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Probability Theory and Related Fields

Small values of Gaussian processes and functional laws of the iterated logarithm*

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Summary. We estimate small ball probabilities for locally nondeterministic Gaussian processes with stationary increments, a class of processes that includes the fractional Brownian motions. These estimates are used to prove Chung type laws of the iterated logarithm.

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1 Introduction

In this paper we estimate small ball probabilities for locally nondeterministic Gaussian processes with (approximately) stationary increments and use the estimates to prove Chung type laws and to refine the Strassen law of the iterated logarithm for fractional Brownian motion. A main goal is to develop techniques and results for processes with dependent increments. In this section we give some background, and an overview of our results.

Let $\{X(t);\, t \geqq 0\}$ be a centered continuous Gaussian process, let $M(t) = \max_{0 \leqq s \leqq t} |X(s)|$, and write $\mathrm{LL}t = \log_e |\log_e t|$. If $\{X(t)\}$ is a standard Brownian motion, then Chung's law of the iterated logarithm [7] gives the local growth rate

$$\liminf_{t \downarrow 0} M(t)/\sqrt{t(\mathrm{LL}t)^{-1}} \stackrel{\text{a.s.}}{=} \pi/\sqrt{8}.$$

Chung's result relies on estimates of the probabilities

$$P(M(t) < \varepsilon)$$

for small values of ε (for a sharp bound, see [13, p. 1047]).

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Now, let

$$\eta_t(s) = X(st)/(2t\mathrm{LL}t)^{1/2}, \quad 0 \leqq s \leqq 1.$$

For the case when $\{X(t)\}$ is a Brownian motion, Strassen [23] proved that as $t \downarrow 0$ or $t \uparrow \infty$, $\{\eta_t\}$ is a.s. relatively compact in $C[0, 1]$, with cluster set equal to the unit ball in a Reproducing Kernel Hilbert Space (RKHS) connected with Brownian motion. The result was extended to fractional Brownian motion in Oodaira [19]. (Oodaira's proof contains a gap. On page 298 it is not enough to show that for each fixed j, $P(\limsup_r C_r^{(j)}) = 1$. Instead we could use the Cramer–Wold device [14, Theorem 3.1] to prove that the cluster set equals K. However, the results of Sect. 4 below contain much more.)

Csaki [8] established a functional law of the iterated logarithm for Brownian motion which at the same time gives a "rate of convergence" in the Strassen law and extends Chung's result to small $C[0, 1]$ balls centered around general functions in the unit ball of the RKHS. His results correspond to the case $\alpha = 1$ in Sect. 4 below (see also [1, 15]). Professors Kuelbs and Li kindly suggested to us that a combination of our techniques and those of the above mentioned papers would yield similar functional laws for fractional Brownian motion.

In Sect. 2 we obtain the bounds for small ball probabilities for strongly locally nondeterministic processes. We refer to [3, 4, 9, 18], and to Sect. 2 for information on local nondeterminism. An important special case is fractional Brownian motion (fBm). The process $\{X(t);\, t \geqq 0\}$ is a standard fBm if it is a centered continuous Gaussian process with covariance function

$$E\{X(s)X(t)\} = \tfrac{1}{2}\{s^\alpha + t^\alpha - |s-t|^\alpha\}, \tag{1.1}$$

where $0 < \alpha < 2$. We say that α is the index of the fBm. If $\alpha = 1$, then we have ordinary Brownian motion. By [17], a standard fBm with index α may be represented as

$$X(t) = k_\alpha^{-1} \int_{-\infty}^{0} \{(t-s)^{(\alpha-1)/2} - (-s)^{(\alpha-1)/2}\}\, dB(s)$$

$$+ k_\alpha^{-1} \int_0^t (t-s)^{(\alpha-1)/2}\, dB(s), \tag{1.2}$$

for $t > 0$, where $\{B(s){:}\ -\infty < s < \infty\}$ denotes a standard Brownian motion and

$$k_\alpha^2 = \int_{-\infty}^{0} \{(1-s)^{(\alpha-1)/2} - (-s)^{(\alpha-1)/2}\}^2\, ds + \int_0^1 (1-s)^{\alpha-1}\, ds.$$

It is easily seen that fBm is selfsimilar, i.e., that $\{X(s);\, s \geqq 0\}$ and $\{t^{-\alpha/2}X(st);\, s \geqq 0\}$ have the same distribution for any $t > 0$. As discussed in Sect. 2 below fBm is strongly locally nondeterministic.

For fBm our bound for small ball probabilities is that, for $0 < \varepsilon < t^{\alpha/2}$,

$$e^{-Ct\varepsilon^{-2/\alpha}} \leqq P\{M(t) < \varepsilon\} \leqq e^{-ct\varepsilon^{-2/\alpha}} \tag{1.3}$$

for some strictly positive constants c and C (Corollary 2.2).

In Sect. 3 we show that it follows that for fBm's (and for more general strongly locally nondeterministic Gaussian processes)

$$\liminf_{t\downarrow 0} M(t)/\{t^{\alpha/2}(\mathrm{LL}t)^{-\alpha/2}\} \overset{\text{a.s.}}{=} c_\alpha$$

for some positive constant c_α. For fBm's a corresponding result holds for $t\to\infty$,

$$\liminf_{t\to\infty} M(t)/\{t^{\alpha/2}(\mathrm{LL}t)^{-\alpha/2}\} \overset{\text{a.s.}}{=} c'_\alpha.$$

The extensions of the theory of Csaki [8] are considered in Sect. 4. Let $\|f\|_\infty$ denote the sup-norm on $C[0,1]$, and $H_\alpha \subseteq C[0,1]$ the RKHS of the kernel

$$\Gamma(s,t) = \tfrac{1}{2}\{s^\alpha + t^\alpha - |s-t|^\alpha\},\quad 0\leqq s\leqq 1,\quad 0\leqq t\leqq 1.$$

Put

$$K = \{f\in H_\alpha\colon \langle f,f\rangle_\alpha \leqq 1\},$$

where $\langle f,g\rangle_\alpha$ denotes the inner product in H_α. (A slightly more explicit characterization of K can be found in Theorem 4.1 (C) of [10].) According to [19], for η_t given by

$$\eta_t(s) = X(st)/(2t^\alpha \mathrm{LL}t)^{1/2},\quad 0\leqq s\leqq 1, \tag{1.4}$$

the set of functions $\{\eta_t\}$ is a.s. relatively compact, with cluster set K, as $t\downarrow 0$ or $t\uparrow\infty$.

We strengthen this result as follows. If $\langle f,f\rangle_\alpha \leqq 1$, then

$$\liminf (\mathrm{LL}t)^{(\alpha+1)/(\alpha+2)}\|\eta_t - f\|_\infty \overset{\text{a.s.}}{<} \infty.$$

Furthermore,

$$\liminf (\mathrm{LL}t)^{(\alpha+1)/2}\|\eta_t - f\|_\infty \overset{\text{a.s.}}{=} \gamma(f),$$

for some constant $0<\gamma(f)<\infty$, if and only if $\langle f,f\rangle_\alpha < 1$. Chung's LIL is the special case $f\equiv 0$. Finally, we establish that among the functions $f\in H_\alpha$ with $\langle f,f\rangle_\alpha = 1$, there is a dense set for which a.s.

$$0<\liminf (\mathrm{LL}t)^{(\alpha+1)/(\alpha+2)}\|\eta_t - f\|_\infty < \infty.$$

2 The probability that a Gaussian path is flat

In this section we establish bounds of the type (1.3) for the probability that a Gaussian path stays within a narrow strip. Let $\{X(t)\colon t\geqq 0\}$ be a centered and continuous Gaussian process with incremental variance $\sigma_t^2(h) =$

$V(X(t+h)-X(t))$. Let $0<\alpha<2$ and let δ, c_1, c_2 be strictly positive constants. The bounds will follow from the assumptions

$$\sigma_t^2(h) \leqq c_1 h^\alpha, \quad 0 \leqq h \leqq \delta, \quad 0 \leqq t \leqq \delta - h \tag{2.1}$$

and

$$V(X(t+h) \mid X(s): 0 \leqq s \leqq t) \geqq c_2 h^\alpha, \quad 0 \leqq h \leqq \delta, \quad 0 \leqq t \leqq \delta - h. \tag{2.2}$$

The main case when (2.1) is satisfied for suitable δ, c_1 is when $\sigma_t^2(h)=\sigma^2(h)$ does not depend on t, and

$$\sigma^2(h) \sim \text{const} \cdot h^\alpha, \quad h \to 0. \tag{2.3}$$

If (2.1) holds, (2.2) is the same as requiring that $\{X(t)\}$ is strongly locally nondeterministic, see [9].

