defined for almost all sample functions of the Brownian motion and the question arises whether there exists a function f: $C_o([0, 1]) \to \mathbb{R}$ which (i) is continuous on $C_o([0, 1])$ in the supremum norm and (ii) $f(W) = F(W)$ *a.s.*. Note that given $F(W)$, $F(\lambda W)$ is not defined since the measures induced by $\{W\}$ and $\{\lambda W\}$ on $C_o([0, 1])$ are mutually singular and we are free to define $F(\lambda W)$, $|\lambda| \neq 1$, given $F(W)$, in anyway we please. Also note that a continuous extension, if it exists, is unique since W is dense in $C_o([0, 1])$ in the sense that the Wiener measure of a neighborhood of a continuous function is non zero (we will express this by saying that the support of W is $C_o([0, 1])$).

B. *The Case* $n = 1$

Let us start with a sufficient condition: suppose that there exists a signed measure μ on $[0, 1]$ with $\mu(\{0\}) = 0$. Set $K(t) = \mu((t, 1])$, note that $K(1) = 0$, therefore integrating by parts

$$I_1(K) = \int_{[0,1]} K(t) W(dt) = \int_{[0,1]} W(t) \mu(dt) .$$

For $x \in C_o([0, 1])$ set $f(x) = \int_{[0,1]} x_t \mu(dt)$ then $f(x)$ is continuous and linear. Turning to the converse direction: Given $K(t) \in L^2[0, 1]$, we start by showing that a continuous extension, if it exists then it must be linear. Let $-1 \leqq \alpha \leqq 1$, and let W^a and W^b be independent Brownian motions, then

$$\int_{[0,1]} K(t) d(\alpha W_t^a + \sqrt{1-\alpha^2} W_t^b) = \int K(t) \alpha W^a(dt) + \int \sqrt{1-\alpha^2} K(t) W^b(dt) .$$

Therefore, if a continuous version $f(\cdot)$ exists then

$$f(\alpha W^a + \sqrt{1-\alpha^2} W^b) = \alpha f(W^a) + \sqrt{1-\alpha^2} f(W^b) \quad a.s.$$

This holds for W^a, W^b however it implies linearity of $f(\cdot)$ on $C_o[0, 1]$ since the support of the Wiener measure is $C_o([0, 1])$ and $f(\cdot)$ was assumed to be continuous. Now, if $f(\cdot)$ is linear and continuous then by the Riesz representation theorem there exists a measure μ, with $\mu(\{0\}) = 0$ such that

$$f(x) = \int_{[0,1]} x_s \mu(ds)$$

and a straightforward calculation of integration by parts yields

$$f(x) = \int_{[0,1]} \mu((s, 1]) x(ds) .$$

For $n = 2$, measures on $[0, 1]^2$ induce bilinear continuous functionals on $C_o([0, 1]) \times C_o([0, 1])$ via

$$f(x, y) = \int_{[0,1]^2} x(s)y(t)\,\mu(ds, dt)\,,$$

however it is known that not all multilinear continuous functionals on $C_o([0, 1]) \times \cdots \times C_o([0, 1])$ are representable via measures. For the purpose of the construction of multilinear continuous functionals on $C_o([0, 1]) \times \cdots \times C_o([0, 1])$ the natural notion is that of a bimeasure for $n = 2$ and more generally, that of a multimeasure for $n \geqq 2$. A bimeasure $\mu(A_1, A_2)$ is a real valued function on pairs A_1, A_2 of Borel measurable sets on $\mathbb{R}$ such that $\mu(dt_1, A_2)$ is a signed measure in dt_1 for every fixed A_2 and $\mu(A_1, dt_2)$ is a signed measure in dt_2 for every fixed A_1.

In the next section we first summarize very briefly some definitions and known results on multimeasures. For more information and further references cf. [1, 2, 5, 7]. In the remaining part of section 2 we present some specific results which are needed in the later sections. The characterization of kernels $K(t_1, \ldots, t_n)$ which induce Wiener functionals possessing a continuous extension will be derived in section 3. Two problems are considered in section 3, the first is that of the continuous multilinear extension of functionals of the type defined by equation (3.1) and the second, the main result, is that of the continuous extension of functionals $F(W) = I_n(f)$. Multiple Ogawa and Stratonovich integrals are defined and discussed in section 4. The relation between these stochastic integrals and the original one introduced by N. Wiener in [15] is also discussed in this section 4.

The characterization results of section 3 can be extended in several directions. One generalization is to replace $W = \{W(t), t \in [0, 1]\}$ with $\{W(\underline{t}), \underline{t} \in [0, 1]^m\}$ another direction is to replace W with $\{(W_1(t), \ldots, W_d(t)), t \in [0, 1]\}$. Still another possibility is to formulate a "mixed" case combining theorems 3.1 and 3.2. These extensions are not pursued since they follow along the same lines as the results presented in this paper.

In a recent paper [13], H. Sugita introduced the notion of essential continuity of Wiener functionals and considered the characterization of symmetric kernels K for which $I(K)$ is essentially continuous. The setup of [13] is an abstract Wiener space (B, H, μ) where B is a real Banach space, H is a real and separable Hilbert space continuously and densely imbedded in B, μ is a Gaussian measure of B. A Wiener functional $F: B \to \mathbb{R}$ is defined in [13] to be essentially continuous if there exists a Banach space B_1 and a functional $F_1: B_1 \to \mathbb{R}$ such that: (i) (B_1, H, μ) is an abstract Wiener space (ii) $\mu(B_1) = 1$ (iii) $F_1(W) = F(W)$ a.s. μ and (iv) $F_1(\cdot)$ is continuous in the B_1 norm. The notion of essential continuity is intrinsic as it does not depend on the particular Banach space B, note however, that the space B_1 on which $F(W)$ (or $I(K)$) possesses a continuous extension may depend on F (or K). The results of the present paper are less intrinsic since we deal with the continuity on a particular B space (regardless of $F(\cdot)$ or K). Note that continuity in the sense of the present paper implies essential continuity.

