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On the tournament equilibrium set

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Abstract. An example is provided showing that Schwartz's tournament equilibrium set is not identical to the minimal covering set of Dutta.

Introduction

In a recent paper, Schwartz [5] axiomatically characterises a new solution set for majority preference tournaments. This set, called the TEQ set by Schwartz, is based on a recursive definition. Schwartz comments that this feature makes it difficult to check the properties of the TEQ set. Thus, although Schwartz is able to show that the TEQ set is always contained in the Banks set (Banks [1]), and hence in the uncovered set (Fishburn [3], Miller [4]), the relationship between the TEQ set and the minimal covering set of Dutta [2] is not clear. Schwartz remarks that in all the cases that he had examined, these sets turned out to be identical. In this note, I construct a tournament in which the TEQ set is a strict subset of the minimal covering set. However, I do not know whether the TEQ set is always contained in the minimal covering set.

Notation and definition

Let P be a tournament[1] on a nonempty set A. Elements of A are denoted by x, y, z, etc.; finite nonempty subsets of A are represented by α, β, γ.

A choice function is any function C of finite, nonempty subsets of A such that $C(\alpha)$ is contained in α for all α. Solution sets are formulated as definitions of choice functions.

Let α be any subset of A. For any $x, y \in \alpha$, x *covers* y in α iff xPy and for all $z \in \alpha$, $yPz \rightarrow xPz$. The *uncoverd set* of α, is $U(\alpha) = \{x \in \alpha |$ no y in α covers x in $\alpha\}$. A

[1] A tournament is an asymmetric and connected binary relation.

subset β of α is a covering set of α iff *(i)* $U(\beta)=\beta$, and *(ii)* $x \in U(\beta \cup \{x\})$ for no $x \in \alpha-\beta$. The *minimal covering* set of α, denoted $MC(\alpha)$ is the *smallest* covering set. (Dutta [2] showed that a smallest covering set exists).

To define the TEQ set, some more notation is needed. For any α and any choice function C, define a new binary relation $D(C,\alpha)$ as follows. For all $x, y \in \alpha$, $xD(C,\alpha)y$ *iff* $x \in C(t \in \alpha | tPy\}$. A nonempty subset β of α is *C-retentive* in α iff nothing in $\alpha-\beta$ bears $D(C,\alpha)$ to anything in β. β is a minimum C-retentive set in α iff β is C-retentive in α and no proper subset of β is C-retentive in α. For any α, TEQ(α) = the union of minimum TEQ-retentive subsets of α. Although the TEQ set is defined in terms of TEQ-retentiveness, the definition is not circular. It is recursive, the recursion variable being the cardinality of α.

An example

In this section, I construct a specific tournament for which the TEQ set is a strict subset of the MC set. Let $A=\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$. The tournament relation is represented by a tournament matrix in which $a_{ij}=1$ if x_iPx_j, and $a_{ij}=-1$ if x_jPx_j. I adopt the convention that $a_{ii}=0$ for all i

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	0	−1	1	−1	−1	1	−1	−1
x_2	1	0	−1	1	−1	−1	1	−1
x_3	−1	1	0	1	1	−1	−1	−1
x_4	1	−1	−1	0	1	1	1	1
x_5	1	1	−1	−1	0	1	−1	1
x_6	−1	1	1	−1	−1	0	1	1
x_7	1	−1	1	−1	1	−1	0	1
x_8	1	1	1	−1	−1	−1	−1	0

The following table describes the d(TEQ, A) relation.

x_i	$\{t \in A \| tPx_i\}$	TEQ($\{t \in A \| tPx_i\}$)
x_1	x_2, x_4, x_5, x_7, x_8	x_2, x_4, x_5
x_2	x_3, x_5, x_6, x_8	x_3, x_5, x_6
x_3	x_1, x_6, x_7, x_8	x_1, x_6, x_7
x_4	x_2, x_3	x_3
x_5	x_3, x_4, x_7	x_3, x_4, x_7
x_6	x_1, x_4, x_5	x_4
x_7	x_2, x_4, x_6	x_2, x_4, x_6
x_8	x_4, x_5, x_6, x_7	x_4

Not that for no $x_i \in A-\{x_8\}$ is it the case that $x_8D(\text{TEQ}, A)x_i$. Hence, x_8 does not belong to the minimum TEQ-retentive set of A. Indeed, TEQ$(A)=A-\{x_8\}$. However, TEQ(A) is not a covering set of A since $x_8 \in U(A)$. It can be checked that the only covering set of A is A itself. Hence, MC$(A)=A$. So, TEQ(A) is contained in MC(A).

Remark. Note that the Banks set for A is also $A-\{x_8\}$. However, one can construct other tournaments in which the minimal covering set is a strict subset of the Banks set.

References

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