From the representation (1.2) it easily follow that if $\{X(t)\}$ is a fBm of index α, then

$$V(X(t+h) \mid X(s): 0 \leqq s \leqq t) \geqq V\left(k_\alpha^{-1} \int_t^{t+h} (t+h-s)^{(\alpha-1)/2} \, dB(s)\right)$$
$$= k_\alpha^{-2} \alpha^{-1} h^\alpha,$$

and that (2.2) hence holds, with $c_2 = k_\alpha^{-2}\alpha^{-1}$. General conditions for (strong) local nondeterminism are given by Marcus [18] and Berman [5], who show that $\sigma_t^2(h)=\sigma^2(h)$, independent of t, with $\sigma^2(h) \to 0$, $h \to 0$ and $\sigma^2(h)$ concave in $[0, 2\delta]$ is sufficient, and by Berman [3] who requires (2.1) and that the increments of $\{X(t)\}$ are stationary with spectral measure whose absolutely continuous component has a density $f(\lambda)$ which satisfies

$$f(\lambda) > \text{const} \cdot |\lambda|^{-\alpha-1}$$

for large $|\lambda|$.

Let $\eta(x) = \int_{-x}^{x} (2\pi)^{-1/2} \exp(-y^2/2)\, dy$ denote the distribution function of the absolute value of a standard normal random variable. We will use the easily proved inequality

$$\log \eta(x) \geqq \begin{cases} \log Kx & 0 \leqq x \leqq 1, \\ -Ke^{-x^2/2} & 1 \leqq x. \end{cases} \tag{2.4}$$

Here and in the sequel K is a generic positive constant whose value may change from appearance to appearance. Further, write

$$M(t) = \max_{0 \leqq s \leqq t} |X(t) - X(0)|.$$

Theorem 2.1 *Let $\{X(t): t \geqq 0\}$ be a centered, real valued Gaussian process with continuous sample paths. If* (2.2) *holds then there is a constant $c>0$ such that the right-hand inequality in* (2.5) *below is satisfied for $t \leqq \delta$ and $0 < \varepsilon < t^{\alpha/2}$. If*

(2.1) holds then there is a constant $C > 0$ such that the left-hand inequality in (2.5) is satisfied for $t \leqq \delta$ and $\varepsilon \leqq t^{\alpha/2}$. Hence if both (2.1) and (2.2) hold then

$$e^{-Ct\varepsilon^{-2/\alpha}} \leqq P(M(t) \leqq \varepsilon) \leqq e^{-ct\varepsilon^{-2/\alpha}} \tag{2.5}$$

for $t \leqq \delta$ and $0 < \varepsilon < t^{\alpha/2}$.

Proof. We first prove the right-hand inequality. We may assume that $X(0) = 0$. Fix $t \leqq \delta$. By considering the process $\{c_2^{-1/2} t^{-\alpha/2} X(st): 0 \leqq s\}$, instead of the process $\{X(s): 0 \leqq s\}$ we may assume that $t = 1$, $\varepsilon \leqq c_2^{-1/2}$ and that

$$V(X(t+h) \mid X(s)\colon 0 \leqq s \leqq t) \geqq h^\alpha, \quad 0 \leqq h \leqq 1, \quad 0 \leqq t \leqq 1 - h. \tag{2.6}$$

Let n and k be integers with $1 \leqq k \leqq n$. The probability that the modulus of a Gaussian random variable is smaller than a constant increases if the mean is set to zero and the variance is decreased. Hence, since conditional distributions in Gaussian processes are Gaussian it follows from (2.6) that

$$P(|X(k/n)| \leqq \varepsilon \mid X(j/n) = x_j, 1 \leqq j \leqq k-1) \leqq \eta(\varepsilon n^{\alpha/2}),$$

with η as defined just before the theorem. Thus, by repeated conditioning,

$$\begin{aligned} P(M(1) \leqq \varepsilon) &\leqq P\Big(\max_{1 \leqq k \leqq n} |X(k/n)| \leqq \varepsilon\Big) \\ &\leqq \eta(\varepsilon n^{\alpha/2})^n. \end{aligned}$$

Choosing $n = [2c_2^{-1/\alpha}\varepsilon^{-2/\alpha}] \geqq 2$ we get that

$$\begin{aligned} P(M(1) \leqq \varepsilon) &\leqq \eta(2^{\alpha/2} c_2^{-1/2})^{[2c_2^{-1/\alpha}\varepsilon^{-2/\alpha}]} \\ &\leqq \exp(-K(2c_2^{-1/\alpha}\varepsilon^{-2/\alpha} - 1)) \\ &\leqq \exp(-Kc_2^{-1/\alpha}\varepsilon^{-2/\alpha}), \end{aligned}$$

where $K = -\log \eta(2^{\alpha/2} c_2^{-1/2}) > 0$. This proves the right-hand inequality in (2.5).

We next prove the left-hand inequality, by bounding the increments over a dyadic partition. Fix $t \leqq \delta$. This time considering the process $\{c_1^{-1/2} t^{-\alpha/2} X(st): s \geqq 0\}$ instead of $\{X(s): s \geqq 0\}$, we may assume that $t = 1$, $\varepsilon \leqq c_1^{-1/2}$ and that $\sigma_t^2(h) \leqq h^\alpha$, $0 \leqq h \leqq 1$. For $0 < \theta < 1$, put $c(\theta) = (1+\theta)/(1-\theta)$ so that

$$\sum_{n=1}^{\infty} \theta^{|n-n_0|} \leqq c(\theta) \tag{2.7}$$

for any integer n_0. Choose $\theta \in (2^{-\alpha/2}, 1)$ such that $c(\theta) \geqq c_1^{-1/2}$. Further, for $n = 1, 2, \ldots$ and $i = 1, \ldots, 2^n$ let $\Delta_{n,i} = X(i2^{-n}) - X((i-1)2^{-n})$ so that $\Delta_{n,i}$ is normal with zero mean and variance less than $2^{-n\alpha}$. Since any $t \in [0, 1]$ may be written as $t = \sum_{n=1}^{\infty} b_n(t) 2^{-n}$, where each $b_n(t)$ is zero or one,

$$X(t) = \sum_{n=1}^{\infty} b_n(t) \Delta_{n, i(n,t)}, \quad 0 \leqq t \leqq 1,$$

where $1 \leqq i(n, t) \leqq 2^n$. It follows that, for $0 \leqq t \leqq 1$,

$$M(t) \leqq \sum_{n=1}^{\infty} \max_{1 \leqq i \leqq 2^n} |\Delta_{n,i}|.$$

Hence by (2.7), and using [22, Corollary 3] in the second step,

$$\begin{aligned} P(M(1) \leqq c(\theta)\varepsilon) &\geqq P(|\Delta_{n,i}| \leqq \theta^{|n-n_0|}\varepsilon,\ 1 \leqq n,\ 1 \leqq i \leqq 2^n) \\ &\geqq \prod_{n=1}^{\infty} \prod_{i=1}^{2^n} P(|\Delta_{n,i}| \leqq \theta^{|n-n_0|}\varepsilon) \\ &\geqq \prod_{n=1}^{\infty} \eta(\theta^{|n-n_0|} 2^{n\alpha/2}\varepsilon)^{2^n}. \end{aligned} \tag{2.8}$$