2. Preliminaries

A. Let $(X_1, \mathscr{B}_1)$, $(X_2, \mathscr{B}_2)$, ..., $(X_N, \mathscr{B}_N)$ be measurable spaces. The notion of a multimeasure or pseudomeasure is an extension of the notion of a measure and is defined as follows.

Definition. A mapping μ: $\mathscr{B}_1 \times \cdots \times \mathscr{B}_N \to \mathbb{R}$ is said to be a multimeasure if for every i, $1 \leqq i \leqq N$ and fixed $B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_N$ with $B_j \in \mathscr{B}_j$, $\mu(B_1, \ldots, B_{i-1}, F, B_{i+1}, \ldots, B_N)$ is a signed measure in the variable $F \in \mathscr{B}_i$; namely, μ is the difference of two positive *finite* measures in F.

In this paper we will consider the case where $\mathscr{B}$ is the Borel σ-field on [0, 1] and $\mathscr{B}_i = \mathscr{B}$, $i = 1, 2, \ldots, N$. $C([0, 1])$ will denote the real valued continuous functions on [0, 1] and $C_o([0, 1])$ will denote the elements of $C([0, 1])$ which vanish at zero. Let $\{A_1^k, \ldots, A_M^k\}$ denote a measurable partition of X_k.

Definition. Let μ be a multimeasure on $\mathscr{B}_1 \times \cdots \times \mathscr{B}_N$. The Fréchet variation FV^N of μ is defined as

$$\|\mu\|_{FV^N} = \sup \sum_{i_1, \ldots, i_N = 1}^{M} \varepsilon_{i_1} \cdot \varepsilon_{i_2} \cdot \cdots \cdot \varepsilon_{i_N} \mu(A_{i_1}^1, \ldots, A_{i_N}^N) \tag{2.1}$$

where ε_i are -1 or 1, $1 \leqq i \leqq N$ and the supremum is over $\varepsilon \in \{-1, 1\}^N$, and over all finite partitions of X_k.

Since μ is a multimeasure, it follows that $\|\mu\|_{FV^N} < \infty$. The class of multimeasures normed by $\|\cdot\|_{FV^N}$ will be denoted by F^N becomes a Banach space under this norm. A theory of integration with respect to multimeasures follows naturally along lines similar to those of integration with respect to signed measures. Let $f \in L^\infty(X_1)$ then

$$\mu_{f_1, \ldots, f_k} = \int_{X_1 \times \cdots \times X_k} f_1(t_1) \cdot f_2(t_2) \cdot \cdots \cdot f_k(t_k)\, \mu(dt_1, \ldots, dt_k, \cdot) \in F^{N-k} \tag{2.2}$$

and

$$\|\mu_{f_1, \ldots, f_k}\|_{FV^{N-1}} \leqq \|f_1\|_\infty \cdots \|f_k\|_\infty \cdot \|\mu\|_{FV^N}\,. \tag{2.3}$$

Furthermore the integral is independent of the order of integration.

From now on we consider the case where $X_i = [0, 1]$ and $\mathscr{B}_i$ is the Borel sigma field on [0, 1]. Let V^N denote the following collection of measurable functions on $\mathbb{R}^N$:

$$V^N = \left\{ f(t_1, \ldots, t_N) = \sum_{k=1}^{\infty} a_k g_k^{(1)}(t_1) \cdot \cdots \cdot g_k^{(N)}(t_n) \colon \Sigma |a_k| < \infty \ \textit{and}\ \|g_k^{(j)}\|_\infty \leqq 1 \right\}. \tag{2.4}$$

V_c^N and $V_{c,o}^N$ will denote the subset of V^N generated by the functions $g_k^{(j)} \in C([0, 1])$ and $g_k^{(j)} \in C_o([0, 1])$ respectively. We norm V^N (and $V_{c,o}^N$) by setting $\|f\|_{V^N} = \inf\{\Sigma |a_k|\}$ where the infimum is over all representations of f as $f = \sum_{k=1}^{\infty} a_k g_k^{(1)} \cdots g_k^{(N)}$ with $\|g_k^{(j)}\|_\infty \leqq 1$, $g_k^{(j)} \in C([0, 1])$ (and for $V_{c,o}^N$,

$g_k^{(j)} \in C_o([0, 1])$). Then obviously, $\int f\,d\mu$ can be extended to $f \in V^N$, $\mu \in F^N$ and

$$|\int f\,d\mu| \leqq \|f\|_{V^N} \cdot \|\mu\|_{F^N} \leqq \|f\|_{V^N_{c,o}} \cdot \|\mu\|_{F^N};$$

furthermore let f be a function in V^k, then $\mu_f(A_1, \ldots, A_{N-k})$ belongs to F^{N-k} and

$$\|\mu_f\|_{FV^{N-k}} \leqq \|f\|_{V^k} \cdot \|\mu\|_{F^N}. \tag{2.5}$$

Let F_o^N denote the class of N-multimeasures $\mu \in F^N$ such that $\mu(A_1, \ldots, A_N) = 0$ whenever $A_i = \{0\}$ for some i, $1 \leqq i \leqq N$. Then we have the following extension of the Riesz representation theorem (Fréchet [4], cf. theorem 4.12 of [1]):

Theorem 2.1. *The dual to $V^N_{c,o}$ is F_o^N; namely, every bounded multilinear functional on $C_o([0, 1]) \times \cdots \times C_o([0, 1])$ can be represented as a multimeasure.*

B. Some results regarding multimeasures which will be needed later will be briefly presented now. We start with the following result on the approximation of integration of elements in V_c^N.