Assume that $\varepsilon \leqq 1$ and put

$$n_0 = \left[\frac{2\log \varepsilon^{-1}}{\alpha \log 2}\right] + 1.$$

Then

$$1 \leqq \varepsilon 2^{n_0\alpha/2} \leqq 2^{\alpha/2} \quad \text{and} \quad 2^{n_0} \leqq 2\varepsilon^{-2/\alpha}. \tag{2.9}$$

By (2.8) and the first part of (2.9),

$$\log P(M(1) \leqq c(\theta)\varepsilon) \geqq \sum_{n=1}^{\infty} 2^n \log \eta(\theta^{|n-n_0|} 2^{(n-n_0)\alpha/2}). \tag{2.10}$$

Since $\theta 2^{-\alpha/2} \leqq 1$, it follows from the upper inequality in (2.4), and using the second part of (2.9) in the third step, that

$$\begin{aligned} \sum_{n=1}^{n_0-1} 2^n \log \eta((\theta 2^{-\alpha/2})^{n_0-n}) &\geqq \sum_{n=1}^{n_0-1} 2^n \{\log K + (n_0 - n)\log(\theta/2^{\alpha/2})\} \\ &\geqq 2^{n_0}\left\{\log K - \log(2^{\alpha/2}/\theta) \sum_{n=1}^{\infty} n2^{-n}\right\} \\ &\geqq \varepsilon^{-2/\alpha} 2\left\{\log K - \log(2^{\alpha/2}/\theta) \sum_{n=1}^{\infty} n2^{-n}\right\} \\ &= -\varepsilon^{-2/\alpha} K, \end{aligned} \tag{2.11}$$

with the last K-value positive.
Next, using in turn the second part of (2.4), that θ is chosen to make $\theta 2^{\alpha/2} > 1$, and the second inequality in (2.9),

$$\begin{aligned} \sum_{n=n_0}^{\infty} 2^n \log \eta((\theta 2^{\alpha/2})^{n-n_0}) &\geqq -2^{n_0} \sum_{n=0}^{\infty} 2^n K \exp\left\{-\frac{1}{2}(\theta^2 2^{\alpha})^n\right\} \\ &\geqq -\varepsilon^{-2/\alpha} 2K \sum_{n=0}^{\infty} 2^n \exp\left\{-\frac{1}{2}(\theta^2 2^{\alpha})^n\right\} \\ &= -\varepsilon^{-2/\alpha} K. \end{aligned} \tag{2.12}$$

Replacing ε by $\varepsilon/c(\theta) \leqq 1$ it follows from (2.10)–(2.12) that

$$\log P(M(1) \leqq \varepsilon) \geqq -\left(\frac{\varepsilon}{c(\theta)}\right)^{-2/\alpha} K,$$

which proves the left inequality of (2.5), with $C = c(\theta)^{2/\alpha}K$. This concludes the proof. □

It may be noted that the constant c in (2.5) is independent of δ and only depends on α, c_2 and that similarly C is independent of δ and only depends on α, c_1. Simple modifications of the proof show that the left inequality in (2.5) in fact holds for $\varepsilon \leqq \max\{1, t^{\alpha/2}\}$ and $t \leqq \delta$, and that it also holds for arbitrary values of t if the restrictions $0 \leqq h \leqq \delta$, $0 \leqq t \leqq \delta - h$ are removed. Since

$$P(M(t) \leqq \varepsilon) \leqq P(M(s) \leqq \varepsilon)$$

for $0 \leqq s \leqq t$ it follows from the right inequality in (2.5) that for, any t,

$$P(M(t) \leqq \varepsilon) \leqq \exp\{-c(t)\varepsilon^{-2/\alpha}\}$$

for $\varepsilon \leqq t^{\alpha/2}$, with $c(t)$ depending on t but not on ε. The selfsimilarity of fBm's makes it easy to remove the restrictions on t entirely from the result. We state this as a corollary.

Corollary 2.2 *Let $\{X(t): t \geqq 0\}$ be a standard fractional Brownian motion with exponent α. Then there are constants $0 < c \leqq C < \infty$, which are independent of ε and t, such that*

$$e^{-Ct\varepsilon^{-2/\alpha}} \leqq P(M(t) \leqq \varepsilon) \leqq e^{-ct\varepsilon^{-2/\alpha}} \quad \text{for } \varepsilon \leqq t^{\alpha/2}. \tag{2.13}$$

Proof. As discussed before the theorem, fBm's satisfies the hypotheses of Theorem 2.1, so that in particular (2.13) holds for some $t_0 > 0$. The general case then follows at once from the fact that the processes $\{X(s): s \geqq 0\}$ and $\left\{\left(\frac{t_0}{t}\right)^{-\alpha/2} X\left(s\frac{t_0}{t}\right): s \geqq 0\right\}$ have the same distributions. □

The restriction $\varepsilon \leqq t^{\alpha/2}$ cannot be removed entirely from the right inequality in (2.13), for any $c > 0$, since it is known that $1 - P(M(t) \leqq \varepsilon)$ decreases exponentially in $t\varepsilon^{-2/\alpha}$ as $t\varepsilon^{-2/\alpha} \to 0$ while $1 - \exp\{-ct\varepsilon^{-2/\alpha}\} \sim ct\varepsilon^{-2/\alpha}$. However, for our purposes this is a less interesting case. It is also possible to find examples of locally nondeterministic processes which are periodic, and for which the righthand inequality in (2.5) fail as $\varepsilon = \text{const} \cdot t^{\alpha/2} \to \infty$.

Remark. Professor Qi-Man Shao derived the inequalities in (2.13) independently almost simultaneously with the authors using essentially the same arguments.

3 Chung's law of the iterated logarithm

The bounds of the preceding section immediately lead to the easy half of Chung's law of the iterated logarithm for Gaussian processes satisfying (2.1) and (2.2). For the harder half, some form of approximate independence is needed. We restrict our attention to Gaussian processes with stationary increments. For such processes we obtain the necessary independence by splitting up their spectral representation.

Let $X = \{X(t)\colon -\infty < t < \infty\}$ be a real-valued, centered Gaussian process. We assume that $X(0) = 0$ and that X has stationary increments and continuous covariance function

$$R(s,t) = E\{X(s)X(t)\} = \int_{-\infty}^{\infty} (e^{is\lambda} - 1)(e^{-it\lambda} - 1)\Delta(d\lambda), \tag{3.1}$$

where the symmetric spectral measure Δ satisfies

$$\int_{-\infty}^{\infty} \frac{\lambda^2}{1+\lambda^2}\Delta(d\lambda) < \infty.$$

There exists a centered, complex-valued, Gaussian random measure $W(d\lambda)$ such that

$$X(t) = \int_{-\infty}^{\infty} (e^{it\lambda} - 1)W(d\lambda). \tag{3.2}$$

The measures W and Δ are related by the identity

$$E\{W(A)\overline{W(B)}\} = \Delta(A \cap B)$$

for all real Borel sets A and B. Furthermore,

$$W(-A) = \overline{W(A)}.$$

We shall need the following version of Fernique's lemma (see [12, Lemma 1.1, p. 138]).

Lemma 3.1 *Let $\{X(t)\colon t \geqq 0\}$ be a separable, centered, real-valued Gaussian process with incremental variance $\sigma_t^2(h) = V(X(t+h) - X(t))$. Assume that*

$$\sigma_t(h) \leqq \varphi(h), \quad t > 0, \quad h > 0,$$

for some continuous nondecreasing function φ with $\varphi(0) = 0$. Put $M(t) = \sup_{0 \leqq s \leqq t}|X(s) - X(0)|$. For any positive integer $k > 1$ and any positive constants t, x and $\theta(p)$, $p = 1, 2, 3, \ldots$,

$$P\Big(M(t) > x\varphi(t) + \sum_{p=1}^{\infty} \theta(p)\varphi(tk^{-2^p})\Big) \leqq k^2 e^{-x^2/2} + \sum_{p=1}^{\infty} k^{2^{p+1}} e^{-\theta(p)^2/2}.$$

Using this inequality we are able to prove Chung type laws of the iterated logarithm for a large class of Gaussian processes.

Theorem 3.2 *Let $\{X(t)\colon -\infty < t < \infty\}$ be a real-valued, centered Gaussian process with continuous sample paths. Assume that $X(0) = 0$ and that X has* *stationary increments. Also assume that* (2.1) *and* (2.2) *hold and that for some* $l > 0$,

$$\liminf_{|\lambda| \to \infty} |\lambda|^3 \Delta([\lambda, \lambda + l]) > 0. \tag{3.3}$$

Then there exists a positive constant c_X *such that*

$$\liminf_{t \downarrow 0} \frac{M(t)}{t^{\alpha/2}(\mathrm{LL}t)^{-\alpha/2}} \overset{\text{a.s.}}{=} c_X. \tag{3.4}$$

A standard fractional Brownian motion with covariance function (1.1) is easily seen to satisfy the assumptions of Theorem 3.2. We have the following result.

Theorem 3.3 *Let* $\{X(t)\colon t \geqq 0\}$ *be a standard fractional Brownian motion with exponent* α. *There exist positive constants* c_α *and* c'_α *such that*

$$\liminf_{t \downarrow 0} \frac{M(t)}{t^{\alpha/2}(\mathrm{LL}t)^{-\alpha/2}} \overset{\text{a.s.}}{=} c_\alpha \tag{3.5}$$

and

$$\liminf_{t \to \infty} \frac{M(t)}{t^{\alpha/2}(\mathrm{LL}t)^{-\alpha/2}} \overset{\text{a.s.}}{=} c'_\alpha. \tag{3.6}$$

Remark. For standard Brownian motion (the case $\alpha = 1$) we have $c_1 = c'_1 = \pi/\sqrt{8}$. We have not been able to compute c_α or c'_α for $\alpha \neq 1$.

Proof of Theorem 3.2 Throughout, it is sufficient to consider t-values which make the iterated logarithm positive. We first show that

$$\liminf_{t \downarrow 0} M(t)/\psi(t) \overset{\text{a.s.}}{\geqq} c^{\alpha/2} > 0, \tag{3.7}$$

for $\psi(t) = t^{\alpha/2}(\log\log t^{-1})^{-\alpha/2}$ and c given by (2.5). Let $\varepsilon > 0$ and $\gamma > 1$, and for $k = 1, 2, \ldots$ put $t_k = \gamma^{-k}$, $\beta = (c/(1+\varepsilon))^{\alpha/2}$. Then, by (2.5),

$$\sum^{\infty} P(M(t_k)/\psi(t_k) \leqq \beta) \leqq \sum^{\infty} (\log \gamma^k)^{-(1+\varepsilon)} < \infty,$$

where the sums are over all k large enough to make $k \log \gamma > 1$ and $\beta(\log\log\gamma^k)^{-\alpha/2} < 1$. Hence, by the Borel–Cantelli lemma, $M(t_k) \geqq \beta\psi(t_k)$ for all k greater than some $k_0 = k_0(\omega)$. Further, for $k \geqq k_0$ and $t_{k+1} \leqq t < t_k$,

$$M(t) \geqq M(t_{k+1}) \geqq \beta\psi(t_{k+1}) \geqq \beta\psi(t)\psi(t_{k+1})/\psi(t_k).$$

Hence

$$\liminf_{t \downarrow 0} M(t)/\psi(t) \overset{\text{a.s.}}{\geqq} \beta\gamma^{-\alpha/2}. \tag{3.8}$$

Since ε and γ may be chosen arbitrarily close to 0 and 1, respectively, this proves (3.7).

Next, we prove that

$$\liminf_{t \downarrow 0} M(t)/\psi(t) \overset{\text{a.s.}}{\leqq} C^{\alpha/2} < \infty \tag{3.9}$$

for C given by (2.5). This time we choose

$$\beta = C^{\alpha/2}, \quad t_k = k^{-k}, \quad d_k = k^{k+(2-\alpha)/2}.$$

It follows from (2.5) that

$$\sum^{\infty} P(M(t_k)/\psi(t_k) \leqq \beta) \geqq \sum (k \log k)^{-1} = \infty, \tag{3.10}$$

where the sums are over all $k \geqq 2$ large enough to make $\beta(\mathrm{LL}t_k)^{-\alpha/2} < 1$. If the events in the first sum were independent, this would conclude the proof. However, they are not.

We shall use the spectral representation (3.2) to get the necessary independence. It follows from (3.1) that

$$\sigma_t^2(h) = 2 \int_{-\infty}^{\infty} (1 - \cos(h\lambda))\Delta(d\lambda).$$

Under assumption (2.1) there exists a constant $K > 0$ such that for all $t > 1$,

$$\int_{|\lambda| \geqq t} \Delta(d\lambda) \leqq Kt^{-\alpha}$$

and

$$\int_{|\lambda| \leqq t} \lambda^2 \Delta(d\lambda) \leqq Kt^{2-\alpha}.$$

(See the truncation inequalities on p. 209 of [16]). Define for $k = 1, 2, \ldots$ and $-\infty < t < \infty$,

$$X_k(t) = \int_{|\lambda| \in (d_{k-1}, d_k]} (e^{it\lambda} - 1) W(d\lambda), \tag{3.11}$$

$$\tilde{X}_k(t) = X(t) - X_k(t). \tag{3.12}$$

By standard Borel–Cantelli arguments, (3.9) follows if we prove that

$$\sum^{\infty} P\left(\max_{0 \leqq t \leqq t_k} |X_k(t)|/\psi(t_k) \leqq \beta \right) = \infty \tag{3.13}$$

and

$$\sum^{\infty} P\left(\max_{0 \leqq t \leqq t_k} |\tilde{X}_k(t)|/\psi(t_k) > \varepsilon \right) < \infty, \quad \text{for any } \varepsilon > 0, \tag{3.14}$$

since the events in (3.13) are independent.

Here (3.13) follows from (3.10) since

$$P\left(\max_{0 \leqq t \leqq t_k} |X_k(t)|/\psi(t_k) \leqq \beta\right) \geqq P(M(t_k)/\psi(t_k) \leqq \beta),$$

according to [2, Corollary 4].
For $0 \leqq h \leqq t_k$,

$$V(\tilde{X}_k(h)) = 2 \int_{|\lambda| \notin (d_{k-1}, d_k]} (1 - \cos(h\lambda))\Delta(d\lambda)$$

$$\leqq t_k^2 \int_{|\lambda| \leqq d_{k-1}} \lambda^2 \Delta(d\lambda) + 4 \int_{|\lambda| \geqq d_k} \Delta(d\lambda)$$

$$\leqq K k^{-\alpha k - \alpha(2-\alpha)/2}.$$

Put $\delta = \alpha(2 - \alpha)/2$. For a suitable constant K (that does not depend on k)

$$\varphi_k(h)^2 = K \min\{h^\alpha, k^{-\alpha k - \delta}\} \geqq V(\tilde{X}_k(h)) \tag{3.15}$$

for $0 \leqq h \leqq t_k$.
We shall now apply Fernique's lemma to the process $\tilde{X}_k$. Put $x_k = (8 \log k)^{1/2}$.
Given $\varepsilon > 0$ define

$$\theta_k(p) = \varepsilon(p+1)^{-2}\psi(t_k)/\varphi_k(t_k k^{-2^p})$$

for $p = 1, 2, \ldots$ For large enough k,

$$\theta_k(p) > 4(\log k)^{1/2} 2^{p/2} \quad \text{for all } p \geqq 1,$$

in addition to

$$x_k \varphi_k(t_k) + \sum_{p=1}^{\infty} \theta_k(p) \varphi_k(t_k k^{-2^p}) < \varepsilon\psi(t_k).$$