Proposition 2.2. *Let $f(t_1, \ldots, t_N) \in V_c^N$, let π_m be a sequence of partitions of $[0, 1]$ such that π_{m+1} is a refinement of π_m,* and $|\pi_m| = \sup_{t_j \in \pi_m} \{|t_{j+1} - t_j|\}$, *with $|\pi_m| \to 0$. Set*

$$f^{\pi_m}(t_1, \ldots, t_N) = f\,(t_1^{\pi_m}, \ldots, t_N^{\pi_m})$$

where t^{π_m} denotes the partition point nearest to t from below. Then, as $m \to \infty$

$$\int_{[0,1]^N} f^{\pi_m} d\mu \to \int_{[0,1]^N} f\,d\mu. \tag{2.6}$$

Proof. Since $f \in V_c^N$, f has an expansion

$$f(t_1, \ldots, t_N) = \sum_{i=1}^{\infty} \alpha_i h_1^{(i)}(t_1) \cdots h_N^{(i)}(t_N)$$

with $\|h_j^{(i)}\|_\infty \leqq 1$ and $\Sigma|\alpha_i| < \infty$, with all $h_j^{(i)}$ continuous on $[0, 1]$ and $|\int f d\mu| \leqq \|\mu\|_{FV^N} \cdot \Sigma|\alpha_i|$. Note that $f^{\pi_m} \in V^N$ but in general $f^{\pi_m} \notin V_c^N$. Now,

$$\begin{aligned} &f(t_1, \ldots, t_N) - f(t_1^\pi, \ldots, t_N^\pi) \\ &= \sum_{j=0}^{N-1} \{f(t_1, \ldots, t_{j-1}, t_j^\pi, \ldots, t_N^\pi) - f(t_1, \ldots, t_j, t_{j+1}^\pi, \ldots t_N^\pi)\} \\ &=: \sum_{j=0}^{N-1} f_{j,\pi}(t_1, \ldots, t_N), \end{aligned}$$

and
$$f_{j,\pi}(t_1, \ldots, t_N) = \sum_{i=1}^{\infty} \sum_{j=1}^{N} \alpha_i h_1^{(i)}(t_1) \cdots h_{j-1}^{(i)}(t_{j-1})[h_j^{(i)}(t_j^\pi) - h_j^{(i)}(t_j)] \cdot \cdot h_{j+1}^{(i)}(t_{j+1}^\pi) \cdots h_N^{(i)}(t_N^\pi).$$

Therefore $\|f_{j,\pi}\|_{V^N} \leqq 2N\Sigma|\alpha_i|$ and

$$|\int f_{j,\pi} d\mu| \leqq \left(\sum_{i=1}^{\infty} |\alpha_i| \cdot \sup_{t_j} |h_j^{(i)}(t_j^\pi) - h_j^{(i)}(t_j)|\right) \cdot \|\mu\|_{FV^N}.$$

Note that the right hand side converges to zero as $|\pi| \to 0$ by the continuity (hence uniform continuity) of $h^i_j(t)$ and by dominated convergence applied to the sum. This completes the proof.

By the following result $(t \wedge s) = \min(t, s) \in V^2_{c,o}$, the proof of lemma 2.3 is straightforward and therefore omitted.

Lemma 2.3. *Assume that*

$$K(t_1, \ldots, t_N) = (t_1 \wedge t_2) \cdots (t_{2k-1} \wedge t_{2k}) f(t_{2k+1}, \ldots, t_N) \tag{2.7}$$

where $f(t_{2k+1}, \ldots, t_N) \in V^{N-2k}_{c,o}$, then $K(t_1, \ldots, t_N) \in V^N_{c,o}$.

There are several possible definitions for the notion of "the trace of order k, $k \leqq [n/2]$" associated with n-kernels (cf. [9, 12, 13]). For the purpose of the present paper we define (cf. [9]):

Definition. The symmetric kernel $K(t_1, \ldots, t_n)$ in $L^2([0,1]^n)$ will be said to possess a multiple trace of order k, $k \leqq [n/2]$ if for all complete orthonormal sequences on $L^2([0,1])$, $(\alpha = 1, \ldots, k)$ $\{\phi^\alpha_i, i \geqq 1\}$ the infinite sum

$$\sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \int_{[0,1]^{2k}} K(t_1, \ldots, t_n) \phi^1_{i_1}(t_1) \phi^1_{i_1}(t_2) \cdots \phi^k_{i_k}(t_{2k-1}) \phi^k_{i_k}(t_{2k}) dt_1 \ldots dt_{2k} \tag{2.8}$$

converges in $L^2[0,1]^{n-2k}$ as $N \to \infty$ and moreover the limit is independent of the choice of the complete orthonormal sequence. The limit will be denoted $\text{trace}^k K$ and $\tilde{H}^n_{tr}$ will denote the set of symmetric square integrable kernels possessing traces for all $k \leqq [n/2]$.

Proposition 2.4. *Let $K(t_1, \ldots, t_N)$ be a symmetric L^2 kernel satisfying*

$$K(t_1, \ldots, t_N) = \mu((t_1, 1], \ldots, (t_N, 1])$$

for all $(t_1, \ldots, t_N) \in [0,1]^N$, then the traces $\text{trace}^k K$, $k \leqq [n/2]$, exist and satisfy

$$\text{trace}^k K = \int_{[0,1]^{2k}} (\theta_1 \wedge \theta_2) \cdots (\theta_{2k-1} \wedge \theta_{2k}) \mu(d\theta_1, \ldots, d\theta_{2k}, (t_1, 1], \ldots, (t_{N-2k}, 1]). \tag{2.9}$$

Proof. We have

$$\sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \left| \int_{[0,1]^{2k}} K(t_1, \ldots, t_n) \phi^1_{i_1}(t_1) \phi^1_{i_1}(t_2) \cdots \phi^k_{i_k}(t_{2k-1}) \phi^k_{i_k}(t_{2k}) dt_1 \cdots dt_{2k} \right|$$

$$= \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \left| \int_{[0,1]^{2k}} \left(\int_0^{t_1} \phi^1_{i_1}(s) ds \right) \left(\int_0^{t_2} \phi^1_{i_1}(s) ds \right) \cdots \left(\int_0^{t_{2k-1}} \phi^k_{i_k}(s) ds \right) \left(\int_0^{t_{2k}} \phi^k_{i_k}(s) ds \right) \cdot \right.$$

$$\left. \cdot \mu(dt_1, dt_2, \ldots, dt_{2k-1}, dt_{2k}, (t_{2k+1}, 1], \ldots, (t_n, 1]) \right|$$

$$\leqq \| \mu(\cdot, (t_{2k+1}, 1], \ldots, (t_n, 1]) \|_{FV^{n-2k}}$$

$$\times \left(\sum_{i-1}^{\infty} \sup_t \left| \int_0^t \phi^1_i(s) ds \right|^2 \right) \cdots \left(\sum_{i=1}^{\infty} \sup_t \left| \int_0^t \phi^k_i(s) ds \right|^2 \right) < \infty ,$$

because for every $t \in [0, 1]$ we have $\sum_{i=1}^{\infty} \left| \int_0^t \phi_i^\alpha(s) ds \right|^2 = t$ and the convergence of this series is uniform in t due to Dini's convergence theorem. Consequently, the sum in (2.8) can be computed as an iterated sum. Now, using Lemma 2.3 we deduce that the trace^k K exists and (2.9) holds.