Since

$$\sum_{k=1}^{\infty} k^2 e^{-x_k^2/2} + \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} k^{2^{p+1}} e^{-8(\log k) 2^p} < \infty,$$

it follows from Fernique's lemma that

$$\sum_{k=1}^{\infty} P\left(\sup_{0 \leqq s \leqq t_k} |\tilde{X}_k(s)| > \varepsilon\psi(t_k)\right) < \infty.$$

We have thus established that

$$c^{\alpha/2} \leqq \liminf_{t \downarrow 0} M(t)/\psi(t) \leqq C^{\alpha/2} \quad \text{a.s.}$$

A zero–one law [21, Theorem 2.1] guarantees that the liminf is constant (it is here we use assumption (3.3)). □

Proof of Theorem 3.3 Let $X = \{X(t): t \geqq 0\}$ be a standard fBm of index α. The covariance function has the representation

$$R(s, t) = \frac{1}{2}\{|s|^\alpha + |t|^\alpha - |s - t|^\alpha\} \tag{3.16}$$

$$= c(\alpha) \int_{-\infty}^{\infty} (e^{is\lambda} - 1)(e^{-it\lambda} - 1)|\lambda|^{-(\alpha+1)} d\lambda. \tag{3.17}$$

The hypotheses of Theorem 3.2 are thus satisfied and X obeys a Chung law at time $t = 0$.

The proof of the Chung law at time $t = \infty$ is identical to the proof of the law at time $t = 0$. We change the definition of ψ to $\psi(t) = t^{\alpha/2}(\log\log t)^{-\alpha/2}$, invert the expressions defining t_k, and change $k + 1$ to $k - 1$ in all expressions. To establish that the tail σ-algebra at time $t = \infty$ is trivial we use the fact that $\{X(t): t > 0\}$ and $\{t^\alpha X(t^{-1}): t > 0\}$ are equivalent processes. The zero–one law at time $t = \infty$ therefore follows from Pitt and Tran's zero–one law at time $t = 0$. □

Example 3.4 Let $Y = \{Y(t): -\infty < t < \infty\}$ be a real-valued stationary Gaussian process with mean zero and covariance function

$$R(s, t) = e^{-|s-t|^\alpha},$$

where $0 < \alpha < 2$. The spectral measure $\Delta(d\lambda)$ has a density $\Delta(\lambda)$ which satisfies

$$\lim_{|\lambda| \to \infty} |\lambda|^{\alpha+1} \Delta(\lambda) = c(\alpha)$$

for some positive constant $c(\alpha)$. The hypotheses of Theorem 3.2 are therefore satisfied for the process $X(t) = Y(t) - Y(0)$.

Example 3.5 The spectral measure $\Delta(d\lambda)$ need not be absolutely continuous. Consider, for example, the real-valued stationary Gaussian process $\{Y(t): -\infty < t < \infty\}$ defined by

$$Y(t) = \frac{\sqrt{8}}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} (\xi_n \cos((2n-1)t) + \eta_n \sin((2n-1)t)),$$

where (ξ_n) and (η_n) are i.i.d. standard normal.

$$R(s, t) = 1 - \frac{2}{\pi}|s - t| \quad \text{for} \quad -\pi < s - t < \pi.$$

It follows from Marcus' result that Y is strongly locally nondeterministic with $\alpha = 1$. The process $X(t) = Y(t) - Y(0)$ therefore satisfies the assumptions of Theorem 3.2.

4 Functional laws of the iterated logarithm

We shall need some well known facts about reproducing kernel Hilbert spaces. (For reference, see [6, 14]). Consider a separable Banach space E with dual E^* and a centered Gaussian measure μ on E. Let π denote the canonical map of E^* into $L^2(E, \mu)$, i.e., the restriction to E^* of the canonical map of the space $\mathscr{L}^2(E, \mu)$ of μ-square-integrable functions into the quotient space $L^2(E, \mu)$. We shall let E_μ^* denote the closure in $L^2(\mu)$ of $\pi(E^*)$. For any $\eta \in E_\mu^*$, the measure $\eta(x)\mu(dx)$ has a barycenter

$$\Delta(\eta) = \int_E x\eta(x)\mu(dx) \in E, \tag{4.1}$$

where the integral may be interpreted either in the Pettis or the Bochner sense. The map $\eta \to \Delta(\eta)$ is linear and injective. The reproducing kernel Hilbert space H_μ of μ is the range $\Delta(E_\mu^*) \subset E$ with the inner product

$$\langle \Delta\eta, \Delta\xi \rangle_\mu = \int_E \eta(x)\xi(x)\mu(dx), \quad \eta, \xi \in E_\mu^*. \tag{4.2}$$

If we put $\hat{\Delta} = \Delta \circ \pi$, then $\hat{\Delta}(E^*)$ is dense in H_μ. We shall write $\|f\|_\mu^2 = \langle f, f \rangle_\mu$ for $f \in H_\mu$.

The following inequalities are well known (see [6] or [1]).

Proposition 4.1 *Let V be a convex, symmetric, measurable subset of E. For all $f \in H_\mu$ and $\xi \in E^*$,*

$$\mu(f + V) \leqq \mu(V)\exp\left\{ -\frac{1}{2}\|f\|_\mu^2 + \frac{1}{2}\|f - \hat{\Delta}(\xi)\|_\mu^2 + \sup_{x \in V} \xi(x) \right\}.$$

Furthermore,

$$\mu(f + V) \geqq \mu(V)\exp\left\{ -\frac{1}{2}\|f\|_\mu^2 \right\}.$$

Combining these two inequalities we can establish the following uniform version of a result proved by Borell [6, Theorem 2.3].

Proposition 4.2 *Let V be a convex, symmetric, bounded, measurable subset of E of positive μ-measure. If $f \in H_\mu$, then*

$$\lim_{t \to \infty} t^{-2}\{\log \mu(tf + V) - \log \mu(V)\} = -\tfrac{1}{2}\|f\|_\mu^2. \tag{4.3}$$

Furthermore, the convergence is uniform over all such sets V of diameter less than 1, *and the limit is a lower bound for all t.*

Proof. By Proposition 4.1 we have for every $t > 0$ and $\xi \in E^*$,

$$t^{-2}\{\log \mu(tf + V) - \log \mu(V)\} \geqq -\tfrac{1}{2}\|f\|_\mu^2$$

and

$$t^{-2}\{\log \mu(tf+V)-\log \mu(V)\} \leqq -\tfrac{1}{2}\|f\|_{\mu}^{2}+\tfrac{1}{2}\|f-\hat{\Delta}(\xi)\|_{\mu}^{2}+t^{-1}\sup_{x\in V}\xi(x).$$

The result now follows from the fact that $\|f-\hat{\Delta}(\xi)\|_{\mu}^{2}$ can be made arbitrarily small. □

Next let $\{X(s): s \geqq 0\}$ denote standard fractional Brownian motion of index α. As in the introduction let

$$\eta_t(s) = X(st)/(2t^{\alpha}\mathrm{LL}t)^{1/2}, \quad 0 \leqq s \leqq 1. \tag{4.4}$$

and let $H_\alpha \subseteq C[0,1]$ be the RKHS of the kernel

$$\Gamma(s,t) = \tfrac{1}{2}\{s^{\alpha}+t^{\alpha}-|s-t|^{\alpha}\}, \quad 0 \leqq s \leqq 1, \quad 0 \leqq t \leqq 1.$$

H_α is the RKHS corresponding to the centered Gaussian measure μ on the Banach space $C[0,1]$ induced by $\{X(s): 0 \leqq s \leqq 1\}$. As before let $\langle f, g\rangle_\alpha$ be the inner product in H_α and let $\|f\|_\infty$ be the sup-norm on $C[0,1]$. If $f \in H_\alpha$, then $|f(s)-f(t)|^2 \leqq |s-t|^{\alpha}\langle f,f\rangle_\alpha$.

We first take care of the case $\langle f,f\rangle_\alpha < 1$.

Theorem 4.3 *Let $\langle f,f\rangle_\alpha < 1$. As $t \downarrow 0$*

$$\liminf\, (\mathrm{LL}t)^{(\alpha+1)/2}\|\eta_t - f\|_\infty \overset{\text{a.s.}}{=} \gamma(f), \tag{4.5}$$

where $\gamma(f)$ is a constant satisfying

$$2^{-1/2}c^{\alpha/2}(1-\langle f,f\rangle_\alpha)^{-\alpha/2} \leqq \gamma(f) \leqq 2^{-1/2}C^{\alpha/2}(1-\langle f,f\rangle_\alpha)^{-\alpha/2}. \tag{4.6}$$

Here c and C denote the positive constants in (2.13). *The same result, possibly with a different constant $\gamma(f)$, holds for $t \to \infty$.*

The case $\langle f,f\rangle_\alpha = 1$ is more delicate. Let C^* denote the dual of the Banach space $C[0,1]$. Combining the analysis of Kuelbs et al. [15] of i.i.d. Banach space valued Gaussian vectors with our technique of treating fractional Brownian motion as if the process had independent increments we get the following result.

Theorem 4.4 (I) *Let $\langle f,f\rangle_\alpha = 1$. As either $t \downarrow 0$ or $t \uparrow \infty$,*

$$\liminf (\mathrm{LL}t)^{(\alpha+1)/2}\|\eta_t - f\|_\infty \overset{\text{a.s.}}{=} \infty, \tag{4.7}$$

whereas

$$\liminf (\mathrm{LL}t)^{(\alpha+1)/(\alpha+2)}\|\eta_t - f\|_\infty \overset{\text{a.s.}}{<} \infty. \tag{4.8}$$

(II) *If $\langle f,f\rangle_\alpha = 1$ and $f \in \hat{\Delta}(C^*)$, then*

$$\liminf (\mathrm{LL}t)^{(\alpha+1)/(\alpha+2)}\|\eta_t - f\|_\infty \overset{\text{a.s.}}{>} 0. \tag{4.9}$$

(III) *If* $\langle f,f\rangle_\alpha = 1$ *and* $f \notin \hat{\Lambda}(C^*)$, *then*

$$\liminf (\mathrm{LL}t)^{(\alpha+1)/(\alpha+2)} \|\eta_t - f\|_\infty \overset{\text{a.s.}}{=} 0. \tag{4.10}$$

In the proofs of both Theorem 4.3 and 4.4 we shall need the following two lemmas which we adapt from [1].

Lemma 4.5 *Let* $\varphi: \mathbb{N} \to \mathbb{R}^+$ *be such that*

(i) φ *is decreasing and* $\varphi(k) \to 0$,
(ii) $k\varphi(k)$ *is eventually strictly increasing,*
(iii) $(\log k)^{-1}(\log \varphi(k)) \to 0$,
(iv) $(\log k)^{\alpha+1}(\varphi(k))^\alpha \to 0$.

Put $t_k = \exp(-k\varphi(k))$. *Then*

(1) *for all* $a > 1$, $\sum_k \exp\{-a\mathrm{LL}t_k\} < \infty$
(2) $t_{k+1}/t_k \to 1$,
(3) $(\mathrm{LL}t_k)^{\alpha+1}(t_k - t_{k+1})^\alpha / t_k^\alpha \to 0$.

Lemma 4.6 *For* $0 < s < t < u < e^{-1}$ *and* $f \in H_\alpha$,

$$(\mathrm{LL}t)^{(\alpha+1)/2}\|\eta_t - f\|_\infty \geqq \left(\frac{s\mathrm{LL}u}{u\mathrm{LL}s}\right)^{\alpha/2}(\mathrm{LL}s)^{(\alpha+1)/2}\|\eta_s - f\|_\infty$$
$$-(\mathrm{LL}u)^{(\alpha+1)/2}\left(\frac{u-s}{u}\right)^{\alpha/2}\{\|f\|_\infty + \langle f,f\rangle_\alpha^{1/2}\}.$$

The proofs are very similar to the proofs of deAcosta's Lemmas 5.2 and 5.3, respectively.

Proof of Theorem 4.3 We shall only consider case $t \downarrow 0$. Let $\varepsilon > 0$. Put

$$\beta = (c/(1+\varepsilon))^{\alpha/2}(1 - \langle f,f\rangle_\alpha)^{-\alpha/2}, \tag{4.11}$$

where c is given by (2.13). Let $t_k = \exp(-k\varphi(k))$, with φ as in Lemma 4.5. Consider the events

$$A_k = \{\|t_k^{-\alpha/2}X((\cdot)t_k) - (2\mathrm{LL}t_k)^{1/2}f\|_\infty \leqq \beta(\mathrm{LL}t_k)^{-\alpha/2}\}.$$

By Proposition 4.2 and Corollary 2.2 we have for any $\delta > 0$ and $k > k_0(\delta)$,

$$\begin{aligned}
&(\mathrm{LL}t_k)^{-1}\log P(A_k)\\
&\leqq (\mathrm{LL}t_k)^{-1}\log P(M(1) \leqq \beta(\mathrm{LL}t_k)^{-\alpha/2}) - \langle f,f\rangle_\alpha + \delta\\
&\leqq -(1+\varepsilon)(1 - \langle f,f\rangle_\alpha) - \langle f,f\rangle_\alpha + \delta\\
&= -[1 + \varepsilon(1 - \langle f,f\rangle_\alpha) - \delta].
\end{aligned}$$

For such k,

$$P(A_k) \leqq \exp\{-[1 + \varepsilon(1 - \langle f,f\rangle_\alpha) - \delta]\mathrm{LL}t_k\}. \tag{4.12}$$

Since we can choose $\delta < \varepsilon(1 - \langle f, f \rangle_\alpha)$, it follows from Lemma 4.5(1) that $\sum P(A_k) < \infty$. Hence

$$\liminf_{k \to \infty} (\mathrm{LL}t_k)^{(\alpha+1)/2} \| \eta_{t_k} - f \|_\infty \overset{\text{a.s.}}{\geqq} 2^{-1/2} \beta. \tag{4.13}$$

It now follows from Lemmas 4.5 and 4.6 that

$$\liminf_{t \downarrow 0} (\mathrm{LL}t)^{(\alpha+1)/2} \| \eta_t - f \|_\infty \overset{\text{a.s.}}{\geqq} 2^{-1/2} \beta. \tag{4.14}$$