Let $\pi = \pi_m$ be a finite partition of $[0, 1]$, $\pi = \{0 = t_o \leqq t_1 \leqq \cdots \leqq t_m = 1\}$. Consider the partition of $[0, 1]^2$ induced by $\pi \times \pi$; in particular, consider the elements $d_j(\pi)$, $0 \leqq j \leqq m - 1$ of the partition on the main diagonal of $[0, 1]^2$ where $d_j(\pi) = (t_j^\pi, t_{j+1}^\pi] \times (t_j^\pi, t_{j+1}^\pi]$ and let $|d_j|$ denote the Lebesgue area of d_j. Then we have

Proposition 2.5. *Let $N = 2$ and assume that $K(t_1, t_2)$ is induced by a bimeasure and that π_m is a sequence of refinements satisfying $|\pi_m| \to 0$ as $m \to \infty$ then*

$$\lim_{m \to \infty} \sum \frac{1}{|d_j(\pi_m)|^{1/2}} \int_{d_j(\pi_m)} K(t_1, t_2) dt_1 dt_2 = \operatorname{trace} K \,. \tag{2.10}$$

and K is right continuous.

Remark. For $N \geqq 3$ and $K(t_1, \ldots, t_N)$ induced by a symmetric multimeasure, the same results hold with respect to any two variables with the other $N - 2$ variables fixed.

Proof. Choose for the complete orthonormal sequence on $[0, 1]$ the modified Haar orthonormal sequence which was constructed in [11]. Then the limit (2.10) follows by the same arguments as in [11]. Turning to the proof of right continuity, this property follows from the following Grothendieck inequality which holds for bimeasures but not for multimeasures with $n \geqq 3$. This inequality assures that given a bimeasure $\mu(dt_1, dt_2)$, we can find a regular probability measure ν on $[0, 1]$ such that

$$\left| \int_{[0,1]^2} f_1(t_1) f_2(t_2) \mu(dt_1, dt_2) \right| \leqq G \|\mu\|_{FV} \cdot \|f_1\|_\infty \cdot \left(\int_{[0,1]} f_2^2(t) \nu(dt) \right)^{1/2}$$

for all $C([0, 1])$ functions f_1, f_2 and G is a universal constant ([2, 5]), therefore the inequality also holds for any functions f_1, f_2 which are measurable and bounded. Now assume that $s_n \downarrow s$ and $t_n \downarrow t$. Then

$$K(s_n, t_n) = \mu((s_n, 1], (t_n, 1]) = \int_{[0,1]^2} \mathbf{1}_{(s_n, 1]}(t_1) \mathbf{1}_{(t_n, 1]}(t_2) \mu(dt_1, dt_2) \,,$$

and the right continuity of K follows since

$$\begin{aligned} |K(s, t) - K(s_n, t_n)| &= \left| \int_{[0,1]^2} \{\mathbf{1}_{(s,1]}(t_1) \mathbf{1}_{(t,1]}(t_2) - \mathbf{1}_{(s_n,1]}(t_1) \mathbf{1}_{(t_n,1]}(t_2)\} \mu(dt_1, dt_2) \right| \\ &\leqq \left| \int_{[0,1]^2} \mathbf{1}_{(s, s_n]}(t_1) \mathbf{1}_{(t,1]}(t_2) \mu(dt_1, dt_2) \right| \\ &\quad + \left| \int_{[0,1]^2} \mathbf{1}_{(s_n,1]}(t_1) \mathbf{1}_{(t,t_n]}(t_2) \mu(dt_1, dt_2) \right| \\ &\leqq G \|\mu\|_{FV} \left(\sqrt{\nu((s, s_n])} + \sqrt{\nu((t, t_n])} \right) \to 0 \ as\ n \to \infty \,. \end{aligned}$$

C. The following combinatorial lemma will be needed in the next section:

Lemma 2.6. *For any $n \geq 1$ the product $x_1 \cdots x_n$ belongs to the subspace of multinomials of degree n in the variables $x_1, \ldots, x_n$ spanned by $(\alpha_1 x_1 + \cdots + \alpha_n x_n)^n$, $\alpha \in \mathbb{R}^n$.*

Proof. We proceed by induction on n. For $n = 1$ it is obvious. Suppose that the result holds for n, that means, $x_1 \cdots x_n = \sum_{i=1}^{K} \gamma_i(\alpha_1^i x_1 + \cdots + \alpha_n^i x_n)^n$. Then, to get a similar expression for $x_1 \cdots x_n x_{n+1}$ it suffices to write the multinomial $y^n x$ as a linear combination of multinomials of the form $(\alpha y + x)^{n+1}$, α real numbers. This is possible, because the equation $y^n x = \sum_{i=1}^{n+2} (\alpha_i y + x)^{n+1} \gamma_i$ is equivalent to

$$\sum_{i=1}^{n+2} \alpha_i^k \gamma_i = \begin{cases} \dfrac{1}{n+1} & \text{if } \; k = n \\ 0 & \text{if } \; k = 0, 1, 2, \ldots, n-1, n+1 \end{cases}$$

and if we choose arbitrary numbers $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n+2}$, the $(n+2)^2$-matrix (α_i^k), $1 \leq i \leq n+2$, $0 \leq k \leq n+1$, is invertible, completing the proof of the lemma.