(In Lemma 4.6, let $u = t_k$ and $s = t_{k+1}$ as $k \to \infty$.) This proves half of Theorem 4.3. Next choose

$$\beta = C^{\alpha/2} (1 - \langle f, f \rangle_\alpha)^{-\alpha/2}, \tag{4.15}$$

where C is given by (2.13). As in the proof of Theorem 3.2, put

$$t_k = k^{-k}, \qquad d_k = k^{k + (2-\alpha)/2} \tag{4.16}$$

for $k = 2, 3, \ldots$ Let the processes X_k and $\tilde{X}_k$ be defined by (3.11) and (3.12), respectively. For $\varepsilon > 0$ define the events

$$A_k(\varepsilon) = \{ \| t_k^{-\alpha/2} X((\cdot) t_k) - (2\mathrm{LL}t_k)^{1/2} f \|_\infty \leqq \beta(1+\varepsilon)(\mathrm{LL}t_k)^{-\alpha/2} \},$$

$$B_k(\varepsilon) = \{ \| t_k^{-\alpha/2} X_k((\cdot) t_k) - (2\mathrm{LL}t_k)^{1/2} f \|_\infty \leqq \beta(1+\varepsilon)(\mathrm{LL}t_k)^{-\alpha/2} \},$$

$$C_k(\varepsilon) = \{ \| t_k^{-\alpha/2} \tilde{X}_k((\cdot) t_k) \|_\infty \geqq \varepsilon\beta(\mathrm{LL}t_k)^{-\alpha/2} \}.$$

Clearly,

$$A_k(\varepsilon) \subset B_k(2\varepsilon) \cup C_k(\varepsilon) \subset A_k(3\varepsilon) \cup C_k(\varepsilon). \tag{4.17}$$

In Sect. 3 we proved that

$$\sum_{k=1}^{\infty} P(C_k(\varepsilon)) < \infty. \tag{4.18}$$

By Proposition 4.2 and Corollary 2.2 we have for large k,

$$(\mathrm{LL}t_k)^{-1} \log P(A_k(\varepsilon)) \geqq -(1+\varepsilon)^{-2/\alpha}(1 - \langle f, f \rangle_\alpha) - \langle f, f \rangle_\alpha. \tag{4.19}$$

Since the right-hand side is greater than -1, it follows that

$$\sum_{k=1}^{\infty} P(A_k(\varepsilon)) = \infty. \tag{4.20}$$

Combining (4.17), (4.18), and (4.20) we get

$$\sum_{k=1}^{\infty} P(B_k(2\varepsilon)) = \infty. \tag{4.21}$$

Since the events in (4.21) are independent, it follows from Borel–Cantelli that

$$P\left(\limsup_k B_k(2\varepsilon) \right) = 1. \tag{4.22}$$

Combining (4.22), (4.17), and (4.18) we get

$$P\left(\limsup_{k} A_k(3\varepsilon)\right) = 1. \tag{4.23}$$

In other words,

$$\liminf_{k\to\infty} (\mathrm{LL}t_k)^{(\alpha+1)/2} \| \eta_{t_k} - f \|_\infty \overset{\text{a.s.}}{\leqq} 2^{-1/2}\beta(1+3\varepsilon). \tag{4.24}$$

Since ε can be chosen arbitrarily close to 0, Theorem 4.3 now follows from Pitt and Tran's zero–one law. □

Proof of Theorem 4.4 We shall again only consider the case $t \downarrow 0$. We prove (4.7) in the same way we proved the first half of Theorem 4.3. Let $t_k = \exp(-k\varphi(k))$, with φ as in Lemma 4.5. For any large constant $K > 0$, consider the events

$$A_k = \{ \| t_k^{-\alpha/2} X((\cdot)t_k) - (2\mathrm{LL}t_k)^{1/2} f \|_\infty \leqq K(\mathrm{LL}t_k)^{-\alpha/2} \}.$$

For any $\delta > 0$ and $k > k_0(\delta)$,

$$(\mathrm{LL}t_k)^{-1} \log P(A_k) \leqq -cK^{-2/\alpha} - \langle f,f \rangle_\alpha + \delta. \tag{4.25}$$

Since $\langle f,f \rangle_\alpha = 1$ and we can choose $\delta < cK^{-2/\alpha}$, it follows from Lemma 4.5(1) that $\sum P(A_k) < \infty$. Using Lemmas 4.5 and 4.6 we can conclude that

$$\liminf_{t\downarrow 0} (\mathrm{LL}t)^{(\alpha+1)/2} \| \eta_t - f \|_\infty \overset{\text{a.s.}}{\geqq} 2^{-1/2} K. \tag{4.26}$$

This proves (4.7). In our proof of (4.8), we follow [15]. For $k = 2, 3, \ldots$, let $t_k = k^{-k}$. Put

$$\beta_k = K(\mathrm{LL}t_k)^{-(\alpha+1)/(\alpha+2)}, \qquad f_k = (1 - \beta_k \| f \|_\infty^{-1}) f \tag{4.27}$$

for a suitable, large constant K. It follows from Proposition 4.1 that

$$\begin{aligned} P(\|\eta_{t_k} - f\|_\infty \leqq 2\beta_k) &\geqq P(\|\eta_{t_k} - f_k\|_\infty \leqq \beta_k) \\ &= P(\| t_k^{-\alpha/2} X((\cdot)t_k) - (2\mathrm{LL}t_k)^{1/2} f_k \|_\infty \leqq \beta_k (2\mathrm{LL}t_k)^{1/2}) \\ &\geqq \exp\{ -\langle f_k, f_k \rangle_\alpha \mathrm{LL}t_k \} P(M(1) \leqq \beta_k (2\mathrm{LL}t_k)^{1/2}). \end{aligned}$$

Since $\langle f,f \rangle_\alpha = 1$, it follows from Corollary 2.2 that for large k,

$$\begin{aligned} &\log P(\|\eta_{t_k} - f\|_\infty \leqq 2\beta_k) \\ &\quad \geqq -(1 - \beta_k \| f \|_\infty^{-1})^2 \mathrm{LL}t_k - C\beta_k^{-2/\alpha} (2\mathrm{LL}t_k)^{-1/\alpha} \\ &\quad = -\mathrm{LL}t_k + (2K \| f \|_\infty^{-1} - CK^{-2/\alpha} 2^{-1/\alpha})(\mathrm{LL}t_k)^{1/(\alpha+2)} \\ &\qquad - K^2 \| f \|_\infty^{-2} (\mathrm{LL}t_k)^{-\alpha/(\alpha+2)}. \end{aligned}$$

If K is chosen large enough, then we have for $k > k_0(K)$,

$$P(\|\eta_{t_k} - f\|_\infty \leqq 2\beta_k) \geqq \exp(-\mathrm{LL}t_k) = (k \log k)^{-1}.$$

With this choice of K,

$$\sum P(\|\eta_{t_k} - f\|_\infty \leqq 2\beta_k) = \infty.$$

Arguing as in the second half of the proof of Theorem 4.3 we get

$$\liminf_{k\to\infty} (\mathrm{LL}t_k)^{(\alpha+1)/(\alpha+2)} \|\eta_{t_k} - f\|_\infty \overset{\text{a.s.}}{\leqq} 2K. \tag{4.28}$$

This proves (4.8).

Next we prove (4.9). Assume that $\langle f, f\rangle_\alpha = 1$ and that $f = \hat{\Delta}(\xi)$ for some $\xi \in C^*$. Again we follow [15]. We shall use the notation

$$\|\xi\| = \sup\{\xi(g) \colon g \in C[0,1], \|g\|_\infty \leqq 1\}.$$

Let $t_k = \exp(-k(\log k)^{-(\alpha+2)/\alpha})$ for $k \geqq 2$, and put

$$\beta_k = \varepsilon(\mathrm{LL}t_k)^{-(\alpha+1)/(\alpha+2)} \tag{4.29}$$

for a suitable, small constant $\varepsilon > 0$. By Proposition 4.1, we have for large enough k,

$$\begin{aligned} P(\|\eta_{t_k} - f\|_\infty \leqq \beta_k) &\leqq P(M(1) \leqq (2\mathrm{LL}t_k)^{1/2}\beta_k)\exp\{-\mathrm{LL}t_k + (2\mathrm{LL}t_k)\beta_k\|\xi\|\} \\ &\leqq \exp\{-\mathrm{LL}t_k - (c\varepsilon^{-2/\alpha}2^{-1/\alpha} - 2\varepsilon\|\xi\|)(\mathrm{LL}t_k)^{1/(\alpha+2)}\}. \end{aligned}$$

If ε is chosen small enough, then we have for $k > k_0(\varepsilon)$,

$$P(\|\eta_{t_k} - f\|_\infty \leqq \beta_k) \leqq \exp\left\{-\log k + \frac{\alpha+2}{\alpha}\mathrm{LL}k - K_\varepsilon(\log k)^{1/(\alpha+2)}\right\},$$

for some constant $K_\varepsilon > 0$. With this choice of ε,

$$\sum P(\|\eta_{t_k} - f\|_\infty \leqq \beta_k) < \infty. \tag{4.30}$$

Hence

$$\liminf_{k\to\infty} (\mathrm{LL}t_k)^{(\alpha+1)/(\alpha+2)} \|\eta_{t_k} - f\|_\infty \overset{\text{a.s.}}{\geqq} \varepsilon. \tag{4.31}$$

Since $\varphi(k) = (\log k)^{-(\alpha+2)/\alpha}$ satisfies the assumptions of Lemma 4.5, it follows from Lemmas 4.5 and 4.6 that

$$\liminf_{t\downarrow 0} (\mathrm{LL}t)^{(\alpha+1)/(\alpha+2)} \|\eta_t - f\|_\infty \overset{\text{a.s.}}{\geqq} \varepsilon. \tag{4.32}$$

This completes the proof of (4.9).

We shall finally prove (4.10). Let $t_k = k^{-k}$. Given $\varepsilon > 0$ define β_k by (4.29). Consider the I-functional defined for $f \in H_\alpha$ and $\delta \geqq 0$ by

$$I(f, \delta) = \inf\{\langle g, g\rangle_\alpha \colon g \in H_\alpha, \|f - g\|_\infty \leqq \delta\}. \tag{4.33}$$

Arguing as in Lemma 1 of [11] we see that there is a unique element $f_k \in H_\alpha$ such that

$$\|f-f_k\|_\infty = \beta_k, \qquad I(f, \beta_k) = \langle f_k, f_k \rangle_\alpha. \tag{4.34}$$

It follows from Proposition 4.1 that, for large k,

$$\begin{aligned} P(\|\eta_{t_k} - f\|_\infty \leqq 2\beta_k) &\geqq P(\|\eta_{t_k} - f_k\|_\infty \leqq \beta_k) \\ &\geqq \exp\{-\langle f_k, f_k\rangle \mathrm{LL}t_k\} P(M(1) \leqq \beta_k (2\mathrm{LL}t_k)^{1/2}) \\ &\geqq \exp\{-I(f, \beta_k)\mathrm{LL}t_k - C\beta_k^{-2/\alpha}(2\mathrm{LL}t_k)^{-1/\alpha}\}. \end{aligned}$$

According to Proposition 2 of [15], if $\langle f, f\rangle_\alpha = 1$ and $f \notin \hat{\Delta}(C^*)$, then $\lim_{\delta \downarrow 0} (1 - I(f, \delta))/\delta = \infty$. We conclude that for any large constant K and all $k > k_0(K)$,

$$\begin{aligned} P(\|\eta_{t_k} - f\|_\infty \leqq 2\beta_k) &\geqq \exp\{-(1 - K\beta_k)\mathrm{LL}t_k - C\beta_k^{-2/\alpha}(2\mathrm{LL}t_k)^{-1/\alpha}\} \\ &= \exp\{-\mathrm{LL}t_k + (K\varepsilon - C\varepsilon^{-2/\alpha}2^{-1/\alpha})(\mathrm{LL}t_k)^{1/(\alpha+2)}\}. \end{aligned}$$

Since we can choose $K > C\varepsilon^{-(\alpha+2)/\alpha}2^{-1/\alpha}$, it follows that

$$\sum P(\|\eta_{t_k} - f\|_\infty \leqq 2\beta_k) = \infty. \tag{4.35}$$

Arguing as in the second half of the proof of Theorem 4.3 we get

$$\liminf_{k\to\infty} (\mathrm{LL}t_k)^{(\alpha+1)/(\alpha+2)} \|\eta_{t_k} - f\|_\infty \overset{\text{a.s.}}{\leqq} 2\varepsilon. \tag{4.36}$$

Since ε can be chosen arbitrarily small, this completes the proof of (4.10). □

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References

1. deAcosta, A.: Small deviations in the functional central limit theorem with applications to functional laws of the iterated logarithm. Ann. Probab. **11**, 78–101 (1983)
2. Anderson, T.W.: The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Am. Math. Soc. **6**, 170–176 (1955)
3. Berman S.M.: Gaussian sample functions: uniform dimension and Hölder conditions nowhere. Nagoya Math. J. **46**, 63–86 (1972)
4. Berman S.M.: Local nondeterminism and local times for Gaussian processes. Indiana Univ. J. Math. **23**, 69–94 (1973)
5. Berman S.M.: Gaussian processes with biconvex covariances. J. Multivar. Anal. **8**, 30–44 (1978)
6. Borell, C.: A note on Gauss measures which agree on small balls. Ann. Inst. Henri Poincaré, Sect. B, XIII-3, 231–238 (1977)
7. Chung, K.L.: On the maximum partial sums of sequences of independent random variables. Trans. Am. Math. Soc. **64**, 205–233 (1948)
8. Csáki, E.: A relation between Chung's and Strassen's laws of the iterated logarithm. Z. Wahrscheinlichkeitstheor. Verw. Geb. **54**, 287–301 (1980)

9. Cuzick, J., Du Preez, J.: Joint continuity of Gaussian local times. Ann. Probab. **10**, 810–817 (1982)
10. Goodman, V., Kuelbs, J.: Rates of clustering for some Gaussian self-similar processes. Probab. Theory Relat. Fields **88**, 47–75 (1991)
11. Grill, K.: A lim inf result in Strassen's law of the iterated logarithm. Probab. Theory Relat. Fields **89**, 149–157 (1991)
12. Jain, N.C., Marcus, M.B.: Continuity of subgaussian processes. Adv. Probab. **4**, 81–196 (1978)
13. Jain, N.C., Pruitt, W.E.: The other law of the iterated logarithm. Ann. Probab. **3**, 1046–1049 (1975)
14. Kuelbs, J.: The law of the iterated logarithm and related strong convergence theorems for Banach space valued random variables. (Lect. Notes Math., 539, pp. 224–314) Berlin Heidelberg New York: Springer (1976)
15. Kuelbs, J., Li, W.V., Talagrand, M.: Lim inf results for Gaussian samples and Chung's functional LIL. (to appear 1992)
16. Loeve, M.: Probability Theory I. Berlin Heidelberg New York: Springer 1977
17. Mandelbrot, B.B., van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Review **10**, 422–437 (1968)
18. Marcus, M.B.: Gaussian processes with stationary increments possessing discontinuous sample paths. Pac. J. Math. **26**, 149–157 (1968)
19. Oodaira, H.: On Strassen's version of the law of the iterated logarithm for Gaussian processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **21**, 289–299 (1972)
20. Pitt, L.D.: Local times for Gaussian vector fields. Indiana Univ. J. Math. **27**, 309–330 (1978)
21. Pitt, L.D., Tran, L.T.: Local sample path properties of Gaussian fields. Ann. Probab. **7**, 477–493 (1979)
22. Šidák, Z.: On multivariate normal probabilities of rectangles: their dependence on correlations. Ann. Math. Stat. **39**, 1425–1434 (1968)
23. Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeitstheor. Verw. Geb. **3**, 211–226 (1964)