3. The Characterization of Multiple Stochastic Integrals Possessing a Continuous Extension

The purpose of this section is to present a characterization of multiple stochastic integrals possessing continuous extensions. Let $F(W) = I_n(K)$ be a multiple stochastic integral of a symmetric kernel $K \in L^2([0, 1]^n)$. Let

$$G(W_1, \ldots, W_n) = \int_{[0,1]^n} Q(t_1, \ldots, t_n) W_1(dt_1) \cdots W_n(dt_n) \tag{3.1}$$

where Q is a (not necessarily symmetric) kernel on $L^2([0, 1]^n)$ and the W_i are standard independent Brownian motions. The first result in this section will characterize the kernels Q for which G has a version which is continuous and multilinear on $C_o([0, 1]) \times \cdots \times C_o([0, 1])$. The second result will characterize the symmetric kernels K for which F has a version which is continuous on $C_o([0, 1])$. Returning to the continuous extensions of F and G, as already mentioned earlier, if a continuous extension to F or G exists then it is unique, this is a direct consequence of the fact that W is dense in $C_o([0, 1])$ and $(W_1, \ldots, W_n)$ is dense in $C_o([0, 1]) \times \cdots \times C_o([0, 1])$. Assume that $n = 2$ and $K(t_1, t_2) = Q(t_1, t_2) \equiv 1$; then $F(W) = W^2(1) - 1$ and $G(W_1, W_2) = W_1(1) \cdot W_2(1)$ and consequently, in general, $F(W) \neq G(W, \ldots, W)$, cf. the concluding remark following proposition 4.2 regarding the relationship between these two functionals.

Theorem 3.1. *Let G be as defined above, then the following are equivalent. (a) G possesses an extension on $C_o([0, 1]) \times \cdots \times C_o([0, 1])$ which is continuous and multilinear.*

(*b*) *There exists a multimeasure* μ *on* $[0, 1]^n$ *satisfying* $\mu \in F_o^n$, $\mu(A_1, \ldots, A_n) = 0$ *whenever for some i,* $A_i = \{0\}$ *and*

$$Q(t_1, \ldots, t_n) = \mu((t_1, 1], \ldots, (t_n, 1])$$

almost everywhere with respect to the Lebesgue measure on $[0, 1]^n$.

Proof. Assume that G possesses a continuous version and denote by ϕ the version of the random variable G on $C_o([0, 1]) \times \cdots \times C_o([0, 1])$ given by the continuous extension. By the generalized Riesz-Frechet representation theorem (Theorem 2.1), there exists a multimeasure μ on $[0, 1]^n$ belonging to F_o^n such that

$$\phi(W_1, \ldots, W_n) = \int_{[0,1]^n} W_1(t_1) \cdots W_n(t_n)\mu(dt_1, \ldots, dt_n), \tag{3.2}$$

for all $W_1, \ldots, W_n$ in $C_o([0, 1])$. In order to check the equality (3.1) it suffices to show that

$$G = \int_{[0,1]^n} \mu((t_1, 1], \ldots, (t_n, 1]) W_1(dt_1) \cdots W_n(dt_n) \text{ a.s.} \tag{3.3}$$

Let π_n be a sequence of subdivisions of $[0, 1]$ such that $\lim_m |\pi_m| = 0$ and π_{m+1} is a refinement of π_m for all m. We denote by $W_i^{\pi_m}$ the piecewise linear approximation of the Brownian motion W_i, defined by

$$W_i^{\pi_m}(t) = \sum_{j=1}^{k_m} W_i(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}} (W_i(t_j) - W_i(t_{j-1}))\mathbf{1}_{(t_{j-1}, t_j]}(t), \tag{3.4}$$

where $\pi_m = \{0 = t_o < \cdots < t_{k_m} = 1\}$:
The independence of the Brownian motions W_i, $1 \leqq i \leqq n$ implies that the sequence

$$\int_{[0,1]^n} \mu((t_1, 1], \ldots, ((t_n, 1]) W_1^{\pi_m}(dt_1) \cdots W_n^{\pi_m}(dt_n)$$

converges in probability as m tends to infinity to the right hand side of (3.3). On the other hand, for every m we can apply the integration by parts formula on each coordinate, obtaining by iteration that (3.4) is equal to

$$\int_{[0,1]^n} W_1^{\pi_m}(t_1) \cdots W_n^{\pi_m}(t_n)\mu(dt_1, \ldots, dt_n). \tag{3.5}$$

Finally, using the properties of multimeasures we deduce that (3.5) converges to (3.2) and therefore to G as m tends to infinity. Conversely, assuming that (3.3) holds, (3.2) follows by the same arguments as before and consequently G possesses a multilinear continuous extension which completes the proof.

As an example, consider the case $n = 2$, $Q(t_1, t_2) = \mathbf{1}(t_1 < t_2)$ (cf. p. 27 of [14] and remark 4 in [13]), in this case $G(W_1, W_2) = \int_{[0,1]} W_1(t_2)dW_2(t_2)$. Since in this case Q is not induced by a bimeasure [3], it follows that $G(W_1, W_2)$ does not possess a continuous extension.

Theorem 3.2. *Let* $F = I_n(K)$ *be as defined at the beginning of this section, then the following are equivalent.* (*a*) *F possesses a continuous extension on* $C_o([0, 1])$.

(*b*) *There exists a symmetric multimeasure* μ *on* $[0, 1]^n$ *satisfying* $\mu \in F_o^n$, $\mu(A_1, \ldots, A_n)$ *whenever for some* i, $A_i = \{0\}$ *and* μ *induces* K *namely*

$$K(t_1, \ldots, t_n) = \mu((t_1, 1], \ldots, (t_n, 1]) . \tag{3.6}$$

Proof. (i) We prove first that (a) implies that for the given kernel K,

$$G = \int_{[0,1]^n} K(t_1, \ldots, t_n) W_1(dt_1) \cdots W_n(dt_n)$$

possesses a continuous, symmetric multilinear extension on $C_o([0, 1]) \times \cdots \times C_o([0, 1])$. Denote by f the continuous version of F given by condition (a). To illustrate the idea of the proof consider first the case $n = 2$, and define

$$\phi(W_1, W_2) = f\left(\frac{W_1 + W_2}{\sqrt{2}}\right) - f\left(\frac{W_1 - W_2}{\sqrt{2}}\right), \tag{3.7}$$

for W_1, $W_2 \in C_o([0, 1])$. This functional ϕ is a version of $G = \int_{[0,1]^2} K(t_1, t_2) W_1(dt_1) W_2(dt_2)$ because $\phi(W_1, W_2)$ is equal, almost surely, to

$$\int_{[0,1]^2} K(t_1, t_2)\left(\frac{W_1(dt_1) + W_2(dt_1)}{\sqrt{2}}\right)\left(\frac{W_1(dt_2) + W_2(dt_2)}{\sqrt{2}}\right) -$$
$$- \int_{[0,1]^2} K(t_1, t_2)\left(\frac{W_1(dt_1) - W_2(dt_1)}{\sqrt{2}}\right)\left(\frac{W_1(dt_2) - W_2(dt_2)}{\sqrt{2}}\right) .$$

This functional ϕ is continuous multilinear and symmetric. The multilinear property is proved as in the case $n = 1$ (see sect. 1.B). That means that the fact that the support of the Wiener measure is $C_o([0, 1])$ implies that

$$\phi(\alpha W_1^a + \sqrt{1 - \alpha^2}\, W_1^b, W_2) = \alpha\phi(W_1^a, W_2) + \sqrt{1 - \alpha^2}\,\phi(W_1^b, W_2) \tag{3.8}$$

for any $-1 \leq \alpha \leq 1$. Then, taking into account that ϕ is continuous we deduce from (3.8) that ϕ is linear in the first coordinate. The symmetry of ϕ follows from the property $f(W) = f(-W)$ for all $W \in C_o([0, 1])$, which holds again using the continuity of f and the support property of the Wiener measure. Hence (b) holds for $n = 2$.

The extension of this argument to an arbitrary n is not straightforward. We have to express the product $x_1 \cdots x_n$ as a linear combination of polynomials of the type $(\alpha_1 x_1 + \cdots + \alpha_n x_n)^n$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a vector of norm one. This can be done by means of Lemma 2.6. More precisely, by Lemma 2.6, we can write

$$x_1 \cdots x_n = \sum_{k=1}^{k_o} \lambda_k (\alpha_1^k x_1 + \cdots + \alpha_n^k x_n)^n , \tag{3.9}$$

where $\lambda_k \in \mathbb{R}$ and $|\alpha^k| = 1$ for all $k = 1, \ldots, k_o$.

For any $k = 1, \ldots, k_o$, $y^k = \alpha_1^k W_1 + \cdots + \alpha_n^k W_n$ is again a Brownian motion and we denote by $I_n^{y_k}(K)$ the multiple stochastic integral of the kernel K with respect to y^k. A version of this multiple stochastic integral is provided by $f(y^k)$.

Then we have

$$G = \int_{[0,1]^n} K(t_1, \ldots, t_n) W_1(dt_1) \cdots W_n(dt_n) = \sum_{k=1}^{k_o} \lambda_k I_n^{y_k}(K)$$

$$= \sum_{k=1}^{k_o} \lambda_k f(\alpha_1^k W_1 + \cdots + \alpha_n^k W_n)\,, \quad a.s.$$

Therefore, $\phi(W_1, \ldots, W_n) = \sum_{k=1}^{k_o} \lambda_k f(\alpha_1^k W_1 + \cdots + \alpha_n^k W_n)$ is a continuous version of G. The symmetry and the multilinear property of this function ϕ is proved as in the case $n = 2$.

(ii) By the result of part (i) and Theorem (3.1) the existence of a continuous version implies (3.6). Turning to the proof that (3.6) implies (a), note first that the kernel K given by (3.6) is measurable and bounded by the constant $\|\mu\|_{FV^n}$. Thus we can define the random variable $F = I_n(K)$. We want to show that F has a continuous version on $C_o([0, 1])$. We will prove this property by induction on n. We know that it is true for $n = 1$. Assume that it holds up to $n - 1$.

We claim that

$$\int_{[0,1]^n} W(t_1) \cdots W(t_n) \mu(dt_1, \ldots, dt_n) = \sum_{k=0}^{[n/2]} \alpha_{n,k} I_{n-2k}(\text{trace}^k K)\,, \tag{3.10}$$

where $\alpha_{n,k} = \frac{n!}{(n-2k)!k!2^k}$, and $\text{trace}^k K$ is given by expression (2.9). We recall that (see Lemma 2.3) the function $(t_1 \wedge t_2) \cdots (t_{2k-1} \wedge t_{2k})$ belongs to V_c^{2k} and by (2.5) $\int_{[0,1]^{2k}} (t_1 \wedge t_2) \cdots (t_{2k-1} \wedge t_{2k}) \mu(dt_1, \ldots, dt_{2k}, \cdot)$ defines a $(n - 2k)$-dimensional symmetric multimeasure belonging to F_o^{n-2k}. Therefore, by the induction hypothesis all terms in the right hand side of (3.10) with $k > 0$ possess continuous versions and, consequently, from (3.10) we deduce that $I_n(K)$ possesses a continuous version.

In order to show (3.10) consider the Wiener-Chaos expansion of the random variable $W_{t_1} \cdots W_{t_n}$:

$$W_{t_1} \cdots W_{t_n} = \sum_{k=0}^{[n/2]} \frac{\alpha_{n,k}}{n!} \sum_{\sigma \in \sigma_n} (t_{\sigma(1)} \wedge t_{\sigma(2)}) \cdots (t_{\sigma(2k-1)} \wedge t_{\sigma(2k)})$$

$$\cdot I_{n-2k}(\mathbf{1}_{[0, t_{\sigma(2k+1)}]} \otimes \cdots \otimes \mathbf{1}_{[0, t_{\sigma(n)}]})\,, \tag{3.11}$$

where σ_n denotes the collection of all permutations of the first n integers. The expression (3.11) can be proved by induction on n, using the product formula for multiple stochastic integrals. Notice that every term of the form

$$(t_{\sigma(1)} \wedge t_{\sigma(2)}) \cdots (t_{\sigma(2k-1)} \wedge t_{\sigma(2k)}) \cdot I_{n-2k}(\mathbf{1}_{[0, t_{\sigma(2k+1)}]} \otimes \cdots \otimes \mathbf{1}_{[0, t_{\sigma(n)}]})$$

belongs to $V_{c,o}^n$ almost surely. Indeed, assuming $\sigma = I_d$, in order to simplify the notation, this is due to Lemma 2.3 and the fact that the multiple stochastic integral $I_{n-2k}(\mathbf{1}_{[0, t_{2k+1}]} \otimes \cdots \otimes \mathbf{1}_{[0, t_n]})$ can be written as

$$I_{n-2k-1}(\mathbf{1}_{[0, t_{2k+2}]} \otimes \cdots \otimes \mathbf{1}_{[0, t_n]}) W(t_{2k+1}) + \sum_{j=2k+2}^{n} (t_{2k+1} \wedge t_j)$$

$$\cdot I_{n-2k-2}(\mathbf{1}_{[0, t_{2k+2}]} \otimes \cdots \otimes \hat{\mathbf{1}}_{[0, t_j]} \otimes \cdots \otimes \mathbf{1}_{[0, t_n]})\,,$$

so we can use an induction argument.

Finally, note that the following Fubini type relation

$$\int_{[0,1]^n} (t_1 \wedge t_2) \cdots (t_{2k-1} \wedge t_{2k}) I_{n-2k}(\mathbf{1}_{[0,t_{2k+1}]} \otimes \cdots \otimes \mathbf{1}_{[0,t_n]}) \mu(dt_1, \ldots, dt_n)$$

$$= I_{n-2k}\left(\int_{[0,1]^{2k}} (t_1 \wedge t_2) \cdots (t_{2k-1} \wedge t_{2k}) \mu(dt_1, \ldots, dt_{2k}), (s_{2k+1}, 1], \ldots, (s_n, 1] \right), \tag{3.12}$$

holds (since the function $(t_{2k+1}, \ldots, t_n) \mapsto I_{n-2k}(\mathbf{1}_{[0,t_{2k+1}]} \otimes \cdots \otimes \mathbf{1}_{[0,t_n]}$ can be approximated by step functions and in this way the order of the multiple Wiener integration and the integration with respect to the multimeasure μ may be interchanged). Consequently (3.10) follows from (3.11) by integrating both sides of the equality with respect to μ, proposition 2.4 and (3.12). This completes the proof of the theorem.

4. Multiple Ogawa and Skorohod Integrals

Let $K(t_1, \ldots, t_n)$ be a symmetric $L^2([0, 1]^n)$ kernel, following [10] we can define a *multiple Ogawa integral* as follows. Let $\{e_i, i \geqq 1\}$ be a complete orthonormal system on $L^2([0, 1])$. For any multi-index $\underline{r} = (r_1, \ldots, r_n)$ set

$$K_{\underline{r}} = \int_{[0,1]^n} K(t_1, \ldots, t_n) e_{r_1}(t_1) \ldots e_{r_n}(t_n) dt_1, \ldots, dt_n .$$

Definition. We will say that K is *Ogawa integrable* if the series

$$\sum_{|\underline{r}| \leqq N} K_{\underline{r}} \cdot \left(\int_0^1 e_{r_1}(s) dW_s \right) \cdots \left(\int_0^1 e_{r_n}(s) dW_s \right)$$

converges in L^2 to a limit and the limit does not depend on the particular complete orthonormal system. The limit will be called the multiple Ogawa integral and will be denoted $\overset{o}{\delta}{}^n K$.

It is easily verified that every kernel K in $\tilde{H}^n_{tr}$ is Ogawa integrable and furthermore, the following Hu-Meyer formula [8] holds

$$\overset{o}{\delta}{}^n K = \sum_{k=0}^{[n/2]} \alpha_{n,k} I_{n-2k}(\text{trace}^k K) , \tag{4.1}$$

where $\alpha_{n,k} = n!/(n-2k)!k!2^k$. Note that in the particular case of a kernel K associated with a multimeasure μ, $K \in \tilde{H}^n_{tr}$ by Proposition 2.4 and from (3.10) and (4.1) it follows that

$$\overset{o}{\delta}{}^n K = \int_{[0,1]^n} W(t_1) \cdots W(t_n) \mu(dt_1, \ldots, dt_n) . \tag{4.2}$$

Moreover, using the inverse formula to (4.1) (cf. [8]) we obtain

$$
\begin{aligned}
I_n(K) &= \sum_{k=0}^{[n/2]} (-1)^k \alpha_{n,k} \overset{O}{\delta^{n-2k}}(\text{trace}^k K) \\
&= \sum_{k=0}^{[n/2]} (-1)^k \alpha_{n,k} \int_{[0,1]^{n-2k}} W(t_1) \cdots W(t_{n-2k}) \mu^k(t_1, \ldots, t_{n-2k}), \qquad (4.3)
\end{aligned}
$$

where μ^k is the $(n-2k)$-dimensional multimeasure defined by

$$\mu^k(A_1, \ldots, A_{n-2k}) = \int_{[0,1]^{2k}} (\theta_1 \wedge \theta_2) \cdots (\theta_{2k-1} \wedge \theta_{2k}) \mu(d\theta_1, \ldots, d\theta_{2k}, A_1, \ldots, A_{n-2k})$$

The Hu-Meyer formulas ((4.1) and the first line of (4.3)) were introduced in [8] on a somewhat formal basis. The results presented here give a justification to these formulas. For other justifications cf. [9, 12, 13].

The multiple Stratonovich integral can be introduced by generalizing the one-dimensional case, as follows. Let $\pi = \{0 = t_1 \leqq \cdots \leqq t_m = 1\}$ be a finite partition of [0, 1] and consider the piecewise linear approximation of the Wiener process given by (3.4). Set $\Delta_i = (t_i, t_{i+1}]$, $W(\Delta_i) = W_{t_{i+1}} - W_{t_i}$ and $|\Delta_i| = t_{i+1} - t_i$. Define for any finite set of indexes $i_1, \ldots, i_n \in \{1, 2, \ldots, m\}$, and for a given symmetric kernel $K \in L^2([0, 1]^n)$,

$$K_{i_1 \cdots i_n} = \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_n}|} \int_{\Delta_{i_1} \times \cdots \times \Delta_{i_n}} K(t_1, \ldots, t_n) dt_1 \cdots dt_n .$$

Set

$$
\begin{aligned}
S_\pi(K) &= \sum_{i_1, \ldots, i_n = 1}^{m} K_{i_1 \cdots i_n} W(\Delta_{i_1}) \cdots W(\Delta_{i_n}) \\
&= \int_{[0,1]^n} K(t_1, \ldots, t_n) \dot{W}^\pi_{t_1} \cdots \dot{W}^\pi_{t_n} dt_1 \cdots dt_n ,
\end{aligned}
$$

where $\dot{W}^\pi_t$ denotes the derivative of W^π_t with respect to t, which exists except for a finite set of points.

Definition. We will say that K is *Stratonovich integrable* if the limit of the sums $S_\pi(K)$ exists in $L^2(\Omega)$ as $|\pi|$ tends to zero. In that case, this limit will be called the multiple Stratonovich integral of K, and will be denoted by $I_n \circ K$.

Notice that for $n = 1$ any square integrable kernel is Stratonovich integrable and its integral coincides with the Wiener integral. For $n > 1$, the following result (Sole-Utzet, [12]) provides a necessary and sufficient condition for the existence of the Stratonovich integral.

Proposition 4.1 (Sole-Utzet, [12]): *Let K be a symmetric square integrable kernel in* $[0, 1]^n$. *Then K is Stratonovich integrable provided that for any* $1 \leqq k \leqq \left[\frac{n}{2}\right]$ *the following limit exists in* $L^2([0, 1]^{n-2k})$:

$$
\begin{aligned}
\beta_k(K) = \lim_{|\pi| \downarrow 0} \sum_{i_1, \ldots, i_k = 1}^{n} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_k}|} \int_{\Delta^2_{i_1} \times \Delta^2_{i_2} \times \cdots \times \Delta^2_{i_k}} \\
K(t_1, \ldots, t_{2k}, \cdot) dt_1 \cdots dt_{2k} \qquad (4.4)
\end{aligned}
$$

In this case, the Hu-Meyer formula becomes:

$$I_n \circ K = \sum_{k=0}^{[n/2]} \alpha_{n,k} I_{n-2k}(\beta_k(K)) .$$

The relation between the Ogawa and Stratonovich multiple integrals is given by the following result.

Proposition 4.2. *If K belongs to $\tilde{H}^n_{tr}$ then, in addition to being Ogawa integrable, K is also Stratonovich integrable, $\beta_k(K) = \text{trace}^k K$ and $I_n \circ K = \overset{o}{\delta}{}^n K$ where $I_n \circ K$ denotes the multiple Stratonovich integral and $\overset{o}{\delta}{}^n K$ denotes the multiple Ogawa integral.*

Remark. Let $G(W_1, \ldots, W_n)$ and $F(W)$ be as in the statements of Theorems 3.1 and 3.2 respectively, then by (3.3), (3.10) and Proposition 2.2 it follows that $G(W, \ldots, W) = \overset{o}{\delta}{}^n K = I_n \circ K \neq F(W)$ (since obviously $F(W) = I_n(K)$).

Proof. By substituting in equation (2.8) for the complete orthonormal system on [0, 1] the modified Haar orthonormal sequence which was constructed in [11], we can show that the limits (4.4) exist for all $k \leqq \left[\frac{n}{2}\right]$, and $\beta_k(K) = \text{trace}^k K$. Then from Proposition 4.1 and equation (4.1) it follows that K is Stratonovich integrable and the multiple Stratonovich integral $I_n \circ K$ coincides with the Ogawa-type integral $\overset{o}{\delta}{}^n K$.

With the three multiple integrals, $I_n(K)$, $I_n \circ K$ and $\overset{o}{\delta}{}^n K$ considered till now it is, of course, natural to ask which, if any, is the multiple integral which was introduced by N. Wiener in [15]? The discussion of the multiple integral in [15] takes place between the middle of page 917 till the top of page 919. Changing a little the notation of [15], the starting point is a special step function $f(t_1, \ldots, t_n)$, $t_i \in [0, 1]$, "taking only a finite set of finite values, each over a set of values $t_1, \ldots, t_n$, which is a product set of measurable sets in each variable t_k", namely

$$f(t_1, \ldots, t_n) = \sum_{i=1}^{N} c_k \varphi_1^k(t_1) \cdots \varphi_n^k(t_n)$$

where the $\varphi_i^k(t_i)$ are indicator functions of measurable sets. From the lines between equations (77) and (78) it seems quite obvious that the definition of N. Wiener for the multiple integral in this case was

$$\sum_{i=1}^{N} c_k \int_0^1 \varphi_1^k(\theta) dW_\theta \cdot \cdots \cdot \int_0^1 \varphi_n^k(\theta) dW_\theta$$

Next, (equations (80), (81) of [15]), $f(t_1, \ldots, t_n)$ is assumed to be a measurable step-function satisfying

$$|f(t_1, \ldots, t_n)| \leqq |f_1(t_1) \cdot \cdots \cdot f_n(t_n)|, \int_0^1 f_i^2(t) dt \leqq A, \; i = 1, \ldots, n . \tag{4.5}$$

Finally (equations (82)–(87)) the multiple integral is extended to measurable kernels f satisfying (4.5) without being a step function. It follows therefore from [15] and also from Lecture 3 of [16] that the multiple integral drafted by N. Wiener is "in spirit" near to the $\overset{o}{\delta}{}^n$ integral.

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